Generalized Domination in Degenerate Graphs: A Complete Dichotomy of Computational Complexity

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Abstract. The so called (σ, ρ)-domination, introduced by J.A. Telle, is a concept which provides a unifying generalization for many variants of domination in graphs. (A set S of vertices of a graph G is called (σ, ρ)-*dominating* if for every vertex $v \in S$, $|S \cap N(v)| \in \sigma$, and for every $v \notin S$, $|S \cap N(v)| \in \rho$, where σ and ρ are sets of nonnegative integers and $N(v)$ denotes the open neighborhood of the vertex v in G.) It is known that for any two nonempty finite sets σ and ρ (such that $0 \notin \rho$), the decision problem whether an input graph contains a (σ, ρ)-dominating set is NP-complete, but that when restricted to some graph classes, polynomial time solvable instances occur. We show that for every k, the problem performs a complete dichotomy when restricted to k-degenerate graphs, and we fully characterize the polynomial and NP-complete instances. It is further shown that the problem is polynomial time solvable if σ, ρ are such that every k-degenerate graph contains at most one (σ, ρ)-dominating set, and NP-complete otherwise. This relates to the concept of ambivalent graphs previously introduced for chordal graphs.

Subject: Computational complexity, graph algorithms.

1 Introduction and Overview of Results

We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. The open neighborhood of a vertex is denoted by $N(u) = \{v \colon (u, v) \in E(G)\}$. The closed neighborhood of a vertex u is the set $N[u] = N(u) \cup \{u\}$. If $U \subset V(G)$, then $G[U]$ denotes the subgraph of G induced by U.

Let σ, ρ be a pair of nonempty sets of nonnegative integers. A set S of vertices of G is called (σ, ρ)-*dominating* if for every vertex $v \in S$, $|S \cap N(v)| \in \sigma$, and for every $v \notin S$, $|S \cap N(v)| \in \rho$. The concept of (σ, ρ)-domination was introduced by

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J.A. Telle [4,5] (and further elaborated on in [6,2]) as a unifying generalization of many previously studied variants of the notion of dominating sets. In particular, ($\mathbb{N}_0$,$\mathbb{N}$)-dominating sets are ordinary dominating sets, ($\{0\}$,$\mathbb{N}_0$)-dominating sets are independent sets, ($\mathbb{N}_0$,$\{1\}$)-dominating sets are efficient dominating sets, ($\{0\}$,$\{1\}$)-dominating sets are 1-perfect codes (or independent efficient dominating sets), ($\{0\}$,$\{0,1\}$)-dominating sets are strong stable sets, ($\{0\}$,$\mathbb{N}$)-dominating sets are independent dominating sets, ($\{1\}$,$\{1\}$)-dominating sets are total perfect dominating sets, or ($\{r\}$,$\mathbb{N}_0$)-dominating sets are induced r-regular subgraphs ($\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and nonnegative integers, respectively).

We are interested in the complexity of the problem of existence of a (σ,ρ)-dominating set in an input graph, and we denote this problem by $\exists(\sigma,\rho)$-DOMINATION. It can be easily seen that if $0 \in \rho$, then the $\exists(\sigma,\rho)$-DOMINATION problem has a trivial solution $S = \emptyset$. So throughout our paper we suppose that $0 \notin \rho$. We consider only finite sets σ and ρ and use the notation $p_{\min} = \min \sigma$, $p_{\max} = \max \sigma$, $q_{\min} = \min \rho$, and $q_{\max} = \max \rho$.

It is known that for any nontrivial combination of finite sets σ and ρ (considered as fixed parameters of the problem), $\exists(\sigma,\rho)$-DOMINATION is NP-complete [4]. It is then natural to pay attention to restricted graph classes for inputs of the problem. The problem is shown polynomial time solvable for interval graphs in [3], where also the study of its complexity for chordal graphs was initiated. A full dichotomy for chordal graphs was proved in [1], where a direct connection between the complexity of $\exists(\sigma,\rho)$-DOMINATION and the so called ambivalence of the parameter sets σ,ρ was noted. A pair (σ,ρ) is called *ambivalent* for a graph class $\mathcal{G}$ if there exists a graph in $\mathcal{G}$ containing at least two different (σ,ρ)-dominating sets (such a graph will be called (σ,ρ)-*ambivalent*), and the pair (σ,ρ) is called *non-ambivalent* otherwise.

