A Topological Study of Tilings

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Abstract. To tile consists in assembling colored tiles on $\mathbb{Z}^2$ while respecting color matching. Tilings, the outcome of the tile process, can be seen as a computation model. In order to better understand the global structure of tilings, we introduce two topologies on tilings, one *à la* Cantor and another one *à la* Besicovitch. Our topologies are concerned with the whole set of tilings that can be generated by any tile set and are thus independent of a particular tile set. We study the properties of these two spaces and compare them. Finally, we introduce two infinite games on these spaces that are promising tools for the study of the structure of tilings.

1 Introduction

Wang was the first to introduce in [Wan61] the study of tilings with colored tiles. A tile is a unit size square with colored edges. Two tiles can be assembled if their common edge has the same color. A finite set of tiles is called a tile set. To tile consists in assembling tiles from a tile set on the grid $\mathbb{Z}^2$.

One of the first famous problems on tilings was the domino problem: can one decide whether given a tile set, there exists a tiling of the plane generated by this tile set? Berger proved the undecidability of the domino problem by constructing an aperiodic set of tiles, *i.e.*, a tile set that can generate only non-periodic tilings [Ber66]. Simplified proofs can be found in [Rob71] and later [AD96]. The main argument of this proof was to simulate the behavior of a given Turing machine with a tile set, in the sense that the Turing machine M stops on an instance ω if and only if the tile set $\tau_{\langle M,\omega\rangle}$ does not tile the plane. Hanf and later Myers [Mye74, Han74] have strengthened this and constructed a tile set that has only non-recursive tilings.

Later, tilings have been studied for different purposes: some researchers have used tilings as a tool for studying mathematical logical problems [AD96], others have studied the different kinds of tilings that one tile set can produce [CK97, DLS01, Rob71], or defined tools to quantify the regular structure of a tiling [Dur99]. One of the most striking facts concerning tilings is that tilings constitute a Turing equivalent computation model. This computation model is particularly

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relevant as a model of computation on the plane. Notions of reductions that have led to notions of universality for tilings and completeness for tile sets have been introduced in [LW07]. It is difficult to quantify this completeness property and other ones as periodicity, quasiperiodicity: one would like to be able to measure how common such a property is, in order to determine when and how they occur, or to give a size to the different sets of tilings with a certain property, or to say if a tile set is more likely to produce tilings with a certain property. One of our ultimate goals is to be able to say that if a set of tilings (generated from a given tile set or a family of tile sets) is large enough, then it necessarily contains a tiling with such and such properties. Naturally, topological tools on tilings would be the first step in this direction; first step that we aim at developing in this paper.

We introduce two metrics on tilings which have the particularity to be independent of a particular tile set, *i.e.*, we can measure a distance between two tilings that are not necessarily generated by the same tile set. These two metrics are similar to two traditional metrics used in the study of cellular automata: the so-called Cantor and Besicovitch metrics. The former gives more importance to the local structure around $(0, 0)$ and the later measures the asymptotic difference of information contained in the two tilings. They give rise to two natural topologies on the set of tilings.

The topological study of subsets of reals is inherently linked to the study of infinite games. In some of these games, such as Banach-Mazur games, a strong connection exists between the existence of a winning strategy and the co-meager property of the set on which the game is played. This connection allows one to show that some sets are meager. Others games, such as Gale-Stewart games, yield a hierarchy of winning strategies for one of the two players - we say in this case that the game is determined - depending on the structure of the sets on which the games are played. Having such a game-topological study of tilings, instead of subsets of reals, can lead to a better understanding of the structure of the tilings generated by a tile set.

This paper is organized as follows: we first recall basic notions on tilings, define the two topologies that we use and prove basic properties of these topologies. Then we study in a deeper way the structure of our topological spaces. We conclude by introducing two types of games on tilings, which help us to prove in a simpler way results of the previous section, and open a new direction for the study of the structure of tilings.

2 Topologies on Tilings

2.1 Tilings

In this paper we use the following terminologies: a *tile set* S is an initial subset $\{1, \ldots, n\}$ of $\mathbb{N}$. To map consists in placing the numbers of S on the grid $\mathbb{Z}^2$. A mapping generated by S is called a S*-mapping*. It is associated to a mapping function $f_A \in S^{\mathbb{Z}^2}$ that gives the tile of S at position (x, y) in A. We call $\mathfrak{M}$ the set of all mappings, *i.e.*, $\mathfrak{M} \equiv \{\{1, \ldots, n\}^{\mathbb{Z}^2}\}_{n \geq 1}$.

An S-pattern without constraints, or just *S-pattern*, is an S-mapping defined on a finite subset of $\mathbb{Z}^2$.

