9

Mean Values for Random Sets

For a stationary random closed set Z in $\mathbb{R}^d$, the volume density or specific volume was defined in Section 2.4 by

$$\overline{V}_d(Z) = \frac{\mathbb{E}\,\lambda(Z \cap B)}{\lambda(B)}, \tag{9.1}$$

where $B \subset \mathbb{R}^d$ can be an arbitrary Borel set with $0 < \lambda(B) < \infty$. This important parameter describes the mean volume of the random set per unit volume of the space. It is obtained by a double averaging, stochastic and spatial. The straightforward definition (9.1) has the advantage that it immediately exhibits $\lambda(Z \cap B)/\lambda(B)$ as an unbiased estimator for the specific volume. The situation becomes less simple if one wants to take other quantitative aspects of point sets into account. For example, in several applications one is interested in the mean surface area (the mean perimeter in the plane) per unit volume. Clearly, one cannot just proceed as in the case of (9.1), since the surface area of $Z \cap B$ is in general not defined. Evidently, we must restrict the realizations of the random set Z as well as the 'observation window' B. For that reason, we shall assume in the following that the realizations of the closed random set Z belong to the extended convex ring $\mathcal{S}$, the sets of which have the property that the intersection with any convex body is a finite union of convex bodies. Moreover, the observation window will be a compact convex set W with positive volume. In that case, $Z \cap W$ has a well-defined surface area. However, part of it generally comes from $Z \cap \operatorname{bd} W$ and not from the boundary of Z. To overcome boundary effects caused by the window W, the definition of densities for functionals other than the volume will require additional devices, for example, limit procedures.

The main purpose of Section 9.2 is the specification of a class of stationary random sets Z (with locally polyconvex realizations) and a class of functionals φ (defined on polyconvex sets) such that the limit

$$\overline{\varphi}(Z) := \lim_{r \to \infty} \frac{\mathbb{E}\,\varphi(Z \cap rW)}{V_d(rW)} \tag{9.2}$$

exists for every convex body W with $V_d(W) > 0$. The parameter $\overline{\varphi}(Z)$ is called the φ-density of Z. Important (but not the only) examples of functionals satisfying the assumptions are the intrinsic volumes (or Minkowski functionals) $V_0, \dots, V_{d-1}$. In this way, the V_j-density, or specific jth intrinsic volume, $\overline{V}_j(Z)$, is defined for a large class of stationary random sets. Included are the specific surface area, $2\overline{V}_{d-1}(Z)$, and the specific Euler characteristic, $\overline{V}_0(Z)$.

For the same class of functionals φ, and for stationary particle processes X with polyconvex grains and satisfying a suitable integrability condition, the φ-density, which was defined in Section 4.1 by

$$\overline{\varphi}(X) = \gamma \int_{\mathcal{C}_0} \varphi \, \mathrm{d}\mathbb{Q},$$

can be represented in the form

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{\mathbb{E} \sum_{C \in X} \varphi(C \cap rW)}{V_d(rW)},$$

which is analogous to (9.2).

For Boolean models with convex grains and satisfying suitable invariance assumptions, the existence of the specific intrinsic volumes can be obtained in a more direct way, as a consequence of explicit formulas. These formulas will be derived, together with some other results on Boolean models, in Section 9.1. They show, in particular, how the specific intrinsic volumes of a stationary, isotropic Boolean model with convex grains can be computed from the specific intrinsic volumes of the underlying Poisson particle process, and conversely. Especially, the intensity of the underlying particle process can, in principle, be determined from the specific intrinsic volumes of the union set. This seems surprising at first sight, but is, of course, nothing but another manifestation of the strong independence properties of Poisson processes. In the derivation, the integral geometric results of Chapters 5 and 6 will play an important role.

Instead of (9.2), it may even happen, under suitable assumptions, that the limit

$$\lim_{r \to \infty} \frac{\varphi(Z \cap rW)}{V_d(rW)}$$

exists $\mathbb{P}$-almost surely and is a constant, which is then equal to $\overline{\varphi}(Z)$. This ergodic approach to densities is described in Section 9.3.

As soon as densities of various functionals for stationary random sets are defined, the problem arises to estimate these densities from observations of realizations of the random set within a bounded sampling window, or from observations in a lower-dimensional section. In Section 9.4, results from integral geometry are employed to derive various formulas which are useful in this respect.

Mathematical principles of further estimation procedures are the topic of Section 9.5. This section gives selected examples and is not meant as a systematic exposition of estimation methods.

9.1 Formulas for Boolean Models

In our treatment of germ-grain models in Section 4.3, we have already emphasized the particular role played by the Boolean models. Recall that a Boolean model in $\mathbb{R}^d$ is a random closed set of the form

$$Z = \bigcup_{K \in X} K,$$

where X is a Poisson particle process. The Boolean model Z is stationary (isotropic) if and only if the underlying particle process X is stationary (isotropic).

In this section, we shall show how some characteristic parameters of random closed sets specialize in the case of stationary (and possibly isotropic) Boolean models, in particular those with convex grains, and then appear in rather explicit formulas. We begin with evaluating in closed form the capacity functional and the contact distribution functions H and $H_{[0,u]}$, introduced in Section 2.4.

Let Z be a Boolean model, generated as the union set of the Poisson particle process X with intensity measure Θ. If Z and thus X is stationary, the decomposition of Θ yields the intensity γ and the grain distribution $\mathbb{Q}$ of X. We shall call Θ, γ and $\mathbb{Q}$ also the intensity measure, the intensity, and the grain distribution, respectively, of Z. For stationary Z, we assume that $\gamma > 0$.

According to Theorem 3.6.3, the capacity functional of the Boolean model Z satisfies the equation

$$T_Z(C) = 1 - e^{-\Theta(\mathcal{F}_C)} \tag{9.3}$$

for all $C \in \mathcal{C}$. Now we assume that Z is stationary. As in the proof of Theorem 4.1.2, we then have

$$\Theta(\mathcal{F}_C) = \gamma \int_{\mathcal{C}_0} V_d(K - C)\, \mathbb{Q}(\mathrm{d}K). \tag{9.4}$$

In general, this integral cannot be simplified further. If, however, Z is a Boolean model with convex grains and if $C \in \mathcal{K}'$, then the volume $V_d(K - C)$ can, according to (14.20), be expressed in terms of mixed volumes, in the form

$$V_d(K - C) = \sum_{j=0}^{d} \binom{d}{j} V(K[j], -C[d-j]).$$

For $C = rB^d$, $r > 0$, this is the Steiner formula (14.5).

The contact distribution function H_M of a random closed set Z with respect to the structuring element $M \in \mathcal{K}'$ with $0 \in M$ is, according to Section 2.4, given by

$$H_M(r) = 1 - \frac{\mathbb{P}(0 \notin Z - rM)}{\mathbb{P}(0 \notin Z)}$$

for $r \geq 0$, if $\mathbb{P}(0 \notin Z) > 0$. For a stationary Boolean model Z with generating Poisson particle process X we always have

$$\mathbb{P}(0 \notin Z) = 1 - T_Z(\{0\}) = \mathrm{e}^{-\overline{V}_d(X)} > 0, \tag{9.5}$$

by (9.3) and (9.4).

Theorem 9.1.1. *Let Z be a stationary Boolean model in $\mathbb{R}^d$ with intensity γ and grain distribution $\mathbb{Q}$. Then*

$$T_Z(C) = 1 - \exp\left(-\gamma \int_{\mathcal{C}_0} V_d(K - C)\, \mathbb{Q}(\mathrm{d}K)\right), \quad C \in \mathcal{C}.$$

For the structuring element $M \in \mathcal{K}'$ with $0 \in M$, the contact distribution function is given by

$$H_M(r) = 1 - \exp\left(-\gamma \int_{\mathcal{C}_0} [V_d(K - rM) - V_d(K)]\, \mathbb{Q}(\mathrm{d}K)\right), \quad r \geq 0.$$

If Z has convex grains, then, for $M \in \mathcal{K}'$,

$$T_Z(M) = 1 - \exp\left(-\gamma \sum_{k=0}^{d} \binom{d}{k} \int_{\mathcal{K}_0} V(-M[k], K[d-k])\, \mathbb{Q}(\mathrm{d}K)\right)$$

and

$$H_M(r) = 1 - \exp\left(-\gamma \sum_{k=1}^{d} \binom{d}{k} r^k \int_{\mathcal{K}_0} V(-M[k], K[d-k])\, \mathbb{Q}(\mathrm{d}K)\right).$$

In particular, in this case the spherical contact distribution function is given by

$$H(r) = 1 - \exp\left(-\sum_{k=1}^{d} \kappa_k r^k \overline{V}_{d-k}(X)\right), \quad r \geq 0,$$

and for $u \in S^{d-1}$, the linear contact distribution function is given by

$$H_{[0,u]}(r) = 1 - \exp\left(-\gamma r \int_{\mathcal{K}_0} V_{d-1}(K|u^{\perp})\, \mathbb{Q}(\mathrm{d}K)\right), \quad r \geq 0.$$

If, moreover, Z is isotropic and $M \in \mathcal{K}'$, then

$$T_Z(M) = 1 - \exp\left(-\sum_{k=0}^{d} c_{0,d}^{k,d-k} V_k(M) \overline{V}_{d-k}(X)\right),$$

where the constants are given by (5.5).

(In the formula for $H_{[0,u]}$, the integrand $V_{d-1}(K|u^{\perp})$ is the $(d-1)$-dimensional volume of the orthogonal projection of K onto $u^{\perp}$.)

Proof. The first two assertions about the capacity functional have already been proved. From these, the formulas for $H_M(r)$ follow because of

$$H_M(r) = 1 - \frac{1 - T_Z(rM)}{1 - T_Z(\{0\})}, \qquad r \geq 0.$$

The special form of $H(r)$ in the case of convex grains is obtained, for $M = B^d$, from (14.20), and the expression for $H_{[0,u]}(r)$ follows from

$$V_d(K + r[0,u]) = V_d(K) + rV_{d-1}(K|u^\perp).$$

Now suppose that Z is also isotropic, so that $\mathbb{Q}$ is rotation invariant. Let $M \in \mathcal{K}'$. In the equation

$$\Theta(\mathcal{F}_M) = \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{F}_M}(K + x)\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K)$$

we can replace K in the integrand by ϑK with a rotation $\vartheta \in SO_d$; this does not change the integral, since $\mathbb{Q}$ is rotation invariant. Then we integrate over all $\vartheta \in SO_d$ with respect to the invariant measure ν and apply Fubini's theorem and the principal kinematic formula (Theorem 5.1.3). For $M, K' \in \mathcal{K}'$ we have $\mathbf{1}_{\mathcal{F}_M}(K') = V_0(M \cap K')$ (since V_0 is the Euler characteristic). This gives

$$\begin{aligned}
\Theta(\mathcal{F}_M) &= \gamma \int_{SO_d} \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{F}_M}(\vartheta K + x)\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K)\, \nu(\mathrm{d}\vartheta) \\
&= \gamma \int_{\mathcal{K}_0} \int_{SO_d} \int_{\mathbb{R}^d} V_0(M \cap (\vartheta K + x))\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)\, \mathbb{Q}(\mathrm{d}K) \\
&= \gamma \sum_{k=0}^{d} c_{0,d}^{k,d-k} V_k(M) \int_{\mathcal{K}_0} V_{d-k}(K)\, \mathbb{Q}(\mathrm{d}K) \\
&= \sum_{k=0}^{d} c_{0,d}^{k,d-k} V_k(M) \overline{V}_{d-k}(X),
\end{aligned}$$

which completes the proof. □

The contact distribution function H_M of a stationary random closed set Z can be generalized in various directions. First, one can skip the stationarity and consider the distribution of the M-distance $d_M(x, Z)$ of a point x to Z, provided $x \notin Z$. Then, one can take into account not only distances but also directions, contact points and other local geometric information which can be measured from outside Z. Such generalized contact distributions are discussed in Section 11.2 and in the corresponding Notes. They give us more information about the random set Z; in some cases, they even determine the distribution of Z. An example of that kind is presented in Section 9.5.

As an introduction to the main topic of this section, we consider the (already defined) specific volume

$$\overline{V}_d(Z) = \frac{\mathbb{E}\,\lambda(Z \cap W)}{\lambda(W)}$$

for the case of a stationary Boolean model Z (with general compact grains). Here W may be an arbitrary Borel set with $\lambda(W) > 0$. We can find a connection with the volume density $\overline{V}_d(X)$ of the underlying particle process X. In fact,

$$\begin{aligned}\overline{V}_d(Z) &= \mathbb{P}(0 \in Z) = 1 - \mathbb{P}(0 \notin Z)\\ &= 1 - \mathbb{P}(\operatorname{card}(X \cap \mathcal{C}_{\{0\}}) = 0) = 1 - \mathrm{e}^{-\Theta(\mathcal{C}_{\{0\}})}\end{aligned}$$

and

$$\begin{aligned}\Theta(\mathcal{C}_{\{0\}}) &= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{C}_{\{0\}}}(K + x)\,\lambda(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}K)\\ &= \gamma \int_{\mathcal{C}_0} V_d(K)\,\mathbb{Q}(\mathrm{d}K)\\ &= \overline{V}_d(X).\end{aligned}$$

Thus, we have found that

$$\overline{V}_d(Z) = 1 - \mathrm{e}^{-\overline{V}_d(X)}. \tag{9.6}$$

This equality should have come as a surprise: it says that the volume density $\overline{V}_d(X)$ of the particle process X can be determined from the volume density of the union set. This is surprising, since in a given realization of Z one cannot identify the generating particles, due to overlapping, and some particles may even be covered totally by others. The reason for the existence of the exact relation (9.6) lies in the strong independence properties of Poisson processes. The elegant connection between quantitative properties of a stationary Boolean model and its underlying particle process is not restricted to the volume, as we shall soon see.

Let Z be a (not necessarily stationary) Boolean model, generated by the Poisson process X with intensity measure Θ. For simplicity, we assume that the particles in X are a.s. convex, although the following results hold true for polyconvex particles, under an additional integrability condition (see the remark at the end of this section). Motivated by practical applications (in small dimensions), we assume that a **sampling window**, a convex body W with $V_d(W) > 0$, is given in which $Z \cap W$ can be observed. Our aim is to study random variables of the type

$$\varphi(Z \cap W),$$

with suitable functionals φ replacing the volume. In this way, we want to find out which information on Z and its underlying particle process can be obtained from measuring the realizations of Z within a bounded observation window W. Under appropriate assumptions, this will lead in a natural way to densities of Z and to relations between such densities defined for Z and similar parameters defined for the underlying particle process.

Since we intend to investigate sets arising as unions of convex bodies, we allow measurable functions φ defined on the convex ring $\mathcal{R}$ and having a simple behavior under unions. Therefore, $\varphi : \mathcal{R} \to \mathbb{R}$ is assumed to be **additive**, that is, to satisfy

$$\varphi(K \cup L) = \varphi(K) + \varphi(L) - \varphi(K \cap L) \tag{9.7}$$

for $K, L \in \mathcal{R}$ and $\varphi(\emptyset) = 0$. We further assume that φ is conditionally bounded. Here, we call a function $\varphi : \mathcal{R} \to \mathbb{R}$ **conditionally bounded** if, for each $K \in \mathcal{K}'$, the function φ is bounded on the set $\{L \in \mathcal{K}' : L \subset K\}$. When φ is translation invariant and additive, it is sufficient for this to assume that φ is bounded on the set $\{L \in \mathcal{K}' : L \subset C^d\}$. If φ is given as a functional on $\mathcal{K}'$ and is continuous and additive (the latter means that (9.7) holds whenever $K, L, K \cup L \in \mathcal{K}'$), then Groemer's extension theorem (Theorem 14.4.2) says that the functional φ has an additive extension (which we denote by the same symbol) to the convex ring $\mathcal{R}$. By Theorem 14.4.4, the extension is measurable and, due to the continuity on $\mathcal{K}'$, it is also conditionally bounded. The intrinsic volumes V_j, $j = 0, \dots, d$, are prototypes of measurable, additive and conditionally bounded functionals $\varphi : \mathcal{R} \to \mathbb{R}$; they are also motion invariant.

For a Boolean model Z with convex grains, $Z \cap W$ is a polyconvex set, hence $\varphi(Z \cap W)$ is defined and yields a random variable. We want to investigate how its expectation is related to the characteristics of the underlying particle process, that is, to the intensity measure Θ of X. In applications, such relations may be used to fit a Boolean model to given data, or to estimate densities of functionals for the particle process, in particular its intensity (in the stationary case), from measurements at realizations of the union set.

To begin with the computation of $\mathbb{E}\,\varphi(Z \cap W)$, for an additive, conditionally bounded and measurable function φ, let ν be the random number of particles of X hitting W, and let $M_1, \dots, M_\nu$ be these particles (with any numbering). Then the inclusion–exclusion principle (14.47) gives

$$\begin{aligned}
\varphi(Z \cap W) &= \varphi\left(\bigcup_{K \in X} K \cap W\right) \\
&= \sum_{k=1}^{\nu} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le \nu} \varphi(W \cap M_{i_1} \cap \dots \cap M_{i_k}) \\
&= \sum_{k=1}^{\nu} \frac{(-1)^{k-1}}{k!} \sum_{(K_1, \dots, K_k) \in X^k_{\neq}} \varphi(W \cap K_1 \cap \dots \cap K_k).
\end{aligned}$$

Here $X^k_{\neq}$ is the set of pairwise distinct k-tuples from X. In the last line, we may extend the first summation to ∞, since $\varphi(\emptyset) = 0$.

Since φ is conditionally bounded, there exists a number c (depending on W) with $|\varphi(L)| \le c$ for all $L \in \mathcal{K}'$ with $L \subset W$. This gives

$$\begin{aligned} |\varphi(Z \cap W)| &\le \sum_{k=1}^{\nu} \frac{1}{k!} \left| \sum_{(K_1,\ldots,K_k) \in X^k_{\neq}} \varphi(W \cap K_1 \cap \ldots \cap K_k) \right| \\ &\le \sum_{k=1}^{\nu} \binom{\nu}{k} c \le 2^{\nu} c = 2^{\mathrm{card}(X \cap \mathcal{K}_W)} c. \end{aligned}$$

Since $\mathrm{card}(X \cap \mathcal{K}_W)$ has a Poisson distribution,

$$\begin{aligned} \mathbb{E}\, 2^{\mathrm{card}(X \cap \mathcal{K}_W)} &= \sum_{k=0}^{\infty} 2^k\, \mathbb{P}(\mathrm{card}(X \cap \mathcal{K}_W) = k) \\ &= \mathrm{e}^{-\Theta(\mathcal{K}_W)} \sum_{k=0}^{\infty} \frac{[2\Theta(\mathcal{K}_W)]^k}{k!} \\ &= \mathrm{e}^{-\Theta(\mathcal{K}_W)} \mathrm{e}^{2\Theta(\mathcal{K}_W)} = \mathrm{e}^{\Theta(\mathcal{K}_W)} < \infty. \end{aligned}$$

It follows that $\varphi(Z \cap W)$ is integrable. By the dominated convergence theorem, we can interchange expectation and summation. Using Theorem 3.1.3 and Corollary 3.2.4, we obtain

$$\begin{aligned} &\mathbb{E}\, \varphi(Z \cap W) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}\, \mathbb{E} \sum_{(K_1,\ldots,K_k) \in X^k_{\neq}} \varphi(W \cap K_1 \cap \ldots \cap K_k) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}} \cdots \int_{\mathcal{K}} \varphi(W \cap K_1 \cap \ldots \cap K_k)\, \Theta(\mathrm{d}K_1) \cdots \Theta(\mathrm{d}K_k). \end{aligned}$$

So far, we have not used stationarity. But if we now assume that Z is stationary, we can use the decomposition of the intensity measure and get

$$\begin{aligned} &\mathbb{E}\, \varphi(Z \cap W) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^k} \varphi(W \cap (K_1 + x_1) \cap \ldots \cap (K_k + x_k)) \\ &\quad \times \lambda^k(\mathrm{d}(x_1, \ldots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k). \end{aligned}$$

We summarize the results in the following theorem.

Theorem 9.1.2. *Let Z be a Boolean model in $\mathbb{R}^d$ with convex grains, let $W \in \mathcal{K}'$ and $\varphi : \mathcal{R} \to \mathbb{R}$ be a measurable, additive and conditionally bounded functional. Then we have*

$$\mathbb{E}\,|\varphi(Z \cap W)| < \infty$$

and

$$\begin{aligned}&\mathbb{E}\,\varphi(Z \cap W) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}} \cdots \int_{\mathcal{K}} \varphi(W \cap K_1 \cap \ldots \cap K_k)\, \Theta(\mathrm{d}K_1) \cdots \Theta(\mathrm{d}K_k).\end{aligned} \tag{9.8}$$

If Z is stationary, then

$$\begin{aligned}&\mathbb{E}\,\varphi(Z \cap W) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \Phi(W, K_1, \ldots, K_k)\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k)\end{aligned}$$

with

$$\begin{aligned}&\Phi(W, K_1, \ldots, K_k) \\ &:= \int_{(\mathbb{R}^d)^k} \varphi(W \cap (K_1 + x_1) \cap \ldots \cap (K_k + x_k)) \lambda^k(\mathrm{d}(x_1, \ldots, x_k)).\end{aligned}$$

To proceed further, in the stationary case, we need to compute the integrals

$$\int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \Phi(W, K_1, \ldots, K_k)\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k).$$

This is possible for special choices of φ, using the translative integral formulas from Section 6.4.

Let us first consider the volume again, $\varphi = V_d$. For convex bodies $K, K_1, \ldots, K_k$, we have

$$\begin{aligned}&\Phi(W, K_1, \ldots, K_k) \\ &= \int_{(\mathbb{R}^d)^k} V_d(W \cap (K_1 + x_1) \cap \ldots \cap (K_k + x_k))\, \lambda^k(\mathrm{d}(x_1, \ldots, x_k)) \\ &= V_d(W) V_d(K_1) \cdots V_d(K_k).\end{aligned}$$

This follows from (6.15), but is also a direct consequence of Fubini's theorem. Thus, we obtain

$$\mathbb{E} V_d(Z \cap W) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} V_d(W) \overline{V}_d(X)^k = V_d(W) \left(1 - \mathrm{e}^{-\overline{V}_d(X)}\right).$$

This is nothing but relation (9.6) again.

Now we consider the intrinsic volume V_{d-1}, which is half the surface area (for convex bodies with interior points). Again from (6.15) or (5.15), we obtain

$$\int_{(\mathbb{R}^d)^k} V_{d-1}(K_0 \cap (K_1 + x_1) \cap \ldots \cap (K_k + x_k))\, \lambda^k(\mathrm{d}(x_1, \ldots, x_k))$$
$$= \sum_{i=0}^{k} V_d(K_0) \cdots V_d(K_{i-1}) V_{d-1}(K_i) V_d(K_{i+1}) \cdots V_d(K_k).$$

Therefore, we get

$$\begin{aligned} &\mathbb{E} V_{d-1}(Z \cap W) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \left[V_{d-1}(W) \overline{V}_d(X)^k + k V_d(W) \overline{V}_{d-1}(X) \overline{V}_d(X)^{k-1} \right] \\ &= V_d(W) \overline{V}_{d-1}(X) \sum_{k=1}^{\infty} \frac{[-\overline{V}_d(X)]^{k-1}}{(k-1)!} + V_{d-1}(W) \left(1 - \mathrm{e}^{-\overline{V}_d(X)}\right), \end{aligned}$$

hence

$$\mathbb{E} V_{d-1}(Z \cap W) = V_d(W) \overline{V}_{d-1}(X) \mathrm{e}^{-\overline{V}_d(X)} + V_{d-1}(W) \left(1 - \mathrm{e}^{-\overline{V}_d(X)}\right).$$

In contrast to the case of the volume, the quotient

$$\frac{\mathbb{E} V_{d-1}(Z \cap W)}{V_d(W)} = \overline{V}_{d-1}(X) \mathrm{e}^{-\overline{V}_d(X)} + \frac{V_{d-1}(W)}{V_d(W)} \left(1 - \mathrm{e}^{-\overline{V}_d(X)}\right)$$

still depends on the observation window W. This influence disappears for increasing W. More precisely, we have

$$\lim_{r \to \infty} \frac{\mathbb{E} V_{d-1}(Z \cap rW)}{V_d(rW)} = \overline{V}_{d-1}(X) \mathrm{e}^{-\overline{V}_d(X)}.$$

The limit on the left side is denoted by $\overline{V}_{d-1}(Z)$ and called the **specific surface area** or the **density of the surface area** of Z (not caring about the factor $1/2$). Such limits exist under more general assumptions, as we shall study in the next section.

