A Complete Characterization of Nash-Solvability of Bimatrix Games in Terms of the Exclusion of Certain 2×2 Subgames*

Endre Boros[1], Khaled Elbassioni[2], Vladimir Gurvich[1], Kazuhisa Makino[3], and Vladimir Oudalov[1]

[1] RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003
{boros,gurvich,voudalov}@rutcor.rutgers.edu
[2] Max-Planck-Institut für Informatik, Stuhlsatzenhausweg 85, 66123 Saarbrücken, Germany
elbassio@mpi-sb.mpg.de
[3] Graduate School of Information Science and Technology, University of Tokyo, Tokyo, 113-8656, Japan
makino@mist.i.u-tokyo.ac.jp

Abstract. In 1964 Shapley observed that a matrix has a saddle point whenever every 2×2 submatrix of it has one. In contrast, a bimatrix game may have no Nash equilibrium (NE) even when every 2×2 subgame of it has one. Nevertheless, Shapley's claim can be generalized for bimatrix games in many ways as follows. We partition all 2×2 bimatrix games into fifteen classes $C = \{c_1, \dots, c_{15}\}$ depending on the preference pre-orders of the two players. A subset $t \subseteq C$ is called a NE-theorem if a bimatrix game has a NE whenever it contains no subgame from t. We suggest a general method for getting all minimal (that is, strongest) NE-theorems based on the procedure of joint generation of transversal hypergraphs given by a special oracle. By this method we obtain all (six) minimal NE-theorems.

1 Introduction, Main Concepts and Results

1.1 Bimatrix Games and Nash Equilibria

Let X_1 and X_2 be finite sets of strategies of players 1 and 2. Pairs of strategies $x = (x_1, x_2) \in X_1 \times X_2 = X$ are called *situations.* A *bimatrix game* $U = (U_1, U_2)$ is a pair of real-valued matrices $U_i : X \to \mathbb{R},\ i = 1, 2$, with common set of entries X. Value $U_i(x)$ is interpreted as utility function (also called profit or payoff) of player $i \in \{1, 2\}$ in the situation x. A situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is called a Nash equilibrium (NE) if

$$U_1(x_1', x_2) \le U_1(x_1, x_2)\ \forall x_1' \in X_1 \text{ and } U_2(x_1, x_2') \le U_2(x_1, x_2)\ \forall x_2' \in X_2;$$

* The second author gratefully acknowledges the partial support of DIMACS, the NSF Center for Discrete Mathematics and Theoretical Computer Science.

E.A. Hirsch et al. (Eds.): CSR 2008, LNCS 5010, pp. 99–109, 2008.

in other words, if no player can make a profit by choosing a new strategy if the opponent keeps the old one. A bimatrix game U is called a *zero sum or matrix* game if $U_1(x) + U_2(x) = 0$ for every $x \in X$. In this case the game is well-defined by one of two matrices, say, by U_1, and a NE is called a *saddle point* (SP).

1.2 Locally Minimal SP-Free Matrix and NE-Free Bimatrix Games

Standardly, we define a subgame as the restriction of U to a subset $X' = X'_1 \times X'_2 \subseteq X_1 \times X_2 = X$, where $X'_1 \subseteq X_1$ and $X'_2 \subseteq X_2$. In 1964 Shapley [8] noticed that a matrix has a saddle point whenever each of its 2×2 submatrices has one. Obviously, in this case, every submatrix has a SP, too. In other words, all minimal SP-free matrices are of size 2×2. Moreover, all locally minimal SP-free matrices are of size 2×2, too; in other words, every SP-free matrix of larger size has a row or column whose elimination still results in an SP-free submatrix; see [1]. Other generalizations of Shapley's theorem can be found, for example, in [6,7]. Let us also notice that a 2×2 matrix has no SP if and only if one of its diagonals is strictly larger than the other.

The "naive generalization" of Shapley's claim to bimatrix games fails: a 3×3 game might have no NE even if each its 2×2 subgame has one; moreover, for each $n \geq 3$ a $n \times n$ bimatrix game might have no NE even if every its subgame has one; see Example 1 in [6] or [1] and also examples given below. However, all locally minimal NE-free games admit the following explicit characterization [1].

For the sake of brevity, let us denote situation (x_1^i, x_2^j) by $x_{i,j}$, where $X_1 = \{x_1^1, x_1^2, \ldots\}$ and $X_2 = \{x_2^1, x_2^2, \ldots\}$.