It is shown in [1] that for finite sets σ,ρ, $\exists(\sigma,\rho)$-DOMINATION is polynomial time solvable for chordal graphs if the pair (σ,ρ) is non-ambivalent (for chordal graphs), and it is NP-complete otherwise. It should be noted that the characterization which is given in [1] is nonconstructive in the sense that the authors did not provide a structural description of ambivalent (or non-ambivalent) pairs σ,ρ (and there is indication that such a description will not be simple).

In this paper we consider the connection between ambivalence and computational complexity of $\exists(\sigma,\rho)$-DOMINATION for k-degenerate graphs. A graph G is called *k-degenerate* (with k being a positive integer) if every induced subgraph of G has a vertex of degree at most k. For example, trees are exactly connected 1-degenerate graphs, every outerplanar graph is 2-degenerate, and every planar graph are 5-degenerate. An ordering of vertices $v_1, v_2, \ldots, v_n$ is called a *k-degenerate ordering* if every vertex v_i has at most k neighbors among the vertices $v_1, v_2, \ldots, v_{i-1}$. It is well known that a graph is k-degenerate if and only if it allows a k-degenerate ordering of its vertices.

It is known [6] that for trees (and for graphs of bounded treewidth), $\exists(\sigma,\rho)$-DOMINATION can be solved in polynomial time. Thus we assume $k \geq 2$ throughout the paper. We prove that also in the case of k-degenerate graphs, ambivalence and NP-hardness of $\exists(\sigma,\rho)$-DOMINATION go hand in hand.

Theorem 1. *For finite sets σ, ρ, $\exists(\sigma,\rho)$-*DOMINATION *is polynomial (linear) time solvable for k-degenerate graphs if the pair (σ, ρ) is non-ambivalent for k-degenerate graphs (moreover, the problem can be solved by an algorithm which is polynomial not only in the size of the graph, but also in $p_{\max}$ and $q_{\max}$), and it is NP-complete otherwise.*

Unlike the case of chordal graphs, for k-degenerate graphs we are able to describe a complete and constructive classification of ambivalent and non-ambivalent pairs.

Theorem 2. *Let σ, ρ be finite sets, and $k \geq 2$. If $p_{\min} > k$, then no k-degenerate graph has a (σ, ρ)-dominating set. If $p_{\min} \leq k$, then the pair (σ, ρ) is non-ambivalent for k-degenerate graphs if and only if one of the following two conditions holds:*

1. $(\sigma \cup \rho) \cap \{0, 1, \ldots, k\} = \{0\}$,
2. *for every $p \in \sigma$ and every $q \in \rho$, $|p - q| > k$.*

The last section of the paper is devoted to planar graphs. These undoubtedly form one of the most interesting k-degenerate classes of graphs ($k = 5$ in this case). Here we end up with several open problems. We are able to prove the NP-hardness part of an analog of Theorem 1.

Theorem 3. *For finite sets σ, ρ, $\exists(\sigma,\rho)$-*DOMINATION *is NP-complete for planar graphs if the pair (σ, ρ) is ambivalent for planar graphs.*

However, we do not know if non-ambivalence implies polynomial time recognition algorithm in this case. We are able to classify ambivalent and non-ambivalent pairs for some special pairs of sets σ and ρ, e.g., one-element sets, but even in this case the proof of non-ambivalence is nonconstructive and does not yield an algorithm.

Theorem 4. *Let σ, ρ be one-element sets, $\sigma = \{p\}$, $\rho = \{q\}$, and $0 \neq q$. If $p > 5$, then no planar graph has a (σ, ρ)-dominating sets. And if $p \leq 5$, then the pair (σ, ρ) is non-ambivalent for planar graphs if and only if $q - p > 3$ or $p - q > 2$.*

2 Classification of Ambivalent and Non-ambivalent Pairs for *k*-Degenerate Graphs

In this section we present a structural characterization of ambivalent and non-ambivalent pairs of sets (σ, ρ) for k-degenerate graphs. We also describe an algorithm which (in case of a non-ambivalent pair (σ, ρ)) constructs the unique (σ, ρ)-dominating set (if it exists) in an input k-degenerate graph. We start with the following simple statement.