We repeat that so far we have obtained the two relations

$$\begin{aligned} \overline{V}_d(Z) &= 1 - \mathrm{e}^{-\overline{V}_d(X)}, \\ \overline{V}_{d-1}(Z) &= \overline{V}_{d-1}(X) \mathrm{e}^{-\overline{V}_d(X)}, \end{aligned} \tag{9.9}$$

connecting specific intrinsic volumes of the stationary Boolean model Z with corresponding densities of the underlying particle process X.

We return to the case of a general additive functional φ (continuous on $\mathcal{K}'$). An explicit formula can still be obtained if we assume that the particle process X and thus the Boolean model Z is isotropic.

Let Z be a stationary, isotropic Boolean model (always with convex grains). Since the grain distribution $\mathbb{Q}$ of X is rotation invariant, we can insert rotations, integrate over the rotation group, apply Fubini's theorem, and then use the iteration of Hadwiger's general integral geometric theorem (Theorem 5.1.2). In this way, we obtain

$$\begin{aligned}
&\int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \Phi(W, K_1, \dots, K_k)\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k) \\
&= \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{SO_d} \int_{\mathbb{R}^d} \cdots \int_{SO_d} \int_{\mathbb{R}^d} \varphi(W \cap (\vartheta_1 K_1 + x_1) \cap \ldots \cap (\vartheta_k K_k + x_k)) \\
&\quad \times \lambda(\mathrm{d}x_1)\, \nu(\mathrm{d}\vartheta_1) \cdots \lambda(\mathrm{d}x_k)\, \nu(\mathrm{d}\vartheta_k)\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k) \\
&= \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \sum_{\substack{r_0, \dots, r_k = 0 \\ r_0 + \ldots + r_k = kd}}^{d} c^d_{d-r_0}\, \varphi_{r_0}(W) \prod_{i=1}^{k} c_d^{r_i} V_{r_i}(K_i)\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k).
\end{aligned}$$

This gives

$$\begin{aligned}
&\mathbb{E}\, \varphi(Z \cap W) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{\substack{r_0, \dots, r_k = 0 \\ r_0 + \ldots + r_k = kd}}^{d} c^d_{d-r_0}\, \varphi_{r_0}(W) \prod_{i=1}^{k} c_d^{r_i} \overline{V}_{r_i}(X) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{m=0}^{d} c^d_{d-m}\, \varphi_m(W) \sum_{\substack{m_1, \dots, m_k = 0 \\ m_1 + \ldots + m_k = kd-m}}^{d} \prod_{i=1}^{k} c_d^{m_i} \overline{V}_{m_i}(X) \\
&= \varphi(W) \left(1 - \mathrm{e}^{-\overline{V}_d(X)}\right) \\
&\quad + \sum_{m=1}^{d} c^d_{d-m}\, \varphi_m(W) \underbrace{\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{\substack{m_1, \dots, m_k = 0 \\ m_1 + \ldots + m_k = kd-m}}^{d} \prod_{i=1}^{k} c_d^{m_i} \overline{V}_{m_i}(X)}_{S}.
\end{aligned}$$

We rearrange the last two sums according to the number, say s, of indices among $m_1, \dots, m_k$ that are smaller than d; here $s \in \{1, \dots, m\}$. This gives

$$\begin{aligned}
S &= \sum_{s=1}^{m} \sum_{r=0}^{\infty} \binom{r+s}{r} \frac{(-1)^{r+s-1}}{(r+s)!} \overline{V}_d(X)^r \sum_{\substack{m_1, \dots, m_s = 0 \\ m_1 + \ldots + m_s = sd-m}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} \overline{V}_{m_i}(X) \\
&= -\mathrm{e}^{-\overline{V}_d(X)} \sum_{s=1}^{m} \frac{(-1)^s}{s!} \sum_{\substack{m_1, \dots, m_s = 0 \\ m_1 + \ldots + m_s = sd-m}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} \overline{V}_{m_i}(X).
\end{aligned}$$

Thus we have obtained the following result.

Theorem 9.1.3. *Let Z be a Boolean model in $\mathbb{R}^d$, generated by a stationary, isotropic Poisson process X of convex particles. If $\varphi : \mathcal{R} \to \mathbb{R}$ is an additive functional which is continuous on $\mathcal{K}'$, then, for any $W \in \mathcal{K}'$ with $V_d(W) > 0$,*

$$\mathbb{E}\,\varphi(Z \cap W) = \varphi(W)\left(1 - \mathrm{e}^{-\overline{V}_d(X)}\right)$$
$$-\mathrm{e}^{-\overline{V}_d(X)} \sum_{m=1}^{d} c_{d-m}^{d}\,\varphi_m(W) \sum_{s=1}^{m} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=0\\ m_1+\dots+m_s=sd-m}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} \overline{V}_{m_i}(X).$$

A remarkable fact here is that the functional φ and its derived functionals φ_m are applied, on the right side, only to the sampling window W. For given φ and W, the expectation $\mathbb{E}\,\varphi(Z \cap W)$ *depends only on the densities of the intrinsic volumes of the generating particle process X*. Conversely, this means that no information about the stationary isotropic particle process X beyond its specific intrinsic volumes can be obtained from expectations of measurements $\varphi(Z \cap W)$. All the densities $\overline{V}_i(X)$ already occur if we choose for φ the intrinsic volumes $V_0, \dots, V_d$.

For that reason, we now concentrate on $\varphi = V_j$, the jth intrinsic volume. By the Crofton formula (5.6), we have

$$(V_j)_m = c_j^{d-m} c_d^{m+j} V_{m+j},$$

with $V_{m+j} = 0$ if $m + j > d$. Inserting this (and renaming the first summation index), we obtain

$$\mathbb{E}\,V_j(Z \cap W) = V_j(W)\left(1 - \mathrm{e}^{-\overline{V}_d(X)}\right)$$
$$-\mathrm{e}^{-\overline{V}_d(X)} \sum_{m=j+1}^{d} c_j^{m} V_m(W) \sum_{s=1}^{m-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j\\ m_1+\dots+m_s=sd+j-m}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} \overline{V}_{m_i}(X).$$

Here we can replace W by rW with $r > 0$ and then let r tend to infinity. We obtain the following result.

Theorem 9.1.4. *Let Z be a Boolean model in $\mathbb{R}^d$, generated by a stationary, isotropic Poisson process X of convex particles. The limit*

$$\overline{V}_j(Z) := \lim_{r\to\infty} \frac{\mathbb{E}\,V_j(Z \cap rW)}{V_d(rW)}$$

exists and is given by

$$\overline{V}_j(Z) = \mathrm{e}^{-\overline{V}_d(X)}\left[\overline{V}_j(X) - \sum_{s=2}^{d-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j+1\\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \prod_{i=1}^{s} c_j^{m_i} \overline{V}_{m_i}(X)\right]$$

if $j = 0, \dots, d-1$ *and*

$$\overline{V}_d(Z) = 1 - \mathrm{e}^{-\overline{V}_d(X)}.$$

The cases $j = d$ and $j = d-1$ have been obtained earlier without the isotropy assumption.

We call $\overline{V}_j(Z)$ the **density of the** jth **intrinsic volume**, or the **specific** jth **intrinsic volume**, of the Boolean model Z. In the next section, we shall introduce such densities for much more general random sets.

For Boolean models, Theorem 9.1.4 can be used to determine the densities $\overline{V}_i(X)$ of the underlying particle process from the densities $\overline{V}_j(Z)$ of the union set. We demonstrate this only in dimensions two and three. Here we use classical notation:

$$\begin{aligned} d = 2 \qquad & A = V_2, \text{ area} \\ & L = 2V_1, \text{ perimeter} \\ & \chi = V_0, \text{ Euler characteristic} \end{aligned}$$

$$\begin{aligned} d = 3 \qquad & V = V_3, \text{ volume} \\ & S = 2V_2, \text{ surface area} \\ & M = \pi V_1, \text{ integral of mean curvature} \\ & \chi = V_0, \text{ Euler characteristic.} \end{aligned}$$

We obtain the following relations: For $d = 2$,

$$\begin{aligned} \overline{A}(Z) &= 1 - \mathrm{e}^{-\overline{A}(X)}, \\ \overline{L}(Z) &= \mathrm{e}^{-\overline{A}(X)}\overline{L}(X), \\ \overline{\chi}(Z) &= \mathrm{e}^{-\overline{A}(X)}\left(\overline{\chi}(X) - \frac{1}{4\pi}\overline{L}(X)^2\right). \end{aligned}$$

For $d = 3$,

$$\begin{aligned} \overline{V}(Z) &= 1 - \mathrm{e}^{-\overline{V}(X)}, \\ \overline{S}(Z) &= \mathrm{e}^{-\overline{V}(X)}\overline{S}(X), \\ \overline{M}(Z) &= \mathrm{e}^{-\overline{V}(X)}\left(\overline{M}(X) - \frac{\pi^2}{32}\overline{S}(X)^2\right), \\ \overline{\chi}(Z) &= \mathrm{e}^{-\overline{V}(X)}\left(\overline{\chi}(X) - \frac{1}{4\pi}\overline{M}(X)\overline{S}(X) + \frac{\pi}{384}\overline{S}(X)^3\right). \end{aligned}$$

In either case, if all the parameters on the left side are known, then all parameters on the right side are known.

In particular, the densities on the left side determine $\overline{\chi}(X)$, which is the intensity γ of X. We point out, however, that the determination of the intensity $\overline{\chi}(X)$ requires the determination of the densities of all the $d+1$ intrinsic volumes of Z.

If we drop the isotropy assumption, hence consider a stationary Boolean model Z with convex grains and $\varphi = V_j$, we can combine Theorem 9.1.2 with the iterated translative formula (6.15). We obtain

$$\begin{aligned}
&\mathbb{E}\, V_j(Z \cap W) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \sum_{\substack{m_0,\dots,m_k=j \\ m_0+\dots+m_k=kd+j}}^{d} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} V^{(j)}_{m_0,\dots,m_k}(W, K_1, \dots, K_k) \\
&\quad \times \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k).
\end{aligned}$$

Again, we replace W by rW, divide by $V_d(rW)$ and let $r \to \infty$. Then, due to the homogeneity properties of the mixed functionals (see Theorem 6.4.1), all summands on the right side with $m_0 < d$ disappear asymptotically. For $m_0 = d$, we can use the decomposability property of the mixed functionals (Theorem 6.4.1) and get, with essentially the same arguments as in the isotropic case,

$$\begin{aligned}
\overline{V}_j(Z) &= \lim_{r\to\infty} \frac{\mathbb{E}\, V_j(Z \cap rW)}{V_d(rW)} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \sum_{\substack{m_1,\dots,m_k=j \\ m_1+\dots+m_k=(k-1)d+j}}^{d} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} V^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k) \\
&\quad \times \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k) \\
&= \sum_{s=1}^{d-j} \sum_{r=0}^{\infty} \binom{r+s}{r} \frac{(-1)^{r+s-1}}{(r+s)!} \overline{V}_d(X)^r \gamma^s \\
&\qquad \sum_{\substack{m_1,\dots,m_s=j \\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} V^{(j)}_{m_1,\dots,m_s}(K_1, \dots, K_s)\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_s) \\
&= -\mathrm{e}^{-\overline{V}_d(X)} \sum_{s=1}^{d-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j \\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X, \dots, X) \\
&= \mathrm{e}^{-\overline{V}_d(X)} \left(\overline{V}_j(X) - \sum_{s=2}^{d-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j+1 \\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X, \dots, X) \right).
\end{aligned}$$

The densities of X appearing here are special cases of the **mixed densities** defined by

$$\overline{V}^{(j)}_{m_1,\dots,m_s}(X, \dots, X, K_{k+1}, \dots, K_s)$$

$$:= \gamma^k \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} V^{(j)}_{m_1,\dots,m_s}(K_1,\dots,K_k,K_{k+1},\dots,K_s)\,\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k).$$

Hence, we arrive at the following result.

Theorem 9.1.5. *For a stationary Boolean model Z in $\mathbb{R}^d$ with convex grains, the limit*

$$\overline{V}_j(Z) := \lim_{r\to\infty} \frac{\mathbb{E}\, V_j(Z \cap rW)}{V_d(rW)}$$

exists and satisfies

$$\overline{V}_d(Z) = 1 - \mathrm{e}^{-\overline{V}_d(X)},$$

$$\overline{V}_{d-1}(Z) = \mathrm{e}^{-\overline{V}_d(X)}\overline{V}_{d-1}(X),$$

and

$$\overline{V}_j(Z) = \mathrm{e}^{-\overline{V}_d(X)}\left(\overline{V}_j(X) - \sum_{s=2}^{d-j}\frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j+1\\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X)\right)$$

for $j = 0,\dots,d-2$.

For $d = 2$, only the formula for the Euler characteristic V_0 is different from the isotropic case, and we have

$$\overline{A}(Z) = 1 - \mathrm{e}^{-\overline{A}(X)},$$

$$\overline{L}(Z) = \mathrm{e}^{-\overline{A}(X)}\overline{L}(X),$$

$$\overline{\chi}(Z) = \mathrm{e}^{-\overline{A}(X)}\left(\gamma - \overline{A}(X,-X)\right),$$

where

$$\overline{A}(X,-X) := \gamma^2 \int_{\mathcal{K}_0}\int_{\mathcal{K}_0} A(K,-M)\,\mathbb{Q}(\mathrm{d}K)\,\mathbb{Q}(\mathrm{d}M).$$

Here, we made use of the fact that the mixed functional $V^{(0)}_{1,1}(K,M)$ in the plane equals twice the mixed area $A(K,-M)$ of K and $-M$. It is obvious that the formulas can no longer be used directly for the estimation of γ. Hence, we need more (local) information for the statistical analysis of non-isotropic Boolean models; this will be discussed in Section 9.5.

As an immediate generalization of Theorem 9.1.5, we can replace the intrinsic volumes $V_j(Z \cap W)$ by (additively extended) mixed volumes $V(Z \cap W[j], M[d-j])$, $j = 1,\dots,d-1$, for $M \in \mathcal{K}'$. Applying Theorem 9.1.2 to the functional φ given by

$$\varphi(K) = \binom{d}{j} V(K[j], -M[d-j]) = V^{(0)}_{j,d-j}(K,M)$$

and using (6.15), we obtain the following result. Since the proof is identical to the previous one, we skip it.

Theorem 9.1.6. *Let Z be a stationary Boolean model in $\mathbb{R}^d$ with convex grains, $j \in \{1, \ldots, d-1\}$ and $M \in \mathcal{K}'$. Then the limit*

$$\overline{V}^{(0)}_{j,d-j}(Z, M) := \binom{d}{j} \lim_{r\to\infty} \frac{\mathbb{E}\, V(Z \cap rW[j], -M[d-j])}{V_d(rW)}$$

exists, is independent of W and satisfies

$$\overline{V}^{(0)}_{j,d-j}(Z, M) = \mathrm{e}^{-\overline{V}_d(X)} \Bigg(\overline{V}^{(0)}_{j,d-j}(X, M) \\ - \sum_{s=2}^{d-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\ldots,m_s=j+1 \\ m_1+\ldots+m_s=(s-1)d+j}}^{d-1} \overline{V}^{(0)}_{m_1,\ldots,m_s,d-j}(X, \ldots, X, M) \Bigg).$$

For $j = d-1$, the theorem yields

$$\overline{V}^{(0)}_{d-1,1}(Z, M) = \mathrm{e}^{-\overline{V}_d(X)} \overline{V}^{(0)}_{d-1,1}(X, M).$$

We can transform this into a local formula for area measures, using (14.23). Namely, we can rewrite (14.23) as

$$V^{(0)}_{d-1,1}(K, M) = \int_{S^{d-1}} h^*(M, -u)\, S_{d-1}(K, \mathrm{d}u) \tag{9.10}$$

(where h^* denotes the centered support function, see Section 4.6) and remark that, by additive extension in each variable, (9.10) holds for $K, M \in \mathcal{R}$. Since the vector space generated by the functions $h^*(M, \cdot)$, $M \in \mathcal{K}'$, is dense in the space of centered, continuous functions on S^{d-1} and since area measures have centroid 0, we deduce that the weak limit

$$\overline{S}_{d-1}(Z, \cdot) := \lim_{r\to\infty} \frac{\mathbb{E}\, S_{d-1}(Z \cap rW, \cdot)}{V_d(rW)}. \tag{9.11}$$

exists and satisfies the local density formula

$$\overline{S}_{d-1}(Z, \cdot) = \mathrm{e}^{-\overline{V}_d(X)} \overline{S}_{d-1}(X, \cdot)\,. \tag{9.12}$$

Here, according to (4.43),

$$\overline{S}_{d-1}(X, \cdot) := \gamma \int_{\mathcal{K}_0} S_{d-1}(K, \cdot)\, \mathbb{Q}(\mathrm{d}K).$$

Alternatively, (9.12) and the existence of the limit (9.11) can be obtained directly, with a proof similar to that of the previous results. For this, we use

Theorem 9.1.2 with $\varphi(K) := S_{d-1}(K, A)$, for a fixed Borel set $A \subset S^{d-1}$, together with the translative formula for area measures,

$$\int_{\mathbb{R}^d} S_{d-1}(K \cap (M + x), \cdot)\, \lambda(\mathrm{d}x) = V_d(M) S_{d-1}(K, \cdot) + V_d(K) S_{d-1}(M, \cdot).$$

For $K, M \in \mathcal{K}'$, this formula can either be deduced from the more general results in Section 6.4 or proved directly, using approximation by polytopes.

Note that (9.12) is a local version of (9.9).

We could also use (9.10) to obtain a formula for a local version of $\overline{V}^{(0)}_{j,d-j}(Z, M)$ for $j = 1$. This would involve the limit of the centered, additively extended support function

$$\overline{h}(Z, \cdot) := \lim_{r\to\infty} \frac{\mathbb{E}\, h^*(Z \cap rW, \cdot)}{V_d(rW)}$$

and expresses $\overline{h}(Z, \cdot)$ in terms of (mean values of) iterated versions of the mixed support functions, which appear in Theorem 6.4.6. We mention only the planar case, where the formula is simple. For $d = 2$,

$$\overline{h}(Z, \cdot) = \mathrm{e}^{-\overline{A}(X)} \overline{h}(X, \cdot), \tag{9.13}$$

where

$$\overline{h}(X, \cdot) := \gamma \int_{\mathcal{K}_0} h^*(K, \cdot)\, \mathbb{Q}(\mathrm{d}K).$$

Remark. Starting with Theorem 9.1.2, the results of this section remain true for Boolean models with polyconvex grains, if the additively extended functionals are used and the grain distribution satisfies (9.17).

Notes for Section 9.1 are included in the Notes for Section 9.4.

9.2 Densities of Additive Functionals

In the previous section we have seen that for stationary isotropic Boolean models Z and for arbitrary convex bodies W with positive volume the limit

$$\lim_{r\to\infty} \frac{\mathbb{E}\, V_j(Z \cap rW)}{V_d(rW)} = \overline{V}_j(Z)$$

always exists. In this way, the specific jth intrinsic volume $\overline{V}_j(Z)$ can be defined. The existence of the limit for Boolean models was deduced from explicit formulas. They yielded, at the same time, a representation of this density of the jth intrinsic volume of the Boolean model Z in terms of densities of the underlying particle process X.

Our aim in this section is to show the existence of corresponding densities for more general random closed sets and for rather general functionals. Essentially, the realizations of the random closed sets will locally belong to the convex ring $\mathcal{R}$, consisting of all finite unions of convex bodies in $\mathbb{R}^d$. The functionals to be considered will share with the intrinsic volumes the property of additivity.

The existence proof for the limit will be prepared by two lemmas. We make use of the unit cube $C^d = [0,1]^d$ and the half-open unit cube $C_0^d := [0,1)^d$. The **upper right boundary**

$$\partial^+ C^d := C^d \setminus C_0^d$$

is the union of d facets of C^d and hence belongs to the convex ring $\mathcal{R}$.

For $z \in \mathbb{Z}^d$, we put

$$C_z := C^d + z, \qquad C_{0,z} := C_0^d + z, \qquad \partial^+ C_z := \partial^+ C^d + z.$$

Then

$$\mathbb{R}^d = \bigcup_{z \in \mathbb{Z}^d} C_{0,z}$$

is a disjoint decomposition of $\mathbb{R}^d$.

Let φ be a real function on the convex ring $\mathcal{R}$, and let $K \in \mathcal{R}$. Since $\emptyset \neq K \cap C_{0,z} = K \cap C_{0,y}$ for $z, y \in \mathbb{Z}^d$ implies $z = y$, we can define

$$\varphi(K \cap C_{0,z}) := \varphi(K \cap C_z) - \varphi(K \cap \partial^+ C_z).$$

Lemma 9.2.1. *If $\varphi : \mathcal{R} \to \mathbb{R}$ is an additive function and $K \in \mathcal{R}$ is a polyconvex set, then*

$$\varphi(K) = \sum_{z \in \mathbb{Z}^d} \varphi(K \cap C_{0,z}).$$

Proof. We give two proofs for this crucial lemma. The first one employs the extension theorem 14.4.3. This allows us to work with relatively open polytopes and, therefore, with disjoint decompositions. The second proof does not use the extension theorem and has, therefore, a basic idea which is slightly less obvious.

First proof. Let $K \in \mathcal{R}$. For a polytope $P \in \mathcal{P}$ we define

$$\psi(P) := \varphi(K \cap P).$$

Then ψ is an additive functional on convex polytopes. By Theorem 14.4.3, it has a unique extension to an additive function on $U(\mathcal{P}_{ro})$, the system of finite unions of relatively open polytopes. We denote this extension also by ψ. Since $C_{0,z} = C_z \setminus \partial^+ C_z$ and all sets here belong to $U(\mathcal{P}_{ro})$, the additivity of ψ gives

$$\psi(C_{0,z}) = \psi(C_z) - \psi(\partial^+ C_z).$$

Moreover, $\psi(P) = 0$ for all convex polytopes P with $K \cap P = \emptyset$. We can choose a finite set $S \subset \mathbb{Z}^d$ such that

$$K \subset Q := \bigcup_{z \in S} C_{0,z}$$

and that $\operatorname{cl} Q$ is convex. Then

$$\begin{aligned}
\varphi(K) &= \varphi(K \cap \operatorname{cl} Q) = \psi(\operatorname{cl} Q) = \psi(Q) \\
&= \sum_{z \in \mathbb{Z}^d} \psi(C_{0,z}) \\
&= \sum_{z \in \mathbb{Z}^d} [\psi(C_z) - \psi(\partial^+ C_z)] \\
&= \sum_{z \in \mathbb{Z}^d} [\varphi(K \cap C_z) - \varphi(K \cap \partial^+ C_z)].
\end{aligned}$$

This concludes the first proof.

Second proof. We denote by $<$ the lexicographic order on $\mathbb{Z}^d$, that is,

$$(z^1, \ldots, z^d) < (y^1, \ldots, y^d)$$

if and only if $z^i = y^i$ for $i < k$ and $z_k < y_k$, for some $k \in \{1, \ldots, d\}$. Then

$$\partial^+ C_z = C_z \cap \bigcup_{z < y \in \mathbb{Z}^d} C_y$$

for $z \in \mathbb{Z}^d$. With the inclusion–exclusion principle we get (all sums are finite)

$$\begin{aligned}
\sum_{z \in \mathbb{Z}^d} \varphi(K \cap \partial^+ C_z) &= \sum_{z \in \mathbb{Z}^d} \varphi\left(\bigcup_{z < y \in \mathbb{Z}^d} (K \cap C_z \cap C_y) \right) \\
&= \sum_{z \in \mathbb{Z}^d} \sum_{k=1}^{\infty} (-1)^{k-1} \sum_{z < y_1 < \ldots < y_k} \varphi(K \cap C_z \cap C_{y_1} \cap \ldots \cap C_{y_k}) \\
&= -\sum_{k=2}^{\infty} (-1)^{k-1} \sum_{z_1 < \ldots < z_k} \varphi(K \cap C_{z_1} \cap \ldots \cap C_{z_k}).
\end{aligned}$$

This gives

$$\begin{aligned}
\varphi(K) &= \varphi\left(\bigcup_{z \in \mathbb{Z}^d} (K \cap C_z) \right) \\
&= \sum_{z \in \mathbb{Z}^d} \varphi(K \cap C_z) + \sum_{k=2}^{\infty} (-1)^{k-1} \sum_{z_1 < \ldots < z_k} \varphi(K \cap C_{z_1} \cap \ldots \cap C_{z_k}) \\
&= \sum_{z \in \mathbb{Z}^d} \varphi(K \cap C_z) - \sum_{z \in \mathbb{Z}^d} \varphi(K \cap \partial^+ C_z),
\end{aligned}$$

as asserted. $\square$

We recall that a function $\varphi : \mathcal{R} \to \mathbb{R}$ is conditionally bounded if it is bounded on $\{L \in \mathcal{K}' : L \subset K\}$, for each $K \in \mathcal{K}'$. In particular, if φ is continuous on $\mathcal{K}'$, it is conditionally bounded.