Given an integer $n \geq 2$ and a bimatrix game U with $|X_1| \geq n$ and $|X_2| \geq n$, let us say that U has the *canonical strong improvement n-cycle C_n^0* if each situation $x_{1,1}, x_{2,2}, \ldots, x_{n-1,n-1}, x_{n,n}$ (respectively, $x_{1,2}, x_{2,3}, \ldots, x_{n-1,n}, x_{n,1}$) is a unique largest in its row with respect to U_2 (in its column with respect to U_1) and is the second largest, not necessarily, unique, in its column with respect to U_1 (in its row with respect to U_2). Any other strong improvement n-cycle C_n is obtained from the canonical one C_0 by arbitrary permutations of the rows of X_1 and columns of X_2.

It is easy to see that if an $n \times n$ bimatrix game U has a strong improvement cycle then U has no NE, yet, every proper subgame obtained from U by elimination of either one row or one column has a NE. In other words, U is a *locally minimal* NE-free bimatrix game. Moreover, the inverse holds, too.

Theorem 1. *([1]). A bimatrix game U is a locally minimal NE-free game if and only if U is of size $n \times n$ for some $n \geq 2$ and it contains a strong improvement n-cycle.* □

Thus, locally minimal NE-free games can be arbitrary large. Several examples are given in Figures 2 - 6, where each game has the canonical strong improvement cycle. Although it seems difficult to characterize or recognize the minimal NE-free games (see [1]), yet, the above characterization of the locally-minimal ones will be sufficient for us.

1.3 Pre-orders

Given a set Y and a mapping $P : Y^2 \to \{<, >, =\}$ that assigns one of these three symbols to every ordered pair $y, y' \in Y$, we say that y is *less or worse* than y' if $y < y'$, respectively, y is *more or better* than y' if $y > y'$, and finally, y and y' are *equivalent or they make a tie* if $y = y'$. Furthermore, P is called a *pre-order* if the following standard properties (axioms) hold for all $y, y', y'' \in Y$:

symmetry: $y < y' \Leftrightarrow y' > y, \quad y = y' \Leftrightarrow y' = y,$ and $y = y;$
transitivity: $y < y' \;\;\&\;\; y' < y'' \Rightarrow y < y'', \quad y < y' \;\;\&\;\; y' = y'' \Rightarrow y < y'',$
$y = y' \;\;\&\;\; y' < y'' \Rightarrow y < y'', \quad y = y' \;\;\&\;\; y' = y'' \Rightarrow y = y'',$

A pre-order without ties is called a *(linear or complete) order.*

We use standard notation: $y \leq y'$ if $y < y'$ or $y = y'$ and $y \geq y'$ if $y > y'$ or $y = y'$. Obviously, transitivity and symmetry still hold:
$y \leq y' \;\;\&\;\; y' < y'' \Rightarrow y < y'', \quad y < y' \;\;\&\;\; y' \leq y'' \Rightarrow y < y'',$
$y \leq y' \;\;\&\;\; y' \leq y'' \Rightarrow y \leq y'',$ and $y \leq y' \Leftrightarrow y' \geq y.$

In Figures 1-6 we use the following notation: an arrow from y to y' for $y > y'$, a line with two dashes for $y = y'$, and an arrow with two dashes for $y \geq y'$.

1.4 Configurations; Fifteen 2-Squares

Let us notice that to decide whether a situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is a NE in U, it is sufficient to know only two pre-orders: in the row x_1 with respect to U_2 and in column x_2 with respect to U_1.

Given X_1 and X_2, let us assign a pre-order P_{x_i} over X_{3-i} to each $x_i \in X_i$; $i = 1, 2$, and call the obtained preference profile $P = \{P_{x_1}, P_{x_2} \mid x_1 \in X_1,\ x_2 \in X_2\}$ a *configuration or bi-pre-order.*

Naturally, every bimatrix game $U = (U_1, U_2)$ defines a unique configuration $P = P(U)$, where P_{x_i} is the pre-order over X_{3-i} defined by $U_i;\ i = 1, 2$. Clearly, each configuration is realized by infinitely many bimatrix games. Yet, it is also clear that to get all NE in game U it is enough to know its configuration $P(U)$.

For brevity, we will refer to a 2×2 configuration as a 2-*square.* Up to permutations and transpositions, there exist only fifteen different types of 2-squares. They are listed in Figure 1 together with the corresponding bimatrix games (for the first 6 squares). Four 2-squares c_1, c_2, c_3, c_4 have no ties; another four, c_5, c_6, c_7, c_8 and the next five, c_9, c_{10}, c_{11}, c_{12}, c_{13}, have, respectively, one and two ties each; finally, c_{14} and c_{15} have 3 and 4 ties.

