7

Integral Geometric Transformations

Mean value formulas with respect to invariant measures, as treated in the preceding two chapters, are a central topic of integral geometry. Another one is transformation formulas for integrals over various spaces of geometric objects. The need for such results in stochastic geometry can be demonstrated by simple examples. Consider, for instance, two independent, identically distributed random hyperplanes in $\mathbb{R}^d$. Suppose the distribution is such that the intersection of the two hyperplanes is almost surely a $(d-2)$-flat. What is the distribution of this random $(d-2)$-flat? Or, consider $k \leq d$ independent, identically distributed random points in $\mathbb{R}^d$, and suppose their distribution is such that they almost surely span a $(k-1)$-flat. What is its distribution? In the cases where the original distributions are derived from invariant measures (by restriction, for example), the answers can be obtained from simple cases of the transformation formulas of this chapter. Generally, these transformation formulas relate integrations over tuples of flats, with respect to invariant measures, to integrations over other sets of flats (or other geometric objects) that are obtained by geometric operations, such as intersection or span. As an example, consider the integral of a function depending on d points. It may happen that the function depends only on the hyperplane spanned (almost everywhere) by the points. Then it may have a simplifying effect to integrate first over the d-tuples of points lying in a fixed hyperplane, and then over all hyperplanes. In principle, the required transformation formulas are just versions of the transformation rule for multiple integrals under differentiable mappings. However, since the mappings are defined by geometric operations, the Jacobians have geometric interpretations, and therefore direct geometric arguments are often simpler and more perspicuous than the use of special parametrizations.

The transformation formulas to be proved have various applications in stochastic geometry, for example in the investigation of convex hulls of random points (Chapter 8), the study of random mosaics (Chapter 10), or in the foundations of stereology. We do not aim at presenting the integral geometric transformation formulas in their greatest generality, but rather give typical and basic examples. This will be done in Sections 7.2 and 7.3. The first section provides simple rules for invariant measures on flag spaces.

7.1 Flag Spaces

In this section, we consider pairs of linear or affine subspaces, one contained in the other.

Let $p, q \in \{0, \ldots, d\}$, and let $L \in G(d,p)$ be a fixed p-dimensional linear subspace. We denote by $G(L,q)$ the space of all q-dimensional linear subspaces contained in L if $q \le p$, respectively containing L if $q > p$. In a similar way, for an affine subspace $E \in A(d,p)$, the space $A(E,q)$ of q-flats contained in E, respectively containing E, is defined. These spaces are described in detail in Section 13.2. There also the invariant measures ν_q^L on $G(L,q)$ and μ_q^E on $A(E,q)$ are constructed. These measures will be used in the following.

Now we turn to spaces of pairs of linear subspaces or flats. For $0 \le p, q \le d$ with $p \ne q$ we define

$$G(d,p,q) := \{(L,M) \in G(d,p) \times G(d,q) : L \subset M\}, \qquad \text{if } p < q,$$

$$G(d,p,q) := \{(L,M) \in G(d,p) \times G(d,q) : L \supset M\}, \qquad \text{if } p > q,$$

and

$$A(d,p,q) := \{(E,F) \in A(d,p) \times A(d,q) : E \subset F\}, \qquad \text{if } p < q,$$

$$A(d,p,q) := \{(E,F) \in A(d,p) \times A(d,q) : E \supset F\}, \qquad \text{if } p > q.$$

In an obvious way, these definitions could be extended to more than two linear or affine subspaces. Spaces of the type $G(d,p,q)$ or $A(d,p,q)$ are called **flag spaces**. The flag space $G(d,p,q)$, for example, is evidently a homogeneous SO_d-space. Defining

$$\begin{aligned} \beta_{p,q} : SO_d &\to \ G(d,p,q), \\ \vartheta \ &\mapsto (\vartheta L_p, \vartheta L_q) \end{aligned}$$

where $(L_p, L_q) \in G(d,p,q)$ is arbitrary but fixed, and

$$\nu_{p,q} := \beta_{p,q}(\nu),$$

we obtain a rotation invariant probability measure $\nu_{p,q}$ on $G(d,p,q)$. Thus, by definition,

$$\int_{G(d,p,q)} f \, \mathrm{d}\nu_{p,q} = \int_{SO_d} f(\vartheta L_p, \vartheta L_q) \, \nu(\mathrm{d}\vartheta) \tag{7.1}$$

for every nonnegative measurable function f on $G(d,p,q)$. We shall first show that this measure can be computed, as one might expect, by iterated integrations over p- and q-dimensional subspaces.

Theorem 7.1.1. *If $0 \le p < q \le d-1$ and if $f : G(d,p,q) \to \mathbb{R}$ is a nonnegative measurable function, then*

$$\int_{G(d,p,q)} f \, \mathrm{d}\nu_{p,q} = \int_{G(d,q)} \int_{G(M,p)} f(L,M) \, \nu_p^M(\mathrm{d}L) \, \nu_q(\mathrm{d}M)$$
$$= \int_{G(d,p)} \int_{G(L,q)} f(L,M) \, \nu_q^L(\mathrm{d}M) \, \nu_p(\mathrm{d}L).$$

Proof. Measurability follows, for example, from the fact that the mapping $(M,B) \mapsto \nu_p^M(B)$, $M \in G(d,q)$, $B \in \mathcal{B}(G(d,p))$, is a kernel; for this, see Lemma 13.2.2. Let $(L_p, L_q) \in G(d,p,q)$. In the subsequent chain of equalities we use, in this order, the definition of ν_q as the image measure of ν under β_q, the invariance property (13.12), the definition of $\nu_p^{L_q}$, Fubini's theorem, the equality $L_q = \rho L_q$ for $\rho \in SO(L_q)$, and the right invariance of ν. We obtain

$$\int_{G(d,q)} \int_{G(M,p)} f(L,M) \, \nu_p^M(\mathrm{d}L) \, \nu_q(\mathrm{d}M)$$
$$= \int_{SO_d} \int_{G(\vartheta L_q,p)} f(L, \vartheta L_q) \, \nu_p^{\vartheta L_q}(\mathrm{d}L) \, \nu(\mathrm{d}\vartheta)$$
$$= \int_{SO_d} \int_{G(L_q,p)} f(\vartheta L', \vartheta L_q) \, \nu_p^{L_q}(\mathrm{d}L') \, \nu(\mathrm{d}\vartheta)$$
$$= \int_{SO_d} \int_{SO(L_q)} f(\vartheta \rho L_p, \vartheta L_q) \, \nu_{L_q}(\mathrm{d}\rho) \, \nu(\mathrm{d}\vartheta)$$
$$= \int_{SO(L_q)} \int_{SO_d} f(\vartheta \rho L_p, \vartheta L_q) \, \nu(\mathrm{d}\vartheta) \, \nu_{L_q}(\mathrm{d}\rho)$$
$$= \int_{SO(L_q)} \int_{SO_d} f(\vartheta L_p, \vartheta L_q) \, \nu(\mathrm{d}\vartheta) \, \nu_{L_q}(\mathrm{d}\rho)$$
$$= \int_{SO_d} f(\vartheta L_p, \vartheta L_q) \, \nu(\mathrm{d}\vartheta).$$

In an analogous manner (though with a difference since $p < q$) we get

$$\int_{G(d,p)} \int_{G(L,q)} f(L,M) \, \nu_q^L(\mathrm{d}M) \, \nu_p(\mathrm{d}L)$$
$$= \int_{SO_d} \int_{G(\vartheta L_p,q)} f(\vartheta L_p, M) \, \nu_q^{\vartheta L_p}(\mathrm{d}M) \, \nu(\mathrm{d}\vartheta)$$
$$= \int_{SO_d} \int_{G(L_p,q)} f(\vartheta L_p, \vartheta M') \, \nu_q^{L_p}(\mathrm{d}M') \, \nu(\mathrm{d}\vartheta)$$
$$= \int_{SO_d} \int_{SO(L_p^\perp)} f(\vartheta L_p, \vartheta \rho L_q) \, \nu_{L_p^\perp}(\mathrm{d}\rho) \, \nu(\mathrm{d}\vartheta)$$

$$= \int_{SO(L_p^\perp)} \int_{SO_d} f(\vartheta L_p, \vartheta\rho L_q)\, \nu(\mathrm{d}\vartheta)\, \nu_{L_p^\perp}(\mathrm{d}\rho)$$

$$= \int_{SO(L_p^\perp)} \int_{SO_d} f(\vartheta L_p, \vartheta L_q)\, \nu(\mathrm{d}\vartheta)\, \nu_{L_p^\perp}(\mathrm{d}\rho)$$

$$= \int_{SO_d} f(\vartheta L_p, \vartheta L_q)\, \nu(\mathrm{d}\vartheta).$$

This, together with (7.1), completes the proof of Theorem 7.1.1. □

Remark. Let $0 \le p < q \le d-1$. The special case $f(L,M) = \mathbf{1}_A(L)$ with $A \in \mathcal{B}(G(d,p))$ in Theorem 7.1.1 yields

$$\nu_p(A) = \int_{G(d,q)} \nu_p^M(A)\, \nu_q(\mathrm{d}M), \tag{7.2}$$

and similarly one has

$$\nu_q(A) = \int_{G(d,p)} \nu_q^L(A)\, \nu_p(\mathrm{d}L) \tag{7.3}$$

for $A \in \mathcal{B}(G(d,q))$.

Remark. Let $p, q \in \{0, \dots, d-1\}$. For the Radon transform

$$R_{pq} : \mathbf{C}(G(d,p)) \to \mathbf{C}(G(d,q))$$

defined by (6.35), Theorem 7.1.1 implies the symmetry relation

$$\int_{G(d,q)} (R_{pq}f) g \,\mathrm{d}\nu_q = \int_{G(d,p)} f(R_{qp}g)\,\mathrm{d}\nu_p$$

for $f \in \mathbf{C}(G(d,p))$ and $g \in \mathbf{C}(G(d,q))$.

Theorem 7.1.1 can be generalized. For example, let integers $r < p < q < d$ or $q < p < r < d$ be given. For $L_0 \in G(d,r)$ and $L_2 \in G(d,q)$, let $G(L_0, L_2, p)$ denote the space of all p-dimensional subspaces L_1 with $L_0 \subset L_1 \subset L_2$ if $r < p < q$, respectively with $L_2 \subset L_1 \subset L_0$ if $q < p < r$. It carries a unique probability measure $\nu_p^{L_0, L_2}$ which is invariant under the rotations that map L_0 into itself and L_2 into itself. Let $L_0 \in G(d,r)$ be fixed. If $f : G(d,p,q) \to \mathbb{R}$ is a nonnegative measurable function, then

$$\int_{G(L_0,p)} \int_{G(L_1,q)} f(L_1, L_2)\, \nu_q^{L_1}(\mathrm{d}L_2)\, \nu_p^{L_0}(\mathrm{d}L_1)$$
$$= \int_{G(L_0,q)} \int_{G(L_0,L_2,p)} f(L_1, L_2)\, \nu_p^{L_0,L_2}(\mathrm{d}L_1)\, \nu_q^{L_0}(\mathrm{d}L_2). \tag{7.4}$$

This can be proved along similar lines to above.