Lemma 9.2.2. *Let the function $\varphi : \mathcal{R} \to \mathbb{R}$ be translation invariant, additive and conditionally bounded. Then*

$$\lim_{r\to\infty} \frac{\varphi(rW)}{V_d(rW)} = \varphi(C_0^d)$$

for every $W \in \mathcal{K}'$ with $V_d(W) > 0$.

Proof. Let $W \in \mathcal{K}'$ and $0 \in \operatorname{int} W$, without loss of generality. For $K \in \mathcal{K}$ and $z \in \mathbb{Z}^d$ we put

$$\varphi(K, z) := \varphi(K \cap C_{0,z}). \tag{9.14}$$

Lemma 9.2.1 shows that

$$\varphi(rW) = \sum_{z\in\mathbb{Z}^d} \varphi(rW, z) \qquad \text{for } r > 0.$$

Define

$$Z_r^1 := \{z \in \mathbb{Z}^d : C_z \cap rW \neq \emptyset,\ C_z \not\subset rW\}$$

and

$$Z_r^2 := \{z \in \mathbb{Z}^d : C_z \subset rW\}.$$

Then

$$\lim_{r\to\infty} \frac{|Z_r^1|}{V_d(rW)} = 0, \qquad \lim_{r\to\infty} \frac{|Z_r^2|}{V_d(rW)} = 1, \tag{9.15}$$

where $|A|$ denotes the number of elements of a set A. The limit relations follow from the fact that one easily shows the existence of numbers $r_0 > s, t > 0$ such that

$$z \in Z_r^1 \Rightarrow C_z \subset (r+s)W \setminus (r-s)W$$

and

$$(r-t)W \subset \bigcup_{z\in Z_r^2} C_z$$

for $r \geq r_0$.

By assumption,

$$|\varphi(rW, z)| = |\varphi(rW - z, 0)| \leq b$$

with some constant b independent of z, W and r. This gives

$$\frac{1}{V_d(rW)} \left| \sum_{z\in Z_r^1} \varphi(rW, z) \right| \leq b \frac{|Z_r^1|}{V_d(rW)} \to 0 \qquad \text{for } r \to \infty.$$

From this we deduce that

$$\begin{aligned}\lim_{r\to\infty}\frac{\varphi(rW)}{V_d(rW)} &= \lim_{r\to\infty}\frac{1}{V_d(rW)}\sum_{z\in\mathbb{Z}^d}\varphi(rW,z)\\ &= \lim_{r\to\infty}\frac{1}{V_d(rW)}\sum_{z\in Z_r^2}\varphi(rW,z)\\ &= \varphi(C_0^d)\lim_{r\to\infty}\frac{|Z_r^2|}{V_d(rW)}\\ &= \varphi(C_0^d).\end{aligned}$$

This proves the lemma. □

Mean Values of Additive Functionals for Random Sets

Now we introduce a suitable class of random closed sets for which the existence of densities for rather general functionals can be shown. Recall that the **extended convex ring** in $\mathbb{R}^d$ is defined by

$$\mathcal{S} := \{F \subset \mathbb{R}^d : F \cap K \in \mathcal{R} \ \text{ for } K \in \mathcal{K}\}.$$

The elements of $\mathcal{S}$ are called **locally polyconvex sets**. Thus a locally polyconvex set has the property that its intersection with any convex body is a finite union of convex bodies.

If $M \in \mathcal{R}$ is a nonempty polyconvex set, there are a number $m \in \mathbb{N}$ and convex bodies $K_1, \dots, K_m \in \mathcal{K}'$ such that $M = K_1 \cup \ldots \cup K_m$. The smallest number m with this property is denoted by $N(M)$. We also put $N(\emptyset) = 0$. By Lemma 4.3.1, the function $N : \mathcal{R} \to \mathbb{N}_0$ is measurable. Now we can define the random closed sets which will be admitted in the following.

Definition 9.2.1. *A* **standard random set** *in* $\mathbb{R}^d$ *is a random closed set* Z *in* $\mathbb{R}^d$ *with the following properties:*

(a) *The realizations of* Z *are a.s. locally polyconvex.*
(b) Z *is stationary.*
(c) Z *satisfies the integrability condition*

$$\mathbb{E}\, 2^{N(Z\cap C^d)} < \infty. \tag{9.16}$$

Important examples of standard random sets are the Boolean models Z with convex grains. As we have seen in Section 9.1, they satisfy (9.16).

We are now in a position to prove the existence of densities of suitable functionals for standard random sets.

Theorem 9.2.1. *Let Z be a standard random set, let the function $\varphi : \mathcal{R} \to \mathbb{R}$ be translation invariant, additive, measurable and conditionally bounded. Let $W \in \mathcal{K}'$ be such that $V_d(W) > 0$. Then the limit*

$$\overline{\varphi}(Z) := \lim_{r\to\infty} \frac{\mathbb{E}\,\varphi(Z \cap rW)}{V_d(rW)}$$

exists and satisfies

$$\overline{\varphi}(Z) = \mathbb{E}\,\varphi(Z \cap C_0^d).$$

Hence, $\overline{\varphi}(Z)$ is independent of W.

Proof. Without loss of generality, we can assume that $W \subset C^d$. For given $\omega \in \Omega$, there is a representation

$$Z(\omega) \cap W = \bigcup_{i=1}^{N_W(\omega)} K_i(\omega) \qquad \text{with } K_i(\omega) \in \mathcal{K}',$$

where $N_W(\omega) := N(Z(\omega) \cap W)$. By the inclusion–exclusion principle,

$$\begin{aligned} &\varphi(Z(\omega) \cap W) \\ &= \sum_{k=1}^{N_W(\omega)} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le N_W(\omega)} \varphi(K_{i_1}(\omega) \cap \ldots \cap K_{i_k}(\omega)). \end{aligned}$$

Since φ is conditionally bounded, there is a constant b such that

$$\mathbb{E}\,|\varphi(Z \cap W)| \le b\,\mathbb{E} \sum_{k=1}^{N_W} \binom{N_W}{k} \le b\,\mathbb{E}\,2^{N(Z\cap W)} \le b\,\mathbb{E}\,2^{N(Z\cap C^d)},$$

since $N(Z(\omega) \cap W) \le N(Z(\omega) \cap C^d)$. By assumption, the right side is finite, hence $\varphi(Z \cap W)$ is integrable. For a polyconvex set $M \in \mathcal{R}$, the integrability of $\varphi(Z \cap M)$ then follows from additivity, using the inclusion–exclusion principle again. Therefore, we can define a functional $\phi : \mathcal{R} \to \mathbb{R}$ by

$$\phi(M) := \mathbb{E}\,\varphi(Z \cap M) \qquad \text{for } M \in \mathcal{R}.$$

Then ϕ is additive, translation invariant (as follows from the stationarity of Z) and conditionally bounded (as follows from the last estimate above). Now the assertion of the theorem follows from Lemma 9.2.2. $\square$

With suitable conditions on φ and Z, the preceding result would also hold for general stationary random closed sets with values in $\mathcal{F}$. However, the useful functionals φ on $\mathcal{R}$ that satisfy the assumptions of the theorem have, with the exception of the volume, no reasonable extension to all of $\mathcal{C}$; therefore, the restriction to the convex ring seems appropriate (but see the Notes for Section 9.4, for other set classes).

For $\varphi = V_d$, the density $\overline{V}_d$ was already defined in Section 2.4, for example, by

$$\overline{V}_d(Z) = \frac{\mathbb{E}V_d(Z \cap W)}{V_d(W)},$$

for any Borel set W with $V_d(W) > 0$. Thus, the introduction of the specific volume does not require a limit procedure, and the assertion of Theorem 9.2.1 is trivial, since $V_d(Z \cap \partial^+ C^d) = 0$.

The quantity $\overline{\varphi}(Z)$ in Theorem 9.2.1 is called the **φ-density** of Z. The most important functionals φ are the intrinsic volumes $V_0, \dots, V_{d-1}$. The density $\overline{V}_j(Z)$ is also called the **specific** jth **intrinsic volume** of Z. In particular, $2\overline{V}_{d-1}(Z)$ is the **specific surface area** of Z. In Section 9.4, we shall give an alternative interpretation of $\overline{V}_j(Z)$, as a Radon–Nikodym derivative of the expectation of the curvature measure $\Phi_j(Z, \cdot)$ (which is a stationary random measure on $\mathbb{R}^d$) with respect to the Lebesgue measure λ. This representation will allow us in Section 11.1 to introduce specific intrinsic volumes (as functions on $\mathbb{R}^d$) also for non-stationary random closed sets.

Further functionals φ to which Theorem 9.2.1 can be applied are:

- the mixed volumes, $\varphi(K) := V(K[j], M[d-j])$, for fixed $M \in \mathcal{R}$ and $j \in \{1, \dots, d-1\}$,
- the surface area measure, $\varphi(K) := S_{d-1}(K, A)$, for $A \in \mathcal{B}(S^{d-1})$,
- the centered support function, $\varphi(K) := h^*(K, u)$, for $u \in S^{d-1}$.

Letting $A \in \mathcal{B}(S^{d-1})$ vary, we thus get, under the assumptions of Theorem 9.2.1, a finite Borel measure $\overline{S}_{d-1}(Z, \cdot)$ on S^{d-1}, the **specific surface area measure** or the **mean normal measure** of Z. By (14.15), $\overline{S}_{d-1}(Z, \cdot)$ is always nonnegative, and it is centered. If $\overline{S}_{d-1}(Z, \cdot)$ is not concentrated on a subsphere, it is (by Theorem 14.3.1) the surface area measure of a unique convex body in $\mathcal{K}_0$, which we call the **Blaschke body** $B(Z)$ of Z. Further, letting $u \in S^{d-1}$ vary, we get a centered continuous function $\overline{h}(Z, \cdot)$ on S^{d-1}, the **specific support function** of Z. (The continuity can be shown with the aid of (6.28), cf. Goodey and Weil [280, p. 339].) The function $\overline{h}(Z, \cdot)$ is a support function for $d = 2$, but in general not for $d \geq 3$.

Mixed volumes $V(K[j], M[d-j])$ are only special cases of mixed functionals $V^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k)$, as studied in Section 6.4. Since the latter are additive in each component and have an additive extension to $\mathcal{R}$, Theorem 9.2.1 also yields densities of mixed functionals for standard random sets Z. Later, we shall need one series of these mixed densities for Z, but in a local version. Namely, for $j \in \{0, \dots, d\}$, $k \in \{j, \dots, d\}$, $M \in \mathcal{K}'$ and $A \in \mathcal{B}$, the functional

$$\varphi_A : K \mapsto \Phi^{(j)}_{k,d-k+j}(K, M; \mathbb{R}^d \times A)$$

satisfies the assumptions of Theorem 9.2.1. The density $\overline{\varphi_A}(Z)$, as a function of A, is a (signed) measure, which we denote by $\overline{\Phi}^{(j)}_{k,d-k+j}(Z, M; \cdot)$, thus

$$\overline{\Phi}^{(j)}_{k,d-k+j}(Z,M;A) = \lim_{r\to\infty} \frac{\mathbb{E}\,\Phi^{(j)}_{k,d-k+j}(Z\cap rW,M;\mathbb{R}^d\times A)}{V_d(rW)},$$

with a window W as before. We call $\overline{\Phi}^{(j)}_{k,d-k+j}(Z,M;\cdot)$ the **specific** (j,k)th **mixed measure** of Z (for given M). We notice that

$$\overline{\Phi}^{(j)}_{j,d}(Z,M;\cdot) = \overline{V}_j(Z)\,\Phi_d(M,\cdot),$$

$$\overline{\Phi}^{(j)}_{d,j}(Z,M;\cdot) = \overline{V}_d(Z)\,\Phi_j(M,\cdot).$$

We finally remark that we can now give a new interpretation of the intensity of a stationary process of k-flats that was introduced in Theorem 4.4.2. Let Z_X be the union set of a stationary k-flat process X with intensity γ. For $r>0$ and $W\in\mathcal{K}$ with $V_d(W)>0$ we have, by the additivity of V_k,

$$\frac{1}{V_d(rW)}\,\mathbb{E}V_k(Z_X\cap rW) = \frac{1}{V_d(rW)}\,\mathbb{E}\sum_{E\in X}\lambda_E(rW) = \gamma,$$

by Theorem 4.4.3. Thus, the left side is independent of r, hence for $r\to\infty$ it converges to $\overline{V}_k(Z_X)$, even without the integrability condition (9.16). Therefore, we have

$$\gamma = \overline{V}_k(Z_X).$$

Mean Values for Particle Processes

For a stationary particle process X, the φ-density $\overline{\varphi}(X)$ was already introduced in Section 4.1, and different representations were established. Further representations in the case of additive functionals can now be obtained in analogy to Theorem 9.2.1. Of the stationary particle process X in $\mathcal{R}$ to be considered we need the integrability condition

$$\int_{\mathcal{R}_0} 2^{N(C)}V_d(C+B^d)\,\mathbb{Q}(\mathrm{d}C) < \infty, \tag{9.17}$$

where $\mathbb{Q}$ is the grain distribution of X. If the particles are convex, then condition (9.17) reduces to the original condition (4.4).

Theorem 9.2.2. *Let X be a stationary particle process in $\mathbb{R}^d$ with particles in $\mathcal{R}$ and with grain distribution $\mathbb{Q}$ satisfying* (9.17)*. Let $\varphi:\mathcal{R}\to\mathbb{R}$ be translation invariant, additive, measurable and conditionally bounded. Then φ is $\mathbb{Q}$-integrable, and*

$$\overline{\varphi}(X) = \lim_{r\to\infty}\frac{1}{V_d(rW)}\,\mathbb{E}\sum_{C\in X}\varphi(C\cap rW)$$

holds for all $W\in\mathcal{K}$ with $V_d(W)>0$. Moreover,

$$\overline{\varphi}(X) = \mathbb{E}\sum_{C\in X}\varphi(C\cap C^d_0).$$

Proof. For given $C \in \mathcal{R}_0$, let

$$Z := \{z \in \mathbb{Z}^d : C \cap C_z \neq \emptyset\}.$$

For $z \in Z$ we have $C_z \subset C + \sqrt{d}B^d$, hence

$$|Z| = \lambda\left(\bigcup_{z \in Z} C_z\right) \leq V_d(C + \sqrt{d}B^d) \leq kV_d(C + B^d), \tag{9.18}$$

if k is chosen such that $\sqrt{d}B^d$ can be covered by k unit balls.

Let $W \in \mathcal{K}'$ and $r > 0$. By

$$\varphi_x(M) := \varphi((M + x) \cap rW), \qquad M \in \mathcal{R},$$

for given $x \in \mathbb{R}^d$, an additive functional φ_x is defined. By Lemma 9.2.1 we get, using the notation of (9.14), that

$$\varphi((C + x) \cap rW) = \varphi_x(C) = \sum_{z \in Z} \varphi_x(C, z). \tag{9.19}$$

As in the proof of Theorem 9.2.1, the additivity and translation invariance of φ lead to an estimate

$$|\varphi_x(C, z)| \leq b2^{N(C)}, \tag{9.20}$$

with

$$b := c(d) \sup_{L \in \mathcal{K},\, L \subset C^d} |\varphi(L)| < \infty$$

and $c(d)$ depending only on d. Together with (9.18), (9.19) and (9.20) this gives

$$|\varphi((C + x) \cap rW)| \leq kb2^{N(C)}V_d(C + B^d).$$

Since the right side is $\mathbb{Q}$-integrable, this yields (with $x = 0$ and $r \to \infty$) the $\mathbb{Q}$-integrability of φ. Further, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |\varphi((C + x) \cap rW)|\, \lambda(\mathrm{d}x) &\leq \sum_{z \in Z} \int_{\mathbb{R}^d} |\varphi_x(C, z)|\, \lambda(\mathrm{d}x) \\ &\leq |Z| b2^{N(C)} V_d(rW + C^d) \\ &\leq kbV_d(rW + C^d)2^{N(C)}V_d(C + B^d)\end{aligned}$$

and hence

$$\int_{\mathcal{R}_0} \int_{\mathbb{R}^d} |\varphi((C + x) \cap rW)|\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}C) < \infty.$$

Therefore, we can apply the Campbell theorem (Theorem 3.1.2), and with Theorem 4.1.1 we obtain

$$\mathbb{E} \sum_{C \in X} \varphi(C \cap rW) = \gamma \int_{\mathcal{R}_0} \int_{\mathbb{R}^d} \varphi((C + x) \cap rW)\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}C).$$

Here we can decompose

$$\int_{\mathbb{R}^d} \varphi((C+x) \cap rW)\, \lambda(\mathrm{d}x) = I_1(r) + I_2(r)$$

with

$$I_\nu(r) := \sum_{z \in Z} \int_{A_r^\nu - z} \varphi_x(C, z)\, \lambda(\mathrm{d}x), \quad \nu = 1, 2,$$

$$A_r^1 := \{x \in \mathbb{R}^d : (C^d + x) \cap rW \neq \emptyset,\ C^d + x \not\subset rW\},$$
$$A_r^2 := \{x \in \mathbb{R}^d : C^d + x \subset rW\}.$$

We have

$$\lim_{r\to\infty} \frac{\lambda(A_r^1)}{V_d(rW)} = 0, \quad \lim_{r\to\infty} \frac{\lambda(A_r^2)}{V_d(rW)} = 1.$$

With (9.20) we get

$$|I_1(r)| \leq |Z| b 2^{N(C)} \lambda(A_r^1)$$

and hence

$$\lim_{r\to\infty} \frac{I_1(r)}{V_d(rW)} = 0.$$

Further, we have

$$I_2(r) = \sum_{z \in Z} \varphi(C, z) \lambda(A_r^2) = \varphi(C) \lambda(A_r^2)$$

by Lemma 9.2.1 and thus

$$\lim_{r\to\infty} \frac{I_2(r)}{V_d(rW)} = \varphi(C).$$

This yields

$$\frac{|I_1(r) + I_2(r)|}{V_d(rW)} \leq k b 2^{N(C)} V_d(C + B^d) \frac{\lambda(A_r^1)}{V_d(rW)} + |\varphi(C)|.$$

By (9.17) and the $\mathbb{Q}$-integrability of φ we now obtain, using the dominated convergence theorem,

$$\lim_{r\to\infty} \frac{1}{V_d(rW)} \mathbb{E} \sum_{C \in X} \varphi(C \cap rW) = \gamma \int_{\mathcal{R}_0} \varphi(C)\, \mathbb{Q}(\mathrm{d}C) = \overline{\varphi}(X),$$

which is the first assertion of the theorem.

We put

$$\phi(K) := \mathbb{E} \sum_{C \in X} \varphi(C \cap K) \qquad \text{for } K \in \mathcal{R}.$$

For $K \in \mathcal{K}$, the random variable $\sum_{C \in X} \varphi(C \cap K)$ is integrable, as shown, hence by additivity it is also integrable for $K \in \mathcal{R}$. The functional ϕ is additive, translation invariant and conditionally bounded. Now Lemma 9.2.2 yields the second assertion of the theorem. $\square$

As in the case of random closed sets Z, the natural candidates for the functional φ are the intrinsic volumes, the mixed volumes, the surface area measure, and the centered support function. These choices lead to the **specific intrinsic volumes** $\overline{V}_j(X)$ ($j \in \{0,\dots,d\}$) and to the mean values $\overline{V}(X[j], M[d-j])$ ($M \in \mathcal{R}$, $j \in \{1,\dots,d-1\}$), $\overline{S}_{d-1}(X,\cdot)$ and $\overline{h}(X,\cdot)$. Other examples are the densities of mixed measures or mixed functionals. We shall need the **specific** (j,k)th **mixed measure** $\overline{\Phi}^{(j)}_{k,d-k+j}(X,M;\cdot)$ of X (and M), which either arises as an outcome of Theorem 9.2.2 or can be defined directly by

$$\overline{\Phi}^{(j)}_{k,d-k+j}(X,M;\cdot) := \gamma \int_{\mathcal{R}_0} \Phi^{(j)}_{k,d-k+j}(C,M;\mathbb{R}^d \times \cdot)\, \mathbb{Q}(\mathrm{d}C).$$

Again, we have

$$\overline{\Phi}^{(j)}_{j,d}(X,M;\cdot) = \overline{V}_j(X)\, \Phi_d(M,\cdot),$$
$$\overline{\Phi}^{(j)}_{d,j}(X,M;\cdot) = \overline{V}_d(X)\, \Phi_j(M,\cdot).$$

The mixed densities

$$\overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X) \quad \text{and} \quad \overline{V}^{(0)}_{m_1,\dots,m_s,d-j}(X,\dots,X,M)$$

were introduced in Section 9.1, for Poisson particle processes X, by a multiple integral with respect to $(\gamma\mathbb{Q})^s$. Their definition immediately extends to general point processes on $\mathcal{K}'$ (or $\mathcal{R}'$). By an iterated application of Theorem 9.2.2, one gets

$$\overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X) = \lim_{r_1\to\infty} \cdots \lim_{r_s\to\infty} \frac{1}{V_d(r_1W)\cdots V_d(r_sW)}$$
$$\times\, \mathbb{E} \sum_{(C_1,\dots,C_s)\in X_1\times\dots\times X_s} V^{(j)}_{m_1,\dots,m_s}(C_1\cap r_1W,\dots,C_s\cap r_sW),$$

where $X_1,\dots,X_s$ are independent copies of X. A similar limit relation holds for $\overline{V}^{(0)}_{m_1,\dots,m_s,d-j}(X,\dots,X,M)$. For Poisson processes, we can use Corollary 3.2.4 to obtain

$$\overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X) = \lim_{r_1\to\infty} \cdots \lim_{r_s\to\infty} \frac{1}{V_d(r_1W)\cdots V_d(r_sW)}$$
$$\times\, \mathbb{E} \sum_{(C_1,\dots,C_s)\in X^s_{\neq}} V^{(j)}_{m_1,\dots,m_s}(C_1\cap r_1W,\dots,C_s\cap r_sW).$$

Also for stationary particle processes X, we shall give in Section 9.4 an alternative interpretation of $\overline{V}_j(X)$ as a Radon–Nikodym derivative with respect to the Lebesgue measure λ, namely of the expectation of the stationary random measure

$$\Phi_j(X,\cdot) := \sum_{K\in X} \Phi_j(K,\cdot).$$

This representation will be used in Section 11.1 to introduce specific intrinsic volumes for non-stationary particle processes X, again as functions on $\mathbb{R}^d$.

We remark that the measure $\overline{S}_{d-1}(X,\cdot)$ on the unit sphere S^{d-1} is always nonnegative (by (14.15)) and centered and therefore, if it is not concentrated on a subsphere, is the surface area measure of a unique convex body in $\mathcal{K}_0$, the **Blaschke body** $B(X)$, which was introduced in Section 4.6. There, we assumed that the particles $K \in X$ are convex, but now we see that polyconvex particles can be allowed. For polyconvex particles, $\overline{h}(X,\cdot)$ is a continuous function. If the particles are convex (or if $d = 2$), $\overline{h}(X,\cdot)$ is the support function of a unique convex body in $\mathcal{K}_0$, the **mean body** $M(X)$, which was also introduced and studied in Section 4.6.

Remark. For the introduction of the densities $\overline{\varphi}(Z)$, $\overline{\varphi}(X)$ of translation invariant functionals φ for random sets Z and particle processes X, we have not used the stationarity of Z or X to its full extent. In fact, for a particle process X, for example, we need only the invariance of the expectations

$$\mathbb{E} \sum_{K \in X+t} \varphi(K \cap W)$$

under all translations by $t \in \mathbb{R}^d$, for all windows W. This, in turn, is satisfied if the process X is **weakly stationary**, which means that its intensity measure is translation invariant. The process X is called **weakly isotropic** if its intensity measure is rotation invariant. The analogous terminology is used for processes of flats. For Poisson processes, there is no difference between stationarity and weak stationarity (isotropy and weak isotropy), by Theorem 3.2.1. We remark that most mean value formulas to be proved later for stationary (stationary and isotropic) particle processes X require only that X be weakly stationary (weakly stationary and weakly isotropic). For simplicity, however, we shall stay in the framework of stationarity and isotropy. A similar remark refers to random sets Z, where instead of stationarity it is mostly only needed that the expectations

$$\mathbb{E}\, \varphi((Z+t) \cap W)$$

are invariant under all translations by $t \in \mathbb{R}^d$.