A *Wang tile* is an oriented unit size square with colored edges from C, where C is a finite set of colors. A *Wang tile set*, or just *tile set*, is a finite set of different tiles. To tile consists in placing the tiles of a given tile set on the grid $\mathbb{Z}^2$ such that two adjacent tiles share the same color on their common edge. A tiling P generated by a tile set τ is called a τ-tiling. It is associated to a tiling function f_P where $f_P(x, y)$ gives the tile at position (x, y) in P.

An S-mapping $A \in \mathfrak{M}$ is a Wang tiling if there exist a Wang tile set τ, a τ-tiling P and a bijective function $h : S \to \tau$ such that $h \circ f_A(x, y) = f_P(x, y)$. By this, we mean that A *works* as P. We define $\mathfrak{T}$ as the subset of mappings of $\mathfrak{M}$ which are Wang tilings.

Different kinds of tile sets and tilings have been identified: a periodic tiling is a tiling such that there exist two integers a and b such that for any (x, y), the tiles at position (x, y) and $(x+a, y+b)$ are the same; a tile set is periodic if it generates a periodic tiling; a tiling is finite if it is a pattern; a tiling P is universal if for any tile set τ, P simulates at least one τ-tiling. For more precisions on simulation and universality we refer the reader to [LW07].

Different tools are used to quantify the regular structure of a tiling. One of these is the quasiperiodic function. For a tiling P, the quasiperiodic function of P, denoted g_P, is the function that given an integer n, gives the smallest integer s which has the following property: if m is a square pattern of P of size n (the length of the sides of the square), then m appears in any square of size s in $\mathbb{Z}^2$. Thus, the quasiperiodic function of a tiling quantifies the regularity of appearance of the patterns in the tiling. Some tilings do not have a quasiperiodic function defined for every n. We say that a tiling is quasiperiodic if it has a quasiperiodic function defined for every n. An important result in [Dur99] is that any tile set, that can tile the plane, can generate a quasiperiodic tiling of the plane.

2.2 A Besicovitch Topology

The first metric we introduce is a metric similar to the cellular automata metric *à la* Besicovitch. A $\{n \times n'\}$ τ-pattern m can be seen has a sequence of numbers placed in a rectangle of size $n \times n'$; we have the number k in position (x, y) if $f_m(x, y) = t_k$ where t_k is the k^{th} tile of τ. Then intuitively any reordering of the tiles of τ gives the same pattern. Thus, we would like to say that the distance between two patterns m and m' is c if the proportion of different tiles between m and m' is at most c up to a reordering of the tiles. We formalize this notion in the following definitions, by defining a metric for any pattern without constraints:

Definition 1. *Let S and S' be two initial subsets of $\mathbb{N}$ such that $|S| \leq |S'|$. Let A be a $\{n \times n\}$ S-pattern, B be a $\{n \times n\}$ S'-pattern, and g be a one-to-one function from S to S'. We define the metric related to g by:* $\delta_g(A, B) = \frac{\#\{\,(x,y)\mid g\circ f_A(x,y)\neq f_B(x,y)\,\}}{n^2}$.

If A is an S-pattern and B is an S'-pattern such that $|S| \geq |S'|$, then $\delta_g(A,B)$ is defined to be equal to $\delta_g(B,A)$.

We define the absolute metric by: $\delta(A,B) = \min_{g \in S'^S}\{\delta_g(A,B)\}$.

From this definition of distance between patterns, we have that δ is symmetric and satisfies the triangle inequality.

Now that we have defined a metric between patterns, we can generalize it to tilings of the whole plane, since a tiling can be seen as the infinite union of patterns of ever increasing sizes:

Definition 2. *Let A be an S-tiling and let B be an S'-tiling. For any function $g \in S'^S$, we define the tiling metric d_g by: $d_g(A,B) = \limsup_{i\to\infty} \delta_g(A_i,B_i)$, where the A_i (*resp. *B_i) are the $\{n \times n\}$ S-patterns (*resp. *S'-patterns) centered around $(0,0)$ in A (*resp. *B).*

We define the absolute tiling metric d by: $d(A,B) = \min_{g \in S'^S}\{d_g(A,B)\}$.