A result analogous to Theorem 7.1.1 is valid for affine subspaces. It can be deduced from this theorem.

Theorem 7.1.2. *If $0 \le p < q \le d-1$ and if $f : A(d,p,q) \to \mathbb{R}$ is a nonnegative measurable function, then*

$$\int_{A(d,q)} \int_{A(F,p)} f(E,F)\, \mu_p^F(\mathrm{d}E)\, \mu_q(\mathrm{d}F)$$
$$= \int_{A(d,p)} \int_{A(E,q)} f(E,F)\, \mu_q^E(\mathrm{d}F)\, \mu_p(\mathrm{d}E). \tag{7.5}$$

Proof. For measurability, we again refer to Lemma 13.2.2. In the subsequent chain of equations we use (13.9), (13.13), Theorem 7.1.1, (13.14), again (13.9), and several times the theorem of Fubini. In this way, we get

$$\int_{A(d,q)} \int_{A(F,p)} f(E,F)\, \mu_p^F(\mathrm{d}E)\, \mu_q(\mathrm{d}F)$$
$$= \int_{G(d,q)} \int_{L^\perp} \int_{A(L+t,p)} f(E, L+t)\, \mu_p^{L+t}(\mathrm{d}E)\, \lambda_{d-q}(\mathrm{d}t)\, \nu_q(\mathrm{d}L)$$
$$= \int_{G(d,q)} \int_{L^\perp} \int_{G(L,p)} \int_{M^\perp \cap L} f(M+x+t, L+t)$$
$$\times\, \lambda_{q-p}(\mathrm{d}x)\, \nu_p^L(\mathrm{d}M)\, \lambda_{d-q}(\mathrm{d}t)\, \nu_q(\mathrm{d}L)$$
$$= \int_{G(d,q)} \int_{G(L,p)} \int_{L^\perp} \int_{M^\perp \cap L} f(M+x+t, L+x+t)$$
$$\times\, \lambda_{q-p}(\mathrm{d}x)\, \lambda_{d-q}(\mathrm{d}t)\, \nu_p^L(\mathrm{d}M)\, \nu_q(\mathrm{d}L)$$
$$= \int_{G(d,q)} \int_{G(L,p)} \int_{M^\perp} f(M+z, L+z)\, \lambda_{d-p}(\mathrm{d}z)\, \nu_p^L(\mathrm{d}M)\, \nu_q(\mathrm{d}L)$$
$$= \int_{G(d,p)} \int_{G(M,q)} \int_{M^\perp} f(M+z, L+z)\, \lambda_{d-p}(\mathrm{d}z)\, \nu_q^M(\mathrm{d}L)\, \nu_p(\mathrm{d}M)$$
$$= \int_{G(d,p)} \int_{M^\perp} \int_{G(M,q)} f(M+z, L+z)\, \nu_q^M(\mathrm{d}L)\, \lambda_{d-p}(\mathrm{d}z)\, \nu_p(\mathrm{d}M)$$
$$= \int_{G(d,p)} \int_{M^\perp} \int_{A(M+z,q)} f(M+z, F)\, \mu_q^{M+z}(\mathrm{d}F)\, \lambda_{d-p}(\mathrm{d}z)\, \nu_p(\mathrm{d}M)$$
$$= \int_{A(d,p)} \int_{A(E,q)} f(E,F)\, \mu_q^E(\mathrm{d}F)\, \mu_p(\mathrm{d}E).$$

This completes the proof of Theorem 7.1.2. □

Remark. Analogously to (7.2) one obtains, for $0 \le p < q \le d-1$, a representation of the invariant measure μ_p in the form

$$\mu_p(A) = \int_{A(d,q)} \mu_p^F(A)\, \mu_q(\mathrm{d}F),$$

for $A \in \mathcal{B}(A(d,p))$. There is, however, no representation corresponding to (7.3), because the measure μ_p^F, where $F \in A(d,q)$ and $p < q$, is not finite.

Note for Section 7.1

Theorems 7.1.1 and 7.1.2 can also be deduced from the essential uniqueness of invariant measures on homogeneous spaces (Theorem 13.3.1).

7.2 Blaschke–Petkantschin Formulas

We recall that for $E \in A(d,q)$ we denote by λ_E the q-dimensional Lebesgue measure on E, considered as a measure on all of $\mathbb{R}^d$, thus

$$\lambda_E(A) = \lambda_q(A \cap E) \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^d).$$

For several applications one needs integral geometric transformations of a kind for which the following is a typical example. Suppose we have to integrate a function of q-tuples of points in $\mathbb{R}^d$, where $q \in \{1, \dots, d-1\}$, with respect to the product measure λ^q. In some cases it may simplify the computation to integrate first over the q-tuples of points in a fixed q-dimensional linear subspace L, with respect to the product measure λ_L^q, and then to integrate over all linear subspaces L, with respect to the invariant measure ν_q on $G(d,q)$. The case $q = 1$ corresponds essentially to the well-known computation of a volume integral in terms of polar coordinates. The Jacobian appearing in the general transformation formula has a simple geometric meaning. A similar transformation formula exists for affine, instead of linear, subspaces. Results of this type are called **Blaschke–Petkantschin formulas**. We prepare the proof of these formulas by a lemma which extends the polar coordinate formula.

We denote by $d(x,L)$ the distance of the point $x \in \mathbb{R}^d$ from the subspace $L \subset \mathbb{R}^d$.

Lemma 7.2.1. *If $r \in \{0, \dots, d-1\}$ and $L \in G(d,r)$ is a fixed linear subspace, then*

$$\int_{\mathbb{R}^d} f \, \mathrm{d}\lambda = \frac{\omega_{d-r}}{2} \int_{G(L,r+1)} \int_M f d(\cdot, L)^{d-r-1} \, \mathrm{d}\lambda_M \, \nu_{r+1}^L(\mathrm{d}M)$$

for every nonnegative measurable function f on $\mathbb{R}^d$.

Proof. We denote by L_u the positive hull of L and a vector u. Using spherical coordinates in $L^\perp$ and Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} f(x) \, \lambda(\mathrm{d}x) \\
&= \int_L \int_{L^\perp} f(x_0 + x_1) \lambda_{L^\perp}(\mathrm{d}x_1) \, \lambda_L(\mathrm{d}x_0)
\end{aligned}$$

$$
\begin{aligned}
&= \int_L \int_0^\infty \int_{S^{d-1}\cap L^\perp} f(x_0+\rho u)\rho^{d-r-1}\,\sigma_{d-r-1}(\mathrm{d}u)\,\mathrm{d}\rho\,\lambda_L(\mathrm{d}x_0)\\
&= \int_{S^{d-1}\cap L^\perp} \int_L \int_0^\infty f(x_0+\rho u)\rho^{d-r-1}\,\mathrm{d}\rho\,\lambda_L(\mathrm{d}x_0)\,\sigma_{d-r-1}(\mathrm{d}u)\\
&= \int_{S^{d-1}\cap L^\perp} \int_{L_u} f(x)d(x,L)^{d-r-1}\,\lambda_{L_u}(\mathrm{d}x)\,\sigma_{d-r-1}(\mathrm{d}u)\\
&= \frac{\omega_{d-r}}{2}\int_{G(L,r+1)} \int_M f(x)d(x,L)^{d-r-1}\,\lambda_M(\mathrm{d}x)\,\nu^L_{r+1}(\mathrm{d}M).
\end{aligned}
$$

This was the assertion. □

We recall (from Section 4.4 or Section 14.1) that for $q \in \{1,\dots,d\}$ and $x_1,\dots,x_q \in \mathbb{R}^d$ we denote by $\nabla_q(x_1,\dots,x_q)$ the q-dimensional volume of the parallelepiped spanned by the vectors $x_1,\dots,x_q$. For $q+1$ points $x_0,x_1,\dots,x_q \in \mathbb{R}^d$,

$$
\Delta_q(x_0,\dots,x_q) := \frac{1}{q!}\nabla_q(x_1-x_0,\dots,x_q-x_0) \tag{7.6}
$$

is the q-dimensional volume of the convex hull of $\{x_0,\dots,x_q\}$.

The following result is known as the **linear Blaschke–Petkantschin formula**.

Theorem 7.2.1. *If $q \in \{1,\dots,d\}$ and if $f : (\mathbb{R}^d)^q \to \mathbb{R}$ is a nonnegative measurable function, then*

$$
\int_{(\mathbb{R}^d)^q} f\,\mathrm{d}\lambda^q = b_{dq}\int_{G(d,q)}\int_{L^q} f\nabla_q^{d-q}\,\mathrm{d}\lambda^q_L\,\nu_q(\mathrm{d}L) \tag{7.7}
$$

with

$$
b_{dq} := \frac{\omega_{d-q+1}\cdots\omega_d}{\omega_1\cdots\omega_q}. \tag{7.8}
$$

Proof. The subsequent proof, which is adapted from Miles [525], proceeds by induction. For $q=1$, the assertion reduces to Lemma 7.2.1 (case $r=0$) and hence is true. We assume that the assertion has been proved for some $q \ge 1$ and all dimensions d. In the inductive step we make use of the fact that for $x_1,\dots,x_q \in L \in G(d,q)$ and $x_{q+1} \in \mathbb{R}^d$ one has

$$
\nabla_{q+1}(x_1,\dots,x_{q+1}) = \nabla_q(x_1,\dots,x_q)d(x_{q+1},L). \tag{7.9}
$$

Below we abbreviate $(x_1,\dots,x_q)$ by $\mathbf{x}$. First we use, besides Fubini's theorem, the induction hypothesis and Lemma 7.2.1, to obtain

$$
I := \int_{(\mathbb{R}^d)^{q+1}} f\,\mathrm{d}\lambda^{q+1}
$$

$$= \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^q} f(\mathbf{x}, x) \, \lambda^q(\mathrm{d}\mathbf{x}) \, \lambda(\mathrm{d}x)$$

$$= b_{dq} \int_{\mathbb{R}^d} \int_{G(d,q)} \int_{L^q} f(\mathbf{x}, x) \nabla_q(\mathbf{x})^{d-q} \, \lambda_L^q(\mathrm{d}\mathbf{x}) \, \nu_q(\mathrm{d}L) \, \lambda(\mathrm{d}x)$$

$$= b_{dq} \int_{G(d,q)} \int_{L^q} \nabla_q(\mathbf{x})^{d-q} \int_{\mathbb{R}^d} f(\mathbf{x}, x) \, \lambda(\mathrm{d}x) \, \lambda_L^q(\mathrm{d}\mathbf{x}) \, \nu_q(\mathrm{d}L)$$

$$= \frac{b_{dq}\omega_{d-q}}{2} \int_{G(d,q)} \int_{L^q} \nabla_q(\mathbf{x})^{d-q} \int_{G(L,q+1)} \int_M f(\mathbf{x}, x) \, d(x, L)^{d-q-1}$$

$$\times \lambda_M(\mathrm{d}x) \, \nu_{q+1}^L(\mathrm{d}M) \, \lambda_L^q(\mathrm{d}\mathbf{x}) \, \nu_q(\mathrm{d}L).$$