Notes for Section 9.2 are included in the Notes for Section 9.4.

9.3 Ergodic Densities

In the previous section we have seen that for suitable closed random sets Z and functions φ a density $\overline{\varphi}(Z)$ can be defined by

$$\overline{\varphi}(Z) := \lim_{r\to\infty} \frac{\mathbb{E}\, \varphi(Z \cap rW)}{V_d(rW)}.$$

It is a natural question whether a corresponding limit exists also pointwise, that is, without taking the expectation. More precisely, we would like to know under which conditions the limit

$$\overline{\varphi}(Z,\omega) := \lim_{r\to\infty} \frac{\varphi(Z(\omega)\cap rW)}{V_d(rW)}$$

exists for almost all $\omega \in \Omega$. Existence assumed, $\overline{\varphi}(Z,\cdot)$ is a random variable, and we would expect that it satisfies

$$\mathbb{E}\,\overline{\varphi}(Z,\cdot) = \overline{\varphi}(Z).$$

Particularly interesting are the random closed sets Z for which $\overline{\varphi}(Z,\cdot)$ is almost surely equal to a constant, thus satisfying

$$\overline{\varphi}(Z,\cdot) = \overline{\varphi}(Z) \qquad \text{a.s.}$$

For such a random set Z, the φ-density $\overline{\varphi}(Z)$ can be estimated from a single realization $Z(\omega)$, by measuring

$$\frac{\varphi(Z(\omega)\cap W)}{V_d(W)}$$

in a large window W. Results of the type

$$\overline{\varphi}(Z) = \lim_{r\to\infty} \frac{\varphi(Z\cap rW)}{V_d(rW)}, \tag{9.21}$$

where the left side is a constant and the right side is a limit of random variables, are known as ergodic theorems. More precisely, one talks of individual ergodic theorems if the equality holds almost surely, and of statistical ergodic theorems if on the right one has L^p-convergence, for suitable p. We restrict ourselves here to individual ergodic theorems. Such an ergodic result holds for random closed sets Z satisfying certain independence properties, for instance, for ergodic random closed sets, as will be explained below. If the density $\overline{\varphi}(Z)$ can be obtained in the form (9.21), one talks of an ergodic density.

The program thus sketched will now be made precise. We shall, however, not give complete proofs, but for one crucial theorem rely on the literature. In order that the results be applicable not only to random sets, but also to point processes, the following considerations will adopt a more general point of view.

Let $(\Omega,\mathbf{A},\mathbb{P})$, as always, be the underlying probability space. A bijective map $T:\Omega\to\Omega$ with the property that T and T^{-1} are measurable and leave the probability measure $\mathbb{P}$ invariant (that is, satisfy $\mathbb{P}(TA)=\mathbb{P}(T^{-1}A)=\mathbb{P}(A)$ for all $A\in\mathbf{A}$), is called an **automorphism**. We assume that for $(\Omega,\mathbf{A},\mathbb{P})$ a set $\mathcal{T}=\{T_x : x\in\mathbb{Z}^d\}$ of automorphisms satisfying $T_xT_y=T_{x+y}$ for $x,y\in\mathbb{Z}^d$ is given; thus the set $\mathcal{T}$ together with the composition is an abelian group. We denote by $\mathbf{T}\subset\mathbf{A}$ the σ-algebra of all events invariant under $\mathcal{T}$, thus

$$\mathbf{T} := \{A \in \mathbf{A} : T_x A = A \text{ for all } x \in \mathbb{Z}^d\}.$$

A family $(\xi_K)_{K\in\mathcal{R}}$ of real random variables on $(\Omega, \mathbf{A})$ is called a **stochastic process with parameter space** $\mathcal{R}$. Since here the parameter $K \in \mathcal{R}$ plays the role of the time (for stochastic processes with continuous time), we also talk of a 'spatial process'. The spatial process $(\xi_K)_{K\in\mathcal{R}}$ is called **additive** if for $K, K' \in \mathcal{R}$ almost surely

$$\xi_{K\cup K'} + \xi_{K\cap K'} = \xi_K + \xi_{K'}$$

holds and, in addition, $\xi_\emptyset = 0$. It is called $\mathcal{T}$**-covariant** if for all $K \in \mathcal{R}$ and all $x \in \mathbb{Z}^d$ the equation

$$\xi_{K+x}(T_x\omega) = \xi_K(\omega)$$

holds for almost all $\omega \in \Omega$. Further, $(\xi_K)_{K\in\mathcal{R}}$ is called **bounded** if there exists an integrable random variable $\eta \geq 0$ with

$$|\xi_K| \leq \eta \qquad \text{a.s. for all } K \in \mathcal{K} \text{ with } K \subset C^d. \tag{9.22}$$

In the following theorem, $\mathbb{E}(\cdot \mid \mathbf{T})$ denotes the conditional expectation with respect to the σ-algebra $\mathbf{T}$ of $\mathcal{T}$-invariant events.

Theorem 9.3.1. *Let $(\xi_W)_{W\in\mathcal{R}}$ be an additive, $\mathcal{T}$-covariant, bounded stochastic process with parameter space $\mathcal{R}$. Then, for $W \in \mathcal{K}$ with $0 \in \operatorname{int} W$, the relation*

$$\lim_{r\to\infty} \frac{\xi_{rW}}{V_d(rW)} = \mathbb{E}\left(\xi_{C^d} - \xi_{\partial^+ C^d} \mid \mathbf{T}\right)$$

holds a.s.

Proof. First we proceed as in the proof of Lemma 9.2.2 and also use the same notation. Let W be as above and assume, without loss of generality, that $W \subset C^d$. For $z \in \mathbb{Z}^d$ and $K \in \mathcal{K}$ we put

$$\xi_{K,z} := \xi_{K\cap C_z} - \xi_{K\cap\partial^+ C_z}.$$

Then, by Lemma 9.2.1, for $r > 0$ we have

$$\begin{aligned}
\xi_{rW}(\omega) &= \sum_{z\in\mathbb{Z}^d} \xi_{rW,z}(\omega) \\
&= \sum_{z\in Z_r^1} \xi_{rW,z}(\omega) + \sum_{z\in Z_r^2} \xi_{rW,z}(\omega) \\
&= \sum_{z\in Z_r^1} \xi_{rW,z}(\omega) + \sum_{z\in Z_r^2} \left[\xi_{C^d}(T_{-z}\omega) - \xi_{\partial^+ C^d}(T_{-z}(\omega))\right].
\end{aligned}$$

If (9.22) holds, we obtain as in the proof of Theorem 9.2.1 that

$$|\xi_{rW,z}| \leq |\xi_{rW\cap C_z}| + |\xi_{rW\cap\partial^+ C_z}| \leq c_d\, \eta \circ T_{-z}$$

with a constant c_d, hence

$$\left| \sum_{z \in Z_r^1} \xi_{rW,z}(\omega) \right| \leq c_d \sum_{z \in Z_r^1} \eta(T_{-z}\omega).$$

Now we apply a version of the individual ergodic theorem, for which we refer to Tempel'man [755, Th. 6.1]. If ζ is an integrable random variable on Ω and $(Z_k)_{k\in\mathbb{N}}$ is an increasing sequence of sets $Z_k \subset \mathbb{Z}^d$, satisfying certain assumptions, then

$$\lim_{k\to\infty} \frac{1}{|Z_k|} \sum_{z\in Z_k} \zeta(T_{-z}\omega) = \mathbb{E}(\zeta \mid \mathbf{T})(\omega)$$

holds for almost all $\omega \in \Omega$.

We apply this theorem, first, to $\zeta = \eta$ and $Z_k = Z^1_{r_k} \cup Z^2_{r_k}$, respectively $Z_k = Z^2_{r_k}$, where $(r_k)_{k\in\mathbb{N}}$ is an increasing real sequence with $r_k \to \infty$. The required assumptions on the sequence $(Z_z)_{k\in\mathbb{N}}$ are satisfied in either case. Observing (9.15), we obtain

$$\lim_{k\to\infty} \frac{1}{V_d(r_kW)} \sum_{z\in Z^1_{r_k}} \eta(T_{-z}\omega) = 0$$

for almost all ω, hence also

$$\lim_{k\to\infty} \frac{1}{V_d(r_kW)} \sum_{z\in Z^1_{r_k}} \xi_{r_kW,z}(\omega) = 0.$$

Second, with $\zeta = \xi_{C^d} - \xi_{\partial^+C^d}$ and $Z_k = Z^2_{r_k}$ and with the result just obtained, we get

$$\begin{aligned} \lim_{k\to\infty} \frac{\xi_{r_kW}(\omega)}{V_d(r_kW)} &= \lim_{k\to\infty} \frac{1}{V_d(r_kW)} \sum_{z\in Z^2_{r_k}} [\xi_{C^d}(T_{-z}\omega) - \xi_{\partial^+C^d}(T_{-z}\omega)] \\ &= \mathbb{E}\left(\xi_{C^d} - \xi_{\partial^+C^d} \mid \mathbf{T}\right)(\omega) \end{aligned}$$

for almost all ω. This yields the assertion, also for the limit $r \to \infty$ (cf. Tempel'man, *loc.cit.* §8). $\square$

The quadruple $(\Omega, \mathbf{A}, \mathbb{P}, \mathcal{T})$ underlying our considerations is often called a **dynamical system**. This system is called **ergodic** if $\mathbb{P}(A) \in \{0,1\}$ for all $A \in \mathbf{T}$. In the ergodic case we have

$$\mathbb{E}\left(\xi_{C^d} - \xi_{\partial^+C^d} \mid \mathbf{T}\right) = \mathbb{E}\left(\xi_{C^d} - \xi_{\partial^+C^d}\right) \qquad \text{a.s.},$$

thus the limit in Theorem 9.3.1 is almost surely constant. The system $(\Omega, \mathbf{A}, \mathbb{P}, \mathcal{T})$ is called **mixing** if the automorphisms $T_x \in \mathcal{T}$ have the asymptotic independence property

$$\lim_{||x||\to\infty} \mathbb{P}(A \cap T_x B) = \mathbb{P}(A)\mathbb{P}(B) \tag{9.23}$$

for all $A, B \in \mathbf{A}$. Every mixing system is ergodic, since (9.23) with $A \in \mathbf{T}$ and $B = A$ implies $\mathbb{P}(A) = \mathbb{P}(A)^2$. The next lemma shows that it is sufficient to check (9.23) for a restricted class of sets.

Lemma 9.3.1. *The dynamical system $(\Omega, \mathbf{A}, \mathbb{P}, \mathcal{T})$ is mixing if there is a semialgebra $\mathbf{A}_0 \subset \mathbf{A}$ generating $\mathbf{A}$ and satisfying*

$$\lim_{||x||\to\infty} \mathbb{P}(A \cap T_x B) = \mathbb{P}(A)\mathbb{P}(B) \tag{9.24}$$

for all $A, B \in \mathbf{A}_0$.

Proof. Suppose such a semialgebra $\mathbf{A}_0$ exists. The algebra $\mathbf{A}_1$ generated by $\mathbf{A}_0$ consists of all finite disjoint unions of sets from $\mathbf{A}_0$. Therefore, (9.24) holds also for $A, B \in \mathbf{A}_1$. Now let $A, B \in \mathbf{A}$. For given $\epsilon > 0$ there are elements $A', B' \in A_1$ with $\mathbb{P}(A \bigtriangleup A') \le \epsilon$ and $\mathbb{P}(B \bigtriangleup B') \le \epsilon$ (see, for example, Chow and Teicher [175, p. 23]). From $\mathbb{P}((A\cap B) \bigtriangleup (A' \cap B')) \le \mathbb{P}(A \bigtriangleup A') + \mathbb{P}(B \bigtriangleup B')$ and the $\mathcal{T}$-invariance of $\mathbb{P}$ we obtain

$$\mathbb{P}((A \cap T_x B) \bigtriangleup (A' \cap T_x B')) \le 2\epsilon$$

for all $x \in \mathbb{Z}^d$. This gives

$$|\mathbb{P}(A \cap T_x B) - \mathbb{P}(A)\mathbb{P}(B)| \le |\mathbb{P}(A' \cap T_x B') - \mathbb{P}(A')\mathbb{P}(B')| + 4\epsilon,$$

which yields the assertion. $\square$

The preceding general considerations will now be applied to more concrete situations. Let Z be a stationary random closed set in $\mathbb{R}^d$. We can choose $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mathbb{P}_Z)$ as the underlying probability space $(\Omega, \mathbf{A}, \mathbb{P})$ and $\mathcal{T}$ as the group of the ordinary lattice translations of $\mathbb{R}^d$. Here, $T_x F := F + x$ for $F \in \mathcal{F}$ and $T_x \in \mathcal{T}$. Since Z is stationary, the probability measure $\mathbb{P}_Z$ is invariant under all translations $T_x \in \mathcal{T}$. We call the random closed set Z **mixing**, respectively **ergodic**, if the dynamical system $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mathbb{P}_Z, \mathcal{T})$ has this property. The following theorem expresses the mixing property of Z in terms of the capacity functional T_Z.

Theorem 9.3.2. *The stationary random closed Z in $\mathbb{R}^d$ is mixing if and only if*

$$\lim_{||x||\to\infty} (1 - T_Z(C_1 \cup T_x C_2)) = (1 - T_Z(C_1))(1 - T_Z(C_2)) \tag{9.25}$$

holds for all $C_1, C_2 \in \mathcal{C}$.

Proof. By Lemma 2.2.2, the system

$$\mathbf{A}_0 := \{\mathcal{F}^{C_0}_{C_1,\dots,C_k} : C_0, \dots, C_k \in \mathcal{C},\ k \in \mathbb{N}_0\}$$

is a semialgebra, which by Lemma 2.1.1 generates the σ-algebra $\mathcal{B}(\mathcal{F})$.

Let $A, B \in \mathbf{A}_0$, say

$$A = \mathcal{F}^{C_0}_{C_1,\dots,C_p}, \quad B = \mathcal{F}^{D_0}_{D_1,\dots,D_q}.$$

First we assume that $p, q \geq 1$. Using (2.2) and (2.3), we obtain

$$\begin{aligned}
&\mathbb{P}_Z(A \cap T_x B) \\
&= \mathbb{P}_Z\left(\mathcal{F}^{C_0 \cup T_x D_0}_{C_1,\dots,C_p,T_x D_1,\dots,T_x D_q}\right) \\
&= \sum_{r=0}^{p}\sum_{s=0}^{q}(-1)^{r+s-1} \sum_{\substack{0=i_0<i_1<\dots<i_r\leq p\\ 0=j_0<j_1<\dots<j_s\leq q}} T_Z\left(\bigcup_{\nu=0}^{r} C_{i_\nu} \cup \bigcup_{\mu=0}^{s} T_x D_{j_\mu}\right) \\
&= \sum_{r=0}^{p}\sum_{s=0}^{q}(-1)^{r+s} \sum_{\substack{0=i_0<i_1<\dots<i_r\leq p\\ 0=j_0<j_1<\dots<j_s\leq q}} \left(1 - T_Z\left(\bigcup_{\nu=0}^{r} C_{i_\nu} \cup T_x \bigcup_{\mu=0}^{s} D_{j_\mu}\right)\right).
\end{aligned}$$

This shows that (9.25) implies

$$\begin{aligned}
&\lim_{||x||\to\infty} \mathbb{P}_Z(A \cap T_x B) \\
&= \sum_{r=0}^{p}\sum_{s=0}^{q}(-1)^{r+s} \sum_{\substack{0=i_0<i_1<\dots<i_r\leq p\\ 0=j_0<j_1<\dots<j_s\leq q}} \left(1 - T_Z\left(\bigcup_{\nu=0}^{r} C_{i_\nu}\right)\right)\left(1 - T_Z\left(\bigcup_{\mu=0}^{s} D_{j_\mu}\right)\right) \\
&= \mathbb{P}_Z(A)\mathbb{P}_Z(B).
\end{aligned}$$

The argument is similar if $p = 0$ or $q = 0$, where, for example, $\mathbb{P}(\mathcal{F}^{C_0}) = 1 - T_Z(C_0)$ has to be used. From Lemma 9.3.1 it now follows that Z is mixing. The converse direction is clear. $\square$

We remark that Theorem 9.3.2 and its proof verbally carry over to the case where $\mathbb{R}^d$ is replaced by $E = \mathcal{F}'(\mathbb{R}^d)$ as base space and the operation of $\mathcal{T}$ on $\mathcal{F}(E)$ is defined by $T_x F := F + x$ (with $F + x := \{A + x : A \in F\}$).

Now we apply Theorem 9.3.1 to the situation described in Theorem 9.2.1. Let Z be a stationary random closed set with values in $\mathcal{S}$. If $\varphi : \mathcal{R} \to \mathbb{R}$ is translation invariant, additive, measurable and conditionally bounded and if Z satisfies the integrability condition of Theorem 9.2.1, then

$$\varphi_K(Z) := \varphi(Z \cap K), \qquad K \in \mathcal{R},$$

defines an additive, $\mathcal{T}$-covariant and bounded stochastic process with parameter space $\mathcal{R}$. The $\mathcal{T}$-covariance of $(\varphi_K)_{K\in\mathcal{R}}$ follows from the translation invariance of φ, and the boundedness is a consequence of the integrability condition on Z, since φ is conditionally bounded. Recall that we work with the canonical probability space $(\Omega, \mathbf{A}, \mathbb{P}) = (\mathcal{F}, \mathcal{B}(\mathcal{F}), \mathbb{P}_Z)$, so that

$$\mathbf{T} = \{A \in \mathcal{B}(\mathcal{F}) : A + x = A \text{ for all } x \in \mathbb{Z}^d\},$$

where $A+x := \{F+x : F \in A\}$. In order to stay within the general framework of this section, we nevertheless continue to use notations such as $Z(\omega)$. From Theorem 9.3.1 we obtain the following result.

Theorem 9.3.3. *Let Z be a standard random set and let $\varphi : \mathcal{R} \to \mathbb{R}$ be additive, translation invariant, measurable and conditionally bounded. Then, for $W \in \mathcal{K}'$ with $0 \in \operatorname{int} W$, the limit*

$$\overline{\varphi}(Z, \omega) := \lim_{r\to\infty} \frac{\varphi(Z(\omega) \cap rW)}{V_d(rW)}$$

exists, for almost all $\omega \in \Omega$, and this limit is independent of W. Further,

$$\overline{\varphi}(Z, \cdot) = \mathbb{E}(\widetilde{\varphi}(Z) \mid \mathbf{T}) \qquad a.s.,$$

where $\widetilde{\varphi}(S) := \varphi(S \cap C^d) - \varphi(S \cap \partial^+ C^d)$ for $S \in \mathcal{S}$.

If Z is ergodic, then

$$\overline{\varphi}(Z, \cdot) = \overline{\varphi}(Z) \qquad a.s.$$

The last assertion follows from the fact that in the ergodic case we have

$$\overline{\varphi}(Z, \cdot) = \mathbb{E}(\widetilde{\varphi}(Z)) = \mathbb{E}\left[\varphi(Z \cap C^d) - \varphi(Z \cap \partial^+ C^d)\right] \qquad \text{a.s.}$$

and that this is equal to $\overline{\varphi}(Z)$, by Theorem 9.2.1.

Theorem 9.3.1 can also be applied to the situation of Theorem 9.2.2. Let X be a stationary particle process in $\mathbb{R}^d$. We consider the dynamical system $(\mathsf{N}, \mathcal{N}, \mathbb{P}_X, \mathcal{T})$ with $\mathsf{N} = \mathsf{N}(\mathcal{F}'(\mathbb{R}^d))$, $\mathcal{N} = \mathcal{N}(\mathcal{F}'(\mathbb{R}^d))$, where $\mathbb{P}_X$ is the distribution of X and $\mathcal{T} = \{T_x : x \in \mathbb{Z}^d\}$ is defined by $(T_x\eta)(B) := \eta(B - x)$ for $B \in \mathcal{B}(\mathcal{F}')$ and $\eta \in \mathsf{N}$. Because of the stationarity of X, the probability measure $\mathbb{P}_X$ is invariant under the mappings $T_x \in \mathcal{T}$. The invariant σ-algebra $\mathbf{T}$ is given by

$$\mathbf{T} = \{A \in \mathcal{N} : T_x A = A \text{ for all } x \in \mathbb{Z}^d\},$$

where $T_x A := \{T_x\eta : \eta \in A\}$. The particle process X is called **mixing**, respectively **ergodic**, if the dynamical system $(\mathsf{N}, \mathcal{N}, \mathbb{P}_X, \mathcal{T})$ has this property.

Now suppose that X and the functional φ satisfy the assumptions of Theorem 9.2.2. Then by

$$\varphi_K(X) := \sum_{C \in X} \varphi(C \cap K), \qquad K \in \mathcal{R},$$

we define an additive, $\mathcal{T}$-covariant and bounded stochastic process with parameter space $\mathcal{R}$. This is verified similarly to above, as well as the following result.

Theorem 9.3.4. *Let X be a stationary particle process in $\mathbb{R}^d$ with particles in $\mathcal{R}$ and with grain distribution $\mathbb{Q}$ satisfying* (9.17). *Let $\varphi : \mathcal{R} \to \mathbb{R}$ be additive, translation invariant, measurable and conditionally bounded. Then for $W \in \mathcal{K}'$ with $0 \in \operatorname{int} W$ the limit*

$$\overline{\varphi}(X, \omega) := \lim_{r\to\infty} \frac{1}{V_d(rW)} \sum_{C \in X(\omega)} \varphi(C \cap rW)$$

exists for almost all $\omega \in \Omega$, and this limit is independent of W. Further,

$$\overline{\varphi}(X, \cdot) = \mathbb{E}(\widetilde{\varphi}(X) \mid \mathbf{T}) \qquad a.s.,$$

where the function $\widetilde{\varphi}$ is defined by

$$\widetilde{\varphi}(\eta) := \sum_{C \in \operatorname{supp} \eta} [\varphi(C \cap C^d) - \varphi(C \cap \partial^+ C^d)], \quad \eta \in \mathsf{N}.$$

If X is ergodic, then

$$\overline{\varphi}(X, \cdot) = \overline{\varphi}(X) \qquad a.s.$$

At least for the most important examples of stationary random closed sets, respectively particle processes, we want to show that they are mixing and thus ergodic.

Theorem 9.3.5. *Stationary Boolean models are mixing.*

Proof. For the stationary Boolean model Z with intensity γ and grain distribution $\mathbb{Q}$, the capacity functional is, according to Theorem 9.1.1, given by

$$1 - T_Z(C) = \mathrm{e}^{-\gamma \int_{\mathcal{C}_0} V_d(C-K)\,\mathbb{Q}(\mathrm{d}K)}, \quad C \in \mathcal{C}.$$

For $C_1, C_2 \in \mathcal{C}$ we have

$$V_d((C_1 \cup T_x C_2) - K) = V_d((C_1 - K) \cup (C_2 - K + x)).$$

For given $K \in \mathcal{C}_0$ and sufficiently large $||x||$ we get $(C_1 - K) \cap (C_2 - K + x) = \emptyset$, hence

$$\lim_{||x||\to\infty} V_d((C_1 \cup T_x C_2) - K) = V_d(C_1 - K) + V_d(C_2 - K).$$

Further,

$$V_d((C_1 \cup T_x C_2) - K) \le V_d(C_1 - K) + V_d(C_2 - K).$$

The dominated convergence theorem yields (9.25) and thus the assertion. $\square$

The preceding theorem allows us, in particular, to interpret the intrinsic volume densities of a stationary Boolean model Z with grains in $\mathcal{R}$ as ergodic densities. In the case of convex grains, the integrability condition (9.16) is satisfied automatically, since for the Poisson particle process X that generates Z we have, for $K \in \mathcal{K}$,

$$\mathbb{E}\, 2^{N(Z\cap K)} \leq \mathbb{E}\, 2^{X(\mathcal{F}_K)} = \sum_{k=0}^{\infty} 2^k \mathrm{e}^{-\Theta(\mathcal{F}_K)} \frac{\Theta(\mathcal{F}_K)^k}{k!} = \mathrm{e}^{\Theta(\mathcal{F}_K)} < \infty.$$

Hence, for any convex body $W \in \mathcal{K}$ with $V_d(W) > 0$ and for $j = 0, \ldots, d$ we conclude from Theorems 9.3.3 and 9.3.5 that

$$\overline{V}_j(Z) = \lim_{r\to\infty} \frac{V_j(Z \cap rW)}{V_d(rW)} \qquad \text{a.s.}$$

A counterpart to Theorem 9.3.5 is true for particle processes.