Since δ satisfies the triangle inequality, and since reflexivity and symmetry are obvious, then d is a pseudometric on $\mathfrak{M}$. The natural way to obtain a metric on $\mathfrak{M}$ is to introduce an equivalence relation $\equiv_d$ defined by: $A \equiv_d B \Leftrightarrow d(A,B) = 0$. One can see that $\equiv_d$ is an equivalence relation. We can now consider the quotient space $\mathfrak{M}/\equiv_d$ where a typical element is the equivalence class $[A] = \{\, B \,|\, d(A,B) = 0 \,\}$. By adding to this space the metric d we obtain a metric space that we call $\mathfrak{M}_B$. In this paper, $[A]$ denotes the equivalence class of the particular mapping A; an element of $\mathfrak{M}_B$ is designated by a capital letter in bold, *e.g.*, $\mathbf{A}$; and we say "let $A \in \mathfrak{M}_B$" in the sense that we consider a mapping of the equivalence class of A, *i.e.*, a mapping in $[A]$. Similarly, we define the space $\mathfrak{T}/_{\equiv_d}$ where a typical element is $[P] = \{\, Q \,|\, d(P,Q) = 0 \,\}$. By adding to this space the metric d we obtain the metric space $\mathfrak{T}_B$. Of course, we have $\mathfrak{T}_B \subset \mathfrak{M}_B$. An element of $\mathfrak{T}_B$ is an equivalence class of $\mathfrak{M}_B$ that can contain mappings that are not Wang tilings, but which work "almost" like Wang tilings since the local constraint is respected almost everywhere.

From this definition, we have the two following results: for any two mappings $A, B \in \mathfrak{M}$, the distance between A and B is in $[0,1[$ and for any mapping $C \in \mathfrak{M}$ (*resp.* any tiling $P \in \mathfrak{T}$) and any $\epsilon \in [0,1[$, there exists a tiling D (*resp.* Q) such that $d(C,D) \geq 1-\epsilon$ (*resp.* $d(P,Q) \geq 1-\epsilon$). Therefore, for any tiling A, we can build a tiling B such that the distance between A and B is almost one.

To obtain a topological space, since $\mathfrak{M}_B$ is a metric space, we use the natural topology induced by the metric where the open sets are the balls $B(A,r)$, where A is a mapping. We use the subset topology on $\mathfrak{T}_B$.

The Besicovitch metric is one of the traditional metrics used for tilings. The other one is the Cantor one which gives more importance to the finite patterns centered around the origin.

2.3 A Cantor Metric

We define and study another *traditional* metric adapted for tilings, a metric *à la* Cantor. The metric studied above, is a metric that allows one to understand the behavior of a tiling in the whole $\mathbb{Z}^2$ grid. The distance between two tilings is small if their behavior is close. Another way to measure distance between tilings is to consider the greatest common pattern centered around $(0,0)$ that they share. We first define the function p as the function $p : \mathbb{N} \to \mathbb{Z}^2$ such that $p(0) = (0,0)$, $p(1) = (0,1)$, $p(2) = (1,1)$, $p(3) = (1,0) \dots$ and p keeps having the behavior of a spiral afterward. This function allows us to enumerate the tiles of a given tiling. We define the *prefix-patterns* of a tiling P:

Definition 3. *Let P be a tiling. The* prefix-pattern *of size n of P is the pattern m defined on the finite subset $D = \{p(0), \dots, p(n)\} \subsetneq \mathbb{Z}^2$ with the pattern function $f_m(x,y) = f_P(x,y)$ if $(x,y) \in D$. We denote the prefix-pattern m by a finite sequence of tiles ordered by the function p: $m = \{f_m \circ p(0), f_m \circ p(1), \dots, f_m \circ p(n-1)\}$. If m is a prefix-pattern of size n, then $m.t_1$, the concatenation of m and the tile t_1, is the prefix-pattern of size $n+1$ such that $m.t_1 = \{f_m \circ p(0), f_m \circ p(1), \dots, f_m \circ p(n-1), t_1\}$.*

The set of prefix-patterns can be enumerated by a tree T. The rules of the construction of T are the following: at level 0 in T, we have the empty prefix-pattern; at level 1 we have an unique prefix-pattern $\{1\}$. Then a pattern m at the level i, composed of j different tiles, has $j+1$ sons: $m.1, m.2, \dots, m.(j+1)$.

One can see that for any tile set S and any prefix-pattern $m = \{m_1, \dots, m_n\}$ generated by S, there exists an unique bijective function $e_m : S \to \{1, \dots, |S|\}$ such that the prefix-pattern $e_m(m) \stackrel{\text{def}}{=} \{e_m(m_1), e_m(m_2), \dots, e_m(m_n)\}$ is an element of T. $e_m(m)$ is said to be the *canonical form* of m. In T, an infinite branch corresponds to a mapping of the plane. Thus, to any S-mapping A there exists a unique bijective function e_A such that the set $e_A(A) \stackrel{\text{def}}{=} \{e_A \circ f_A \circ p(0), e_A \circ f_A \circ p(1), \dots\}$ corresponds to an infinite branch of T. $e_A(A)$ is said to be the *canonical form* of A. We say that m is a prefix-pattern of A if $e_m(m)$ is a prefix-pattern of $e_A(A)$. We can now define a metric *à la* Cantor:

Definition 4. *Let A be an S-mapping and B be an S'-mapping. We define the Cantor metric δ_C as: $\delta_C(A,B) = 2^{-i}$ where i is the size of the greatest common prefix-pattern of $e_A(A)$ and $e_B(B)$, i.e., the highest level in T where $e_A(A)$ and $e_B(B)$ are equal.*

If $e_A(A) = e_B(B)$ then $\delta_C(A,B) = 0$.