Applying Theorem 7.1.1 for interchanging the integrations over q- and $(q+1)$-dimensional subspaces and then using (7.9), we get

$$I = \frac{b_{dq}\omega_{d-q}}{2} \int_{G(d,q+1)} \int_{G(M,q)} \int_{L^q} \int_M f(\mathbf{x}, x) \nabla_q(\mathbf{x})^{d-q} \, d(x, L)^{d-q-1}$$

$$\times \lambda_M(\mathrm{d}x) \, \lambda_L^q(\mathrm{d}\mathbf{x}) \, \nu_q^M(\mathrm{d}L) \, \nu_{q+1}(\mathrm{d}M)$$

$$= \frac{b_{dq}\omega_{d-q}}{2} \int_{G(d,q+1)} \int_M \int_{G(M,q)} \int_{L^q} f(\mathbf{x}, x) \nabla_{q+1}(\mathbf{x}, x)^{d-q-1} \nabla_q(\mathbf{x})$$

$$\times \lambda_L^q(\mathrm{d}\mathbf{x}) \, \nu_q^M(\mathrm{d}L) \, \lambda_M(\mathrm{d}x) \, \nu_{q+1}(\mathrm{d}M).$$

Now we apply the induction hypothesis again, to a q-fold integration over the $(q+1)$-dimensional space M and the function $f(\cdot, x)\nabla_{q+1}(\cdot, x)^{d-q-1}$. We obtain

$$I = \frac{b_{dq}\omega_{d-q}}{2b_{(q+1)q}} \int_{G(d,q+1)} \int_M \int_{M^q} f(\mathbf{x}, x) \nabla_{q+1}(\mathbf{x}, x)^{d-q-1}$$

$$\times \lambda_M^q(\mathrm{d}\mathbf{x}) \, \lambda_M(\mathrm{d}x) \, \nu_{q+1}(\mathrm{d}M)$$

$$= b_{d(q+1)} \int_{G(d,q+1)} \int_{M^{q+1}} f \nabla_{q+1}^{d-q-1} \, \mathrm{d}\lambda_M^{q+1} \, \nu_{q+1}(\mathrm{d}M),$$

which is the assertion for a $(q+1)$-fold integration. □

Before proceeding further, we want to explain in which situations we talk of a formula of ‘Blaschke–Petkantschin type’; thus, we try to describe the common feature of these transformations. The starting point is an integration over a product (possibly with one factor only) of measure spaces of geometric objects (points or flats, as a rule), mostly homogeneous spaces with their invariant measures. Almost everywhere, the integration variable, which is a tuple of geometric objects, determines a new geometric object (for example, by span or intersection). We call this new object the ‘pivot’. The initial integration is then decomposed into an outer and an inner integration. The outer integration space is the space of all possible pivots, with a natural measure; often it is a homogeneous space. For a given pivot, the inner integration space consists of the tuples of the initial integration space which determine precisely this pivot; as a rule, it is a product of homogeneous spaces.

Lemma 7.2.1 was already of this type. The initial integration is over $\mathbb{R}^d$. The integration variable $x \in \mathbb{R}^d$ determines (almost everywhere) the $(q+1)$-subspace which is spanned by x and the fixed q-subspace L. This $(q+1)$-subspace is the pivot. The outer integration space is the space $G(L, q+1)$ of all $(q+1)$-subspaces containing L. For M in this space, the inner integration space is equal to M. In the case of Theorem 7.2.1, the initial integration is over $(\mathbb{R}^d)^q$, and the pivot is the q-subspace spanned by the integration variable $(x_1, \dots, x_q) \in (\mathbb{R}^d)^q$. Hence, the outer integration space is the Grassmannian $G(d,q)$ of all q-subspaces. For $L \in G(d,q)$, the inner integration space is the product L^q.

There are also extensions of formulas of Blaschke–Petkantschin type where the pivot is not uniquely determined by the integration variable, but only associated with it in some way. For example, in the situation of Theorem 7.2.1, a pivot associated with $(x_1, \dots, x_q)$ could be a subspace of fixed dimension $s \geq q$ containing $x_1, \dots, x_q$, or the span of $x_1, \dots, x_q$ and of a fixed subspace. One can also combine both possibilities; this gives the following generalization of the linear Blaschke–Petkantschin formula. Here we denote by $\nabla_{q,r}(x_1, \dots x_q, L_0)$ the $(q+r)$-dimensional volume spanned by the vectors $x_1, \dots, x_q \in \mathbb{R}^d$ and an orthonormal basis of the subspace $L_0 \in G(d,r)$.

Theorem 7.2.2. *Let $q \geq 1$, $r \geq 0$ and s be integers with $q + r \leq s \leq d$, let $L_0 \in G(d,r)$. If $f : (\mathbb{R}^d)^q \to \mathbb{R}$ is a nonnegative measurable function, then*

$$\int_{(\mathbb{R}^d)^q} f \, \mathrm{d}\lambda^q = \frac{b_{(d-r)q}}{b_{(s-r)q}} \int_{G(L_0,s)} \int_{L^q} f \nabla_{q,r}(\cdot, L_0)^{d-s} \, \mathrm{d}\lambda_L^q \, \nu_s^{L_0}(\mathrm{d}L). \tag{7.10}$$

Proof. First we consider the case $s = q + r$, that is, the formula

$$\int_{(\mathbb{R}^d)^q} f \, \mathrm{d}\lambda^q = b_{(d-r)q} \int_{G(L_0,q+r)} \int_{L^q} f \nabla_{q,r}(\cdot, L_0)^{d-q-r} \, \mathrm{d}\lambda_L^q \, \nu_{q+r}^{L_0}(\mathrm{d}L). \tag{7.11}$$

Its proof proceeds by induction, in a similar manner to Theorem 7.2.1. The case of $q = 1$ is again provided by Lemma 7.2.1. In the induction step, one applies Lemma 7.2.1 to a fixed subspace L of dimension $q + r$ and then uses the interchange formula (7.4) instead of Theorem 7.1.1. After observing that

$$\nabla_{q+1,r}(x_1, \dots, x_{q+1}, L_0) = \nabla_{q,r}(x_1, \dots, x_q, L_0) d(x_{q+1}, L) \quad \text{for } L_0 \subset L$$

and applying Fubini's theorem, the induction hypothesis is applied to a q-fold integration over a $(q+r+1)$-dimensional subspace M containing L_0. Apart from these changes, the proof is the same as before. Thus, the formula (7.11) is proved.

To prove (7.10), we assume that $q + r \leq s \leq d$ and start with the integral

$$I := \int_{G(L_0,s)} \int_{L^q} f \, \mathrm{d}\lambda_L^q \, \nu_s^{L_0}(\mathrm{d}L).$$

We apply (7.11) to the integral over L^q; here $\dim L = s$ and $L_0 \subset L$. Then we use the interchange formula (7.4) and Fubini's theorem. This yields

$$\begin{aligned}
I &= b_{(s-r)q} \int_{G(L_0,s)} \int_{G(L_0,L,q+r)} \int_{M^q} f \nabla_{q,r}(\cdot, L_0)^{s-q-r} \\
&\quad \times \mathrm{d}\lambda_M^q \, \nu_{q+r}^{L_0,L}(\mathrm{d}M) \, \nu_s^{L_0}(\mathrm{d}L) \\
&= b_{(s-r)q} \int_{G(L_0,q+r)} \int_{G(M,s)} \int_{M^q} f \nabla_{q,r}(\cdot, L_0)^{s-q-r} \, \mathrm{d}\lambda_M^q \, \nu_s^M(\mathrm{d}L) \, \nu_{q+r}^{L_0}(\mathrm{d}M) \\
&= b_{(s-r)q} \int_{G(L_0,q+r)} \int_{M^q} \int_{G(M,s)} f \nabla_{q,r}(\cdot, L_0)^{s-q-r} \, \nu_s^M(\mathrm{d}L) \, \mathrm{d}\lambda_M^q \, \nu_{q+r}^{L_0}(\mathrm{d}M) \\
&= b_{(s-r)q} \int_{G(L_0,q+r)} \int_{M^q} f \nabla_{q,r}(\cdot, L_0)^{s-q-r} \, \mathrm{d}\lambda_M^q \, \nu_{q+r}^{L_0}(\mathrm{d}M) \\
&= \frac{b_{(s-r)q}}{b_{(d-r)q}} \int_{(\mathbb{R}^d)^q} f \nabla_{q,r}(\cdot, L_0)^{s-d} \, \mathrm{d}\lambda^q,
\end{aligned}$$

by another application of (7.11). Replacing f by $f \nabla_{q,r}(\cdot, L_0)^{d-s}$, we obtain the assertion. □

The original Blaschke–Petkantschin formula is a source of a series of further integral geometric transformations, of which we shall give some examples. First we derive transformation formulas for integrals over tuples of linear subspaces. The cases where the sum of the dimensions of the subspaces is at most d or larger than d have to be distinguished.

In the subsequent theorem, the initial integration space is $\prod_{i=1}^q G(d, r_i)$, with $\sum_{i=1}^q r_i =: p \le d$, and the pivot determined by a q-tuple of subspaces is their linear span. Correspondingly, the outer integration space is $G(d,p)$, and for $L \in G(d,p)$, the inner integration space is $\prod_{i=1}^q G(L, r_i)$.

In the following, we shall have to use the subspace determinant $[\cdot, \ldots, \cdot]$ defined in Section 14.1. If $q \in \mathbb{N}$, $r_1, \ldots, r_q \in \{1, \ldots, d-1\}$ and $(L_1, \ldots, L_q) \in G(d, r_1) \times \ldots \times G(d, r_q)$, we write

$$[L_1, \ldots, L_q] =: [L_1, \ldots, L_q]_{\mathbf{r}},$$

where $\mathbf{r} := (r_1, \ldots, r_q)$ serves as a multi-index. If L_0 is a fixed linear subspace, we also write

$$[L_1, \ldots, L_q, L_0] =: [L_1, \ldots, L_q, L_0]_{\mathbf{r}}.$$

Thus, for $\mathbf{r} := (r_1, \ldots, r_q)$, the functions $[\cdot, \ldots, \cdot]_{\mathbf{r}}$ and $[\cdot, \ldots, \cdot, L_0]_{\mathbf{r}}$ are both defined on $G(d, r_1) \times \ldots \times G(d, r_q)$.