Theorem 9.3.6. *Stationary Poisson particle processes in $\mathbb{R}^d$ are mixing.*

Proof. Given the stationary Poisson particle process X, we consider, as before Theorem 9.3.4, the dynamical system $(\mathsf{N}(E), \mathcal{N}(E), \mathbb{P}_X, \mathcal{T})$ for the base space $E = \mathcal{F}'(\mathbb{R}^d)$. By Lemma 3.1.4, $Z := \operatorname{supp} X$ defines a locally finite random closed set in E. For $T_x \in \mathcal{T}$ and $F \in \mathcal{F}(E)$ we let $T_x F := F + x$ (with $F + x := \{A + x : A \in F\}$). We show that the dynamical system $(\mathcal{F}(E), \mathcal{B}(\mathcal{F}(E)), \mathbb{P}_Z, \mathcal{T})$ is mixing. Since the operations of $\mathcal{T}$ on $\mathsf{N}(E)$ respectively $\mathcal{F}(E)$ commute with the mapping $i : \eta \mapsto \operatorname{supp} \eta$ of Lemma 3.1.4, we can then deduce that also $(\mathsf{N}(E), \mathcal{N}(E), \mathbb{P}_X, \mathcal{T})$ is mixing, which is the assertion.

As remarked after the proof of Theorem 9.3.2, that theorem holds also for $\mathcal{F}(\mathbb{R}^d)$ instead of $\mathbb{R}^d$. In this form it will be used in the following.

The capacity functional of the random closed set Z is given by

$$T_Z(C) = \mathbb{P}(C \cap \operatorname{supp} X \neq \emptyset) = \mathbb{P}(X(C) \neq 0)$$

for $C \in \mathcal{C}(E)$. If Θ denotes the intensity measure of the Poisson process X, then

$$1 - T_Z(C) = \mathrm{e}^{-\Theta(C)}. \tag{9.26}$$

In order to show (9.25) (in its generalized form), let $C_1, C_2 \in \mathcal{C}(E)$, thus these are compact subsets of $\mathcal{F}'(\mathbb{R}^d)$. There are (according to the proof of Lemma 2.3.1) compact subsets K_1, K_2 of $\mathbb{R}^d$ with $C_i \subset \mathcal{F}_{K_i}$, $i = 1, 2$. For $x \in \mathbb{Z}^d$ we have $T_x C_2 \subset \mathcal{F}_{K_2+x}$, hence

$$\begin{aligned}\Theta(C_1 \cap T_x C_2) &\leq \Theta(\mathcal{F}_{K_1, K_2+x}) \\ &= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{F}_{K_1, K_2+x}}(C + t)\, \lambda(\mathrm{d}t)\, \mathbb{Q}(\mathrm{d}C),\end{aligned}$$

by Theorem 4.1.1. For given $C \in \mathcal{C}_0$ and sufficiently large $||x||$, there is no t satisfying $(C + t) \cap K_1 \neq \emptyset$ and $(C + t) \cap (K_2 + x) \neq \emptyset$; therefore,

$$\lim_{||x||\to\infty} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{F}_{K_1,K_2+x}}(C+t)\,\lambda(\mathrm{d}t) = 0.$$

Moreover,

$$\int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{F}_{K_1,K_2+x}}(C+t)\,\lambda(\mathrm{d}t) \le V_d(K_1 - C),$$

and the function $C \mapsto V_d(K_1 - C)$ is $\mathbb{Q}$-integrable, by (4.4). The dominated convergence theorem yields

$$\lim_{||x||\to\infty} \Theta(C_1 \cap T_x C_2) = 0$$

and thus

$$\lim_{||x||\to\infty} \mathrm{e}^{-\Theta(C_1 \cup T_x C_2)} = \mathrm{e}^{-\Theta(C_1)}\mathrm{e}^{-\Theta(C_2)}.$$

Now (9.26) and Theorem 9.3.2 yield the assertion. □

If X is, in particular, a stationary Poisson particle process in $\mathcal{R}$ that satisfies (9.17), then for $W \in \mathcal{K}$ with $V_d(W) > 0$ we obtain

$$\overline{V}_j(X) = \lim_{r\to\infty} \frac{1}{V_d(rW)} \sum_{C\in X} V_j(C \cap rW) \qquad \text{a.s.}$$

Note for Section 9.3

First uses of ergodic theorems in stochastic geometry were made by Miles [517]; see also [521, 523]. In special situations, he has proved a number of convergence results for ‘increasing observation windows’. A unified and general treatment of such convergence theorems was given by Nguyen and Zessin [584], building on work of Tempel’man [755]. In Section 9.3, we followed their approach. In their application to Boolean models, however, Nguyen and Zessin did not mention that the conditional expectation obtained as a limit function is almost surely constant, as a consequence of the mixing property of stationary Poisson processes. The importance of mixing properties in stochastic geometry was pointed out by Cowan [180, 181]. A simple proof of the mixing property of stationary Boolean models was given by Wieacker [815]. In using the capacity functional in establishing mixing properties, we here followed Heinrich [324]; there one also finds further information on germ-grain models. For ergodic theory in general, we refer to Krengel [428].

9.4 Intersection Formulas and Unbiased Estimators

The following is a typical question arising from practical applications of random sets. Suppose that the realizations $Z(\omega)$ of a standard random set Z can be observed in a window, say, a compact convex set W with $V_d(W) > 0$. By ‘observation’ we mean that, in principle, values such as $V_j(Z(\omega) \cap W)$ can be measured. We want to use the random variables $V_j(Z \cap W)/V_d(W)$ to estimate the densities $\overline{V}_j(Z)$. In general, however, $V_j(Z \cap W)/V_d(W)$ will depend on W and will not be an unbiased estimator for $\overline{V}_j(Z)$. To estimate the bias, we have to determine the expectation of $V_j(Z \cap W)$. Under suitable assumptions on the random set Z, this can be achieved by means of integral geometry. From the obtained set of expectations, one can then also derive unbiased estimators for the densities of the intrinsic volumes.

Analogous situations arise for stationary processes X of polyconvex particles or k-dimensional flats. In both cases, the total jth intrinsic volumes of the visible parts in a sampling window W,

$$\sum_{F \in X(\omega)} V_j(F \cap W),$$

are observable for certain realizations $X(\omega)$ of X and we need the corresponding expectation to derive unbiased estimators for $\overline{V}_j(X)$.

Finally, a problem, also motivated by practical applications, consists in the estimation of densities of a stationary random set Z or a stationary process X of particles or k-flats from measurements in lower-dimensional sections. Here, stochastic versions of the Crofton formulas yield an answer.

Intersection Formulas for Random Sets

We begin this program with an extension of the local translative formula (5.17) to standard random sets.

Theorem 9.4.1. *Let Z be a standard random set in $\mathbb{R}^d$, let $W \in \mathcal{K}'$ and $j \in \{0, \dots, d\}$. Then*

$$\mathbb{E}\, \Phi_j(Z \cap W, \cdot) = \sum_{k=j}^{d} \overline{\Phi}^{(j)}_{k,d-k+j}(Z, W; \cdot), \tag{9.27}$$

If Z is isotropic, then

$$\mathbb{E}\, \Phi_j(Z \cap W, \cdot) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \overline{V}_k(Z) \Phi_{d-k+j}(W, \cdot), \tag{9.28}$$

where the constants are given by (5.5).

Proof. Let $B \in \mathcal{B}(\mathbb{R}^d)$ be bounded. The function

$$\begin{aligned} &\mathbb{R}^d \times \Omega \to \mathbb{R} \\ &(x, \omega) \ \mapsto \Phi_j(Z(\omega) \cap W \cap (B^d + x), B) \end{aligned}$$

is measurable, by Theorems 14.2.2 and 14.4.4. It is also integrable with respect to the product measure $\lambda \otimes \mathbb{P}$. This follows as in the proof of Theorem 9.2.1

(using $\Phi_j(K,B) \leq V_j(K)$ for convex bodies) if we additionally assume that $W \subset C^d$. This assumption is not a restriction of generality, since in the arguments the cube C^d can clearly be replaced by a larger cube.

For $x \in \mathbb{R}^d$ and $r > 0$, we deduce from the translation covariance of Φ_j and the stationarity of Z that

$$\begin{aligned}\mathbb{E}\,\Phi_j(Z \cap W \cap (rB^d + x), B) &= \mathbb{E}\,\Phi_j((Z - x) \cap (W - x) \cap rB^d, B - x)\\ &= \mathbb{E}\,\Phi_j(Z \cap (W - x) \cap rB^d, B - x).\end{aligned}$$

Using Fubini's theorem and the invariance properties of λ, we get

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d} \Phi_j(Z \cap W \cap (rB^d + x), B)\,\lambda(\mathrm{d}x)\\ &= \mathbb{E}\int_{\mathbb{R}^d} \Phi_j(Z \cap (W + x) \cap rB^d, B + x)\,\lambda(\mathrm{d}x).\end{aligned}$$

We apply the local translative formula (5.17) (for polyconvex sets, see Theorem 5.2.4) to either side (with one of the sets A, B in the quoted formula equal to $\mathbb{R}^d$) and obtain

$$\sum_{k=j}^{d} \mathbb{E}\,\Phi^{(j)}_{k,d-k+j}(Z \cap W, rB^d; B \times \mathbb{R}^d) = \sum_{k=j}^{d} \mathbb{E}\,\Phi^{(j)}_{k,d-k+j}(Z \cap rB^d, W; \mathbb{R}^d \times B).$$

Now we divide both sides by $V_d(rB^d)$ and let r tend to infinity. Because of

$$\Phi^{(j)}_{k,d-k+j}(Z \cap W, rB^d; B \times \mathbb{R}^d) = r^{d-k+j}\Phi^{(j)}_{k,d-k+j}(Z \cap W, B^d; B \times \mathbb{R}^d)$$

and the decomposability property (Theorem 6.4.1), the left side tends to $\mathbb{E}\,\Phi_j(Z \cap W, B)$ and, by Theorem 9.2.1, the right side tends to

$$\sum_{k=j}^{d} \overline{\Phi}^{(j)}_{k,d-k+j}(Z, W; B).$$

If Z is isotropic,

$$\begin{aligned}&\overline{\Phi}^{(j)}_{k,d-k+j}(Z, W; B)\\ &= \lim_{r\to\infty} \frac{1}{V_d(rB^d)}\mathbb{E}\,\Phi^{(j)}_{k,d-k+j}(Z \cap rB^d, W; \mathbb{R}^d \times B)\\ &= \lim_{r\to\infty} \frac{1}{V_d(rB^d)}\mathbb{E}\int_{SO_d} \Phi^{(j)}_{k,d-k+j}(\vartheta Z \cap rB^d, W; \mathbb{R}^d \times B)\,\nu(\mathrm{d}\vartheta)\\ &= \lim_{r\to\infty} \frac{1}{V_d(rB^d)} c^{k,d-k+j}_{j,d}\,\mathbb{E}\,V_k(Z \cap rB^d)\Phi_{d-k+j}(W, B).\end{aligned}$$

Here we have used Fubini's theorem and Theorem 6.4.2. This completes the proof. □

The special case $j = d$ of formula (9.27) reduces to (2.20); it holds for arbitrary stationary random closed sets.

We note two consequences of Theorem 9.4.1. Let Z be a standard random set in $\mathbb{R}^d$ and $B \subset \mathbb{R}^d$ a bounded Borel set. We choose a convex body $W \in \mathcal{K}'$ with $B \subset \operatorname{int} W$. Then $\Phi_j(Z \cap W, B) = \Phi_j(Z, B)$. Since $\Phi^{(j)}_{k,d-k+j}(C, W; \mathbb{R}^d \times B) = 0$ for $k > j$ and $\Phi^{(j)}_{j,d}(C, W; \mathbb{R}^d \times B) = V_j(C)\lambda(B)$, for arbitrary $C \in \mathcal{R}$, we obtain

$$\overline{\Phi}^{(j)}_{k,d-k+j}(Z, W; B) = 0, \;\; k > j, \quad \text{and} \quad \overline{\Phi}^{(j)}_{j,d}(Z, W; B) = \overline{V}_j(Z)\lambda(B).$$

Therefore, Theorem 9.4.1 implies the following result, which was already announced in Section 9.2.

Corollary 9.4.1. *If Z is a standard random set in $\mathbb{R}^d$ and $B \subset \mathbb{R}^d$ is a bounded Borel set, then*

$$\mathbb{E}\,\Phi_j(Z, B) = \overline{V}_j(Z)\lambda(B)$$

for $j \in \{0, \dots, d\}$.

For $j = d - 1$, Theorem 9.4.1 yields

$$\mathbb{E}\,\Phi_{d-1}(Z \cap W, B) = \overline{V}_{d-1}(Z)\lambda(W \cap B) + \overline{V}_d(Z)\Phi_{d-1}(W, B),$$

and Corollary 9.4.1 reads

$$\mathbb{E}\,\Phi_{d-1}(Z, B) = \overline{V}_{d-1}(Z)\lambda(B).$$

Since $\Phi_{d-1}(C, \cdot) \geq 0$ for $C \in \mathcal{R}$, by (14.15), both results and the limit relation

$$\overline{V}_{d-1}(Z) = \lim_{r \to \infty} \frac{\mathbb{E} V_{d-1}(Z \cap rW)}{V_d(rW)}$$

hold for stationary random closed sets Z with values in $\mathcal{S}$, even without the integrability condition (9.16), but the corresponding expressions may be infinite.

As a further consequence of Theorem 9.4.1, we note the global case of (9.28), which is the formula

$$\mathbb{E} V_j(Z \cap W) = \sum_{k=j}^{d} c^{k,d-k+j}_{j,d} \overline{V}_k(Z) V_{d-k+j}(W).$$

It holds for isotropic Z, but also in the non-isotropic case, if W is a ball or if W is replaced by a randomly rotated version ϑW, with ϑ uniform and independent of Z, and if in addition the expectation over ϑ is taken on the left side. In that case, we can even consider functionals without motion invariance.

Theorem 9.4.2. *Let Z be a standard random set in $\mathbb{R}^d$, let $\varphi : \mathcal{R} \to \mathbb{R}$ be additive, translation invariant and continuous on $\mathcal{K}'$, let $W \in \mathcal{K}'$. Further, let θ be a random rotation with distribution ν and independent of Z. Then*

$$\mathbb{E}_\nu \, \mathbb{E} \, \varphi(Z \cap \theta W) = \sum_{k=0}^{d} V_k(W)\overline{\varphi_{d-k}}(Z).$$

Proof. Similar to the proof of Theorem 9.4.1, one shows that

$$(x, \vartheta, \omega) \mapsto \varphi(Z(\omega) \cap \vartheta W \cap (B^d + x))$$

is $\lambda \otimes \nu \otimes \mathbb{P}$-integrable. The translation invariance of φ and the stationarity of Z show that

$$\mathbb{E}_\nu \, \mathbb{E} \, \varphi(Z \cap \theta W \cap (rB^d + x)) = \mathbb{E}_\nu \, \mathbb{E} \, \varphi(Z \cap (\theta W - x) \cap rB^d).$$

Integration over $\mathbb{R}^d$ and Fubini's theorem give

$$\begin{aligned} &\mathbb{E}_\nu \, \mathbb{E} \int_{\mathbb{R}^d} \varphi(Z \cap \theta W \cap (rB^d + x)) \, \lambda(\mathrm{d}x) \\ &= \mathbb{E} \int_{SO_d} \int_{\mathbb{R}^d} \varphi(Z \cap (\vartheta W - x) \cap rB^d) \, \lambda(\mathrm{d}x) \, \nu(\mathrm{d}\vartheta). \end{aligned}$$

In the first integral, we can replace B^d by ρB^d with $\rho \in SO_d$ and then integrate over all $\rho \in SO_d$ with respect to ν. This gives

$$\mathbb{E}_\nu \, \mathbb{E} \int_{G_d} \varphi(Z \cap \theta W \cap grB^d) \, \mu(\mathrm{d}g) = \mathbb{E} \int_{G_d} \varphi(Z \cap gW \cap rB^d) \, \mu(\mathrm{d}g).$$

Now Theorem 5.1.2 (Hadwiger's general integral geometric theorem) yields

$$\sum_{k=0}^{d} \mathbb{E}_\nu \, \mathbb{E} \, \varphi_{d-k}(Z \cap \theta W) V_k(rB^d) = \sum_{k=0}^{d} \mathbb{E} \, \varphi_{d-k}(Z \cap rB^d) V_k(W). \tag{9.29}$$

Recall that

$$\varphi_{d-k}(K) = \int_{A(d,k)} \varphi(K \cap E) \, \mu_k(\mathrm{d}E)$$

for $K \in \mathcal{K}$. This definition can also be used for $K \in \mathcal{R}$ and then provides the additive extension of φ_{d-k} to the convex ring $\mathcal{R}$. Since φ is continuous on $\mathcal{K}'$ and therefore conditionally bounded, also φ_{d-k} is conditionally bounded. Hence, Theorem 9.2.1 applies, and the density

$$\overline{\varphi_{d-k}}(Z) = \lim_{r \to \infty} \frac{\mathbb{E} \, \varphi_{d-k}(Z \cap rB^d)}{V_d(rB^d)}$$

exists. Therefore, dividing the obtained equation (9.29) by $V_d(rB^d)$ and letting r tend to infinity, we obtain the assertion. □

The special choice $\varphi = V_j$, where

$$\varphi_{d-k} = (V_j)_{d-k} = c_{j,d}^{k,d-k+j} V_{d-k+j},$$

gives

$$\mathbb{E}_\nu \, \mathbb{E} \, V_j(Z \cap \theta W) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} V_k(W) \overline{V}_{d-k+j}(Z),$$

for $j = 0, \ldots, d$.

Intersection Formulas for Particle Processes

Now we consider similar intersection formulas for particle processes. For simplicity, we restrict ourselves to convex particles, although under suitable integrability conditions the results are also valid for point processes in the convex ring $\mathcal{R}$. For convex particles, the only integrability condition needed is (4.1), which by Theorem 4.1.2 is equivalent to the integrability of the intrinsic volumes with respect to the grain distribution.

Theorem 9.4.3. *Let X be a stationary process of convex particles in $\mathbb{R}^d$, let $j \in \{0, \ldots, d\}$ and $W \in \mathcal{K}'$. Then*

$$\mathbb{E} \sum_{K \in X} \Phi_j(K \cap W, \cdot) = \sum_{k=j}^{d} \overline{\Phi}_{k,d-k+j}^{(j)}(X, W; \cdot). \tag{9.30}$$

If X is isotropic, then

$$\mathbb{E} \sum_{K \in X} \Phi_j(K \cap W, \cdot) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \overline{V}_k(X) \Phi_{d-k+j}(W, \cdot). \tag{9.31}$$

Proof. Let $B \in \mathcal{B}(\mathbb{R}^d)$. With Campbell's theorem (Theorem 3.1.2) and the decomposition of Theorem 4.1.1, we obtain

$$\mathbb{E} \sum_{K \in X} \Phi_j(K \cap W, B) = \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \Phi_j((K + x) \cap W, B) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}K),$$

where γ and $\mathbb{Q}$ are, respectively, the intensity and the grain distribution of X. Now the translative formula (5.17) immediately yields (9.30).

For isotropic X, we can in addition integrate over rotations of the particles K and then either apply the kinematic formula for curvature measures (Theorem 5.3.2) or the rotation formula for mixed measures (Theorem 6.4.2) to get (9.31). $\square$

The consequences of this result are similar to those of Theorem 9.4.1. Namely, for $j = d$ and $j = d - 1$, (9.30) reduces to the simple relations

$$\mathbb{E} \sum_{K \in X} \Phi_d(K \cap W, \cdot) = \overline{V}_d(X) \Phi_d(W, \cdot),$$

$$\mathbb{E} \sum_{K \in X} \Phi_{d-1}(K \cap W, \cdot) = \overline{V}_{d-1}(X) \Phi_d(W, \cdot) + \overline{V}_d(X) \Phi_{d-1}(W, \cdot).$$

If B is a bounded Borel set and W is large enough such that $B \subset \operatorname{int} W$, then (9.30) implies

$$\mathbb{E} \sum_{K \in X} \Phi_j(K, B) = \overline{V}_j(X) \lambda(B).$$

Since both sides are nonnegative, they define locally finite measures and the equality holds for arbitrary Borel sets B.

Corollary 9.4.2. *Let X be a stationary process of convex particles in $\mathbb{R}^d$ and let $B \in \mathcal{B}(\mathbb{R}^d)$ be a Borel set. Then*

$$\mathbb{E} \sum_{K \in X} \Phi_j(K, B) = \overline{V}_j(X) \lambda(B).$$

for $j \in \{0, \ldots, d\}$.

The global case of (9.31), which can be written as

$$\mathbb{E} \sum_{K \in X} V_j(K \cap W) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} V_k(W) \overline{V}_{d-k+j}(X), \tag{9.32}$$

holds for isotropic X or for general X if either W is a ball or if we average over random rotations of W. The following result is the analog of Theorem 9.4.2.

Theorem 9.4.4. *Let X be a stationary process of convex particles in $\mathbb{R}^d$, let $\varphi : \mathcal{R} \to \mathbb{R}$ be additive, translation invariant and continuous on $\mathcal{K}'$, let $W \in \mathcal{K}'$. If θ is a random rotation with distribution ν and independent of Z, then*

$$\mathbb{E}_\nu \mathbb{E} \sum_{K \in X} \varphi(K \cap \theta W) = \sum_{k=0}^{d} V_k(W) \overline{\varphi_{d-k}}(X).$$

Proof. In complete analogy to the proof of Theorem 9.4.2, we obtain

$$\sum_{k=0}^{d} \mathbb{E}_\nu \mathbb{E} \sum_{K \in X} \varphi_{d-k}(K \cap \theta W) V_k(rB^d) = \sum_{k=0}^{d} \mathbb{E} \sum_{K \in X} \varphi_{d-k}(K \cap rB^d) V_k(W).$$

Dividing by $V_d(rB^d)$, letting $r \to \infty$, and using Theorem 9.2.2, we complete the proof. $\square$

Processes of Flats

Instead of particle processes, we now consider k-flat processes in $\mathbb{R}^d$. For $k \in \{0, \dots, d-1\}$, let X be a stationary k-flat process with intensity γ and directional distribution $\mathbb{Q}$, let $j \in \{0, \dots, k\}$ and $W \in \mathcal{K}'$. In analogy to the corresponding notion for particle processes, we define the **specific** (j,k)th **mixed measure** $\overline{\Phi}^{(j)}_{k,d-k+j}(X, W; \cdot)$ of X (for given W) by

$$\overline{\Phi}^{(j)}_{k,d-k+j}(X, W; \cdot) := \gamma \int_{G(d,k)} \Phi^{(j)}_{k,d-k+j}(L, W; B_L \times \cdot)\, \mathbb{Q}(\mathrm{d}L),$$

where $B_L \subset L$ is the ball with center 0 and $\lambda_k(B_L) = 1$.

Theorem 9.4.5. *Let X be a stationary k-flat process in $\mathbb{R}^d$, $k \in \{0, \dots, d-1\}$, and let $j \in \{0, \dots, k\}$, $W \in \mathcal{K}'$ and $B \subset \mathbb{R}^d$ a Borel set. Then*

$$\mathbb{E} \sum_{E \in X} \Phi_j(E \cap W, B) = \overline{\Phi}^{(j)}_{k,d-k+j}(X, W; B).$$

If X is isotropic, then

$$\mathbb{E} \sum_{E \in X} \Phi_j(E \cap W, B) = \gamma c^{k,d-k+j}_{j,d} \Phi_{d-k+j}(W, B). \tag{9.33}$$

Proof. Using Campbell's theorem, the decomposition of Theorem 4.4.2, the local determination of curvature measures and the translative Crofton formula from Theorem 6.4.3, we obtain

$$\begin{aligned} &\mathbb{E} \sum_{E \in X} \Phi_j(E \cap W, B) \\ &= \gamma \int_{G(d,k)} \int_{L^\perp} \Phi_j(W \cap (L + x), B \cap (L + x))\, \lambda_{L^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}L) \\ &= \gamma \int_{G(d,k)} \Phi^{(j)}_{k,d-k+j}(L, W; B_L \times B)\, \mathbb{Q}(\mathrm{d}L) \\ &= \overline{\Phi}^{(j)}_{k,d-k+j}(X, W; B). \end{aligned}$$

In the isotropic case, we can perform an additional integration over all rotations of L and then use either the Crofton formula for curvature measures or the rotation formula for mixed measures. □

The relations (9.33) provide $k + 1$ interpretations of the intensity γ, including those given by (4.27) and Theorem 4.4.3.