Fifteen 2-squares have $0, 2, 1, 1, 1, 2, 1, 2, 3, 2, 2, 2, 2, 3,$ and 4 NE, respectively. Thus, only c_1 has no NE. Shapley's theorem asserts that each c_1-free zero-sum game (or configuration) has a NE. Let us note that 2-squares c_1 - c_6 are frequent in the literature. For example, the non-zero-sum bimatrix games realizing c_2 and c_4 may represent classical "family dispute" and "prisoner's dilemma"; respectively, c_5 and c_6 illustrate the concepts of the "promise" and "threat".

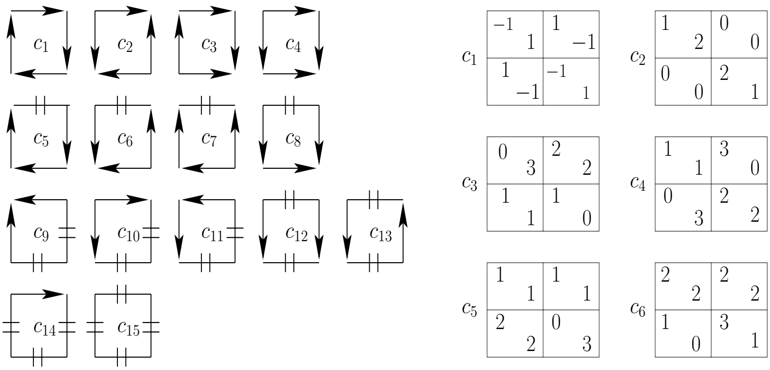


Fig. 1. Fifteen 2-squares

1.5 Dual or Transversal Hypergraphs

Let C be a finite set whose elements we denote by $c \in C$. A *hypergraph* H (*on the ground set* C) is a family of subsets $h \subseteq C$ that are called the *edges* of H. A hypergraph H is called Sperner if containment $h \subseteq h'$ holds for no two distinct edges of H. Given two hypergraphs T and E on the common ground set C, they are called *transversal* or *dual* if the following properties hold:

(i) $t \cap e \neq \emptyset$ for every $t \in T$ and $e \in E$;

(ii) for every subset $t' \subseteq C$ such that $t' \cap e \neq \emptyset$ for each $e \in E$ there exists an edge $t \in T$ such that $t \subseteq t'$;

(iii) for every subset $e' \subseteq C$ such that $e' \cap t \neq \emptyset$ for each $t \in T$ there exists an edge $e \in E$ such that $e \subseteq e'$.

Property (i) means that edges of E and T are transversal, while (ii) and (iii) mean that T contains all minimal transversals to E and E contains all minimal transversals to T, respectively. It is well-known, and not difficult to see, that (ii) and (iii) are equivalent whenever (i) holds. Although for a given hypergraph $\mathcal{H}$ there exist infinitely many dual hypergraphs, yet, only one of them, which we will denote by $\mathcal{H}^d$, is Sperner. Thus, within the family of Sperner hypergraphs duality is well-defined; moreover, it is an involution, that is, equations $T = E^d$ and $E = T^d$ are equivalent. It is also easy to see that dual Sperner hypergraphs have the same set of elements. For example, the following two hypergraphs are dual:

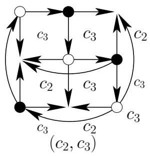
$$E' = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6)\}, \tag{1}$$

$$T' = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_6, c_9), (c_1, c_3, c_9)\}; \tag{2}$$

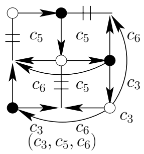
as well as the following two:

$$E = \{(c_1), (c_2, c_3), (c_5, c_9), (c_3, c_5, c_6), (c_2, c_4, c_5, c_6)\}, \tag{3}$$

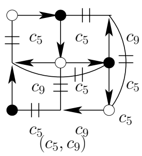
$$T = \{(c_1, c_2, c_5), (c_1, c_3, c_5), (c_1, c_2, c_3, c_9), (c_1, c_2, c_6, c_9), (c_1, c_3, c_4, c_9), (c_1, c_3, c_6, c_9)\}. \tag{4}$$



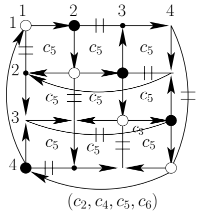
1, 2	2, 1	0, 0
0, 0	1, 2	2, 1
2, 1	0, 0	1, 2



0, 1	1, 0	0, 0
0, 0	0, 2	2, 1
1, 1	0, 0	1, 2



0, 1	1, 0	0, 0
0, 0	0, 1	1, 0
1, 0	0, 0	0, 1



	12	13	14	23	24	34
12	c_5	c_6	c_5	c_5	c_4	c_5
13	c_4	c_2	c_6	c_6	c_2	c_4
14	c_5	c_4	c_5	c_5	c_6	c_5
23	c_5	c_4	c_5	c_5	c_6	c_5
24	c_6	c_2	c_4	c_4	c_2	c_6
34	c_5	c_6	c_5	c_5	c_4	c_5