Theorem 7.2.3. *Let* $r_1, \dots, r_q \in \{1, \dots, d-1\}$ *be integers with* $r_1 + \dots + r_q =: p \le d$, *and put* $\mathbf{r} := (r_1, \dots, r_q)$. *If* $f : G(d, r_1) \times \dots \times G(d, r_q) \to \mathbb{R}$ *is a nonnegative measurable function, then*

$$\int_{G(d,r_1)\times\ldots\times G(d,r_q)} f \,\mathrm{d}(\nu_{r_1} \otimes \ldots \otimes \nu_{r_q})$$
$$= b \int_{G(d,p)} \int_{G(L,r_1)\times\ldots\times G(L,r_q)} f\, [\cdot, \dots, \cdot]_{\mathbf{r}}^{d-p} \,\mathrm{d}(\nu^L_{r_1} \otimes \ldots \otimes \nu^L_{r_q})\, \nu_p(\mathrm{d}L)$$

with

$$b := b_{dp} \prod_{j=1}^{q} \frac{b_{pr_j}}{b_{dr_j}}. \tag{7.12}$$

Proof. We begin with a preparatory remark. If $r \in \{1, \dots, d-1\}$ and if $h : G(d, r) \to \mathbb{R}$ is a nonnegative measurable function, then

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} h(\mathrm{lin}\,\{x_1, \dots, x_r\}) \prod_{j=1}^{r} \mathbf{1}_{B^d}(x_j)\, \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_r)$$
$$= \kappa_d^r \int_{G(d,r)} h(L)\, \nu_r(\mathrm{d}L).$$

In fact, choosing for h the indicator function of a Borel set, one can use the left side to define a finite measure on $G(d, r)$. Since it is rotation invariant, it must be a multiple of the invariant measure ν_r. The factor can then be determined by choosing $h = 1$.

Now we define, almost everywhere on $(\mathbb{R}^d)^p$, a function g by

$$g(x_1^1, \dots, x_{r_1}^1, \dots, x_1^q, \dots, x_{r_q}^q)$$
$$:= f(\mathrm{lin}\,\{x_1^1, \dots, x_{r_1}^1\}, \dots, \mathrm{lin}\,\{x_1^q, \dots, x_{r_q}^q\}) \prod_{j=1}^{q} \prod_{i=1}^{r_j} \mathbf{1}_{B^d}(x_i^j).$$

Applying Fubini's theorem and q times the preceding remark, we obtain

$$I := \int_{(\mathbb{R}^d)^p} g \,\mathrm{d}\lambda^p = \kappa_d^p \int_{G(d,r_1)\times\ldots\times G(d,r_q)} f \,\mathrm{d}(\nu_{r_1} \otimes \ldots \otimes \nu_{r_q}).$$

On the other hand, Theorem 7.2.1 gives

$$I = b_{dp} \int_{G(d,p)} \int_{L^p} g \nabla_p^{d-p} \,\mathrm{d}\lambda_L^p\, \nu_p(\mathrm{d}L).$$

We abbreviate $(x_{r_1}^j, \dots, x_{r_j}^j) =: \mathbf{x}_j$ and $\mathrm{lin}\,\{x_{r_1}^j, \dots, x_{r_j}^j\} =: \mathrm{lin}\,\mathbf{x}_j$ for $j = 1, \dots, q$, then

$$I = b_{dp} \int_{G(d,p)} \int_{L^{r_1}} \cdots \int_{L^{r_q}} g(\mathbf{x}_1, \ldots, \mathbf{x}_q) \nabla_p(\mathbf{x}_1, \ldots, \mathbf{x}_q)^{d-p} \times \lambda_L^{r_q}(\mathrm{d}\mathbf{x}_1) \cdots \lambda_L^{r_1}(\mathrm{d}\mathbf{x}_q)\, \nu_p(\mathrm{d}L).$$

From the definitions of $[\cdot, \ldots, \cdot]_{\mathbf{r}}$ and ∇_r it follows that

$$\nabla_p(\mathbf{x}_1, \ldots, \mathbf{x}_q) = \nabla_{r_1}(\mathbf{x}_1) \cdots \nabla_{r_q}(\mathbf{x}_q)[\operatorname{lin} \mathbf{x}_1, \ldots, \operatorname{lin} \mathbf{x}_q]_{\mathbf{r}}.$$

We insert this in the last integrand. Then, for fixed $L \in G(d,p)$, we use Theorem 7.2.1 (with $(\mathbb{R}^d, q)$ replaced by (L, r_1)) to transform the integration involving $\mathbf{x}_1$ into an integration over $G(L, r_1)$ and, for fixed $L_1 \in G(L, r_1)$, an integration with respect to the measure $\lambda_{L_1}^{r_1}$. The integral

$$\int_{L_1^{r_1}} \mathbf{1}_{(B^d \cap L_1)^{r_1}}(\mathbf{x}_1) \nabla_{r_1}(\mathbf{x}_1)^{d-r_1}\, \lambda_{L_1}^{r_1}(\mathrm{d}\mathbf{x}_1) = I(r_1, r_1, d-r_1)$$

occurring here can be evaluated by means of Theorem 8.2.2. In a similar way the integrations involving $\mathbf{x}_j$, $j = 2, \ldots, q$, are treated. Now the assertion follows. □

In the following generalization of Theorem 7.2.3, a fixed subspace is given, and the pivot determined by a tuple of subspaces is the linear span of these and the given one.

Theorem 7.2.4. *Let* $r_1, \ldots, r_q \in \{1, \ldots, d-1\}$ *and* $r_0 \in \{0, \ldots, d-1\}$ *be integers with*

$$r_1 + \ldots + r_q =: p \leq d - r_0;$$

put $\mathbf{r} := (r_1, \ldots, r_q)$. *Let* $L_0 \in G(d, r_0)$ *be a fixed subspace. If* $f : G(d, r_1) \times \ldots \times G(d, r_q) \to \mathbb{R}$ *is a nonnegative measurable function, then*

$$\int_{G(d,r_1) \times \ldots \times G(d,r_q)} f \,\mathrm{d}(\nu_{r_1} \otimes \ldots \otimes \nu_{r_q})$$
$$= c \int_{G(L_0, p+r_0)} \int_{G(L,r_1) \times \ldots \times G(L,r_q)} f\, [\cdot, \ldots, \cdot, L_0]_{\mathbf{r}}^{d-p-r_0} \times \mathrm{d}(\nu_{r_1}^L \otimes \ldots \otimes \nu_{r_q}^L)\, \nu_{p+r_0}^{L_0}(\mathrm{d}L)$$

with

$$c := b_{(d-r_0)p} \prod_{j=1}^{q} \frac{b_{(p+r_0)r_j}}{b_{dr_j}}.$$

Proof. The proof is the obvious extension of the previous one. Instead of applying Theorem 7.2.1 first, we employ (7.11), with (q, r) replaced by (p, r_0). After using the identity

$$\nabla_{p,r_0}(\mathbf{x}_1, \ldots, \mathbf{x}_q, L_0) = \nabla_{r_1}(\mathbf{x}_1) \cdots \nabla_{r_q}(\mathbf{x}_q)[\operatorname{lin} \mathbf{x}_1, \ldots, \operatorname{lin} \mathbf{x}_q, L_0]_{\mathbf{r}},$$

the rest of the proof is the same. □

The preceding theorems have counterparts where linear spans are replaced by intersections. In that case, we consider linear subspaces $L_1, \dots, L_q \subset \mathbb{R}^d$ with $\sum_{i=1}^q \dim L_i \geq (q-1)d$. In Section 14.1 we define $[L_1, \dots, L_q] = [L_1^\perp, \dots, L_q^\perp]$. In particular, if $L_1, \dots, L_q$ are hyperplanes through 0 and if u_i is a unit normal vector of L_i for $i = 1, \dots, q$, then $[L_1, \dots, L_q]$ is the q-dimensional volume of the parallelepiped spanned by $u_1, \dots, u_q$, also denoted by $\nabla_q(u_1, \dots, u_q)$. As before, we write $[L_1, \dots, L_q] = [L_1, \dots, L_q]_{\mathbf{s}}$ with $\mathbf{s} := (s_1, \dots, s_q)$ if $\dim L_i = s_i$, $i = 1, \dots, q$.

In the next theorem, the initial integration space is $\prod_{i=1}^q G(d, s_i)$, with $\sum_{i=1}^q s_i \geq (q-1)d$, and the pivot determined by a q-tuple of subspaces is their intersection.

Theorem 7.2.5. *Let $s_1, \dots, s_q \in \{1, \dots, d-1\}$ be integers satisfying*

$$s_1 + \dots + s_q - (q-1)d =: m \geq 0;$$

put $\mathbf{s} := (s_1, \dots, s_q)$. If $f : G(d, s_1) \times \dots \times G(d, s_q) \to \mathbb{R}$ is a nonnegative measurable function, then

$$\begin{aligned} &\int_{G(d,s_1)\times\dots\times G(d,s_q)} f \, \mathrm{d}(\nu_{s_1} \otimes \dots \otimes \nu_{s_q}) \\ &= \bar{b} \int_{G(d,m)} \int_{G(L,s_1)\times\dots\times G(L,s_q)} f \, [\cdot, \dots, \cdot]_{\mathbf{s}}^m \, \mathrm{d}(\nu_{s_1}^L \otimes \dots \otimes \nu_{s_q}^L) \, \nu_m(\mathrm{d}L) \end{aligned}$$

with

$$\bar{b} := b_{d(d-m)} \prod_{j=1}^q \frac{b_{(d-m)(d-s_j)}}{b_{d(d-s_j)}}. \tag{7.13}$$

Proof. We put $d - s_j =: r_j$ and $\mathbf{r} := (r_1, \dots, r_q)$. For $M_j \in G(d, r_j)$, $j = 1, \dots, q$, we set

$$f^\perp(M_1, \dots, M_q) := f(M_1^\perp, \dots, M_q^\perp).$$

By Theorem 7.2.3,

$$\begin{aligned} &\int_{G(d,r_1)\times\dots\times G(d,r_q)} f^\perp \, \mathrm{d}(\nu_{r_1} \otimes \dots \otimes \nu_{r_q}) \\ &= b \int_{G(d,d-m)} \int_{G(L,r_1)\times\dots\times G(L,r_q)} f^\perp [\cdot, \dots, \cdot]_{\mathbf{r}}^m \, \mathrm{d}(\nu_{r_1}^L \otimes \dots \otimes \nu_{r_q}^L) \, \nu_{d-m}(\mathrm{d}L) \end{aligned}$$

with b given by (7.12). Now we observe that the mapping $L \mapsto L^\perp$ maps the space $G(d, k)$ to $G(d, d-k)$ and transforms the measure ν_k into ν_{d-k} Moreover, for a fixed subspace M, it maps the space $G(M, k)$ onto $G(M^\perp, d-k)$ and transforms the measure ν_k^M into $\nu_{d-k}^{M^\perp}$, as follows from the uniqueness of these invariant measures. Hence, the last equation is equivalent to the assertion. □

In the same way, one obtains from Theorem 7.2.4 the following generalization of the preceding result. Here a fixed subspace is given, and the pivot determined by a tuple of subspaces is the intersection of these and the given one.