As before, we get a simpler result if we apply an independent uniform random rotation to the sampling window. We state it only in the global version.

Theorem 9.4.6. *Let X be a stationary k-flat process of intensity γ in $\mathbb{R}^d$, let $k \in \{1, \dots, d-1\}$, $j \in \{0, \dots, k\}$ and $W \in \mathcal{K}'$. If θ is a random rotation with distribution ν and independent of X, then*

$$\mathbb{E}_\nu \, \mathbb{E} \sum_{E \in X} V_j(E \cap \theta W) = \gamma c_{j,d}^{k,d-k+j} V_{d-k+j}(W).$$

If X is isotropic or if W is a ball, the result holds without the expectation $\mathbb{E}_\nu$.

Crofton Formulas

Theorems 9.4.1 and 9.4.3 also immediately yield Crofton formulas for random sets and particle processes. If we talk of a standard random set Z or a stationary particle process X in some affine subspace E, the stationarity (and possibly isotropy) of Z and X refers to E, and densities of intrinsic volumes have to be computed in E. For the following results, we denote by

$$\overline{V}^{(j)}_{d-k+j,k}(Y, K) := \overline{\Phi}^{(j)}_{d-k+j,k}(Y, K; \mathbb{R}^d)$$

the **specific** $(j, d-k+j)$th **mixed functional** of the random set or particle process Y and $K \in \mathcal{K}'$.

Theorem 9.4.7. *Let Z be a standard random set in $\mathbb{R}^d$, let $E \in A(d,k)$ be a k-dimensional flat, where $k \in \{1, \dots, d-1\}$, $B_E \subset E$ a ball with $\lambda_k(B_E) = 1$, and let $j \in \{0, \dots, k\}$. Then $Z \cap E$ is a standard random set in E, and*

$$\overline{V}_j(Z \cap E) = \overline{V}^{(j)}_{d-k+j,k}(Z, B_E).$$

If Z is isotropic, then $Z \cap E$ is isotropic and

$$\overline{V}_j(Z \cap E) = c_{j,d}^{k,d-k+j}\, \overline{V}_{d-k+j}(Z).$$

Proof. We omit the (not difficult) proof that $Z \cap E$ is, with respect to E, again a standard random set (and isotropic, if Z is isotropic). For that reason, the density $\overline{V}_j(Z \cap E)$ exists. Theorem 9.4.1 yields

$$\mathbb{E} V_j(Z \cap B_E) = \sum_{m=d-k+j}^{d} \overline{V}^{(j)}_{m,d-m+j}(Z, B_E) \tag{9.34}$$

where only terms with $m \geq d-k+j$ appear since $\overline{V}^{(j)}_{m,d-m+j}(Z, B_E) = 0$ for $m < d-k+j$. Since Z is stationary, we can assume that $0 \in E$ and hence $rB_E \subset E$ for $r > 0$. In (9.34), we replace B_E by rB_E and divide the equation by $V_k(rB_E)$. For $r \to \infty$, the left side tends to $\overline{V}_j(Z \cap E)$, since $V_j(Z \cap rB_E) = V_j(Z \cap E \cap rB_E)$ (and the intrinsic volumes do not depend on the dimension of the surrounding space). Since $\overline{V}^{(j)}_{m,d-m+j}(Z, rB_E)$ is homogeneous of degree $d-m+j$ in r, the right side tends to $\overline{V}^{(j)}_{d-k+j,k}(Z, B_E)$.

As in earlier proofs, the result for isotropic Z follows from the rotation formula in Theorem 6.4.2. ∎

In analogy to Theorem 9.4.7, the following Crofton formula for particle processes can be stated. For simplicity, we assume that the resulting intersection processes $X \cap E$ are simple, though it is not difficult to extend the results to the general case.

Theorem 9.4.8. *Let X be a stationary process of convex particles in $\mathbb{R}^d$, let $E \in A(d,k)$ be a k-dimensional flat, where $k \in \{1,\dots,d-1\}$, $B_E \subset E$ a ball with $\lambda_k(B_E) = 1$, and let $j \in \{0,\dots,k\}$. Then the intersection process $X \cap E$ is a stationary process of convex particles with respect to E, and*

$$\overline{V}_j(X \cap E) = \overline{V}^{(j)}_{d-k+j,k}(X, B_E).$$

If X is isotropic, then $X \cap E$ is isotropic and

$$\overline{V}_j(X \cap E) = c^{k,d-k+j}_{j,d} \overline{V}_{d-k+j}(X).$$

Proof. It is clear that $X \cap E$ is a stationary process of convex particles in E (and isotropic if X is isotropic). In view of the stationarity of X, we may assume that $0 \in B_E$. From Theorem 9.4.3, and since $\overline{V}^{(j)}_{m,d-m+j}(X, B_E) = 0$ for $m < d-k+j$, we get

$$\begin{aligned}\mathbb{E} \sum_{K' \in X \cap E} V_j(K' \cap B_E) &= \mathbb{E} \sum_{K \in X} V_j(K \cap B_E) \\ &= \sum_{m=d-k+j}^{d} \overline{V}^{(j)}_{m,d-m+j}(X, B_E).\end{aligned}$$

We replace B_E by rB_E with $r > 0$ and divide by $V_k(rB_E)$. For $r \to \infty$, by Theorem 9.2.2, applied in E, the left side converges to $\overline{V}_j(X \cap E)$, and the right side converges to $\overline{V}^{(j)}_{d-k+j,k}(X, B_E)$.

For the result in the isotropic case, we use Theorem 6.4.2 again. □

As an example, let X be a stationary and isotropic process of line segments in $\mathbb{R}^d$. For a hyperplane $E \in A(d, d-1)$ we obtain from the preceding theorem

$$\overline{\chi}(X \cap E) = c^{d-1,1}_{0,d} \overline{V}_1(X).$$

For $v \in S^{d-1}$, we let $E := v^\perp$ and put $\gamma(v) := \overline{\chi}(X \cap E)$. Observing that $c^{d-1,1}_{0,d} = 2\kappa_{d-1}/d\kappa_d$, we get

$$\gamma(v) = \frac{2\kappa_{d-1}}{d\kappa_d} \overline{V}_1(X).$$

This is also obtained from (4.40), since in the isotropic case the spherical directional distribution φ is given by

$$\varphi = \frac{1}{\sigma(S^{d-1})}\sigma = \frac{1}{d\kappa_d}\sigma$$

(with the spherical Lebesgue measure σ), and

$$\int_{S^{d-1}} |\langle u, v\rangle| \, \sigma(\mathrm{d}u) = 2\kappa_{d-1}.$$

Unbiased Estimators

The intersection formulas proved so far can be used, in an obvious way, to provide estimators for the specific intrinsic volumes, which are unbiased or asymptotically unbiased.

We first discuss the situation for a standard random set Z. Let $j \in \{0, \dots, d\}$. Since the estimation of the specific volume $\overline{V}_d(Z)$ is of a special and simple nature (and was discussed earlier), we concentrate on the cases $j \le d-1$. An unbiased estimator $\widehat{V}_j$ for $\overline{V}_j(Z)$, based on the observation of Z in a sampling window W with $W \in \mathcal{K}'$ and $V_d(W) > 0$, is immediately given by Corollary 9.4.1, namely

$$\widehat{V}_j := \frac{\Phi_j(Z \cap W, \operatorname{int} W)}{V_d(W)}.$$

For example, for $j = d-1$, this estimator requires us to evaluate the total surface area of the boundary parts of $Z(\omega)$ inside the window W.

Since the evaluation of curvature measures Φ_j with $j < d-1$ is more complicated, it seems natural to use the intrinsic volume $V_j(Z \cap W)$ (normalized by $V_d(W)$) as an estimator. This estimator is, in general, not unbiased. In fact, the bias is given by (9.27), namely through the right side of

$$\mathbb{E} V_j(Z \cap W) = \sum_{k=j}^{d} \overline{V}^{(j)}_{k,d-k+j}(Z, W). \tag{9.35}$$

Writing (9.35), for the sampling window rW, $r > 0$, in the form

$$\frac{\mathbb{E}\, V_j(Z \cap rW)}{V_d(rW)} = \overline{V}_j(Z) + \frac{1}{V_d(W)} \sum_{k=j+1}^{d} r^{j-k} \overline{V}^{(j)}_{k,d-k+j}(Z, W),$$

we see how the mean error tends to 0 for increasing windows W.

In the isotropic case, one can also obtain an unbiased estimator from (9.35). Recall that, for isotropic Z, (9.35) transforms into

$$\mathbb{E} V_j(Z \cap W) = \sum_{k=j}^{d} c^{k,d-k+j}_{j,d} V_k(W) \overline{V}_{d-k+j}(Z), \qquad j = 0, \dots, d.$$

This system of equations can be solved for $\overline{V}_0(Z), \dots, \overline{V}_d(Z)$, since the coefficient matrix is triangular. The resulting formulas are of the form

$$\overline{V}_i(Z) = \mathbb{E}\left(\sum_{m=i}^{d} \beta_{dim}(W) V_m(Z \cap W)\right), \qquad i = 0, \dots, d,$$

hence

$$\widehat{V}_j := \sum_{m=j}^{d} \beta_{djm}(W) V_m(Z \cap W)$$

is an unbiased estimator for $\overline{V}_j(Z)$. As an example, we write down the two-dimensional case, using the notations A, L, χ for area, perimeter and Euler characteristic, respectively:

$$\begin{aligned}
\overline{A}(Z) &= \mathbb{E}\, \frac{A(Z \cap W)}{A(W)}, \\
\overline{L}(Z) &= \mathbb{E}\left[\frac{L(Z \cap W)}{A(W)} - \frac{L(W)A(Z \cap W)}{A(W)^2}\right], \\
\overline{\chi}(Z) &= \mathbb{E}\left[\frac{\chi(Z \cap W)}{A(W)} - \frac{1}{2\pi}\,\frac{L(W)L(Z \cap W)}{A(W)^2}\right. \\
&\qquad \left. + \left(\frac{1}{2\pi}\,\frac{L(W)^2}{A(W)^3} - \frac{1}{A(W)^2}\right) A(Z \cap W)\right].
\end{aligned}$$

The method just described requires the evaluation of all the intrinsic volumes $V_0(Z \cap W), \dots, V_d(Z \cap W)$. A similar method is based on the evaluation of one functional, the Euler characteristic $V_0(Z \cap W)$, but in different sampling windows $r_0 W, \dots, r_d W$. The system of equations then reads

$$\mathbb{E} V_0(Z \cap r_j W) = \sum_{k=0}^{d} c_{0,d}^{k,d-k} r_j^k V_k(W) \overline{V}_{d-k}(Z), \qquad j = 0, \dots, d.$$

If the parameters $r_0, \dots, r_d$ are chosen such that the square matrix with entries $c_{0,d}^{k,d-k} r_j^k V_k(W)$ is regular, this system of equations can again be solved for $\overline{V}_0(Z), \dots, \overline{V}_d(Z)$ and yields unbiased estimators

$$\widehat{V}_j := \sum_{m=0}^{d} \alpha_{djm}(W) V_0(Z \cap r_m W)$$

for $\overline{V}_j(Z)$.

Returning to random sets Z without the isotropy condition, there is also an unbiased estimator for $\overline{V}_j(Z)$ coming from Theorem 9.2.1, namely

$$\widehat{V}_j := V_j(Z \cap C^d) - V_j(Z \cap \partial^+ C^d).$$

This estimator has been described in the stereological literature.

If Z is a stationary (or stationary and isotropic) Boolean model, the estimators described so far in a sampling window W are strongly consistent, for increasing W, due to Theorem 9.3.3. For example,

$$\widehat{V}_j := \frac{V_j(Z \cap rW)}{V_d(rW)} \to \overline{V}_j(Z) \qquad \text{a.s.}$$

as $r \to \infty$.

With respect to stationary particle processes X on $\mathcal{K}'$ (or $\mathcal{R}'$), the situation is completely analogous. We therefore skip the details. The basic result here is Theorem 9.4.3. It provides unbiased estimators

$$\widehat{V}_j := \frac{\sum_{K \in X} \Phi_j(K \cap W, \operatorname{int} W)}{V_d(W)}$$

for $\overline{V}_j(X)$, whereas the estimator

$$\widehat{V}_j := \frac{\sum_{K \in X} V_j(K \cap W)}{V_d(W)}$$

is asymptotically unbiased. A different unbiased estimator is given by

$$\widehat{V}_j := \sum_{K \in X} \left(V_j(K \cap C^d) - V_j(K \cap \partial^+ C^d) \right).$$

Of course, the different representations of φ-densities in Theorem 4.1.3 yield further unbiased or asymptotically unbiased estimators.

If the particles $K \in X$ are polyconvex and uniformly bounded and the window W is large enough, such that $V_d(W \ominus K) > 0$, for $\mathbb{Q}$-almost all K, another simple estimator is given by

$$\widehat{V}_j := \sum_{K \in X,\, K \subset W} \frac{V_j(K)}{V_d(W \ominus K)}.$$

In fact, since

$$\begin{aligned}
&\mathbb{E} \sum_{K \in X,\, K \subset W} \frac{V_j(K)}{V_d(W \ominus K)} \\
&= \gamma \int_{\mathcal{R}_0} \int_{\mathbb{R}^d} \frac{V_j(K + x)}{V_d(W \ominus (K + x))} \mathbf{1}\{K + x \subset W\}\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K) \\
&= \gamma \int_{\mathcal{R}_0} \frac{V_j(K)}{V_d(W \ominus K)} \int_{\mathbb{R}^d} \mathbf{1}\{K + x \subset W\}\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K) \\
&= \gamma \int_{\mathcal{R}_0} V_j(K)\, \mathbb{Q}(\mathrm{d}K) \\
&= \overline{V}_j(X),
\end{aligned}$$

this estimator is unbiased.

For isotropic X, the linear equation method yields unbiased estimators

$$\widehat{V}_j := \sum_{m=j}^{d} \beta_{djm}(W) \sum_{K\in X} V_m(K\cap W),$$

respectively

$$\widehat{V}_j := \sum_{m=j}^{d} \alpha_{djm}(W) \sum_{K\in X} V_0(K\cap r_m W).$$

Notice that the coefficients $\beta_{djm}(W)$ and $\alpha_{djm}(W)$ are the same as in the case of random sets. Therefore, also the given explicit formulas in the planar case transfer immediately to particle processes.

For Poisson processes, Theorem 9.3.5 implies that the estimators are strongly consistent.

Let us now come to applications of the Crofton formulas. We concentrate on stationary and isotropic random sets Z. Then we can work with a fixed plane E. Analogous estimators for non-isotropic sets Z follow, if a random plane E with isotropic distribution (and independent of Z) is chosen. Also, the formulas for particle processes X are totally analogous.

We have seen how the densities $\overline{V}_j(Z)$ of an isotropic standard random set admit asymptotically unbiased or even unbiased estimators. If Z is observed in a k-dimensional section $Z\cap E$, then we can obtain estimators for $\overline{V}_j(Z\cap E)$. Theorem 9.4.7 tells us that these are at the same time (asymptotically) unbiased estimators for the densities $c_{j,d}^{k,d-k+j}\overline{V}_{d-k+j}(Z)$.

As an example, we consider the practically relevant case where $d=3$ and $k=2$. We deal with the three-dimensional densities $\overline{V}$ (volume), $\overline{S}$ (surface area), $\overline{M}$ (integral of mean curvature) and with the two-dimensional densities $\overline{A}$ (area), $\overline{L}$ (boundary length), $\overline{\chi}$ (Euler characteristic). The equations of Theorem 9.4.7 now read

$$\overline{V}(Z) = \overline{A}(Z\cap E), \tag{9.36}$$

$$\overline{S}(Z) = \frac{4}{\pi}\overline{L}(Z\cap E), \tag{9.37}$$

$$\overline{M}(Z) = 2\pi\overline{\chi}(Z\cap E). \tag{9.38}$$

These equations provide an exact theoretical foundation for the ‘fundamental equations of stereology’, which are traditionally written in the form

$$\begin{aligned} V_V &= A_A, \\ S_V &= \frac{4}{\pi}L_A, \\ M_V &= 2\pi\chi_A. \end{aligned}$$

In this way, formula (9.35) and Theorem 9.4.7 provide theoretical justifications for some practical procedures of stereology, at least in those cases where it is reasonable to model probes of real materials by realizations of isotropic standard random sets. From the practical point of view, the consideration of locally polyconvex sets only does not seem to be very restrictive. Of the invariance properties, stationarity is always unrealistic, requiring unbounded sets, but it may well be satisfied approximately at close range. The most critical assumption is that of isotropy. For that reason, the applicability of motion invariant stereology is limited, and the employment of translative integral geometry is appropriate.

Notes for Sections 9.1, 9.2, 9.4

1. The introduction of densities of functionals for random $\mathcal{S}$-sets, intersection formulas as in Section 9.4, and formulas for Boolean models as in Section 9.1, go back to various sources, where they can be found in varying degrees of generality, in part under special assumptions or treated heuristically. We mention the following references, roughly in chronological order: Matheron [462], Davy [198, 199], Miles [530], Miles and Davy [536], Stoyan [739], A.M. Kellerer [390, 391], H.G. Kellerer [392], Weil [787], Wieacker [815], Weil and Wieacker [804], Zähle [826].

The starting point for much of the presentation in Sections 9.2, 9.4 and 9.1 was the work of Weil and Wieacker [804]. We gratefully acknowledge simplifications suggested orally by Markus Kiderlen (proof of Theorem 9.4.1) and Lars Diening (second proof of Lemma 9.2.1).

2. Theorem 9.4.1 and its counterpart for particle processes, Theorem 9.4.3, which provide unbiased estimators for the specific intrinsic volumes without isotropy assumptions, were proved by Weil [787, 788].

3. Special cases of the intersection formulas of Section 9.4 first came up in stereology (see also Note 2 of Subsection 8.4.2). We have treated them here rigorously and generally, for suitable stationary random closed sets or particle processes as the employed models. An alternative approach of stochastic geometry to section stereology consists in working with deterministic (and bounded) structures and investigating them with the aid of random sections. Different distributions of intersection planes that are relevant in this context are discussed in Section 8.4. A presentation of stereological problems and formulas from a geometric point of view is found in Weil [785]. A reader interested in the practical side of stereology is referred to the two volumes of Weibel [778]. More recent developments in the stereology of non-stationary structures are presented in the book by Jensen [379]. For a modern view on stereology in general, we refer to the volume *Stereology for Statisticians* by Baddeley and Jensen [53].

We have restricted ourselves here to standard random sets. Other classes of random closed sets can be treated according to the availability of suitable integral geometric formulas. For example, a counterpart to the second formula of Theorem 9.4.7, for stationary, isotropic random closed sets which are rectifiable manifolds, appears in Mecke [479]. A very general investigation of intersection formulas for random processes of Hausdorff rectifiable closed sets is due to Zähle [822].

4. Applications of Boolean models to various questions of statistical physics (percolation, complex fluids, structure of the universe) were suggested and investigated by K. Mecke [505, 506]; see also Beisbart *et al.* [89], Beisbart *et al.* [88]. Here specific intrinsic volumes (under the name of means of Minkowski functionals) are used as morphological parameters for the description of spatial structures.

5. Concerning the estimation of the specific intrinsic volumes of standard random sets Z, Schmidt and Spodarev [669] proposed a further method, based on the additively extended Steiner formula (14.70). In global form, with $\rho_\epsilon(K) := \rho_\epsilon(K, \mathbb{R}^d)$, the latter says that

$$\rho_\epsilon(K) = \sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} V_j(K)$$

for $\epsilon \geq 0$ and $K \in \mathcal{R}$. Since $K \mapsto \rho_\epsilon(K)$ is additive, translation invariant and locally bounded (it is even continuous on $\mathcal{K}'$), the density

$$\overline{\rho}_\epsilon(Z) := \lim_{r\to\infty} \frac{\mathbb{E}\rho_\epsilon(Z \cap rW)}{V_d(rW)}$$

exists and satisfies

$$\overline{\rho}_\epsilon(Z) = \sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} \overline{V}_j(Z).$$

Choosing pairwise different values $\epsilon_0, \dots, \epsilon_d$ and inverting the system of linear equations yields

$$\overline{V}_j(Z) = \sum_{m=0}^{d} \gamma_{djm} \overline{\rho}_{\epsilon_m}(Z), \qquad j = 0, \dots, d.$$

As estimators of $\overline{\rho}_{\epsilon_m}(Z)$, again the values $\rho_{\epsilon_m}(Z \cap W)/V_d(W)$ in a window W can be used; then

$$\widehat{V}_j := \frac{1}{V_d(rW)} \sum_{m=0}^{d} \gamma_{djm} \rho_{\epsilon_m}(Z \cap rW)$$

is an asymptotically unbiased estimator, as $r \to \infty$. The evaluation of $\rho_{\epsilon_m}(Z \cap W)$ is based on the integral of the index function over $W + \epsilon_m B^d$ (see Note 3 of Section 14.4). A variant, which is also studied in Schmidt and Spodarev [669], is to integrate the index function only over $W \ominus \epsilon_m B^d$ (this method is sometimes called 'minus sampling'), then the corresponding estimator is unbiased. Under additional assumptions, the authors also give a consistent estimation of the asymptotic covariance matrix of these estimators and show that, for germ-grain models satisfying some mixing conditions, a central limit theorem holds.

An algorithmic version of the estimation procedure, for digitized images of random sets, is developed in Klenk, Schmidt and Spodarev [420], and further in Guderlei, Klenk, Mayer, Schmidt and Spodarev [301].

6. Limit theorems. For ergodic standard random sets Z, Theorem 9.3.3 provides an a.s. limit theorem for additive functionals φ of $Z \cap W$, as the sampling window W increases to the whole space. This raises the natural question of more refined results for the corresponding estimators (asymptotic normality, large deviations, etc.). We mention here some of the more recent advances and refer to Molchanov [546] for further and, in particular, earlier results.

Heinrich and Molchanov [329] show a central limit theorem for quite general random measures associated with stationary Boolean models. Their results include the positive extensions of the curvature measures (intrinsic volumes) and also generalize to germ-grain models with suitable ergodicity or mixing conditions.

Pantle, Schmidt and Spodarev [594] study the asymptotic normality of estimators for additive functionals (valuations) for stationary Boolean models. In particular, this includes the (additively extended) intrinsic volumes. Also here, the results extend to more general germ-grain models satisfying a mixing condition.

Heinrich [326] proved a large deviations result for the empirical volume fraction of a stationary Boolean model.

9.5 Further Estimation Problems

In the previous section, we have discussed several methods of estimating important characteristics of stationary random closed sets Z or particle processes X, the specific intrinsic volumes. For stationary and isotropic Boolean models Z, we have also seen in Section 9.1 how measurements on Z can be used to estimate the specific intrinsic volumes of the underlying Poisson process X of particles. These mean values give first quantitative information about Z or X. However, even for a Poisson process X of random balls, which are distributed according to a radius distribution function G on $(0, \infty)$, the specific intrinsic volumes of X, though yielding certain moments of G, in general do not determine the whole distribution.

In the following, we continue these considerations and discuss three particular estimation problems in more detail. The first problem concerns the determination of the intensity γ for a stationary Boolean model Z. We shall describe different estimation methods which work under various assumptions, in particular one which is based on the formulas in Theorem 9.1.5. The second problem is to estimate the radius distribution of a stationary process X of balls in $\mathbb{R}^d$ from measurements of the section process $X \cap L$ in a k-dimensional section plane L, $k \in \{1, \dots, d-1\}$. For $d = 3$, $k = 2$, this is the classical Wicksell problem. In the third problem, we consider a stationary Boolean model with spherical grains and show how the radius distribution can be estimated using generalized contact distributions.

Intensity Estimation for Boolean Models

We have seen in Theorem 9.1.4 that, for a stationary and isotropic Boolean model Z with convex grains, the $(d + 1)$-tuple of specific intrinsic volumes $\overline{V}_0(Z), \dots, \overline{V}_d(Z)$ determines the corresponding $(d+1)$-tuple $\overline{V}_0(X), \dots, \overline{V}_d(X)$ of the underlying Poisson particle process X uniquely (we also emphasized the cases $d = 2$ and $d = 3$). Since the formulas in Theorem 9.1.4 follow a triangular array, although not linear, they can easily be solved for $\overline{V}_0(X), \dots, \overline{V}_d(X)$ yielding equations

$$\overline{V}_j(X) = f_{dj}(\overline{V}_0(Z), \dots, \overline{V}_d(Z)), \qquad j = 0, \dots, d,$$

with rational functions f_{dj}. Using the estimators for $\overline{V}_i(Z)$ from the previous section, we thus obtain estimators for $\overline{V}_0(X), \dots, \overline{V}_d(X)$ (which are no longer unbiased). Since

$$\overline{V}_0(X) = \overline{\chi}(X) = \gamma$$

due to the convexity of the grains, this includes an estimator of the intensity γ. As we have mentioned, Theorem 9.1.4 remains valid for polyconvex grains (under appropriate integrability conditions). If the grains have Euler characteristic one (which in the plane is the case if they are simply connected), then $\overline{\chi}(X) = \gamma$, so we still obtain an estimator for γ. However, if we drop the isotropy of Z, the situation becomes more complicated.