1, 2	2, 1	0, 1	1, 0
1, 0	1, 2	2, 1	0, 1
0, 1	1, 0	1, 2	2, 1
2, 1	0, 1	1, 0	1, 2

Fig. 2. NE-examples

1.6 Hypergraphs of Examples and Theorems

Let $C = \{c_1, \ldots, c_{15}\}$. We call a subset $e \subseteq C$ a *NE-example* if there is a NE-free configuration P such that e is the set of types of 2-squares in P; respectively, a subset $t \subseteq C$ is called a *NE-theorem* if a configuration has a NE whenever it contains no 2-squares from t. Obviously, $e \cap t \neq \emptyset$ for any NE-example e and NE-theorem t, since otherwise e is a counterexample to t. Moreover, it is well-known and easy to see that the hypergraphs of all inclusion-minimal (that is, strongest) NE-examples E_{NE} and NE-theorems T_{NE} are transversal. Let us consider c_1 and four configurations in Figure 2. It is easy to verify that all five contain canonical strong cycles and hence, they are locally minimal (in fact, minimal) NE-free configurations. These five configurations are chosen because they contain few types of 2-squares; the corresponding sets are given in Figure 2; they form the hypergraph E defined by (3). Figure 2 shows that each edge of E is a NE-example.

Let us consider the dual hypergraph T given by (4). We will prove that every edge $t \in T$ is a NE-theorem, thus, showing that the "research is complete", that is, $E = E_{NE}$ and $T = T_{NE}$ are the hypergraphs of all strongest NE-examples and theorems.

Remark 1. Given a family of NE-examples E', the dual hypergraph T' should be viewed as a hypergraph of conjectures rather than theorems. Indeed, some inclusion-minimal examples might be missing in E'; moreover, some examples of E' might be reducible. In this case some conjectures from the dual hypergraph $T' = E'^d$ will fail, being too strong. For instance, let us consider E' given by (1) in which the NE-example (c_2, c_4, c_5, c_6) is missing. (In fact, it is not that easy to obtain a minimal 4×4 example without computer.) Respectively, conjecture (c_1, c_3, c_9) appears in $T' = E'^d$. This conjecture is too strong, so it fails. In $T = T_{NE}$ we substitute for it three weaker (but correct) NE-theorems (c_1, c_3, c_9, c_2), (c_1, c_3, c_9, c_4), and (c_1, c_3, c_9, c_6). Thus, if it seems too difficult to prove a conjecture, one should look for new examples.

1.7 Joint Generation of Examples and Theorems

Of course, this approach can be applied not only to NE-free bimatrix games.

In general, given a set of objects Q (in our case, configurations), list C of subsets (properties) $Q_c \subseteq Q$, $c \in C$ (in our case, c-free configurations), the target subset $Q_0 \subseteq Q$ (configurations that have a NE), we introduce a pair of hypergraphs $E = E(Q, Q_0, C)$ and $T = T(Q, Q_0, C)$ (examples and theorems) defined on the ground set C as follows:

(i) every set of properties assigned to an edge $t \in T$ (a theorem) implies Q_0, that is, $q \in Q_0$ whenever q satisfies all properties of t, or in other words, $\cap_{c \in t} Q_c \subseteq Q_0$; in contrast,

(ii) each set of properties corresponding to the complement $C \setminus e$ of an edge $e \in E$ (an example) does not imply Q_0, i.e., there is an object $q \in Q \setminus Q_0$ satisfying all properties of $C \setminus e$, or in other words, $\cap_{c \notin e} Q_c \not\subseteq Q_0$.

If hypergraphs E and T are dual then we can say that "our understanding of Q_0 in terms of C is perfect", that is, every new example $e' \subset C$ (theorem $t' \subseteq C$) is a superset of some old example $e \in E$ (theorem $t \in T$).

Without loss of generality we can assume that examples of $e \in E$ and theorems $t \in T$) are inclusion-wise minimal in C; or in other words both hypergraphs E and T are Sperner.

Given Q, Q_0 and C, we try to generate hypergraphs E and T jointly [5]. Of course, the oracle may be a problem: Given a subset $C' \subseteq C$, it may be difficult to decide whether C' is a theorem (i.e., if $q \in Q_0$ whenever q satisfies all properties of C') or an example (i.e., if there is a $q \in Q \setminus Q_0$ satisfying all properties of $C \setminus C'$). However, the stopping criterion, $E^d = T$, is well-defined and, moreover, it can be verified in quasi-polynomial time [3].

Remark 2. Let us notice that containment $\cap_{c \in t} Q_c \subseteq Q_0$ might be strict. In other words, theorem t gives sufficient but not always necessary conditions for $q \in Q_0$. We can also say that theorems $t \in T$ give all optimal "inscribed approximations" of $Q_0 \subseteq Q$ in terms of C.