We can see that δ_C is a pseudometric on $\mathfrak{M}$. In fact, d_C is a hypermetric, *i.e.*, a metric such that for any three mappings A, B, C, $d_C(A,C) \leq \max\{d_C(A,B), d_C(B,C)\}$. This is a stronger version of the inequality of the triangle. One can note that in a hypermetric space, any point of an open ball is center of this ball.

To obtain a metric on $\mathfrak{M}$, we say that two tilings P and Q are equivalent, $P \equiv_C Q$, if their Cantor distance is null. Thus, two tilings are equivalent if they represent the same tiling up to a color permutation. We denote $\mathfrak{M}_C$ the space of equivalence classes $\mathfrak{M}/_{\equiv_C}$ equipped with the metric δ_C.

Similarly, we define $\mathfrak{T}/_{\equiv_C}$ that we denote $\mathfrak{T}_C$. The metric δ_C is a metric on $\mathfrak{T}_C$. From this, we can define a topology on $\mathfrak{T}_C$: we say that the set $U_m = \{\, P \mid m \text{ is a prefix-pattern of } P \,\}$ is a clopen set for any prefix-pattern m. This topology gives rise to a different understanding of the topology of tilings than the topological space $\mathfrak{M}_B$ since it gives more importance to the local structure centered around $\{0, 0\}$. Since there is a finite set of pattern of a given size, then we can cover $\mathfrak{M}_B$ and $\mathfrak{T}_B$ with a finite set of open sets. Therefore, $\mathfrak{M}_B$ and $\mathfrak{T}_B$ are precompact, *i.e.*, for all r, there does not exist a finite set of open balls of radius r that covers these spaces.

Since we have a Hausdorff space, because $\mathfrak{M}_C$ is a metric space, and since we have a basis of clopen sets, then $\mathfrak{M}_C$ is a $0-$*dimensional* space.

We finish the definition of the Cantor space by stating some obvious facts about the Cantor metric: if A and B are two mappings, then $d_C(A, B) \in [0, 1/2]$, and for any mapping A, there exists a mapping B such that $d_C(A, B) = 1/2$.

2.4 Basic Properties

The space $\mathfrak{M}_C$ is well-defined since two mappings at distance 0 are in fact the same tiling up to a reordering of the tiles, or, to say it differently, have the same canonical form. The space $\mathfrak{M}_B$ is slightly different, since two tilings at distance 0 can be different. We have to redefine properties for the mappings of $\mathfrak{M}_B$: if a tiling class contains a tiling with a certain property, then all the class has this property, since in fact, all other tilings of the class have "almost" the property. Thus, we can define the following tiling classes: if $\mathbf{P} \in \mathfrak{T}_B$ is a tiling class, we say that $\mathbf{P}$ is: *periodic* if $\mathbf{P}$ contains a periodic tiling, *quasiperiodic* if $\mathbf{P}$ contains a quasiperiodic tiling, *finite* if $\mathbf{P}$ contains a finite tiling (*i.e.,* a pattern), *universal* if $\mathbf{P}$ contains a universal tiling and a tiling with a quasiperiodic function f if $\mathbf{P}$ contains a tiling with a quasiperiodic function f and if $\mathbf{P}$ does not contain a tiling with a quasiperiodic function $g < f$.

We obtain now a space that works almost like the space of all Wang tilings. We can see how the different classes work. The following proposition states that the distance between two periodic tilings of $\mathfrak{T}_B$ is a rational number and gives a characterization of the classes of periodic tilings.