Lemma 1. *Let σ, ρ be finite sets, and let k be a positive integer. If $p_{\min} > k$, then no k-degenerate graph contains a (σ, ρ)-dominating set.*

Proof. Let $v_1, v_2, \dots, v_n$ be a k-degenerate ordering of a k-degenerate graph G. Since $\deg v_n \le k < p_{\min}$, the vertex v_n does not belong to any (σ, ρ)-dominating set. But then every (σ, ρ)-dominating set of G is also a (σ, ρ)-dominating set of the subgraph of G induced by $\{v_1, v_2, \dots, v_{n-1}\}$. By repeating this argument inductively, we conclude that only the empty set can be (σ, ρ)-dominating. And since $0 \notin \rho$, this is impossible. □

Now we assume that $p_{\min} \le k$ and we prove that the conditions given in Theorem 2 are sufficient for non-ambivalence of σ, ρ. Towards this end, we describe greedy algorithms which construct the unique candidate for a (σ, ρ)-dominating set.

Let G be a k-degenerate graph with n vertices, and suppose that $v_1, v_2, \dots, v_n$ is a k-degenerate ordering of the vertices of G. We consider two cases, and in each if them a set S, which is a unique candidate for (σ, ρ)-dominating set in G, is constructed.

Case 1. $(\sigma \cup \rho) \cap \{0, 1, \dots, k\} = \{0\}$.

```
Procedure Construct A;
U := V(G), S := ∅;
while U ≠ ∅ do
    i := max{j : v_j ∈ U};
    S := S ∪ {v_i}, U := U \ N[v_i];
Return S
```

Case 2. $(\sigma \cup \rho) \cap \{0, 1, \dots, k\} \neq \{0\}$, and for every $p \in \sigma$ and every $q \in \rho$, $|p - q| > k$.

```
Procedure Construct B;
U := V(G), S := ∅;
for i := n downto 1 do
    r := |N(v_i) ∩ S|, s := |N(v_i) ∩ U|;
    if there is p ∈ σ such that r ≤ p ≤ r + s then
        S := S ∪ {v_i}, U := U \ {v_i}
    else
        if there is q ∈ ρ such that r ≤ q ≤ r + s then
            U := U \ {v_i}
        else
            Return There is no (σ, ρ)-dominating set, Halt;
Return S
```

Even if the procedures `Construct A` or `Construct B` construct a set S, it is still possible that this set is not (σ, ρ)-dominating. So we have to test for this property:

Procedure `Test`;
for $i := 1$ **to** n **do**
  if $(v_i \in S$ *and* $|N(v_i) \cap S| \notin \sigma)$ *or* $(v_i \notin S$ *and* $|N(v_i) \cap S| \notin \rho)$ **then**
    Return `There is no` (σ,ρ)`-dominating set`, **Halt;**
Return S

The properties of the algorithms are summarized in the next statement.

Lemma 2. *Let σ, ρ be finite sets. Suppose that k is a positive integer, $p_{\min} \leq k$ and either $(\sigma \cup \rho) \cap \{0, 1, \ldots, k\} = \{0\}$, or for every $p \in \sigma$ and every $q \in \rho$, $|p - q| > k$. Then the described algorithms correctly construct the (σ, ρ)-dominating set (if it exists) for any k-degenerate graph G, and this set is unique. The running time is $O((p_{\max} + q_{\max})(n + m))$, where n is the number of vertices of G, and m is the number of its edges.*

Proof. The correctness of the procedure `Construct A` is straightforward. The loop invariant of the procedure is that no vertex of U has a neighbor in S. Hence if $(\sigma \cup \rho) \cap \{0, 1, \ldots, k\} = \{0\}$, then every vertex $v \in U$ with degree no more than $k < q_{\min}$ must belong to every (σ, ρ)-dominating set, and vertices of $N(v)$ can not belong to any such set.