Remark 3. In [4], this approach was illustrated by a simple model problem in which Q is the set of 4-gons, Q_0 is the set of squares, C is a set of six properties of a 4-gon. Two dual hypergraphs of all minimal theorems T and examples E were constructed. In [2], the same approach was applied to a more serious problem related to families of Berge graphs.

1.8 Strengthening NE-Theorems; Main Results

We will prove all six NE-theorems $t \in T_{NE}$. Formally, they cannot be strengthened, since t' is not a NE-theorem whenever $t' \subset t \in T_{NE}$ and the containment $t' \subset t$ is strict. Still, we can get stronger claims in slightly different terms.

Let us notice that for any t the class of t-free configurations (games) is hereditary. Indeed, if a configuration (game) is t-free then every subconfiguration (subgame) of it is t-free, too. Hence, we can restrict ourselves by the locally minimal NE-free examples, which are characterized by Theorem 1.

Now, let us consider NE-theorems (c_1, c_2, c_5), (c_1, c_3, c_5), and (c_1, c_2, c_6, c_9). Formally, since 2-square c_1 has no NE, it must be eliminated. Yet, in a sense, it is the only exception. More precisely, we can strengthen the above three NE-theorems as follows.

Theorem 2. *The 2-square c_1 is a unique locally minimal NE-free configuration that is also (c_2, c_5)- or (c_3, c_5)-, or (c_2, c_6, c_9)-free.*

Furthermore, theorems (c_1, c_3, c_9, c_2), (c_1, c_3, c_9, c_4), (c_1, c_3, c_9, c_6) can be strengthened, too. In fact, we will characterize explicitly the configurations that are locally minimal NE-free and also (c_3, c_9)-free. This family is sparse but still infinite. In particular, we obtain the following result. Let $C(P)$ denote the set of all types of 2-squares of configuration P; furthermore, let $C' = \{c_2, c_4, c_5, c_6, c_7, c_8, c_{13}, c_1\}$ and $C'' = C' \cup \{c_{12}\}$.

Theorem 3. *Let P be a locally minimal NE-free $n \times n$ configuration that is also (c_3, c_9)-free. Then*

(i) n is even unless $n = 1$; *(ii) if $n = 2$ then P is c_1;*

(iii) if $n = 4$ then P is a unique (c_2, c_4, c_5, c_6)-configuration in Figure 2;

(iv) if $n = 6$ then $C(P) = C'$;

(v) if $n = 8$ then $C' \subseteq C(P) \subseteq C''$ and there exist P with $C(P) = C'$;

(vi) finally, if $n \geq 10$ is even then $C(P) = C''$.

It is clear that this statement implies the remaining three NE-theorems: (c_1, c_3, c_9, c_2), (c_1, c_3, c_9, c_4), and (c_1, c_3, c_9, c_6).

2 Proof of Theorems 2 and 3

As we already mentioned, we can restrict ourselves to the locally minimal NE-free configurations. By Theorem 1, each such configuration P is of size $n \times n$ for some $n \geq 2$ and P contains a strong improvement cycle C_n. Without loss of generality we can assume that $C_n = C_n^0$ is canonical. In particular,

$$x_{i,i+1} \geq x_{i,j},\ x_{i,i+1} > x_{j,i+1},\ \text{for } j \neq i,\ x_{j,j} \geq x_{i,j},\ x_{j,j} > x_{j,i+1},\ \text{for } j \neq i+1. \quad (5)$$

Furthermore, if $n = 2$ then 2-square c_1 is a unique NE-free configuration (in fact, c_1 is a strong 2-cycle). Hence, we will assume that $n \geq 3$. Additionally, we assume that P is t-free and consider successively the following subsets t : (c_2, c_5), (c_3, c_5), (c_2, c_6, c_9), and (c_3, c_9). Theorem 2 will follow, since in the first three cases we get a contradiction. For $t = (c_3, c_9)$ we will characterize the corresponding configurations explicitly, thus proving Theorem 3.