Proposition 1. *If P and Q are two different periodic tilings, then there exist $n, m \in \mathbb{N}^*$ such that $d(P, Q) = n/m$. Therefore, if $\mathbf{P} \in \mathfrak{M}_B$ is periodic, then it contains one and only one periodic tiling up to a reordering of the tiles.*

The following proposition shows that two quasiperiodic tilings which belong to the same equivalence class have the same quasiperiodic function:

Proposition 2. *If P and Q are two quasiperiodic tilings such that $P \equiv_d Q$, then $g_P = g_Q$, where g_P and g_Q are the quasiperiodic functions of P and Q.*

We recall a basic notion of tilings: *extraction.* Consider an infinite set of τ-patterns $\{m_1, m_2, \ldots\}$ of ever increasing sizes. We can see them as an infinite tree with the root representing the empty pattern and where a pattern m is a direct son of a pattern n if n is a subpattern of m, and if there does not exist a pattern $m' \neq m$ such that n is a subpattern of m' and m' is a subpattern of m. Thus, we obtain an infinite tree with finite degree. Therefore, by Koenig's lemma, we have at least one infinite branch. In our tree, this branch represents a tiling Q of the plane. This tiling Q is said to be an extraction of the set $\{m_1, m_2, \ldots\}$. Now, we say that Q is extracted from P if there exists an infinite set $\{m_1, m_2, \ldots\}$ of patterns of P such that Q is an extraction of $\{m_1, m_2, \ldots\}$.

Notions of universality and completeness for tilings have been introduced in [LW07]. We relate them to our topological space:

Proposition 3. *Let* $\mathbf{P} \in \mathfrak{M}_B$ *be a universal (*resp. *quasiperiodic, periodic) tiling. Then for any tiling* $A \in \mathbf{P}$*, we can extract from* A *a universal (*resp. *quasiperiodic, periodic) tiling* A'*.*

The previous result shows again that belonging to an equivalence class with a certain property is almost like having this property.

Now we study the different distances that we can obtain between Wang tilings and mappings in our two Besicovitch spaces. We have the following properties:

Theorem 1. *i) There exist a mapping* $A \in \mathfrak{M}_B \setminus \mathfrak{T}_B$ *and* $\epsilon > 0$ *such that the ball* $B(A, \epsilon)$ *does not contain any Wang tilings;*
ii) For any n*, there exists an infinite subset* H *of* $\mathfrak{T}_B$ *such that for any two tilings* P *and* Q *of* H*,* $d(P, Q) \geq 1 - 1/n^2$*.*

As a corollary, we have that the spaces $\mathfrak{M}_B$ and $\mathfrak{T}_B$ are not precompact. We can remark that that there exist prefix-patterns that can not be represented by the local constraint of a Wang tile set. From this, we obtain the following proposition:

Proposition 4. *There exists an open set in* $\mathfrak{M}_C$ *that does not contain any Wang tiling.*

3 Properties of Our Topologies

3.1 Properties of the Metric Spaces

We first study some basic notions of our spaces to have a better understanding of how they work. First of all, since we have metrizable spaces, we have that our spaces are completely Hausdorff and that there are no isolated points neither in $\mathfrak{M}_B$ nor in $\mathfrak{T}_B$. This seems natural for $\mathfrak{M}_B$, and is more interesting in the case of $\mathfrak{T}_B$.

Proposition 5. $\mathfrak{M}_B$*,* $\mathfrak{T}_B$*,* $\mathfrak{M}_C$ *and* $\mathfrak{T}_C$ *are all perfectly normal Hausdorff and perfect.*

The set of tilings is uncountable. But there exist tile sets that generate an uncountable set of tilings but only a countable set of equivalence classes in $\mathfrak{M}_B$. From this, and with the fact that any equivalence class contains an uncountable set of mappings, there arises the question of the cardinality of $\mathfrak{M}_B$. The following proposition shows that even the equivalence classes of $\mathfrak{T}_B$ are uncountable.

Proposition 6. *$\mathfrak{T}_B$ has the cardinality of the continuum.*

The next theorem is important for the understanding of Wang tilings. The metric used is strongly related to the information contained in the tiling. Thus, two tilings are close if the information contained in them is similar. So, theorem 2 can be stated as follows: a countable set of Wang tilings can not approach all the information that Wang tilings can generate. Then, by generalizing this theorem, we obtain a nice corollary.

Theorem 2. *There does not exist a countable set of Wang tilings which is dense in $\mathfrak{T}_B$.*

Corollary 1. *i) For any countable set of Wang tilings H, there exist a tiling P and a natural number n such that $d(P, Q) \geq 1/n$ for any $Q \in H$;*
ii) There exist a Wang tiling P and an integer n such that for any tile set τ, there exists at least one τ-tiling Q such that $d(P, Q) \geq 1/n$.

The next proposition shows how different the two topological spaces $\mathfrak{T}_B$ and $\mathfrak{T}_C$ are. This comes from the fact that one takes a glimpse at the whole tiling since the other one just look at it with blinkers.