The correctness of the procedure `Construct B` follows from the following observation: If for every $p \in \sigma$ and every $q \in \rho$, $|p - q| > k$, then the set $\{r, r + 1, \ldots, s\}$ can not contain elements of both sets σ and ρ, since $s \leq k$. And since the number of S-neighbors of v_i (in the final (σ, ρ)-dominating set S) will end up in the interval $[r, r + s]$, the justification is clear.

It is known that a k-degenerate ordering can be constructed in time $O(n+m)$. Then the estimate of the running time immediately follows from the description of the algorithms. (Note here that we can assume that $p_{\max}, q_{\max} \leq n$ since otherwise we can truncate the sets σ and ρ.) □

To complete the proof of Theorem 2 we have to prove that the conditions given in the theorem are not only sufficient but also necessary. We do so by constructing graphs with at least two different (σ, ρ)-dominating sets. Let σ, ρ be finite sets, and let $k \geq 2$ be a positive integer. Suppose that $p_{\min} \leq k$, $(\sigma \cup \rho) \cap \{0, 1, \ldots, k\} \neq \{0\}$, and there are $p \in \sigma$ and $q \in \rho$ such that $|p - q| \leq k$. We consider 3 different cases.

Case 1. $\max(\sigma \cap \{0, 1, \ldots, k\}) = 0$. Since $(\sigma \cup \rho) \cap \{0, 1, \ldots, k\} \neq \{0\}$, there is a $q \in \rho$ such that $q \leq k$. Then each class of bipartition of the complete bipartite graph $K_{q,q}$ is a (σ, ρ)-dominating set in $K_{q,q}$ and $K_{q,q}$ (and consequently the pair (σ, ρ)) is ambivalent.

Case 2. $1 \in \sigma$. If $p < q$, then we start the construction with the complete bipartite graph $K_{q-p,q-p}$. Let the bipartition of its vertex set be $\{u_1, u_2, \ldots, u_{q-p}\}$ and $\{v_1, v_2, \ldots, v_{q-p}\}$. We further join every pair of vertices u_i and v_i by p different paths of length 2. Let us denote by X the set of the middle vertices of these paths. Since $k \geq 2$ and $q - p \leq k$, the graph constructed is k-degenerate. And it has two different (σ, ρ)-dominating sets: $\{u_1, u_2, \ldots, u_{q-p}\} \cup X$ and $\{v_1, v_2, \ldots, v_{q-p}\} \cup X$.

If $p \geq q$, then the construction starts with two copies of the complete graph K_{p-q+1}, with vertex sets $\{u_1, u_2, \dots, u_{p-q+1}\}$ and $\{v_1, v_2, \dots, v_{p-q+1}\}$. Again, we join every pair of vertices u_i and v_i by q different paths of length 2, and we let X denote the set of the middle vertices of these paths. Since $k \geq 2$ and $p-q \leq k$, the graph constructed in this way is k-degenerate. And it has two different (σ, ρ)-dominating sets: $\{u_1, u_2, \dots, u_{p-q+1}\} \cup X$ and $\{v_1, v_2, \dots, v_{p-q+1}\} \cup X$.

Case 3. $r \in \sigma$ for some $2 \leq r \leq k$. Let H denote the complete graph K_{r+1} with one edge deleted, and let u, v be the endvertices of this edge. We will further refer to these vertices as the *poles* of H. If $p < q$, then we start the construction with two copies of the complete bipartite graph $K_{q-p,q-p}$ with the bipartition of the vertex sets $\{u_1, u_2, \dots, u_{q-p}\}$, $\{v_1, v_2, \dots, v_{q-p}\}$ and $\{x_1, x_2, \dots, x_{q-p}\}$, $\{y_1, y_2, \dots, y_{q-p}\}$, respectively. Then for every $i \in \{1, 2, \dots, q-p\}$, we introduce p copies of H and join one pole of each of them with u_i and v_i by edges, and the other pole with x_i and y_i. Let X be the union of the sets of vertices of all added graphs H. The resulting graph is k-degenerate (first the non-pole vertices of H's have degrees $r \leq k$, after their deletion the pole vertices have degrees $2 \leq k$, and after the deletion of these all the remaining vertices have degrees $p - q \leq k$), and it has two different (σ, ρ)-dominating sets: $\{u_1, u_2, \dots, u_{q-p}\} \cup \{x_1, x_2, \dots, x_{q-p}\} \cup X$ and $\{v_1, v_2, \dots, v_{q-p}\} \cup \{y_1, y_2, \dots, y_{q-p}\} \cup X$.