Theorem 7.2.6. *Let* $s_1, \dots, s_q \in \{1, \dots, d-1\}$ *and* $s_0 \in \{1, \dots, d\}$ *be integers with*

$$s_1 + \dots + s_q - (q-1)d =: m \geq d - s_0;$$

put $\mathbf{s} := (s_1, \dots, s_q)$*. Let* $L_0 \in G(d, s_0)$ *be a fixed subspace. If* $f : G(d, s_1) \times \dots \times G(d, s_q) \to \mathbb{R}$ *is a nonnegative measurable function, then*

$$\begin{aligned}\int_{G(d,s_1)\times\dots\times G(d,s_q)} & f \,\mathrm{d}(\nu_{s_1} \otimes \dots \otimes \nu_{s_q}) \\ = \bar{c} \int_{G(L_0, m+s_0-d)} & \int_{G(L,s_1)\times\dots\times G(L,s_q)} f\, [\cdot, \dots, \cdot, L_0]_{\mathbf{s}}^{m+s_0-d} \\ & \times \mathrm{d}(\nu_{s_1}^L \otimes \dots \otimes \nu_{s_q}^L)\, \nu_{m+s_0-d}^{L_0}(\mathrm{d}L)\end{aligned}$$

with

$$\bar{c} := b_{s_0(d-m)} \prod_{j=1}^{q} \frac{b_{(2d-m-s_0)(d-s_j)}}{b_{d(d-s_j)}}.$$

Now we turn to affine transformation formulas, with affine subspaces instead of linear subspaces. First we derive the **affine Blaschke–Petkantschin formula**. Here the initial integration is over $(\mathbb{R}^d)^{q+1}$, and the pivot is the q-flat affinely spanned (almost everywhere) by the integration variable $(x_0, \dots, x_q) \in (\mathbb{R}^d)^{q+1}$. The outer integration space is the affine Grassmannian $A(d,q)$, and for $E \in A(d,q)$, the inner integration space is the product E^{q+1}. Recall that $\Delta_q(x_0, \dots, x_q)$, as defined by (7.6), denotes the q-dimensional volume of the simplex with vertices $x_0, \dots, x_q$.

Theorem 7.2.7. *If* $q \in \{1, \dots, d\}$ *and if* $f : (\mathbb{R}^d)^{q+1} \to \mathbb{R}$ *is a nonnegative measurable function, then*

$$\int_{(\mathbb{R}^d)^{q+1}} f \,\mathrm{d}\lambda^{q+1} = b_{dq} (q!)^{d-q} \int_{A(d,q)} \int_{E^{q+1}} f \Delta_q^{d-q} \,\mathrm{d}\lambda_E^{q+1}\, \mu_q(\mathrm{d}E) \qquad (7.14)$$

with b_{dq} *given by* (7.8).

Proof. We apply Theorem 7.2.1 and several times the theorem of Fubini:

$$\begin{aligned}&\int_{(\mathbb{R}^d)^{q+1}} f \,\mathrm{d}\lambda^{q+1} \\ &= \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^q} f(x_0, y_1 + x_0, \dots, y_q + x_0)\, \lambda^q(\mathrm{d}(y_1, \dots, y_q))\, \lambda(\mathrm{d}x_0)\end{aligned}$$

$$= b_{dq} \int_{\mathbb{R}^d} \int_{G(d,q)} \int_{L^q} f(x_0, y_1 + x_0, \ldots, y_q + x_0) \nabla_q(y_1, \ldots, y_q)^{d-q}$$
$$\times\, \lambda_L^q(\mathrm{d}(y_1, \ldots, y_q))\, \nu_q(\mathrm{d}L)\, \lambda(\mathrm{d}x_0)$$
$$= b_{dq} \int_{G(d,q)} \int_{L^\perp} \int_L \int_{L^q} f(z + t, y_1 + z + t, \ldots, y_q + z + t)$$
$$\times \nabla_q(y_1, \ldots, y_q)^{d-q}\, \lambda_L^q(\mathrm{d}(y_1, \ldots, y_q))\, \lambda_L(\mathrm{d}z)\, \lambda_{L^\perp}(\mathrm{d}t)\, \nu_q(\mathrm{d}L)$$
$$= b_{dq}(q!)^{d-q} \int_{G(d,q)} \int_{L^\perp} \int_{(L+t)^{q+1}} f(x_0, \ldots, x_q)$$
$$\times \Delta_q(x_0, \ldots, x_q)^{d-q}\, \lambda_{L+t}^{q+1}(\mathrm{d}(x_0, \ldots, x_q))\ \lambda_{L^\perp}(\mathrm{d}t)\, \nu_q(\mathrm{d}L)$$
$$= b_{dq}(q!)^{d-q} \int_{A(d,q)} \int_{E^{q+1}} f(x_0, \ldots, x_q) \Delta_q(x_0, \ldots, x_q)^{d-q}$$
$$\times\, \lambda_E^{q+1}(\mathrm{d}(x_0, \ldots, x_q))\, \mu_q(\mathrm{d}E).$$

Here we have used (13.9). □

Postponing the treatment of affine spans of flats of small dimensions, we now consider the affine counterpart to Theorem 7.2.5. Here, the pivot determined by a q-tuple of flats of large dimensions is their intersection. For an affine subspace E we denote by E^0 the linear subspace parallel to E. For $E_1, \ldots, E_q \subset \mathbb{R}^d$ with $\dim E_i = s_i$ we put $\mathbf{s} := (s_1, \ldots, s_q)$ and

$$[E_1, \ldots, E_q]_{\mathbf{s}} := [E_1^0, \ldots, E_q^0]_{\mathbf{s}},$$

provided the right side is defined.

Theorem 7.2.8. *Let $s_1, \ldots, s_q \in \{1, \ldots, d-1\}$ be integers satisfying*

$$s_1 + \ldots + s_q - (q-1)d =: m \geq 0;$$

put $\mathbf{s} := (s_1, \ldots, s_q)$. If $f : A(d, s_1) \times \ldots \times A(d, s_q) \to \mathbb{R}$ is a nonnegative measurable function, then

$$\int_{A(d,s_1)\times\ldots\times A(d,s_q)} f\, \mathrm{d}(\mu_{s_1} \otimes \ldots \otimes \mu_{s_q})$$
$$= \bar{b} \int_{A(d,m)} \int_{A(E,s_1)\times\ldots\times A(E,s_q)} f\, [\cdot, \ldots, \cdot]_{\mathbf{s}}^{m+1}\, \mathrm{d}(\mu_{s_1}^E \otimes \ldots \otimes \mu_{s_q}^E)\, \mu_m(\mathrm{d}E)$$

with $\bar{b}$ given by (7.13).

Proof. By (13.9) we can write

$$I := \int_{A(d,s_1)\times\ldots\times A(d,s_q)} f\, \mathrm{d}(\mu_{s_1} \otimes \ldots \otimes \mu_{s_q}) \tag{7.15}$$
$$= \int_{G(d,s_1)\times\ldots\times G(d,s_q)} J(L_1, \ldots, L_q)\, (\nu_{s_1} \otimes \ldots \otimes \nu_{s_q})(\mathrm{d}(L_1, \ldots, L_q))$$

with

$$J(L_1,\dots,L_q) = \int_{L_1^\perp\times\dots\times L_q^\perp} f(L_1+t_1,\dots,L_q+t_q)\,(\lambda_{L_1^\perp}\otimes\dots\otimes\lambda_{L_q^\perp})(\mathrm{d}(t_1,\dots,t_q)).$$

Let $L_j \in G(d,s_j)$, $j=1,\dots,q$ and assume, without loss of generality (by Lemma 13.2.1), that these subspaces are in general position. We put $L_1\cap\dots\cap L_q =: L$. For $(t_1,\dots,t_q)\in L_1^\perp\times\dots\times L_q^\perp$ we have

$$(L_1+t_1)\cap\dots\cap(L_q+t_q) = L+\xi(t_1,\dots,t_q)$$

with a unique vector $\xi(t_1,\dots,t_q)\in L^\perp$. This defines a linear map

$$\xi : L_1^\perp\times\dots\times L_q^\perp \to L^\perp.$$

If $\pi_j : L^\perp \to L_j^\perp$ denotes the orthogonal projection, then the inverse map ξ^{-1} is given by $\xi^{-1}(x) = (\pi_1(x),\dots,\pi_q(x))$. Choosing in each space $L_j^\perp$ an orthonormal basis and applying a linear map from $L_1^\perp\times\dots\times L_q^\perp$ to $L^\perp$ that maps the union of these bases to an orthonormal basis of $L^\perp$, we see that

$$J(L_1,\dots,L_q) = [L_1,\dots,L_q]_{\mathbf{s}} \int_{L^\perp} f(L_1+x,\dots,L_q+x)\,\lambda_{L^\perp}(\mathrm{d}x).$$

We insert this in (7.15) and use Theorem 7.2.5. In the subsequent integrals we have $L = L_1\cap\dots\cap L_q$ up to sets of measure zero.

$$\begin{aligned}
I &= \int_{G(d,s_1)\times\dots\times G(d,s_q)} [L_1,\dots,L_q]_{\mathbf{s}} \int_{L^\perp} f(L_1+x,\dots,L_q+x)\,\lambda_{L^\perp}(\mathrm{d}x)\\
&\quad\times(\nu_{s_1}\otimes\dots\otimes\nu_{s_q})(\mathrm{d}(L_1,\dots,L_q))\\
&= \bar b\int_{G(d,m)}\int_{G(L,s_1)\times\dots\times G(L,s_q)}\int_{L^\perp} f(L_1+x,\dots,L_q+x)[L_1,\dots,L_q]_{\mathbf{s}}^{m+1}\\
&\quad\times\lambda_{L^\perp}(\mathrm{d}x)\,(\nu_{s_1}^L\otimes\dots\otimes\nu_{s_q}^L)(\mathrm{d}(L_1,\dots,L_q))\,\nu_m(\mathrm{d}L)\\
&= \bar b\int_{G(d,m)}\int_{L^\perp}\int_{G(L,s_1)\times\dots\times G(L,s_q)} f(L_1+x,\dots,L_q+x)[L_1,\dots,L_q]_{\mathbf{s}}^{m+1}\\
&\quad\times(\nu_{s_1}^L\otimes\dots\otimes\nu_{s_q}^L)(\mathrm{d}(L_1,\dots,L_q))\,\lambda_{L^\perp}(\mathrm{d}x)\,\nu_m(\mathrm{d}L)\\
&= \bar b\int_{G(d,m)}\int_{L^\perp}\int_{A(L+x,s_1)\times\dots\times A(L+x,s_q)} f(E_1,\dots,E_q)[E_1,\dots,E_q]_{\mathbf{s}}^{m+1}\\
&\quad\times(\mu_{s_1}^{L+x}\otimes\dots\otimes\mu_{s_q}^{L+x})(\mathrm{d}(E_1,\dots,E_q))\,\lambda_{L^\perp}(\mathrm{d}x)\,\nu_m(\mathrm{d}L)\\
&= \bar b\int_{A(d,m)}\int_{A(E,s_1)\times\dots\times A(E,s_q)} f(E_1,\dots,E_q)[E_1,\dots,E_q]_{\mathbf{s}}^{m+1}\\
&\quad\times(\mu_{s_1}^{E}\otimes\dots\otimes\mu_{s_q}^{E})(\mathrm{d}(E_1,\dots,E_q))\,\mu_m(\mathrm{d}E).
\end{aligned}$$

Here we have used (13.14) and (13.9). □

In the case of flats of small dimensions, we consider only two flats. Let $E_1 \in A(d,r)$, $E_2 \in A(d,s)$ be flats with dimensions satisfying $r+s \le d-1$. We assume that they are in general position, that is, the dimension of their affine span is equal to $r+s+1$. Under this assumption, the distance between E_1 and E_2 is realized by unique points $x_1 \in E_1$, $x_2 \in E_2$, and the line F through x_1 and x_2 is orthogonal to both, E_1 and E_2. We call F the **ortholine** of E_1 and E_2 and denote the distance $\|x_1 - x_2\|$ by $D(E_1, E_2)$.