We first consider a stationary Boolean model Z with convex grains. According to Theorem 9.1.1, the spherical contact distribution function H of Z is given by

$$H(r) = 1 - \exp\left(-\sum_{k=1}^{d} \kappa_k r^k \overline{V}_{d-k}(X)\right), \qquad r \geq 0.$$

Hence,

$$f(r) := -\ln(1 - H(r)) = \sum_{k=1}^{d} c_k r^k$$

is a polynomial in r with coefficients $c_k := \kappa_k \overline{V}_{d-k}(X)$ (and without constant term). Since $H(r)$ can be expressed in terms of the volume fractions p of Z and $p(r)$ of $Z + rB^d$, simple estimators for $f(r)$ exist (for example, in the planar case by counting pixels in a digitized image of $Z(\omega) \cap W$). If $\widehat{f}_1, \dots, \widehat{f}_m$ are corresponding estimated values of $f(r)$, for different values $r_1, \dots, r_m$, then fitting a polynomial $\widehat{f}$ of degree d (and with $\widehat{f}(0) = 0$) to these values yields estimators for c_k, $k = 1, \dots, d$. Here, $c_d = \gamma$.

Another method, which also requires convex grains, is based on the lower tangent point $\widetilde{z}(C)$, $C \in \mathcal{C}'$, which we have introduced in Section 4.2. Since there we concentrated on the planar case, we shall do this again, although the method can be extended to higher dimensions. Hence, we consider a stationary Boolean model Z in $\mathbb{R}^2$ with convex grains. Let X be the underlying Poisson particle process and γ the intensity. Since $\widetilde{z}$ is a center function, the points $\widetilde{z}(K)$, $K \in X$, constitute a stationary Poisson process $\widetilde{X}$ in $\mathbb{R}^2$ which also has intensity γ (by Theorems 4.2.1 and 4.2.2). Since $\widetilde{z}(K)$ is a boundary point of K, some points of $\widetilde{X}$ lie on the boundary of Z and the others in the interior. Let X' be the thinning of $\widetilde{X}$ consisting of all points $x \in \widetilde{X}$ which lie in the boundary of Z, hence they are observable from Z. These points are the lower tangent points of particles from X which are not covered by any other particle (the case that the lower tangent point x of one particle is also in the boundary of another particle from X has probability 0). Since $X' = \widetilde{X} \cap \operatorname{cl} Z^c$, the (simple) point process X' is stationary. Using the common notation from stereology, we denote the intensity of X' by $\overline{\chi}^+(Z)$.

Theorem 9.5.1. *Let Z be a stationary Boolean model in $\mathbb{R}^2$ with convex grains and X the underlying Poisson particle process with intensity γ. Then*

$$\overline{\chi}^+(Z) = \gamma \mathrm{e}^{-\overline{A}(X)}. \tag{9.39}$$

Proof. In the following proof, we make the identifications explained before Theorems 3.3.5 and 3.5.9. For $K \in \mathcal{K}'$, $\eta \in \mathsf{N}_s(\mathcal{K}')$ and $x \in \mathbb{R}^2$, let

$$Z(K,\eta) := \bigcup_{C \in \eta \setminus \{K\}} C$$

and

$$f(x,K,\eta) := \frac{1}{\pi} \mathbf{1}_{B^2 \cap Z(K,\eta)^c}(x).$$

We apply Theorem 4.2.4 to the particle process X and the center function $\widetilde{z}$. Let $\widetilde{\mathcal{K}} := \{K \in \mathcal{K}' : \widetilde{z}(K) = 0\}$ be the corresponding mark space, $\widetilde{\mathbb{Q}}$ the mark distribution and $(\mathbb{P}^{0,K})_{K \in \widetilde{\mathcal{K}}}$ the regular family occurring in the theorem. We then use Slivnyak's theorem (Theorem 3.5.9) for the stationary marked Poisson process $X_{\widetilde{z}}$. Since we view $\mathbb{P}^{0,K}$ as a measure on $\mathcal{B}(\mathcal{F}(\mathcal{K}'))$, as described in the proof of Theorem 4.2.4, Slivnyak's theorem gives

$$\mathbb{P}^{0,K}(A) = \mathbb{P}(X \cup \{K\} \in A)$$

for $A \in \mathcal{B}(\mathcal{F}(\mathcal{K}'))$ and $\widetilde{\mathbb{Q}}$-almost all $K \in \widetilde{\mathcal{K}}$, hence

$$\int_{\mathsf{N}_s(\mathcal{K}')} g(\eta)\, \mathbb{P}^{0,K}(\mathrm{d}\eta) = \int_{\mathsf{N}_s(\mathcal{K}')} g(\eta \cup \{K\})\, \mathbb{P}_X(\mathrm{d}\eta)$$

for all measurable functions $g \geq 0$. By the definition of $\overline{\chi}^+(Z)$, we thus obtain

$$\begin{aligned}
\overline{\chi}^+(Z) &= \mathbb{E} \sum_{K \in X} f(\widetilde{z}(K), K, X) \\
&= \gamma \int_{\mathbb{R}^2} \int_{\widetilde{\mathcal{K}}} \int_{\mathsf{N}_s(\mathcal{K}')} f(x, K+x, \eta + x)\, \mathbb{P}^{0,K}(\mathrm{d}\eta)\, \widetilde{\mathbb{Q}}(\mathrm{d}K)\, \lambda(\mathrm{d}x) \\
&= \frac{\gamma}{\pi} \int_{\mathbb{R}^2} \int_{\widetilde{\mathcal{K}}} \int_{\mathsf{N}_s(\mathcal{K}')} \mathbf{1}_{B^2}(x) \mathbf{1}_{Z(K,\eta)^c}(0)\, \mathbb{P}^{0,K}(\mathrm{d}\eta)\, \widetilde{\mathbb{Q}}(\mathrm{d}K)\, \lambda(\mathrm{d}x) \\
&= \gamma \int_{\widetilde{\mathcal{K}}} \int_{\mathsf{N}_s(\mathcal{K}')} \mathbf{1}_{Z(K,\eta)^c}(0)\, \mathbb{P}_X(\mathrm{d}\eta)\, \widetilde{\mathbb{Q}}(\mathrm{d}K) \\
&= \gamma \int_{\widetilde{\mathcal{K}}} \int_{\mathsf{N}_s(\mathcal{K}')} \mathbf{1}_{Z_\eta^c}(0)\, \mathbb{P}_X(\mathrm{d}\eta)\, \widetilde{\mathbb{Q}}(\mathrm{d}K) \\
&= \gamma \mathbb{P}(0 \notin Z) \\
&= \gamma \mathrm{e}^{-\overline{A}(X)},
\end{aligned}$$

where $Z_\eta := \bigcup_{K\in\eta} K$ and where we have used that, for fixed K, the relation $K \notin \eta$ holds for $\mathbb{P}_X$-almost all η. □

Combining (9.39) with (9.6) we obtain a simple estimator for the intensity, namely by counting the number $\chi^+(Z \cap W)$ of lower tangent points of Z in the window W and dividing by the area of the uncovered part,

$$\widehat{\gamma} := \frac{\chi^+(Z \cap W)}{A(Z^c \cap W)}.$$

This estimator is ratio-unbiased and strongly consistent, but depends very much on the convexity of the grains.

We next describe a method for stationary Boolean models Z in the plane, which may have arbitrarily shaped compact grains, but they should be connected and their circumradius should be bounded from above by some constant r_0. We make use of the formula

$$\mathbb{P}(Z \cap C = \emptyset) = 1 - T_Z(C) = \mathrm{e}^{-\Theta(\mathcal{F}_C)}$$

for $C \in \mathcal{C}$.

For $\epsilon > 0$, we put

$$\begin{aligned} C_1 &:= [0, 2r_0 + \epsilon] \times [0, \epsilon], \\ C_2 &:= [0, \epsilon] \times [0, 2r_0 + \epsilon], \\ C_0 &:= ([0, 2r_0 + \epsilon] \times \{0\}) \cup (\{0\} \times [0, 2r_0 + \epsilon]). \end{aligned}$$

Then

$$\begin{aligned} &\ln \frac{\mathbb{P}(Z \cap (C_0 \cup C_1 \cup C_2) = \emptyset)\mathbb{P}(Z \cap C_0 = \emptyset)}{\mathbb{P}(Z \cap (C_0 \cup C_1) = \emptyset)\mathbb{P}(Z \cap (C_0 \cup C_2) = \emptyset)} \\ &= \Theta(\mathcal{F}_{C_0 \cup C_1}) + \Theta(\mathcal{F}_{C_0 \cup C_2}) - \Theta(\mathcal{F}_{C_0 \cup C_1 \cup C_2}) - \Theta(\mathcal{F}_{C_0}) \\ &= \Theta(\mathcal{F}^{C_0}_{C_1, C_2}). \end{aligned} \tag{9.40}$$

In order to calculate $\Theta(\mathcal{F}^{C_0}_{C_1,C_2})$, we use Theorem 4.2.1 with the lower left corner z' as center function. Let $\mathbb{Q}'$ be the corresponding mark distribution. Due to our assumptions, we have $r(C) \le r_0$ for $\mathbb{Q}'$-almost all $C \in \mathcal{C}_{z',0} := \{D \in \mathcal{C}' : z'(D) = 0\}$. Therefore, for these C and for $x \in \mathbb{R}^2$, the condition $C + x \in \mathcal{F}^{C_0}_{C_1,C_2}$ is equivalent to $x \in (0, \epsilon]^2$ (here we need the assumption that C is connected). From this, we obtain

$$\Theta(\mathcal{F}^{C_0}_{C_1,C_2}) = \gamma \int_{\mathcal{C}_{z',0}} \int_{\mathbb{R}^2} \mathbf{1}_{(0,\epsilon]^2}(x)\, \lambda(\mathrm{d}x)\, \mathbb{Q}'(\mathrm{d}C) = \gamma\epsilon^2.$$

Since ϵ is known, this can be used for the estimation of γ. Because of

$$\mathbb{P}(Z \cap C = \emptyset) = 1 - \mathbb{P}(0 \in Z - C),$$

one would have to estimate the area densities of $Z - C_0, Z - (C_0 \cup C_1), Z - (C_0 \cup C_2)$ and $Z - (C_1 \cup C_2)$. The resulting estimator only makes sense if the observed area fractions of these four outer parallel sets of Z are smaller than one or even bounded away from one, since otherwise the logarithm of the quotient above is not defined or rather unstable. This implies that the estimation method requires that both the intensity has to be small and the particles need to be small, in comparison to the observation window W.

Now we return to restricted shapes and describe an estimation method based on the formulas of Theorem 9.1.5. We assume a stationary Boolean model Z in $\mathbb{R}^2$ and, since Theorem 9.1.5 was formulated for convex grains, we make the same assumption, although the method also works for simply connected polyconvex grains. We recall the density formulas for this case:

$$\begin{aligned}
\overline{A}(Z) &= 1 - \mathrm{e}^{-\overline{A}(X)}, \\
\overline{L}(Z) &= \mathrm{e}^{-\overline{A}(X)} \overline{L}(X), \\
\overline{\chi}(Z) &= \mathrm{e}^{-\overline{A}(X)} \left(\gamma - \overline{A}(X, -X)\right).
\end{aligned}$$

If the densities on the left side are estimated, we obtain estimators for $\overline{A}(X), \overline{L}(X)$ and $\gamma - \overline{A}(X, -X)$. However, $\overline{A}(X, -X)$ cannot be expressed in terms of $\overline{L}(X)$ or $\overline{A}(X)$. We therefore replace the second equation above by its local counterparts (9.12),

$$\overline{S}_1(Z, \cdot) = \mathrm{e}^{-\overline{A}(X)} \overline{S}_1(X, \cdot),$$

and (9.13),

$$\overline{h}(Z, \cdot) = \mathrm{e}^{-\overline{A}(X)} \overline{h}(X, \cdot).$$

The connection with $\overline{A}(X, -X)$ is given by

$$\begin{aligned}
\overline{A}(X, -X) &= \frac{1}{2}\gamma^2 \int_{\mathcal{K}_0} \int_{\mathcal{K}_0} \int_{S^1} h(K_1, u)\, S_1(-K_2, \mathrm{d}u)\, \mathbb{Q}(\mathrm{d}K_1)\, \mathbb{Q}(\mathrm{d}K_2) \\
&= \frac{1}{2} \int_{S^1} \overline{h}(X, u)\, \overline{S}_1(-X, \mathrm{d}u).
\end{aligned}$$

Hence, we can estimate $\overline{A}(X, -X)$, and therefore also γ, if we can estimate $\overline{h}(X, \cdot)$ and $\overline{S}_1(X, \cdot)$. Fortunately, it is sufficient to estimate only one of these quantities. Namely, the Blaschke body $B(X)$ of X, satisfying $S_1(B(X), \cdot) = \overline{S}_1(X, \cdot)$, is identical with the mean body of X, since in the plane, Blaschke addition coincides with Minkowski addition. It follows that $\overline{h}(X, \cdot) = h(B(X), \cdot)$. Therefore, $\overline{S}_1(X, \cdot)$ determines $B(X)$ and thus $\overline{h}(X, \cdot)$, and conversely.

It is obvious that a corresponding analysis of higher-dimensional Boolean models becomes more and more complicated. We refer to the Notes of this section, for a corresponding analysis of the three-dimensional case.

The Wicksell Problem

Let X be a stationary process of balls in $\mathbb{R}^d$, that is, a stationary particle process with intensity measure concentrated on the set of balls with positive radius. The **radius distribution** $\mathbb{G}$ of X can be defined by

$$\mathbb{G}(A) := \frac{1}{\gamma}\mathbb{E}\sum_{K\in X} \mathbf{1}_B(c(K))\mathbf{1}_A(r(K))$$

for $B \in \mathcal{B}$ with $\lambda(B) = 1$ and $A \in \mathcal{B}(\mathbb{R}^+)$, where $c(K)$ is the center and $r(K)$ is the radius of the ball K, and where $\gamma > 0$ denotes the intensity of X. We assume that $\mathbb{G}(\{0\}) = 0$. Of course, $\mathbb{G}$ is also the image of the grain distribution $\mathbb{Q}$ under the mapping $K \mapsto r(K)$. If we represent X as the marked point process

$$\widetilde{X} := \sum_{K\in X} \delta_{(c(K),r(K))}$$

(with mark space $\mathbb{R}^+$), the radius distribution of X is just the mark distribution of $\widetilde{X}$. For a k-dimensional linear subspace $L \in G(d,k)$, $k \in \{1,\dots,d-1\}$, the section process $X \cap L$ is a stationary process of (k-dimensional) balls; we denote its radius distribution by $\mathbb{G}_L$. We shall now establish a connection between $\mathbb{G}$ and $\mathbb{G}_L$.

For $x \in \mathbb{R}^d$ we use the orthogonal decomposition $x = x_L + x^L$ with $x_L \in L$ and $x^L \in L^\perp$. The Euclidean norm in $\mathbb{R}^d$ is denoted by $\|\cdot\|$. For a ball $K \subset \mathbb{R}^d$, the intersection $K \cap L$ is a ball in L with radius $\sqrt{r(K)^2 - \|c(K)^L\|^2}$, if $\|c(K)^L\| \le r(K)$ (otherwise $K \cap L = \emptyset$). With the section process $X \cap L$ we therefore associate the marked point process

$$\widetilde{X}_L := \sum_{K\in X,\, \|c(K)^L\|\le r(K)} \delta_{\left(c(K)_L,\sqrt{r(K)^2-\|c(K)^L\|^2}\right)}$$

in L with mark space $\mathbb{R}^+$ (assuming here that it is simple); it is stationary in L. The radius distribution $\mathbb{G}_L$ of the section process $X \cap L$ is the mark distribution of $\widetilde{X}_L$. The intensity $\gamma_{X\cap L}$ of $X \cap L$ is also the intensity of $\widetilde{X}_L$. By Theorem 3.5.1, the intensity measure of $\widetilde{X}$ is given by $\gamma\lambda \otimes \mathbb{G}$, and the intensity measure of $\widetilde{X}_L$ is given by $\gamma_{X\cap L}\lambda_L \otimes \mathbb{G}_L$. Therefore, for $B \in \mathcal{B}(L)$ and $A \in \mathcal{B}(\mathbb{R}^+)$ we obtain

$$\begin{aligned}
&\gamma_{X\cap L}\lambda_L(B)\mathbb{G}_L(A)\\
&= \mathbb{E}\sum_{(x,a)\in\widetilde{X}_L} \mathbf{1}_{B\times A}(x,a)\\
&= \mathbb{E}\sum_{(x,a)\in\widetilde{X}} \mathbf{1}_{B\times A}\left(x_L, \sqrt{\max\{0, a^2 - \|x^L\|^2\}}\right)\\
&= \gamma \int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \mathbf{1}_B(x_L)\mathbf{1}_A\left(\sqrt{\max\{0, a^2 - \|x^L\|^2\}}\right)\lambda(\mathrm{d}x)\,\mathbb{G}(\mathrm{d}a)
\end{aligned}$$

$$= \gamma\lambda_L(B) \int_{\mathbb{R}^+} \int_{L^\perp} \mathbf{1}_{[0,a]}(\|z\|)\mathbf{1}_A\left(\sqrt{a^2 - \|z\|^2}\right) \lambda_{L^\perp}(\mathrm{d}z)\, \mathbb{G}(\mathrm{d}a)$$

$$= \gamma\lambda_L(B) \int_{L^\perp} \int_{\mathbb{R}^+} \mathbf{1}_{(\|z\|,\infty)}(a)\mathbf{1}_A\left(\sqrt{a^2 - \|z\|^2}\right) \mathbb{G}(\mathrm{d}a)\, \lambda_{L^\perp}(\mathrm{d}z)$$

$$= \gamma\lambda_L(B)(d-k)\kappa_{d-k} \int_0^\infty \int_t^\infty \mathbf{1}_A\left(\sqrt{a^2 - t^2}\right) \mathbb{G}(\mathrm{d}a) t^{d-k-1}\, \mathrm{d}t.$$

In particular, for $A = [x, \infty)$ with $x > 0$, we get

$$\gamma_{X\cap L}\mathbb{G}_L([x,\infty)) = \gamma(d-k)\kappa_{d-k} \int_0^\infty \mathbb{G}\left(\left[\sqrt{x^2+t^2}, \infty\right)\right) t^{d-k-1}\, \mathrm{d}t. \quad (9.41)$$

With $x \to 0$ we obtain

$$\gamma_{X\cap L} = \kappa_{d-k}\gamma M_{d-k},$$

where M_{d-k} is the $(d-k)$th moment of the radius distribution $\mathbb{G}$.

The **Wicksell corpuscle problem** of stereology is the task to determine, in the case $d = 3$, $k = 2$, the distribution $\mathbb{G}$ from the distribution $\mathbb{G}_L$. If we denote (as is common in stereology) by D_V and D_A the distribution function of $\mathbb{G}$ and $\mathbb{G}_L$, respectively, and if d_V denotes the first moment of $\mathbb{G}$, then (9.41), for $d = 3$, $k = 2$, is equivalent to

$$D_A(r) = 1 - \frac{1}{d_V} \int_0^\infty \left(1 - D_V\left(\sqrt{r^2 + x^2}\right)\right) \mathrm{d}x \qquad \text{for } r > 0.$$

Thus, to determine D_V from D_A, one has (besides the determination of d_V) to solve an Abel type integral equation. An inversion formula exists, but is numerically unstable. In practice, where D_A can only be estimated, this inverse problem presents considerable difficulties.

Boolean Models with Spherical Grains

As a second situation, where an estimation of the radius distribution is possible, we consider a stationary Boolean model Z where the primary grain Z_0 is a random ball with radius distribution $\mathbb{G}$ (again, we assume $\mathbb{G}(\{0\}) = 0$). We recall the contact distribution function H_B (with structuring element B) which we have discussed earlier (see Sections 2.4 and 9.1). For a Boolean model of balls and $t > 0$,

$$\begin{aligned} H_B(t) &= \mathbb{P}(d_B(0,Z) \le t \,|\, 0 \notin Z) \\ &= 1 - \exp\left(-\gamma \sum_{j=1}^{d} \kappa_{d-j} V_j(B) t^j \int_0^\infty r^{d-j}\, \mathbb{G}(\mathrm{d}r)\right). \end{aligned} \quad (9.42)$$

We shall consider a variant of H_B which includes the radii of the observable boundaries of the balls.

Namely, if $0 \notin Z$, the set $d_B(0,Z)B$ touches Z almost surely at a boundary point of precisely one grain. We state this fact, for later use, in a more general form.

Lemma 9.5.1. *Let $X = \{(\xi_1, Z_1), (\xi_2, Z_2), \ldots\}$ be an independently marked Poisson process on $\mathbb{R}^d$ with mark space $\mathcal{K}_0$ and intensity measure*

$$\Theta = \left(\int h \, \mathrm{d}\lambda\right) \otimes \mathbb{Q}.$$

Let $x \in \mathbb{R}^d$. Then

$$\mathbb{P}(0 < d_B(x, \xi_m + Z_m) = d_B(x, \xi_n + Z_n) < \infty) = 0, \quad m \neq n.$$

Proof. Using Campbell's theorem and then Corollary 3.2.4, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{m\neq n} \{0 < d_B(x, \xi_m + Z_m) = d_B(x, \xi_n + Z_n) < \infty\}\right) \\
&\leq \mathbb{E}\left(\frac{1}{2}\sum_{m\neq n} \mathbf{1}\{0 < d_B(x, \xi_m + Z_m) = d_B(x, \xi_n + Z_n) < \infty\}\right) \\
&= \frac{1}{2}\int_{(\mathbb{R}^d\times\mathcal{K}_0)^2} \mathbf{1}\left\{y \in \mathrm{bd}(x - K + d_B(x, z + M)B)\right\} \\
&\quad \times \mathbf{1}\{0 < d_B(x, z + M) < \infty\}\, \Lambda^{(2)}(\mathrm{d}((y, K), (z, M))) \\
&= \frac{1}{2}\int_{\mathcal{K}_0}\int_{\mathcal{K}_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}\left\{y \in \mathrm{bd}(x - K + d_B(x, z + M)B)\right\} h(y)h(z) \\
&\quad \times \mathbf{1}\{0 < d_B(x, z + M) < \infty\}\, \lambda(\mathrm{d}y)\, \lambda(\mathrm{d}z)\, \mathbb{Q}(\mathrm{d}K)\, \mathbb{Q}(\mathrm{d}M).
\end{aligned}$$

The last expression vanishes, since the boundary of a convex body has Lebesgue measure zero. □

We return to our stationary Boolean model Z with spherical grains and assume that $x \notin Z$. Applying the lemma to the underlying (stationary) Poisson process X of balls, we almost surely obtain a unique grain $\widetilde{Z}$ in X with $d_B(x,Z)B \cap \widetilde{Z} \neq \emptyset$. We define $r_B(x,Z)$ as the radius $r(\widetilde{Z})$ of $\widetilde{Z}$. Then the following result holds.