If $p \geq q$, then we start the construction with four copies of the complete graph K_{p-q+1}, with vertex sets $\{u_1, u_2, \dots, u_{p-q+1}\}$, $\{v_1, v_2, \dots, v_{p-q+1}\}$, $\{x_1, x_2, \dots, x_{p-q+1}\}$, and $\{y_1, y_2, \dots, y_{p-q+1}\}$. For every $i \in \{1, 2, \dots, p-q+1\}$, we add q copies of the graph H and join one pole of each of them with u_i and v_i, and the other one with x_i and y_i. Again let X be the union of the sets of vertices of the added copies of H. The resulting graph is k-degenerate, and it has two different (σ, ρ)-dominating sets: $\{u_1, u_2, \dots, u_{p-q+1}\} \cup \{x_1, x_2, \dots, x_{p-q+1}\} \cup X$ and $\{v_1, v_2, \dots, v_{p-q+1}\} \cup \{y_1, y_2, \dots, y_{p-q+1}\} \cup X$.

Unifying the claims of Lemmas 1, 2 and these constructions we have completed the proof of Theorem 2. Also since we presented polynomial time algorithms which construct unique (σ, ρ)-dominating sets (if they exist) for the non-ambivalent pairs (σ, ρ), the polynomial part of Theorem 1 is proved.

To conclude this section, let us point out a property of the constructed graphs which will be used in the next section.

Lemma 3. *For every ambivalent pair (σ, ρ), there is a k-degenerate graph G with at least two different (σ, ρ)-dominating sets, which has a k-degenerate ordering $v_1, v_2, \dots, v_n$ such that for some ℓ, the first ℓ vertices $v_1, \dots, v_\ell$ belong to one and are not included to the other (σ, ρ)-dominating set.*

3 NP-Completeness of $\exists(\sigma, \rho)$-domination for Ambivalent Pairs

It this section we outline the proofs of the NP-hardness part of Theorem 1 and of Theorem 3.

We use a reduction from a special covering problem. Let r be a positive integer. An instance of the COVER BY NO MORE THAN r SETS is a pair (X, M), where X is a nonempty finite set and M is a collection of sets of elements of X. We ask about the existence of a collection $M' \subset M$ of sets such that every element of X belongs to at least one and to at most r sets of M'. The graph $G(X, M)$ of an instance (X, M) is the bipartite graph with the vertex set $X \cup M$ and edge set $\{xm | x \in m \in M\}$. The proof of the following lemma will appear in the full version of the paper.

Lemma 4. *For every fixed $r \geq 1$, the* COVER BY NO MORE THAN r SETS *problem is NP-complete even for instances (X, M) for which the graph $G(X, M)$ is 2-degenerate. It also stays NP-complete if the graph $G(X, M)$ is planar.*

The main technical part of the NP-hardness proof is the construction of a gadget which "enforces" on a given vertex the property of "not belonging to any (σ, ρ)-dominating set", and which guarantees that this vertex has a given number of neighbors in any (σ, ρ)-dominating set in the gadget:

Lemma 5. *Assume that $k \geq 2$ and $p_{\min} \leq k$. Let r be a nonnegative integer. Then there is a rooted graph F with the root u such that:*

1. *there is set $S \subset V(F) \setminus \{u\}$ such that for every $x \in S$, $|N(x) \cap S| \in \sigma$, and for every $x \notin S$, $x \neq u$, $|N(x) \cap S| \in \rho$;*
2. *for every such set S, $|N(u) \cap S| = r$;*
3. *for every set $S \subset V(F)$ such that $u \in S$, either there is $x \in S$, $x \neq u$, for which $|N(x) \cap S| \notin \sigma$, or there is $x \notin S$ for which $|N(x) \cap S| \notin \rho$;*
4. *F has a k-degenerate ordering with u as the first vertex.*

The construction of F (which will appear in the full version of the paper) is technical and requires a lengthy case analysis. A specific variant of the gadget F' for planar graphs is also constructed, and the construction will also appear in the full version of the paper.