In the following theorem, the pivot determined by a pair of flats of small dimensions is their affine span.

Theorem 7.2.9. *Let $r_1, r_2 \in \{0, \dots, d-1\}$ be integers satisfying $r_1+r_2+1 =: p \le d$; put $\mathbf{r} := (r_1, r_2)$. If $f : A(d,r_1) \times A(d,r_2) \to \mathbb{R}$ is a nonnegative measurable function, then*

$$\begin{aligned}&\int_{A(d,r_1)\times A(d,r_2)} f \,\mathrm{d}(\mu_{r_1} \otimes \mu_{r_2}) \\ &= b \int_{A(d,p)} \int_{A(E,r_1)\times A(E,r_2)} f D^{d-p} [\cdot,\cdot]_{\mathbf{r}}^{d-p} \,\mathrm{d}(\mu_{r_1}^E \otimes \mu_{r_2}^E)\, \mu_p(\mathrm{d}E)\end{aligned}$$

with b given by (7.12) *for $q = 2$.*

Proof. It is sufficient to prove the assertion for a function f for which there exists a ball B with $f(E_1, E_2) = 0$ if $E_j \cap \operatorname{int} B = \emptyset$ for at least one $j \in \{1, 2\}$. If this is established, then the general case follows with an application of the monotone convergence theorem.

For $\mathbf{x}_j := (x_0^j, \dots, x_{r_j}^j) \in (\mathbb{R}^d)^{r_j+1}$ we write $\operatorname{aff}\{x_0^j, \dots, x_{r_j}^j\} =: \operatorname{aff}\mathbf{x}_j$, and we define

$$g(\mathbf{x}_1, \mathbf{x}_2) := \prod_{j=1}^{2} \mathbf{1}_{B^{r_j+1}}(\mathbf{x}_j) \lambda_{\operatorname{aff}\mathbf{x}_j}(B)^{-r_j-1} \quad \text{if } \prod_{j=1}^{2} \lambda_{\operatorname{aff}\mathbf{x}_j}(B) \neq 0,$$

and $g(\mathbf{x}_1, \mathbf{x}_2) := 0$ otherwise. To each of the two integrals in

$$\begin{aligned}I := (r_1!)^{r_1-d}(r_2!)^{r_2-d} &\int_{(\mathbb{R}^d)^{r_1+1}} \int_{(\mathbb{R}^d)^{r_2+1}} f(\operatorname{aff}\mathbf{x}_1, \operatorname{aff}\mathbf{x}_2) \\ &\times g(\mathbf{x}_1, \mathbf{x}_2) \Delta_{r_1}(\mathbf{x}_1)^{r_1-d} \Delta_{r_2}(\mathbf{x}_2)^{r_2-d}\, \lambda^{r_2+1}(\mathrm{d}\mathbf{x}_2)\, \lambda^{r_1+1}(\mathrm{d}\mathbf{x}_1)\end{aligned}$$

we apply the affine Blaschke–Petkantschin formula (7.14). This gives

$$I = \prod_{j=1}^{2} b_{dr_j} \int_{A(d,r_1)} \int_{A(d,r_2)} f(E_1, E_2)\, \mu_{r_2}(\mathrm{d}E_2)\, \mu_{r_1}(\mathrm{d}E_1).$$

On the other hand, we can view I as an integral over $(\mathbb{R}^d)^{p+1}$ with respect to the measure λ^{p+1} and apply (7.14) to this. The result can be written as

$$I = b_{dp} \int_{A(d,p)} \int_{E^{r_1+1}} \int_{E^{r_2+1}} f(\operatorname{aff} \mathbf{x}_1, \operatorname{aff} \mathbf{x}_2) g(\mathbf{x}_1, \mathbf{x}_2)$$
$$\times (r_1!)^{r_1-d}(r_2!)^{r_2-d}(p!)^{d-p} \Delta_{r_1}(\mathbf{x}_1)^{r_1-d} \Delta_{r_2}(\mathbf{x}_2)^{r_2-d}$$
$$\times \Delta_p(\mathbf{x}_1, \mathbf{x}_2)^{d-p} \lambda_E^{r_2+1}(\mathrm{d}\mathbf{x}_2) \lambda_E^{r_1+1}(\mathrm{d}\mathbf{x}_1) \mu_p(\mathrm{d}E).$$

Here we employ the (easily established) fact that

$$p!\Delta_p(\mathbf{x}_1, \mathbf{x}_2) = r_1!r_2!\Delta_{r_1}(\mathbf{x}_1)\Delta_{r_2}(\mathbf{x}_2) D(\operatorname{aff} \mathbf{x}_1, \operatorname{aff} \mathbf{x}_2)[\operatorname{aff} \mathbf{x}_1, \operatorname{aff} \mathbf{x}_2].$$

We insert this and then apply (7.14) to the two inner integrals over E^{r_j+1}, $j = 1, 2$. This immediately yields the assertion of the theorem. □

In the following theorem, we restrict ourselves to flats of small dimensions which affinely span the whole space. The pivot determined by a pair (E_1, E_2) of flats in general position will now be the triple (F, x_1, x_2), consisting of the ortholine F of E_1, E_2 and the points x_1, x_2 where F intersects the flats. For a given triple (F, x_1, x_2), the inner integration space is in effect (though written in a more convenient way) the space $A(x_1, F^\perp + x_1, r_1) \times A(x_2, F^\perp + x_2, r_2)$. Here $A(x, F^\perp + x, s)$ denotes the space of s-flats through x and contained in $F^\perp + x$ (recall that $F^\perp := (F^0)^\perp$ is a linear subspace).

Theorem 7.2.10. *Let $r_1, r_2 \in \{0, \dots, d-2\}$ be integers satisfying $r_1 + r_2 = d - 1$. If $f : A(d, r_1) \times A(d, r_2) \to \mathbb{R}$ is a nonnegative measurable function, then*

$$\int_{A(d,r_1)\times A(d,r_2)} f \, \mathrm{d}(\mu_{r_1} \otimes \mu_{r_2})$$
$$= b \int_{A(d,1)} \int_{F^2} \int_{G(F^\perp,r_1)\times G(F^\perp,r_2)} f(L_1 + x_1, L_2 + x_2)[L_1, L_2]^2$$
$$\times (\nu_{r^1}^{F^\perp} \otimes \nu_{r^2}^{F^\perp})(\mathrm{d}(L_1, L_2)) \lambda_F^2(\mathrm{d}(x_1, x_2)) \mu_1(\mathrm{d}F)$$

with b given by (7.12) *for $q = 2$.*

Proof. In the proof, we use repeatedly Fubini's theorem and (13.9). We apply Theorem 7.2.3 and then go over to orthogonal complements:

$$I := \int_{A(d,r_1)\times A(d,r_2)} f \, \mathrm{d}(\mu_{r_1} \otimes \mu_{r_2})$$
$$= \int_{G(d,r_1)\times G(d,r_2)} \int_{L_1^\perp \times L_2^\perp} f(L_1 + x_1, L_2 + x_2)$$
$$\times (\lambda_{L_1^\perp} \otimes \lambda_{L_2^\perp})(\mathrm{d}(x_1, x_2)) (\nu_{r_1} \otimes \nu_{r_2})(\mathrm{d}(L_1, L_2))$$
$$= b \int_{G(d,d-1)} \int_{G(H,r_1)\times G(H,r_2)} \int_{L_1^\perp \times L_2^\perp} f(L_1 + x_1, L_2 + x_2)$$

$$\times [L_1, L_2]\, (\lambda_{L_1^\perp} \otimes \lambda_{L_2^\perp})(\mathrm{d}(x_1, x_2))(\nu_{r_1}^H \otimes \nu_{r_2}^H)(\mathrm{d}(L_1, L_2))\, \nu_{d-1}(\mathrm{d}H)$$

$$= b \int_{G(d,1)} \int_{G(L^\perp, r_1) \times G(L^\perp, r_2)} \int_{L_1^\perp \times L_2^\perp} f(L_1 + x_1, L_2 + x_2)$$

$$\times [L_1, L_2]\, (\lambda_{L_1^\perp} \otimes \lambda_{L_2^\perp})(\mathrm{d}(x_1, x_2))\, (\nu_{r_1}^{L^\perp} \otimes \nu_{r_2}^{L^\perp})(\mathrm{d}(L_1, L_2))\, \nu_1(\mathrm{d}L).$$

Using the direct sum decomposition $L_j^\perp = L \oplus (L_j^\perp \cap L^\perp)$ and writing $x_j = y_j + z_j$ with $y_j \in L$ and $z_j \in L_j^\perp \cap L^\perp$, we obtain

$$I = b \int_{G(d,1)} \int_{G(L^\perp, r_1) \times G(L^\perp, r_2)} \int_{L^2} \int_{(L_1^\perp \cap L^\perp) \times (L_2^\perp \cap L^\perp)}$$

$$f(L_1 + y_1 + z_1, L_2 + y_2 + z_2)[L_1, L_2]\, (\lambda_{L_1^\perp \cap L^\perp} \otimes \lambda_{L_2^\perp \cap L^\perp})(\mathrm{d}(z_1, z_2))$$

$$\times \lambda_L^2(\mathrm{d}(y_1, y_2))\, (\nu_{r_1}^{L^\perp} \otimes \nu_{r_2}^{L^\perp})(\mathrm{d}(L_1, L_2))\, \nu_1(\mathrm{d}L)$$

$$= b \int_{G(d,1)} \int_{L^2} \left\{ \int_{A(L^\perp, r_1) \times A(L^\perp, r_2)} f(E_1 + y_1, E_2 + y_2) \right.$$

$$\left. \times [E_1, E_2]\, (\mu_{r_1}^{L^\perp} \otimes \mu_{r_2}^{L^\perp})(\mathrm{d}(E_1, E_2)) \right\} \lambda_L^2(\mathrm{d}(y_1, y_2))\, \nu_1(\mathrm{d}L).$$