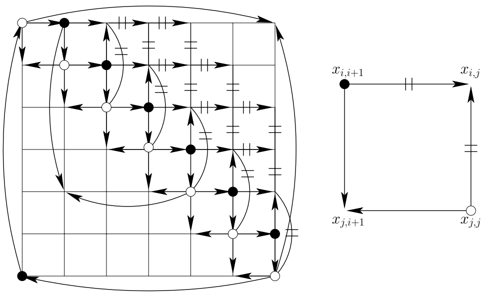


Fig. 3. Locally minimal NE-free and (c_2, c_5)-free configurations do not exist, except c_1

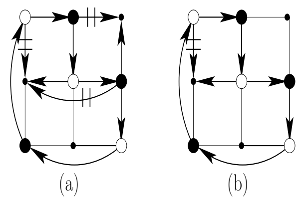


Fig. 4. Locally minimal NE-free and (c_2, c_6, c_9)- or (c_3, c_5)-free configurations do not exist, except c_1

2.1 Locally Minimal NE-Free and (c_2, c_5)-Free Configurations

Let us consider C_n^0 in Figure 3 (where $n = 7$). By (5), $x_{i,i} > x_{i,j}$ (with respect to U_2) whenever $j \neq i$; in particular, $x_{i,i} > x_{i,i-1}$ for $i \in [n] = \{1, \ldots, n\}$, where standardly, $0 \equiv n$. Similarly, $x_{i,i} \geq x_{j,i}$ whenever $j \neq i - 1$ (with respect to U_1); in particular, $x_{i,i} \geq x_{i+1,i}$ for $i \in [n] = \{1, \ldots, n\}$, where standardly, $n + 1 \equiv 1$. Moreover, the latter n inequalities are also strict, since otherwise c_5 would appear.

By similar arguments we show that $x_{i,i+1} > x_{i,i+2}$ and $x_{i,i+1} > x_{i-1,i+1}$ for $i = 1, \ldots, n - 1$; see Figure 3.

Next, let us notice that $x_{i,i} = x_{i-2,i}$ for $i = 2, \ldots n$. Indeed, $x_{i,i} \geq x_{i-2,i}$, since C_n is a strong cycle, and c_2 would appear in case $x_{i,i} > x_{i-2,i}$.

Furthermore, $x_{i,i+2} \geq x_{i,i+3}$ for $i = 1, \ldots, n-3$, since otherwise $x_{i,i+2}$, $x_{i,i+3}$, $x_{i+2,i+2}$, $x_{i+2,i+3}$ would form a c_5.

Next, let us notice that $x_{i,i+3} = x_{i+1,i+3}$ for $i = 1, \ldots, n-3$. Indeed, $x_{i,i+3} \leq x_{i+3,i+3} = x_{i+1,i+3}$, and if $x_{i,i+3} < x_{i+1,i+3}$ then $x_{i,i+1}$, $x_{i,i+3}$, $x_{i+3,i+1}$, $x_{i+3,i+3}$ would form a c_2, by (5).

Similarly, by induction on j, we show that $x_{i,i+j} \geq x_{i,i+j+1}$ and $x_{i,i+j} = x_{i+1,i+j}$ for $1 \leq i \leq n - 3$ and $2 \leq i + j \leq n - 1$.

In particular, $x_{n,n} = x_{n-2,n} = x_{n-3,n} = \ldots = x_{2,n} = x_{1,n}$ in contradiction with the strict inequality $x_{n,n} > x_{1,n}$ obtained before. □

2.2 Locally Minimal NE-Free and (c_2, c_6, c_9)- or (c_3, c_5)-Free Configurations

These two cases are easy. Let us consider C_n^0 in Figures 4 (a) and (b) (where $n = 3$), corresponding respectively to the two cases. By definition, in both cases $x_{2,2} > x_{2,1}$ $x_{1,1} \geq x_{2,1}$. In case (b) we already got a contradiction, since four above situations form c_3 or c_5.

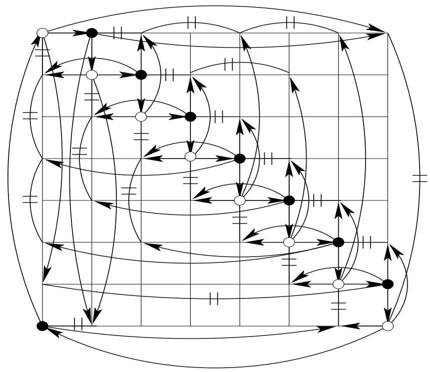


Fig. 5. Locally minimal NE-free and (c_3, c_9)-free configurations

In case (a) we have to proceed a little further. Clearly, $x_{2,3} \geq x_{2,1}$, $x_{1,2} \geq x_{1,3}$, $x_{2,3} > x_{1,3}$, and again we get a contradiction, since situations $x_{1,1}$, $x_{1,3}$, $x_{2,1}$, $x_{2,3}$ form c_9 if two equalities hold, c_6 if exactly one, and c_2 if none. $\square$