Now we complete the proof of Theorem 1. Suppose that σ, ρ are finite sets of integers, $k \geq 2$, and the pair (σ, ρ) is ambivalent for k-degenerate graphs. Let $r = \max\{i \in \mathbb{N}_0 \colon i \notin \rho, i + 1 \in \rho\}$. Since $0 \notin \rho$, r is correctly defined. We are going reduce COVER BY NO MORE THAN t SETS for $t = q_{\max} - r$.

Suppose that (X, M) is an instance of COVER BY NO MORE THAN t SETS such that the graph $G(X, M)$ is 2-degenerate, $X = \{x_1, x_2, \ldots, x_n\}$ and $M = \{s_1, s_2, \ldots, s_m\}$. Let H be a k-degenerate ambivalent rooted graph with root u, such that u belongs to some (σ, ρ)-dominating set and u is not included in some other (σ, ρ)-dominating set, and H has a k-degenerate ordering for which the root is the first vertex. The existence of such a graph was proved in Lemma 3. For every vertex s_i of the graph $G(X, M)$, we take a copy of the graph H and identify its root with s_i. For every vertex x_j of our graph, a copy of the graph F (cf. Lemma 5) is constructed and its root is identified with x_j. Denote the graph obtained in this way by G. Clearly, G is k-degenerate.

We claim that the graph G has a (σ, ρ)-dominating set if and only if (X, M) allows a cover by no more than t sets. Since the graphs H and F depend only on σ and ρ, G has $O(n+m)$ vertices, our reduction is polynomial and the proof will be concluded.

Suppose first that G has a (σ, ρ)-dominating set S. Let $M' = \{s_i \in M : s_i \in S\}$. It follows from the properties of the forcing gadget F that every vertex x_j has exactly r neighbors in the gadget with root x_j and $x_j \notin S$. Then x_j has at least one S-neighbor in the set $\{s_1, s_2, \ldots, s_m\}$, but no more than $t = q_{\max} - r$ such neighbors. So, M' is a cover of X by no more than t sets.

Suppose now that $M' \subseteq M$ is a cover of X by no more than t sets. For every $i = 1, 2, \ldots, m$, we choose a (σ, ρ)-dominating set S_i in the copy of H with the root s_i such that $s_i \in S_i$ if and only if $s_i \in M'$. Let $S'_1, S'_2, \ldots, S'_n$ be (σ, ρ)-dominating sets in the copies of F. Since $\{t+1, t+2, \ldots, q_{\max}\} \subseteq \rho$, $S = S_1 \cup S_2 \cup \cdots \cup S_m \cup S'_1 \cup S'_2 \cup \ldots S'_n$ is a (σ, ρ)-dominating set in G.

The proof of Theorem 3 follows along the same lines. Suppose that (X, M) is an instance of Cover by no more than t sets such that the graph $G(X, M)$ is planar, $X = \{x_1, x_2, \ldots, x_n\}$ and $M = \{s_1, s_2, \ldots, s_m\}$. Let H' be a planar ambivalent rooted graph with root u, such that u belongs to some (σ, ρ)-dominating set and u is not included in some other (σ, ρ)-dominating set. For every vertex s_i of the graph $G(X, M)$, we construct a copy of the graph H' and unify its root with s_i. For every vertex x_j of our graph, a copy of the forcing gadget F' is constructed and the root of F' is identified with x_j. Let G be the resulting graph. Obviously, G is planar and G has a (σ, ρ)-dominating set if and only if (X, M) allows a cover by no more than t sets.