To the integral in braces we apply Theorem 7.2.8 (in $L^\perp$); here $m = 0$ (and thus $\bar{b} = 1$), so that $A(L^\perp, m)$ is identified with $L^\perp$. This gives

$$I = b \int_{G(d,1)} \int_{L^2} \left\{ \int_{L^\perp} \int_{G(L^\perp, r_1) \times G(L^\perp, r_2)} f(L_1 + t + y_1, L_2 + t + y_2) \right.$$

$$\left. \times [L_1, L_2]^2\, (\nu_{r_1}^{L^\perp} \otimes \nu_{r_2}^{L^\perp})(\mathrm{d}(L_1, L_2))\, \lambda_{L^\perp}(\mathrm{d}t) \right\} \lambda_L^2(\mathrm{d}(y_1, y_2))\, \nu_1(\mathrm{d}L)$$

$$= b \int_{G(d,1)} \int_{L^\perp} \int_{(L+t)^2} \int_{G(L^\perp, r_1) \times G(L^\perp, r_2)} f(L_1 + y_1, L_2 + y_2)$$

$$\times [L_1, L_2]^2\, (\nu_{r_1}^{L^\perp} \otimes \nu_{r_2}^{L^\perp})(\mathrm{d}(L_1, L_2))\, \lambda_{L+t}^2(\mathrm{d}(y_1, y_2)) \lambda_{L^\perp}(\mathrm{d}t)\, \nu_1(\mathrm{d}L)$$

$$= b \int_{A(d,1)} \int_{F^2} \int_{G(F^\perp, r_1) \times G(F^\perp, r_2)} f(L_1 + y_1, L_2 + y_2)$$

$$\times [L_1, L_2]^2\, (\nu_{r_1}^{F^\perp} \otimes \nu_{r_2}^{F^\perp})(\mathrm{d}(L_1, L_2))\, \lambda_F^2(\mathrm{d}(y_1, y_2))\, \mu_1(\mathrm{d}F).$$

This completes the proof. □

In Theorem 7.2.10 we have assumed that $r_1 + r_2 + 1$, the dimension of the affine span (if the flats are in general position) is equal to d. If this dimension is less than d, we obtain the corresponding result by first applying Theorem 7.2.9 and then transforming the inner integral by means of Theorem 7.2.10.

Notes for Section 7.2

1. In this note we give an interesting alternative proof of the Blaschke–Petkantschin formula of Theorem 7.2.1. This proof, which is due to Møller [550], is based on a uniqueness result for relatively invariant measures. The method is also briefly described in Barndorff–Nielsen, Blæsild and Eriksen [79, pp. 59–60]. In the following, we use notation and results from Section 13.3.

Second proof of Theorem 7.2.1. To prove (7.7), we need evidently consider only linearly independent q-tuples $(x_1, \dots, x_q)$. We denote by $U \subset (\mathbb{R}^d)^q$ the subspace of linearly independent q-tuples. In the following, we consider the elements of $\mathbb{R}^d$ as column vectors (with respect to some fixed basis) and, correspondingly, $(x_1, \dots, x_q)$ as a (d,q)-matrix; then U is the space of real (d,q)-matrices of rank q. Let $\mathcal{GL}(q)$ be the group of regular (q,q)-matrices with the standard topology; it is locally compact. The same holds true for the direct product $G := \mathcal{SO}(d) \times \mathcal{GL}(q)$, where $\mathcal{SO}(d)$ denotes the group of orthogonal (d,d)-matrices with determinant one. By

$$((D,M),(x_1,\dots,x_q)) \mapsto D(x_1,\dots,x_q)M^t =: (D,M).(x_1,\dots,x_q)$$

for $(D,M) \in G$ and $(x_1,\dots,x_q) \in U$, where M^t denotes the transpose of the matrix M, we define a transitive operation of G on U. It is not difficult to verify that U, with this operation, becomes a homogeneous G-space. Recall that (Sec. 13.3), if a group G operates on a set U, one defines $(g.f)(u) := f(g^{-1}u)$, $u \in U$, for a function f on U.

Now we define two positive linear functionals I_1, I_2 on $\mathbf{C}_c(U)$ by

$$\begin{aligned} I_1(f) &:= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(x_1,\dots,x_q)\,\lambda(\mathrm{d}x_1)\cdots\lambda(\mathrm{d}x_q), \\ I_2(f) &:= \int_{G(d,q)} \int_L \cdots \int_L f(x_1,\dots,x_q)\nabla_q(x_1,\dots,x_q)^{d-q} \\ &\qquad \times \lambda_L(\mathrm{d}x_1)\cdots\lambda_L(\mathrm{d}x_q)\,\nu_q(\mathrm{d}L) \end{aligned}$$

for $f \in \mathbf{C}_c(U)$. For $(D,M) \in G$ we get

$$\begin{aligned} I_1((D,M).f) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f(D^{-1}(x_1,\dots,x_q)M^{-t})\,\lambda(\mathrm{d}x_1)\cdots\lambda(\mathrm{d}x_q) \\ &= |\det M|^d I_1(f), \end{aligned}$$

because λ is rotation invariant, and the linear map $(x_1,\dots,x_q) \mapsto (x_1,\dots,x_q)M$ from the space of (d,q)-matrices into itself has determinant $(\det M)^d$. Further, for $L \in G(d,q)$ we get

$$\begin{aligned} &\int_L \cdots \int_L f(D^{-1}(x_1,\dots,x_q)M^{-t})\nabla_q(x_1,\dots,x_q)^{d-q}\,\lambda_L(\mathrm{d}x_1)\cdots\lambda_L(\mathrm{d}x_q) \\ &= |\det M|^q \int_L \cdots \int_L f(D^{-1}(x_1,\dots,x_q))\nabla_q((x_1,\dots,x_q)M^t)^{d-q} \\ &\quad \times \lambda_L(\mathrm{d}x_1)\cdots\lambda_L(\mathrm{d}x_q) \\ &= |\det M|^d \int_{\vartheta^{-1}L} \cdots \int_{\vartheta^{-1}L} f(x_1,\dots,x_q)\nabla_q(x_1,\dots,x_q)^{d-q} \\ &\quad \times \lambda_{\vartheta^{-1}L}(\mathrm{d}x_1)\cdots\lambda_{\vartheta^{-1}L}(\mathrm{d}x_q), \end{aligned}$$

where $\vartheta \in SO_d$ is the rotation defined by D. The rotation invariance of ν_q now implies

$$I_2((D,M).f) = |\det M|^d I_2(f).$$

Thus the integrals I_1 and I_2 are relatively invariant with the same multiplier. From Theorem 13.3.1 it follows that $I_1 = cI_2$ with a constant c. The value of this constant can be obtained from Theorem 8.2.2. □

2. A general reference for the formulas of Section 7.2 is Santaló's book [662]. His proofs, as much of the original literature, use differential forms. Another flexible tool for obtaining integral geometric transformation formulas is Federer's coarea formula. In contrast to this, our aim was here to give more elementary and geometric proofs, based either on direct integration procedures or invariance arguments.

Results of Blaschke–Petkantschin type can in principle be traced back to Lebesgue [437], who used the transformation rule for multiple integrals to give new proofs for results of Crofton. After an influential lecture course by Herglotz [335] in Göttingen on geometric probabilities, and papers by Blaschke [105] and Varga [762], a systematic and general investigation of such integral geometric transformation formulas was undertaken by Petkantschin [601]. In special forms, most of the results of this section appear already in that paper. The usefulness of Blaschke–Petkantschin type formulas in stochastic geometry was emphasized by Miles. In [531], he gave new proofs and extensions of some results going back to Petkantschin, for example, of Theorem 7.2.2 above. In the style of the present chapter, though less generally, Blaschke–Petkantschin formulas were presented in Schneider and Weil [716].

For a recent application (in particular of Theorem 7.2.3) outside stochastic geometry, we mention E. Milman [540].

3. There are more general versions of Lemma 7.2.1, for integrations over $A(d,q)$ instead of $\mathbb{R}^d$; see Petkantschin [601, formula (49)]. A special case, where the given linear subspace is of dimension zero, reads as follows. Let $q < r \leq d$. For the pivot associated with $E \in A(d,q)$ one can choose a linear r-subspace containing E. Then the outer integration space is $G(d,r)$, and for $L \in G(d,r)$, the inner integration space is $A(L,q)$. The resulting formula is

$$\int_{A(d,q)} f \,\mathrm{d}\mu_q = c \int_{G(d,r)} \int_{A(L,q)} f \, d(\cdot,0)^{d-r} \,\mathrm{d}\mu_q^L \,\nu_r(\mathrm{d}L)$$

with a constant c depending on d,q,r. Applications of the special case $d=3$, $q=1$ are discussed by Cruz–Orive [190].

4. Vertical Sections. The following special case of Theorem 7.2.4 is of interest in stereology. Let $d=3$, $q=1$, $r_1=1$, $r_0=1$ and let $V \in G(3,1)$ be a fixed line. In some applications, the direction of the 'vertical' line V plays a particular role, and two-dimensional planes parallel to V define 'vertical sections'. Theorem 7.2.4 specializes to

$$\int_{G(3,1)} f \,\mathrm{d}\nu_1 = c \int_{G(V,2)} \int_{G(L,1)} f\,[\cdot, V] \,\mathrm{d}\nu_1^L \,\nu_2^V(\mathrm{d}L).$$

This can be interpreted as saying that an isotropic random line through 0 can be generated by first generating a uniform vertical 2-plane L containing V and then in L a random line through 0 with the distribution defined by the inner integral. Such and more general 'vertical uniform random sampling designs' can be of advantage in practical situations where preferred directions are present. They were suggested by Baddeley [46] and further studied in Baddeley [47, 48], Baddeley, Gundersen and Cruz–Orive [52]; see also Kötzer, Jensen and Baddeley [424] and Beneš and Rataj [90, sect. 4.1.3]. A detailed description is found in Baddeley and Jensen [53, ch. 8].