2.3 Locally Minimal NE-Free and (c_3, c_9)-Free Configurations

Let us consider C_n^0 in Figure 5 (where $n = 8$). By (5), for all i we have:

$$x_{i,i} > x_{i,i+1}, x_{i,i} > x_{i,i-1}, x_{i,i} \geq x_{i+1,i}, x_{i,i} \geq x_{i-2,i};$$

$$x_{i,i+1} > x_{i+1,i+1}, x_{i,i+1} > x_{i-1,i+1}, x_{i,i+1} \geq x_{i,i+2}, x_{i,i+1} \geq x_{i,i-1}.$$

Furthermore, it is not difficult to show that

$$x_{i,i} = x_{i+1,i} \text{ and } x_{i,i+1} = x_{i,i+2}, \tag{6}$$

since otherwise c_3 appears, while

$$x_{i,i} > x_{i-2,i} \text{ and } x_{i,i+1} > x_{i,i-1}, \tag{7}$$

since otherwise c_9 appears; see Figure 5.

Standardly, we prove all four claims in (6) and (7) by induction introducing situations in the following (alternating diagonal) order:

$x_{2,1}$, $x_{1,3}$, $\dots$, $x_{i,i-1}$, $x_{i-1,i+1}$, $\dots$, $x_{n,n-1}$, $x_{n-1,1}$, $x_{1,n}$, $x_{n,2}$.

Furthermore, $x_{1,1} = x_{2,1} \geq x_{4,1}$ unless $n < 5$; moreover, $x_{2,1} = x_{4,1}$, since otherwise situations $x_{2,1}$, $x_{4,1}$, $x_{2,4}$, and $x_{4,4}$ form c_3.

Similarly, we prove that $x_{1,3} = x_{1,5}$ unless $n < 5$.

Then let us recall that $x_{4,5} \geq x_{4,1}$ and conclude that $x_{4,5} > x_{4,1}$, since otherwise situations $x_{1,1}$, $x_{4,1}$, $x_{1,5}$, and $x_{4,5}$ form c_9.

In general, it is not difficult to prove by induction that

$$x_{i,i} = x_{i+1,i} = x_{i+3,i} = \ldots = x_{i+2j-1,i}, \text{ while } x_{i-1,i} > x_{i,i} > x_{i+2j,i}; \tag{8}$$

$$x_{i,i+1} = x_{i,i+2} = x_{i,i+4} = \ldots = x_{i,i+2j}, \text{ while } x_{i,i} > x_{i,i+1} > x_{i,i+2j+1}. \tag{9}$$

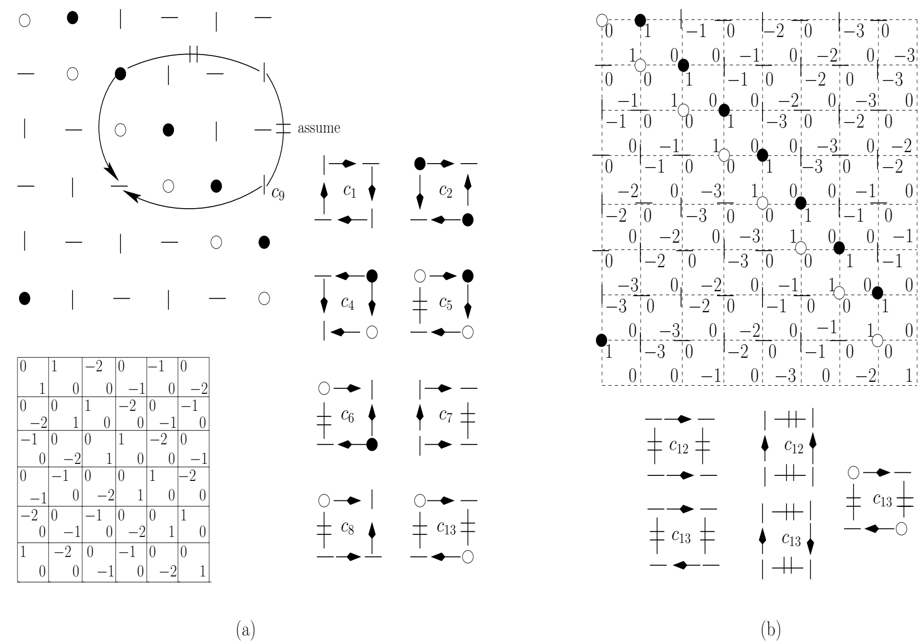


Fig. 6. Two examples from F_6 and F_8: horizontal (respectively, vertical) bars indicate second largest elements with respect to U_1 (respectively U_2)

In both cases each sum is taken $\mod (n)$ (in particular, $n = 0$) and $1 \leq j < n/2$ (in particular, j takes values $1, 2$, and 3 for $n = 7$ and $n = 8$).

If $n > 1$ is odd we immediately get a contradiction, since in this case, by (8), $x_{1,1} = x_{n-1,1}$, while, by (7), $x_{1,1} > x_{n-1,1}$ for all $n > 1$. Yet, for each even n, the family F_n of all locally minimal NE-free and (c_3, c_9)-free configurations is not empty.