Proposition 7. *There exists a countable set of Wang tilings that is dense in τ_C.*

From this, we have that $\mathfrak{M}_B$ is separable, and since it is completely metrizable, we have that $\mathfrak{M}_B$ is a Polish space. The next theorem shows that for any two tilings $A, B \in \mathfrak{T}_B$, there exists a continuous path $c : [0, 1] \to \mathfrak{T}_B$ such that $c(0) = A$ and $c(1) = B$:

Theorem 3. *$\mathfrak{T}_B$ is a path-connected space.*

3.2 Topological Properties

We now study the topological structure of our spaces. Since they are metric spaces, then natural topologies are induced on them by the metrics. We define the following sets:

i) Mapping(S) = $\{ [A] \mid A$ is an S-mapping $\}$,
ii) Wang(S) = $\{ [P] \mid P$ is a Wang S-tiling $\}$.

And for any tile set τ, we define the set: *Wang*(τ) = $\{ [P] \mid P$ is a τ-tiling $\}$.

The following theorem shows that the set of mappings or tilings generated by a tile set is a closed set. Then we give a characterization of the set of Wang tilings that can produce a tile set:

Theorem 4. *Let S and τ be two tile sets. Then Wang(τ) and Mapping(S) are closed sets and Wang(τ) is either a closed discrete set, or a closed non-discrete nowhere-dense set.*

Corollary 2. *i) $\mathfrak{T}_B$ is meager in $\mathfrak{M}_B$;*
ii) $\mathfrak{M}_B \setminus \mathfrak{T}_B$ is dense in $\mathfrak{M}_B$.

We now show that our spaces are Baire spaces:

Theorem 5. *$\mathfrak{T}_B$, $\mathfrak{T}_C$, $\mathfrak{M}_B$ and $\mathfrak{M}_C$ are complete metric spaces, and thus, are Baire spaces.*

In the following section, we introduce games on our topological spaces as tools for the study of the structure of tilings.

4 Games on Tilings

Since the tilings computation model is equivalent to the Turing machines, tilings make possible a geometrical point of view of computability. The different topological tools studied in this paper have shown some interesting aspects of the behavior of computability in tiling spaces. As we have seen, the set of tilings generated by a tile set τ, gives rise to complex subsets of $\mathfrak{M}_B$ and $\mathfrak{M}_C$. These sets can even be uncountable. A natural next step for studying these sets is to consider infinite games on tilings.

Several kinds of infinite games exist and are used in many different fields. Games have been studied for computation models such as pushdown automata (see Serre's PhD thesis [Ser05] for detailed survey). Considering the tilings computation model, we now give definitions for two types of infinite games on tilings.

The first one, Banach-Mazur games [Oxt57], is a play on the topological structure of the space. Different definitions of Banach-Mazur games exist. We propose this one:

Definition 5. *Let X be a topological space and Y a family of subsets of X such that:*

i) any member of Y has nonempty interior;
ii) any nonempty open subset of X contains a member of Y.

Let C be a subset of X. The game proceeds as follows: Player I chooses a subset Y_1 of Y. Player II chooses a subset Y_2 of Y such that $Y_2 \subseteq Y_1$. Then Player I chooses a subset Y_3 of Y such that $Y_3 \subseteq Y_2$ and so on. At the end of the infinite game, we obtain a decreasing sequence of sets: $X \supseteq Y_1 \supseteq Y_2 \supseteq \dots$ such that Player I has chosen the sets with odd indexes and Player II has chosen the sets with even indexes. Player II wins the game if $\bigcup_{n\geq 1} Y_i \subseteq X$.

The study of the different subsets of X such that Player II has a winning strategy is the main application of Banach-Mazur games. Of course, if $C = X$ Player II has a winning strategy. The question is: how big C has to be to allow Player II to have a winning strategy? This gives rise to classical theorem concerning Banach-Mazur games on topological spaces which states: a subset C of X is meager if and only if Player II has a winning strategy for the game on $\{X, X \setminus C\}$. We propose a Banach-Mazur game on the space $\mathfrak{M}_C$:

Definition 6. *Let X be a subset of $\mathfrak{M}_C$ and $C \subseteq X$. The game $\{X, C\}$ is defined as follows: Player I chooses a prefix-pattern m_1 such that m_1 is a prefix-pattern* *of a mapping of X. Player II chooses a prefix-pattern m_2 such that $m_1 \subset m_2$ and such that m_2 is a prefix-pattern of a mapping of X and so on. At the end of the game, we obtain a sequence of prefix-patterns $m_1 \subset m_2 \subset m_3 \ldots$ from which we extract a unique mapping A. Player II wins the game if $A \in C$.*

This amounts to playing the classical Banach-Mazur game with $X = \mathfrak{M}_C$ and $Y \subseteq \{ U_m \,|\, U_m$ is an open set in $\mathfrak{M}_C \}$. In our Cantor space, choosing a prefix-pattern amounts to choosing an open set, since the open sets of $\mathfrak{M}_C$ can be defined from prefix-patterns. Using this tool, we show that the set of Wang tilings is meager in the set of mappings for our topology $\mathfrak{M}_C$:

Theorem 6. *$\mathfrak{T}_C$ is meager in $\mathfrak{M}_C$.*

Proof. We use the game $\{\mathfrak{M}_C, \mathfrak{M}_C \setminus \mathfrak{T}_C\}$. We now have to show that Player II has a winning strategy, *i.e.*, Player II can always chooses integers in such a way that the final mapping can not be a Wang tiling.