4 Ambivalence and Non-ambivalence for Planar Graphs

Since planar graphs are 5-degenerate, Theorem 2 gives sufficient conditions for non-ambivalence, but these conditions are not necessary for planar graphs. In this section we give some new sufficient conditions for non-ambivalence for planar graphs for certain cases of sets σ and ρ, and prove that in some cases these conditions are also necessary. We start with the case $q_{\min} > p_{\max}$.

Lemma 6. *Let σ, ρ be finite sets, and $p_{\min} \leq 5$. If $q_{\min} - p_{\max} > 3$, then the pair (σ, ρ) is non-ambivalent for planar graphs.*

Proof. Assume that $q_{\min} - p_{\max} > 3$ and let G be a planar graph with two different (σ, ρ)-dominating sets S_1 and S_2. Let $X = S_1 \cap S_2$, $Y_1 = S_1 \setminus S_2$ and $Y_2 = S_2 \setminus S_1$. If $x \in Y_1$, then since $x \in S_1$, $|N(x) \cap X| \leq p_{\max}$, and since $x \notin S_2$, $|N(x) \cap S_2| \geq q_{\min}$. So, x has at least 4 neighbors in Y_2. Similarly for $y \in Y_2$. Hence $G[Y_1 \cup Y_2]$ is a planar bipartite graph such that every vertex has degree at least 4, but this is impossible, since planar bipartite graphs are 3-degenerate. □

Now we consider the case $q_{\max} < p_{\min}$.

Lemma 7. *Let $\sigma = \{p\}$ for some $p \leq 5$ and $0 \notin \rho$. If $p - q_{\max} > 2$, then the pair (σ, ρ) is non-ambivalent for planar graphs.*

Proof. Suppose that $p - q_{\max} > 2$ and let G be a planar graph with two different (σ, ρ)-dominating sets S_1 and S_2. Let $X = S_1 \cap S_2$, $Y_1 = S_1 \setminus S_2$ and $Y_2 = S_2 \setminus S_1$. If $x \in Y_1$, then since $x \notin S_2$, $|N(x) \cap X| \leq q_{\max}$, and since $x \in S_1$, $|N(x) \cap S_1| = p$. So x has at least 3 neighbors in Y_1, i.e., $G[Y_1]$ and $G[Y_2]$ are planar graphs with all vertices of degree at least 3. Assume that a vertex $x \in Y_1$ is not adjacent to any vertex of Y_2. Then x has some neighbor $y \in X$. The vertex y must be adjacent to some vertex $z \in Y_2$ because σ contains exactly one element. Hence every vertex of Y_1 is either adjacent to some vertex of Y_2, or is connected with some vertex of Y_2 by a path of length two with the middle vertex from X.

Consider a plane embedding of G. It induces plane embeddings of $G[Y_1]$ and $G[Y_2]$. Let $x \in Y_1$. It is joined by an edge or by a path of length two to some vertex $y \in Y_2$, which belongs to some component H of $G[Y_2]$. This graph H lies completely in one face of $G[Y_1]$. Since all vertices of H have degrees at least 3, the graph H is not outerplanar. Then there is a vertex $z \in V(H)$ which does not belong to the boundary of the external face of H. By repeating the same arguments for z instead of x, we conclude that some component of $G[Y_1]$ lies completely in some internal face of H, and so on. Since the number of components of $G[Y_1]$ and $G[Y_2]$ is finite, this immediately gives a contradiction. □

The conditions given in Lemmas 6 and 7 are not only sufficient but also necessary for one-element sets σ and ρ, and this completes the proof of Theorem 4. The proof of this claim is provided by examples which are omitted here and will be given the full version of the paper. We conclude the section and the paper by explicitly stating some questions left open for planar graphs.

Problem 1. Is $\exists(\sigma, \rho)$-DOMINATION polynomial (NP-complete) when restricted to planar graphs if and only if the pair σ, ρ is non-ambivalent (ambivalent, respectively) for planar graphs? We believe that it would be interesting to solve this problem even for one-element sets σ and ρ.

Problem 2. Complete the characterization of ambivalent pairs σ, ρ for planar graphs.

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