Up to an isomorphism, F_2 (respectively, F_4) consists of a unique configuration: c_1 in Figure 1 (respectively, (c_2, c_4, c_5, c_6) in Figure 2). Two larger examples, from F_6 and F_8, are given in Figures 6 (a) and (b), respectively.

We already know that each configuration $P \in F_{2k}$ must satisfy (5) - (9). Yet, P has one more important property:

$$x_{i,i+2j+1} \neq x_{i,i+2j'+1}, \; x_{i+2j,i} \neq x_{i+2j',i} \tag{10}$$

for all $i \in [n]$ and for all positive distinct $j, j' < n/2$. Indeed, it is easy to see that otherwise c_9 appears; see Figure 6(a).

Let us denote by G_n the family of all configurations satisfying (5) - (10). We already know that $F_n \subseteq G_n$ and $F_n = G_n = \emptyset$ if $n > 1$ is odd. Let us show that $F_n = G_n$ for even n. Obviously, G_4 consists of a unique configuration (c_2, c_4, c_5, c_6) in Figure 2 and $G_2 = \{c_1\}$. Examples of configurations from G_6 and G_8 are given in Figures 6 (a) and (b). It is easy to verify that each configuration of G_n contains eight 2-squares $C' = \{c_2, c_4, c_5, c_6, c_1, c_7, c_8, c_{13}\}$ whenever $n \geq 6$; see Figure 6 (a). Moreover, c_{12} appears, too, when $n \geq 10$.

On the other hand, no configuration $P \in G_n$ contains c_9, c_{10}, c_{11}, c_{14}, or c_{15}, since no 2-square in P can have two adjacent equalities. It is also easy to verify that P cannot contain c_3. Thus, P can contain only nine 2-squares of $C'' = C' \cup \{c_{12}\}$. In particular, each $P \in G_n$ is (c_3, c_9)-free; in other words, $G_n \subseteq F_n$ and, hence, $G_n = F_n$ for each n. This implies Theorem 3 and provides an explicit characterization for family F_n of locally minimal NE-free and (c_3, c_9)-free configurations. □

Remark 4. Interestingly, for even n each configuration $P \in F_n = G_n$ contains the same set of nine 2-squares C'' if $n \geq 10$; for $P \in G_8$ there are two options: C'' or C' (see example in Figure 6 (b), where c_{12} does not appear); for $P \in G_6$ only C'; furthermore, G_4 consists of a unique configuration (c_2, c_4, c_5, c_6) in Figure 2 and G_2 only of c_1; finally, $F_n = G_n$ is empty if $n > 1$ is odd.

Acknowledgments. We are thankful to Kukushkin who promoted the idea of generalizing Shapley's (1964) theorem to bimatrix games and various concepts of solution.

References

1. Boros, E., Gurvich, V., Makino, K.: Minimal and locally minimal games and game forms, Rutcor Research Report 28-2007, Rutgers University
2. Chvatal, V., Lenhart, W.J., Sbihi, N.: Two-colourings that decompose perfect graphs. J. Combin. Theory Ser. B 49, 1–9 (1990)
3. Fredman, M., Khachiyan, L.: On the Complexity of Dualization of Monotone Disjunctive Normal Forms. J. Algorithms 21(3), 618–628 (1996)
4. Gurvich, V.A., Gvishiani, A.D.: Dual set systems and their applications. Izvestiya Akad. Nauk SSSR, ser. Tekhnicheskaya Kibernetika (in Russian) 4, 31–39 (1983); English translation in: Soviet J. of Computer and System Science (formerly Engineering Cybernetics)
5. Gurvich, V., Khachiyan, L.: On generating the irredundant conjunctive and disjunctive normal forms of monotone Boolean functions. Discrete Applied Mathematics 96-97, 363–373 (1999)
6. Gurvich, V., Libkin, L.: Absolutely determined matrices. Mathematical Social Sciences 20, 1–18 (1990)
7. Kukushkin, N.S.: Shapley's 2 × 2 theorem for game forms. Economics Bullletin, `http://economicsbulletin.vanderbilt.edu/2007/volume3/EB-07C70017A.pdf`; see also Technical report, Department of Mathematical Methods for Economic Decision Analysis, Computing Center of Russian Academy of Sciences, `http://www.ccas.ru/mmes/mmeda/ququ/Shapley.pdf`
8. Shapley, L.S.: Some topics in two-person games. In: Drescher, M., Shapley, L.S., Tucker, A.W. (eds.) Advances in Game Theory. Annals of Mathematical Studies, AM52, pp. 1–28. Princeton University Press, Princeton (1964)