This is true since whatever plays Player I at the first round, then Player II can choose a prefix-pattern which does not respect any possible local constraint generated by Wang tilings.

This trivial proof shows the convenience of using games to prove that some subsets are meager. To obtain the same kind of results for $\mathfrak{M}_B$ we define a Banach-Mazur game more adapted to the topology of $\mathfrak{M}_B$:

Definition 7. *Let X be a subset of $\mathfrak{M}_B$ and $C \subseteq X$. A Banach-Mazur game on $\{X, C\}$ is defined as follows: Player I chooses a mapping A_1 of X and an integer n_1; Player II chooses a mapping A_2 of X such that $d(A_1, A_2) \leq 1/n_1$ and an integer $n_2 \geq n_1$; Player I chooses a mapping A_3 of X such that $d(A_3, A_2) \leq 1/n_2$ and an integer $n_3 \geq n_2$, and so on. Player II wins the game if $lim_{i\to\infty} A_i \in C$.*

This game is still equivalent to a classical Banach-Mazur game, and since $\mathfrak{M}_B$ is a topological space, we still have that Player II has a winning strategy if and only if C is co-meager. We now prove the same result for $\mathfrak{M}_B$:

Theorem 7. *$\mathfrak{T}_B$ is meager in $\mathfrak{M}_B$.*

Proof. We will show that Player II has a winning strategy in the game $\{\mathfrak{M}_B, \mathfrak{M}_B \setminus \mathfrak{T}_B\}$. To show this, we first prove the following result: for any tiling P and any open ball $B(P, 1/n)$, there exist a mapping A and an integer $m > n$ such that $B(A, 1/m) \subset B(P, 1/n)$ and $B(A, 1/m) \cap \mathfrak{T}_B = \emptyset$.

Let P be a tiling and n an integer. The idea is to insert error patterns in P. We can build a pattern of size six generated by two tiles such that it can not be represented by a Wang pattern since its construction would imply that the two tiles that compose it are equal. Thus, at least one tile of this pattern can not be represented by Wang tiles. We introduce it in P in such a way that the new mapping A that we obtain is at distance $1/2n$ of P. Because of the error patterns, A can not be a Wang tiling. In the error pattern we have at least one of the six tiles that can not be represented by a Wang tile. Therefore if Q

is a Wang tiling, then $d(Q, A) \geq 1/12n$. Thus, $B(A, 1/12n) \subset B(P, 1/n)$ and $B(A, 1/12n) \cap \mathfrak{T}_B = \emptyset$.

With this result, we can see that the strategy of Player II will be to choose the tiling A and the integer $12n_1$ to be sure to win. Thus, $\mathfrak{T}_B$ is meager.

We introduce another type of games for the study of the complexity of the set of tilings generated by a given tile set: games *à la* Gale-Stewart. Here is a general definition of these games:

Definition 8. *Let A be a nonempty set and $X \subseteq A^{\mathbb{N}}$. We associate with X the following game: Player I chooses an element a_1 of A, Player II chooses an element a_2 of A and so on. Player I wins if $\{a_n\}_{n\in\mathbb{N}} \in X$. We denote $G(A, X)$ this game.*

A traditional question about a game $G(A, X)$ is to know whether one of the two players has a winning strategy, or in the terminology of games, if the game is determined. In [Mar75], Martin has shown that any Borel set is determined. Thus, we have to go beyond the Borelian hierarchy to find subsets complicated enough not to be determined. We would like to use these games on tilings to obtain similar structural complexity results for the set of tilings generated by a tile set. In that direction we give the following definition:

Definition 9. *Let $H \subseteq \mathfrak{M}_B$ and $X \subseteq H$. The Gale-Stewart game $G(H, X)$ is defined as follows: Player I chooses a tile a_1 such that $\{a_1\}$ is a prefix-pattern of a tiling of H; Player II chooses a tile a_2 such that $\{a_1, a_2\}$ is a prefix-pattern of a tiling of H and so on. Player I wins if the tiling $\{a_1, a_2, \ldots\} \in X$.*

If one of the two players has a winning strategy we say that $G(H, X)$ is determined *or that X is* determined in H*. We say that τ is* determined *if Wang(τ) is determined in T_τ, where T_τ is the set of all τ-tilings and τ-patterns, and that τ is* completely undetermined *if for any subset $X \in$ Wang(τ), X is undetermined in Wang(τ).*

The question is to know which kinds of games on tilings are determined, and which ones are not. We give a glimpse in that direction by showing that there exist tile sets determined and other ones completely undetermined:

Theorem 8. *i) There exists a determined tile set;*
ii) There exists a completely undetermined tile set.