Theorem 9.5.2. *Let Z be a stationary Boolean model in $\mathbb{R}^d$ with spherical grains and with intensity γ and radius distribution $\mathbb{G}$. Let $g \geq 0$ be a measurable function on $\mathbb{R}^+ \times \mathbb{R}^+$. Then we have*

$$\begin{aligned}
&\mathbb{E}(g(d_B(0,Z), r_B(0,Z)) \mid 0 \notin Z) \\
&= \gamma \sum_{j=0}^{d-1} (j+1)\kappa_{d-1-j} V_{j+1}(B) \int_0^\infty \int_0^\infty r^{d-1-j} t^j (1 - H_B(t)) g(t,r) \, \mathrm{d}t \, \mathbb{G}(\mathrm{d}r).
\end{aligned}$$

Proof. We use

$$Z = U(X),$$

where $X = \{Z_1, Z_2, \ldots\}$ is a measurable enumeration of the stationary Poisson process of balls underlying Z, with intensity γ and radius distribution $\mathbb{G}$, and where $U(Y)$, for a particle process Y, denotes the union set,

$$U(Y) := \bigcup_{K \in Y} K.$$

For $n \in \mathbb{N}$, we define the events

$$A_n := \{0 < d_B(0, Z_n) < \infty\}$$

and

$$B_n := \{d_B(0, U(X \setminus \{Z_n\})) > d_B(0, Z_n)\}.$$

Then

$$(d_B(0, Z), r_B(0, Z)) = (d_B(0, Z_n), r(Z_n))$$

on $A_n \cap B_n$ and

$$\{0 < d_B(0, Z) < \infty\} = \bigcup_{n=1}^{\infty} (A_n \cap B_n) \qquad \text{a.s.}$$

Using this and Theorem 3.2.5, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\mathbf{1}\{0 < d_B(0, Z) < \infty\} g(d_B(0, Z), r_B(0, Z))\right) \\
&= \mathbb{E} \sum_{n=1}^{\infty} \mathbf{1}_{A_n \cap B_n} g(d_B(0, Z_n), r(Z_n)) \\
&= \mathbb{E}\Bigg(\sum_{K \in X} \mathbf{1}\{0 < d_B(0, K) < \infty\} g(d_B(0, K), r(K)) \\
&\qquad \times \mathbf{1}\{d_B(0, U(X \setminus \{K\})) > d_B(0, K)\}\Bigg) \\
&= \int_{\mathcal{K}'} \mathbf{1}\{0 < d_B(0, K) < \infty\} g(d_B(0, K), r(K)) \\
&\qquad \times \mathbb{P}\left(d_B(0, U(X)) > d_B(0, K)\right) \Theta(\mathrm{d}K) \\
&= \mathbb{P}(0 \notin Z)\gamma \int_0^{\infty} \int_{\mathbb{R}^d} \mathbf{1}\{0 < d_B(0, z + rB^d) < \infty\} g(d_B(0, z + rB^d), r) \\
&\qquad \times (1 - H_B(d_B(0, z + rB^d)))\, \lambda(\mathrm{d}z)\, \mathbb{G}(\mathrm{d}r).
\end{aligned}$$

To the inner integral, we can apply formulas (14.27) (with $K = rB^d$) and (14.25). Using $d_B(0, z + rB^d) = d_B(-z, rB^d)$ and the reflection invariance of λ, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{1}\{0 < d_B(0, z + rB^d) < \infty\} g(d_B(0, z + rB^d), r) \\
&\times (1 - H_B(d_B(0, z + rB^d)))\, \lambda(\mathrm{d}z) \\
&= \int_{\mathbb{R}^d} \mathbf{1}\{0 < d_B(z, rB^d) < \infty\} g(d_B(z, rB^d), r) \\
&\quad \times (1 - H_B(d_B(z, rB^d)))\, \lambda(\mathrm{d}z) \\
&= \sum_{m=0}^{d-1} (d-m)\kappa_{d-m} \int_0^\infty \int_{\mathbb{R}^d \times \mathbb{R}^d} g(t, r)(1 - H_B(t)) t^{d-1-m} \\
&\quad \times \Xi_m(rB^d; B; \mathrm{d}(y, b))\, \mathrm{d}t \\
&= \sum_{j=0}^{d-1} \binom{d-1}{j} dV(B^d[d-1-j], B[j+1]) r^{d-1-j} \int_0^\infty (1 - H_B(t)) t^j g(t, r) \mathrm{d}t.
\end{aligned}$$

Since (9.42) implies $\mathbb{P}(d_B(0, Z) < \infty) = 1$, division by $\mathbb{P}(0 \notin Z)$ and formula (14.18) yield

$$\begin{aligned}
&\mathbb{E}(g(d_B(0, Z), r_B(0, Z)) \,|\, 0 \notin Z) \\
&= \gamma \sum_{j=0}^{d-1} (j+1)\kappa_{d-1-j} V_{j+1}(B) \int_0^\infty \int_0^\infty r^{d-1-j} t^j (1 - H_B(t)) g(t, r)\, \mathrm{d}t\, \mathbb{G}(\mathrm{d}r).
\end{aligned}$$

This proves the theorem. $\square$

For $g(t, r) := \mathbf{1}\{t \leq s\}$, $s \geq 0$, the theorem yields

$$H_B(s) = \int_0^s h_B(t)(1 - H_B(t))\, \mathrm{d}t$$

with

$$h_B(t) := \gamma \sum_{j=0}^{d-1} (j+1)\kappa_{d-1-j} V_{j+1}(B) t^j \int_0^\infty r^{d-1-j}\, \mathbb{G}(\mathrm{d}r).$$

Equation (9.42) shows that $H_B(s) < 1$ and that H_B is a continuous function satisfying $H_B(0) = 0$. Using the monotonicity of H_B, we obtain that

$$\int_0^s h_B(t)\, \mathrm{d}t \leq \frac{H_B(s)}{1 - H_B(s)} < \infty$$

for all $s \geq 0$. Hence, the exponential formula of Lebesgue–Stieltjes calculus (see, for example, Last and Brandt [434, Theorem A4.12]) shows that

$$H_B(s) = 1 - \exp\left\{ - \int_0^s h_B(t)\, \mathrm{d}t \right\}.$$

Consequently, formula (9.42) is contained in Theorem 9.5.2 as a special case.

We now exploit the result for other suitable functions g. Let $W \in \mathcal{K}$ be a sampling window with $\lambda(W) > 0$. Choosing

$$g(t,r) := \frac{f(t)}{h(t,r)} \mathbf{1}_C(r)$$

for a Borel set $C \subset \mathbb{R}^+$, a measurable function $f \geq 0$ and

$$h(t,r) := \sum_{j=0}^{d-1} (j+1)\kappa_{d-1-j} V_{j+1}(B) r^{d-1-j} t^j,$$

we see that

$$\widehat{\mathbb{G}}(C) := \frac{\int_{W \setminus Z} \mathbf{1}_C(r_B(x,Z)) f(d_B(x,Z)) h(d_B(x,Z), r_B(x,Z))^{-1} \lambda(\mathrm{d}x)}{\int_{W \setminus Z} f(d_B(x,Z)) h(d_B(x,Z), r_B(x,Z))^{-1} \lambda(\mathrm{d}x)}$$

is a ratio-unbiased estimator of $\mathbb{G}(C)$. In fact,

$$\begin{aligned} &\mathbb{E} \int_{W \setminus Z} \mathbf{1}_C(r_B(x,Z)) \frac{f(d_B(x,Z))}{h(d_B(x,Z), r_B(x,Z))} \, \lambda(\mathrm{d}x) \\ &= \gamma \lambda(W) \mathbb{P}(0 \notin Z) \int_0^\infty (1 - H_B(t)) f(t) \, \mathrm{d}t \cdot \mathbb{G}(C) \end{aligned}$$

and (putting $C = \mathbb{R}^+$)

$$\mathbb{E} \int_{W \setminus Z} \frac{f(d_B(x,Z))}{h(d_B(x,Z), r_B(x,Z))} \, \lambda(\mathrm{d}x) = \gamma \lambda(W) \mathbb{P}(0 \notin Z) \int_0^\infty (1 - H_B(t)) f(t) \, \mathrm{d}t.$$

It should be emphasized that this estimator uses information outside the sampling window W. Namely, for each $x \in W \setminus Z$, the B-distance $d_B(x,Z)$ and the radius $r_B(x,Z)$ of the grain determined by the corresponding contact point have to be observed and the latter may lie outside W.

However, the above considerations show that the generalized contact distributions which we considered give sufficient information to determine the radius distribution $\mathbb{G}$.

We discuss a special case, where the estimator $\widehat{\mathbb{G}}$ has a simpler form. Namely, we consider a planar Boolean model, choose a square as sampling window and $B = [0,u]$, where the unit vector u is parallel to one side of W. Then $h(t,r) = 2r$, hence

$$\widehat{\mathbb{G}} = \frac{1}{\sum_{i=1}^n w_i} \sum_{i=1}^n w_i \delta_{r_i}. \tag{9.43}$$

Here, $r_1, \dots, r_n$ are the radii of the arcs $C_1, \dots, C_n$ in $\operatorname{bd} Z$ which appear as projections from points in W in direction u. If A_i is the region that 'projects' onto C_i, namely the union of all segments $\overline{xy}$, $x \in W\backslash Z$, $y = x+d_{[0,u]}(x,Z)u \in C_i$, the weights w_i are given by

$$w_i = \frac{1}{2r_i} \int_{A_i} f(d_{[0,u]}(z, C_i)) \, \lambda(\mathrm{d}z), \qquad i = 1, \dots, n.$$

In the simplest case, $f = 1$, the weights are proportional to the area of A_i. On the other hand, if $f(t) = \frac{1}{\epsilon}\mathbf{1}\{t \le \epsilon\}$ with $\epsilon \to 0$, we get in the limit an estimator of the form (9.43) where the weights are proportional to the lengths of the arcs. This estimator is studied in the book by Hall [317].

Notes for Section 9.5

1. The estimation of the parameters of a stationary Boolean model (with or without isotropy) is discussed in the books of Serra [729], Cressie [185], Stoyan, Kendall and Mecke [743] and, in particular, in Molchanov [544, 546]. The method of fitting a polynomial to the logarithm of the (empirical) spherical contact distribution function is known as **minimum contrast method**. As a variant, one can investigate the contact distribution function $H_M(r)$, for a fixed value $r = 1$, say, but for different structuring elements M. For example, in the planar case, M can be chosen to be 0-dimensional (point), 1-dimensional (segment) and 2-dimensional (square). The resulting equations can then be solved for γ. In this way, an estimation method for γ was constructed in Hall [316], which is based on counting the number of cells, edges and vertices of a square lattice which are intersected by the given planar Boolean model.

2. Formula (9.39) for the intensity of the uncovered lower tangent points in Theorem 9.5.1 seems to occur first in Serra [729]. The uncovered lower tangent points are dependent, so they no longer form a Poisson process. However, the following result holds. Consider the stationary Boolean model Z with convex grains and with intensity γ in the half plane

$$\mathbb{R}^2_+ := \{(x^1, x^2) \in \mathbb{R}^2 : x^1 \ge 0\}.$$

The **Laslett transform** $L : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ (depending on Z) shifts the points of $\mathbb{R}^2_+$ to the 'left' as far as possible, treating Z as 'empty space' and its complement as solid. More precisely, $L(x_1, x_2) := (\widehat{x}^1, x^2)$ with

$$\widehat{x}^1 := \lambda_1(([0, x^1] \times \{x^2\}) \cap Z^c).$$

The images of the uncovered lower tangent points of Z under this transformation form the restriction to $\mathbb{R}^2_+$ of a stationary Poisson process with intensity γ. This was first explained in Cressie [185]; a short and elegant proof based on a martingale argument was given by Barbour and Schmidt [78]. They mention that the approach also holds in the d-dimensional setting. A further proof in $\mathbb{R}^d$ was given by Černý [168].

For the estimator $\widehat{\gamma}$ based on the uncovered lower tangent points, asymptotic normality was shown by Molchanov and Stoyan [549].

3. The estimation method for γ based on formula (9.40) is due to Schmitt [670]; it has been extended to non-stationary Boolean models as well (Schmitt [671]).

4. The use of Theorem 9.1.4 for the estimation in stationary and isotropic Boolean models Z is classical (see Molchanov [546]). The procedure is sometimes called the **method of moments** since it yields estimators for all the specific intrinsic volumes $\overline{V}_0(X), \dots, \overline{V}_d(X)$. Because of $\overline{V}_0(X) = \gamma$ (in the case of convex grains), this determines the mean values

$$\int_{\mathcal{K}_0} V_j(K)\,\mathbb{Q}(\mathrm{d}K), \qquad j = 1, \dots, d.$$

If the grains are balls, we thus obtain the first d moments of the distribution of the radii. The extension of the method of moments to non-isotropic Boolean models in the plane is due to Weil [794], based on earlier results in [793]. As we mentioned already, a corresponding analysis in $\mathbb{R}^3$ is still possible. We sketch the corresponding approach from Weil [798].

For $d = 3$, we consider the density equations

$$\begin{aligned}
\overline{V}(Z) &= 1 - \mathrm{e}^{-\overline{V}(X)}, \\
\overline{S}_2(Z, \cdot) &= \mathrm{e}^{-\overline{V}(X)} \overline{S}_2(X, \cdot), \\
\overline{h}(Z, \cdot) &= \mathrm{e}^{-\overline{V}(X)} \left(\overline{h}(X, \cdot) - \overline{h}_2(X, X, \cdot)\right), \\
\overline{\chi}(Z) &= \mathrm{e}^{-\overline{V}(X)} \left(\gamma - \overline{V}^{(0)}_{1,2}(X, X) + \overline{V}^{(0)}_{2,2,2}(X, X, X)\right).
\end{aligned}$$

Here, the first equation is the usual one, and the fourth results from Theorem 9.1.5. The second equation is (9.12), and the third is the three-dimensional analog of (9.13). It involves the **specific mixed support function**

$$\overline{h}_2(X, X, \cdot) := \gamma^2 \int_{\mathcal{K}_0} \int_{\mathcal{K}_0} h^*_2(K, M; \cdot)\,\mathbb{Q}(\mathrm{d}M)\,\mathbb{Q}(\mathrm{d}K)$$

(see Theorem 6.4.6). The first equation serves to remove the exponential expression, so we can assume that the quantities $\overline{S}_2(X, \cdot)$, $\overline{h}(X, \cdot) - \overline{h}_2(X, X, \cdot)$ and $\gamma - \overline{V}^{(0)}_{1,2}(X, X) + \overline{V}^{(0)}_{2,2,2}(X, X, X)$ are determined by the left sides. Using the representation (6.30) of $h^*_2(K, M; \cdot)$ for polytopes K, M, we obtain

$$h^*_2(K, M; u) = \int_{S^2} \int_{S^2} f(-u, v, w)\, S_2(K, \mathrm{d}v)\, S_2(M, \mathrm{d}w), \qquad u \in S^2,$$

with a function f, given explicitly by (6.30). By approximation, this representation extends to all convex bodies K, M, therefore we get

$$\overline{h}_2(X, X; u) = \int_{S^2} \int_{S^2} f(-u, v, w)\, \overline{S}_2(X, \mathrm{d}v)\, \overline{S}_2(X, \mathrm{d}w), \qquad u \in S^2.$$

It follows that $\overline{h}_{2,2}(X, X, \cdot)$ is determined and thus also $\overline{h}(X, \cdot)$. It remains to show that $\overline{V}^{(0)}_{1,2}(X, X)$ and $\overline{V}^{(0)}_{2,2,2}(X, X, X)$ can be expressed in terms of $\overline{h}(X, \cdot)$, $\overline{h}_2(X, X, \cdot)$ and $\overline{S}_2(X, \cdot)$, since then we obtain γ. For the first density, this is easy since (9.10) immediately yields

$$\overline{V}^{(0)}_{1,2}(X, X) = \int_{S^2} \overline{h}(X, u)\, \overline{S}_2(X, \mathrm{d}u).$$

For the second density, it turns out that similarly

$$\overline{V}^{(0)}_{2,2,2}(X,X,X) = \int_{S^2} \overline{h}_2(X,X,-u)\,\overline{S}_2(X,\mathrm{d}u)$$
$$= \int_{S^2}\int_{S^2}\int_{S^2} f(u,v,w)\,\overline{S}_2(X,\mathrm{d}v)\,\overline{S}_2(X,\mathrm{d}w)\,\overline{S}_2(X,\mathrm{d}u)$$

holds.

In Weil [801], these estimation problems for $d=2$ and $d=3$ were reviewed from the point of densities of mixed volumes (compare Theorem 9.1.6). It was shown that, for $d=2$, the densities $\overline{V}_0(Z)$, $\overline{V}(Z[1],M[1])$ for all $M\in\mathcal{K}'$, and $\overline{V}_2(Z)$, determine γ uniquely, whereas in dimension $d=3$, the densities $\overline{V}_0(Z)$, $\overline{V}(Z[1],M[2])$ and $\overline{V}(Z[2],M[1])$, for all $M\in\mathcal{K}'$, as well as $\overline{V}_3(Z)$ are needed. In [801], also the four-dimensional situation was discussed and it was claimed that the intensity γ is determined by the densities of mixed volumes of Z. The proof, however, is incomplete, since a summand $\overline{V}^{(0)}_{2,2}(X,X)$ is missing in the formula for the specific Euler characteristic (see the remarks in Goodey and Weil [280]). Therefore, the four-dimensional case is still open, as are all the higher-dimensional situations.

The approach with local densities (of surface area measures and support functions) or specific mixed volumes can be applied also to non-stationary Boolean models. The specific intrinsic volumes and their local counterparts, the specific surface area measure and the specific support function, then also depend on the location in space. For their definition and further details, see Section 11.1 and the corresponding Note 2.

5. The estimation procedure described before the Wicksell problem requires in practice an estimation of the densities $\overline{h}(Z,\cdot)$ and/or $\overline{S}_1(Z,\cdot)$ from measurements in an observation window. Methods to achieve this are described, for $\overline{h}(Z,\cdot)$ in Weil [794, p. 112 ff], and for $\overline{S}_1(Z,\cdot)$ in Rataj [612], Kiderlen and Jensen [407].

6. The Wicksell corpuscle problem is a classic of stochastic geometry, since Wicksell [813] first treated it and gave an explicit solution of the corresponding Abel type integral equation. The use of marked point processes for the derivation of (9.41) goes back to Mecke and Stoyan [502]. For more details on the Wicksell problem, we refer to Stoyan, Kendall and Mecke [743, sect. 11.4]; see also Ripley [644, sect. 9.4]. Limit distributions of stereological estimators in Wicksell's problem were studied by Heinrich [327]. Zähle [830] treated Wicksell's corpuscle problem in spherical space.

7. Theorem 9.5.2 is a special case of more general results in Hug, Last and Weil [358]. We shall present some of them in Section 11.2. In [358] also various situations are discussed where (generalized) contact distributions of a Boolean model can be used to obtain information on the underlying grain distribution $\mathbb{Q}$. The particular case of spherical grains in the plane, which we presented here, was explained in Weil [802] and is based on work in progress by Hug, Last and Weil.

8. Estimating the intensity of stationary flat processes. Let X be a stationary process of k-flats in $\mathbb{R}^d$ ($k\in\{1,\dots,d-1\}$) with intensity γ . Let $W\in\mathcal{K}$ be a convex sampling window with $V_{d-k}(K)>0$. The 'weighted estimator'

$$\widehat{\gamma} := \sum_{E\in X\cap\mathcal{F}_W} \frac{1}{V_{d-k}(W|E^{\perp})}$$

is an unbiased estimator for the intensity γ, as follows immediately from the Campbell theorem and (4.25). On the other hand, if X has a known directional distribution

$\mathbb{Q}$, then

$$\widehat{\gamma} := \frac{1}{V_{\mathbb{Q}}} \sum_{E \in X \cap \mathcal{F}_W} 1 \qquad \text{with} \qquad V_{\mathbb{Q}} := \int_{G(d,k)} V_{d-k}(W|L^{\perp})\, \mathbb{Q}(\mathrm{d}L)$$

can be used as an unbiased estimator. Schladitz [666] has interpolated between these two extreme cases (of no knowledge and of complete knowledge about the directional distribution), defining an unbiased estimator for the intensity, the 'R-estimator', in the case where the directional distribution of X is known to belong to a given family R of probability measures on $G(d,k)$. She gave sufficient conditions for the R-estimator to be the uniformly best unbiased estimator for the intensity of stationary Poisson k-flat processes with directional distribution in R. For stationary ergodic flat processes, the R-estimator is still uniformly better than the 'naive' one based on Theorem 4.4.3, that is (for $V_d(W) > 0$),

$$\widehat{\gamma} := \frac{1}{V_d(W)} \sum_{E \in X} V_k(E \cap W).$$

9. Estimating the Euler characteristic. The system of formulas (9.36)–(9.38) (as well as the corresponding system in other dimensions) does not include an intersection formula for the specific Euler characteristic $\overline{\chi}(Z)$. In fact, this density, as well as the mean particle number for processes of convex particles, cannot be estimated from the information provided by lower-dimensional sections. To overcome this difficulty, estimators have been suggested that use the information coming jointly from two close parallel hyperplane sections, or from the slab between them. Unbiasedness of these estimators is only guaranteed if the sets under investigation satisfy additional assumptions. We refer to the papers by Ohser and Nagel [588] and by Rataj [616] and to the literature quoted there.

10. Estimating the directional distribution of fiber processes. If X is a stationary fiber process in $\mathbb{R}^d$, with specific length $\overline{V}_1(X)$ and spherical directional distribution φ, then (4.40) says that

$$\overline{V}_0(X \cap v^{\perp}) = \overline{V}_1(X) \int_{S^{d-1}} |\langle u, v\rangle|\, \varphi(\mathrm{d}u)$$

for $v \in S^{d-1}$. If the specific length has already been estimated, this can be used to estimate the directional distribution by means of intersection point counts in hyperplanes $v^{\perp}$. (The function $v \mapsto \overline{V}_0(X \cap v^{\perp})$ is known as the 'rose of intersections', and the even probability measure φ as the 'rose of directions'.) Although the measure φ is uniquely determined by the rose of intersections, there are practical difficulties, since the inversion of the cosine transform is unstable, and only finitely many values of the rose of intersections will be available. Different methods of nonparametric estimation to overcome these difficulties have been described by Kiderlen [403] and by Kiderlen and Pfrang [408].

11. Estimating mean normal measures. Similar problems to those described in the previous note arise if one wants to use (4.41) for the estimation of the directional distribution of a stationary hypersurface process. A related notion is the mean normal measure of a stationary process X of convex particles or of a stationary standard random set Z, denoted by $\overline{S}_{d-1}(X, \cdot)$ and $\overline{S}_{d-1}(Z, \cdot)$, respectively. Since outer normal vectors are used in their definitions, these measures are also called **oriented mean normal measures**, to distinguish them from their even parts, which are called **unoriented mean normal measures** and correspond to the directional distributions of the boundaries. Various estimation procedures for both oriented and unoriented mean normal measures, by means of lower-dimensional sections, have been investigated; we refer to Schneider [704], Kiderlen [404, 406] and the literature quoted there.

12. Estimating particle orientation. If the Blaschke body $B(X)$ of a stationary process X of convex particles is distinctly non-spherical, it reveals anisotropy of X. It may, therefore, be of interest to estimate the Blaschke body by stereological means. Weil [796] has shown that $B(X)$ is uniquely determined by the statistical properties of two-dimensional sections of X, but in practice an estimation based on this fact may be difficult. If only mean particle orientation is of interest, one can replace the Blaschke body by a suitable ellipsoid (equivalently, a positive definite symmetric matrix), which is more accessible to stereological estimation. For a convex body $K \in \mathcal{K}$ with interior points, the **area moment tensor** $T(K)$ is the symmetric tensor of rank two with cartesian coordinates $T_{ij}(K)$ given by

$$T_{ij}(K) := \int_{S^{d-1}} u_i u_j \, S_{d-1}(K, \mathrm{d}u).$$

The eigenvalues and eigendirections of the matrix $(T_{ij}(K))_{i,j=1}^d$ can be used to describe the anisotropy of K. The **specific area moment tensor** of the stationary particle process X (with convex particles, intensity γ and grain distribution $\mathbb{Q}$) is defined by

$$\overline{T}(X) := \gamma \int_{\mathcal{K}_0} T(K) \, \mathbb{Q}(\mathrm{d}K) = \lim_{r \to \infty} \frac{1}{V_d(rW)} \mathbb{E} \sum_{K \in X} T(K \cap rW),$$

for arbitrary $W \in \mathcal{K}$ with $V_d(W) > 0$. It turns out that $\overline{T}(X) = T(B(X))$. A method for estimating $\overline{T}(X)$ from sections with hyperplanes was described by Schneider and Schuster [714].

13. Estimation from digitized images. Practical estimation in two and three dimensions may meet the additional difficulty that only digitized images are available, or the sets under investigation are accessible only via their intersections with sufficiently fine scaled grids. Estimation can then be based, for example, on pixel or voxel configuration counts. Methods for the estimation from digitized images have been developed in several investigations, for the Euler characteristic by Nagel, Ohser and Pischang [573], Ohser, Nagel and Schladitz [589, 590], Kiderlen [405], for specific intrinsic volumes by Lang, Ohser and Hilfer [431], and for directional distributions and oriented mean normal measures by Jensen and Kiderlen [381], Kiderlen and Jensen [407], Gutkowski, Jensen and Kiderlen [302], Ziegel and Kiderlen [835].