Proof. i) To show this, we just have to find a tile set simple enough to generate a determined game. The tile set Easy5composed of a unicolor blue tile and four tile with three sides blue and one red satisfies the theorem. Player I has a winning strategy in the game $G(T_{\text{Easy5}}, \text{Wang}(\text{Easy5}))$: in this game, the goal of player I is to obtain a tiling of the plane, and the goal of player II, while respecting the local constraint of Easy5, is to obtain a situation where Player I can not move anymore. Player I can force the play of Player II by playing always one of the tile with a colored edge to force Player II to play the symmetric of this tile. Thus, the game is determined.

ii) Since Hanf and Myers [Mye74, Han74], we know that there exist tile sets that generate only non-recursive tilings. Let τ be one of them; we consider the game $G(\mathrm{Wang}(\tau), X)$ where X is a subset of $\mathrm{Wang}(\tau)$. If this game is determined, then there exists a winning strategy for one of the two players. Without loss of generality, suppose Player I has a winning strategy. Therefore, whatever Player II plays, Player I has a recursive process that allows him to choose a tile to go in a winning position: this strategy can generate a τ-tiling in a recursive way. This is a contradiction.

5 Concluding Remarks

The topologies and games introduced in this paper have made possible some descriptions of the structure of Wang tilings. This is a first step in the direction of measuring the largeness or meagerness of sets of tilings. One of the many remaining questions is to be able to measure how common universality is.

To reach these goals, the topological study of tilings through games appears as a promising approach.

References

[AD96] Allauzen, C., Durand, B.: The Classical Decision Problem. In: Appendix A: "Tiling problems", pp. 407–420. Springer, Heidelberg (1996)

[Ber66] Berger, R.: The undecidability of the domino problem. Memoirs of the American Mathematical Society 66, 1–72 (1966)

[CD04] Cervelle, J., Durand, B.: Tilings: recursivity and regularity. Theoretical Computer Science 310(1-3), 469–477 (2004)

[CK97] Culik II, K., Kari, J.: On aperiodic sets of Wang tiles. In: Foundations of Computer Science: Potential - Theory - Cognition, pp. 153–162 (1997)

[DLS01] Durand, B., Levin, L.A., Shen, A.: Complex tilings. In: Proceedings of the Symposium on Theory of Computing, pp. 732–739 (2001)

[Dur99] Durand, B.: Tilings and quasiperiodicity. Theoretical Computer Science 221(1-2), 61–75 (1999)

[Dur02] Durand, B.: De la logique aux pavages. Theoretical Computer Science 281(1-2), 311–324 (2002)

[GS53] Gale, F., Stewart, F.M.: Infinite games with perfect information. Ann. Math. Studies 28, 245–266 (1953)

[Han74] Hanf, W.P.: Non-recursive tilings of the plane. I. Journal of Symbolic Logic 39(2), 283–285 (1974)

[LW07] Lafitte, G., Weiss, M.: Universal Tilings. In: Thomas, W., Weil, P. (eds.) STACS 2007. LNCS, vol. 4393, pp. 367–380. Springer, Heidelberg (2007)

[Mar75] Martin, D.A.: 1975. Annals of Math. 102, 363–371 (1975)

[Mye74] Myers, D.: Non-recursive tilings of the plane. II. Journal of Symbolic Logic 39(2), 286–294 (1974)

[Oxt57] Oxtoby, J.C.: Contribution to the theory of games, Vol. III. Ann. of Math. Studies 39, 159–163 (1957)

[Rob71] Robinson, R.: Undecidability and nonperiodicity for tilings of the plane. Inventiones Mathematicae 12, 177–209 (1971)

[Ser05] Serre, O.: Contribution á l'étude des jeux sur des graphes de processus á pile, PhD thesis, Université Paris VII (2005)

[Wan61] Wang, H.: Proving theorems by pattern recognition II. Bell System Technical Journal 40, 1–41 (1961)

[Wan62] Wang, H.: Dominoes and the ∀∃∀-case of the decision problem. In: Proceedings of the Symposium on Mathematical Theory of Automata, pp. 23–55 (1962)