5. Hug and Reitzner [365] have proved and applied the following formula of Blaschke–Petkantschin type. The initial integration space is $(\mathbb{R}^d)^{d+p}$, where $1 \le p \le d$. The pivot determined by $(x_1, \dots, x_{d+p}) \in (\mathbb{R}^d)^{d+p}$ is the pair (H_1, H_2), where $H_1 := \operatorname{aff}\{x_1, \dots, x_d\}$ and $H_2 := \operatorname{aff}\{x_{p+1}, \dots, x_{p+d}\}$. The outer integration space is $A(d, d-1) \times A(d, d-1)$, and for $(H_1, H_2) \in A(d, d-1) \times A(d, d-1)$, the inner integration space is $H_1^p \times (H_1 \cap H_2)^{d-p} \times H_2^p$. The formula reads

$$\int_{(\mathbb{R}^d)^{d+p}} f \, \mathrm{d}\lambda^{d+p} = \left(\frac{d!\kappa_d}{2}\right)^2 \int_{A(d,d-1)\times A(d,d-1)} \int_{H_1^p \times (H_1\cap H_2)^{d-p} \times H_2^p} f$$
$$\times \Delta_d(x_1, \dots, x_d)\Delta_d(x_{p+1}, \dots, x_{p+d})[H_1, H_2]^{p-d}$$
$$\times \mathrm{d}(\lambda_{H_1}^p \otimes \lambda_{H_1\cap H_2}^{d-p} \otimes \lambda_{H_2}^p)\, \mu_{d-1}^2(\mathrm{d}(H_1, H_2)).$$

6. There is a spherical counterpart to the affine Blaschke–Petkantschin formula. The initial integration is over $(S^{d-1})^{q+1}$, where $q \in \{1, \dots, d-1\}$. The pivot is the q-flat affinely spanned (almost everywhere) by the integration variable $(x_0, \dots, x_q) \in (S^{d-1})^{q+1}$. The outer integration space is the affine Grassmannian $A(d, q)$, and for $E \in A(d, q)$ hitting S^{d-1}, the inner integration space is $(S^{d-1} \cap E)^{q+1}$. The result appears in Miles [525, Th. 4], with a short sketch of a proof. A detailed proof could be given similarly to Theorem 8.2.3. A typical application is found in Buchta, Müller and Tichy [134].

A related very general transformation formula, involving spheres and linear instead of affine subspaces, appears together with applications in Arbeiter and Zähle [38, Th. 1].

7. The affine Blaschke–Petkantschin formula of Theorem 7.2.7 can be interpreted as a decomposition of the $(q+1)$-fold product of the Lebesgue measure in $\mathbb{R}^d$. Integration with respect to this product measure is decomposed into integration with respect to the $(q+1)$-fold product of Lebesgue measure in a q-dimensional affine subspace, with a suitable Jacobian, followed by an integration over all q-dimensional affine subspaces. A somewhat similar decomposition is possible if the d-dimensional Lebesgue measure is replaced by the k-dimensional Hausdorff measure on a k-surface, $k < d$. In that case, the relative directions of the intersecting affine subspace and the tangent plane of the k-surface at the intersection points enter into the formula and make it complicated. General formulas of this type were proved by Zähle [829] (see Reitzner [628] for a short proof of a useful special case) and Jensen and Kiêu [382] (using an extended coarea formula by Kiêu [410]). A simplified proof was given in Jensen [379]. Stereological applications were presented by Jensen and Gundersen [380], Jensen, Kiêu and Gundersen [383].

7.3 Transformation Formulas Involving Spheres

In this section we prove two formulas of Blaschke–Petkantschin type (in the sense explained in the previous section), where the pivots are spheres. Correspondingly, the outer integration is over the space of all spheres, or equivalently, over the space of all possible centers and all possible radii, with a very simple measure. The inner integrations are conveniently written in terms of the unit sphere instead of variable spheres.

Theorem 7.3.1. *If* $f : (\mathbb{R}^d)^{d+1} \to \mathbb{R}$ *is a nonnegative measurable function, then*

$$\begin{aligned}\int_{(\mathbb{R}^d)^{d+1}} & f \,\mathrm{d}\lambda^{d+1} \\ &= d! \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} \cdots \int_{S^{d-1}} f(z + ru_0, \dots, z + ru_d) \\ &\quad \times r^{d^2-1} \Delta_d(u_0, \dots, u_d)\, \sigma(\mathrm{d}u_0) \cdots \sigma(\mathrm{d}u_d) \,\mathrm{d}r\, \lambda(\mathrm{d}z).\end{aligned}$$

Proof. It must be shown that the differentiable mapping

$$T : \mathbb{R}^d \times (0, \infty) \times (S^{d-1})^{d+1} \to (\mathbb{R}^d)^{d+1},$$

which is defined by

$$(z, r, u_0, \dots, u_d) \mapsto (z + ru_0, \dots, z + ru_d)$$

and is bijective up to sets of measure zero, has Jacobian given by

$$D(z, r, u_0, \dots, u_d) = d! r^{d^2-1} \Delta_d(u_0, \dots, u_d). \tag{7.16}$$

In the proof, we use the block notation for matrices. We write A^t for the transpose of a matrix A; vectors of $\mathbb{R}^d$ are interpreted as columns. We denote by E_k the $k \times k$ unit matrix. In order to prove (7.16) at a given point $(z, r, u_0, \dots, u_d)$ of $\mathbb{R} \times (0, \infty) \times (S^{d-1})^{d+1}$, we use special local coordinates in a neighborhood of this point. For $i = 0, \dots, d$ we introduce, in a neighborhood of u_i on S^{d-1}, parameters in such a way that the $d \times d$ matrix $(u_i\, \dot{u}_i)$ becomes orthogonal at the considered point; here $\dot{u}_i$ denotes the $d \times (d-1)$ matrix of the partial derivatives of u_i with respect to the corresponding parameters. This can easily be achieved. If for $u \in S^{d-1}$ the matrix $(u\, \dot{u})$ is orthogonal, then

$$\dot{u}^t u = 0, \qquad \dot{u}^t \dot{u} = E_{d-1}, \qquad E_d - \dot{u}\dot{u}^t = uu^t.$$

For $D = D(z, r, u_0, \dots, u_d)$ we therefore get

$$D = \begin{vmatrix} E_d & u_0 & r\dot{u}_0 & 0 & \cdots & 0 \\ \cdot & \cdot & 0 & \cdot & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ E_d & u_d & 0 & \cdot & \cdots & r\dot{u}_d \end{vmatrix}.$$

For $\widetilde{D} := r^{1-d^2} D$ we thus obtain

$$\widetilde{D}^2 = \begin{vmatrix} E_d & \cdot & \cdots & E_d \\ u_0^t & \cdot & \cdots & u_d^t \\ \dot{u}_0^t & 0 & \cdots & 0 \\ 0 & \cdot & & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & \dot{u}_d^t \end{vmatrix} \begin{vmatrix} E_d & u_0 & \dot{u}_0 & 0 & \cdots & 0 \\ \cdot & \cdot & 0 & \cdot & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ E_d & u_d & 0 & \cdot & \cdots & \dot{u}_d \end{vmatrix}$$

$$= \begin{vmatrix} (d+1)E_d & \sum u_i & \dot{u}_0 & \cdots & \dot{u}_d \\ \sum u_i^t & d+1 & 0 & \cdots & 0 \\ \dot{u}_0^t & 0 & E_{d-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dot{u}_d^t & 0 & 0 & \cdots & E_{d-1} \end{vmatrix}$$

$$= \begin{vmatrix} (d+1)E_d - \sum \dot{u}_i \dot{u}_i^t & \sum u_i \\ \sum u_i^t & d+1 \end{vmatrix} = \begin{vmatrix} \sum u_i u_i^t & \sum u_i \\ \sum u_i^t & d+1 \end{vmatrix}$$

$$= \left| \begin{pmatrix} u_0 \cdots u_d \\ 1 \cdots 1 \end{pmatrix} \begin{pmatrix} u_0^t & 1 \\ \vdots & \vdots \\ u_d^t & 1 \end{pmatrix} \right|$$

$$= (d!)^2 \Delta_d^2(u_0, \dots, u_d),$$

as asserted. □

The previous result was based on the fact that $d+1$ points in general position determine a unique sphere through these points. The following counterpart employs the unique sphere touching $d+1$ hyperplanes in general position and contained in the bounded region determined by the hyperplanes. Let $H_0, \dots, H_d \in A(d, d-1)$ by hyperplanes in general position (that is, they don't have a common point, and any d of their normal vectors are linearly independent). There is a unique simplex S such that $H_0, \dots, H_d$ are the facet hyperplanes of S. We denote by P the set of $(d+1)$-tuples of unit vectors positively spanning $\mathbb{R}^d$, that is, not lying in some closed hemisphere of S^{d-1}.

Theorem 7.3.2. *If $f : A(d, d-1)^{d+1} \to \mathbb{R}$ is a nonnegative measurable function, then*

$$\int_{A(d,d-1)^{d+1}} f \, \mathrm{d}\mu_{d-1}^{d+1}$$
$$= \frac{d!}{\omega_d^{d+1}} \int_{\mathbb{R}^d} \int_0^\infty \int_{S^{d-1}} \cdots \int_{S^{d-1}} f(H(u_0, \langle z, u_0 \rangle + r), \dots, H(u_d, \langle z, u_d \rangle + r))$$
$$\times \Delta_d(u_0, \dots, u_d) \mathbf{1}_{\mathsf{P}}(u_0, \dots, u_d) \, \sigma(\mathrm{d}u_0) \cdots \sigma(\mathrm{d}u_d) \, \mathrm{d}r \, \lambda(\mathrm{d}z).$$

Proof. Let $A^*(d,d-1)^{d+1}$ denote the set of $(d+1)$-tuples of hyperplanes in general position. Let $H_0,\dots,H_d \in A^*(d,d-1)^{d+1}$, and let Δ be the simplex determined by these hyperplanes. We denote by z the center of the insphere of Δ, by r its radius, and by $z+ru_i$, $i=0,\dots,d$, the contact points of the insphere with the given hyperplanes. Then $(u_0,\dots,u_d) \in \mathsf{P}$. The mapping

$$(z,r,u_0,\dots,u_d) \mapsto (H(u_0,t_0),\dots,H(u_d,t_d)) \quad \text{with } t_i := \langle z,u_i\rangle + r$$

maps $\mathbb{R}^d \times \mathbb{R}^+ \times \mathsf{P}$ bijectively onto $A^*(d,d-1)^{d+1}$. For fixed $(u_0,\dots,u_d) \in \mathsf{P}$, the mapping $(z,r) \mapsto (t_0,\dots,t_d)$ has Jacobian $d!\Delta_d(u_0,\dots,u_d)$. It follows that

$$\begin{aligned}
&\int_{A(d,d-1)} \cdots \int_{A(d,d-1)} f(H_0,\dots,H_d)\,\mu_{d-1}(\mathrm{d}H_0)\cdots\mu_{d-1}(\mathrm{d}H_d)\\
&= \frac{1}{\omega_d^{d+1}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(H(u_0,\tau_0),\dots,H(u_d,\tau_d))\\
&\quad \times\ \mathrm{d}\tau_0\cdots\mathrm{d}\tau_d\,\sigma(\mathrm{d}u_0)\cdots\sigma(\mathrm{d}u_d)\\
&= \frac{d!}{\omega_d^{d+1}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{\mathbb{R}^d} \int_0^\infty f(H(u_0,\langle z,u_0\rangle + r),\dots,H(u_d,\langle z,u_d\rangle + r))\\
&\quad \times\ \mathrm{d}r\,\lambda(\mathrm{d}z)\mathbf{1}_{\mathsf{P}}(u_0,\dots,u_d)\Delta_d(u_0,\dots,u_d)\,\sigma(\mathrm{d}u_0)\cdots\sigma(\mathrm{d}u_d),
\end{aligned}$$

which gives the assertion, by Fubini's theorem. □

Notes for Section 7.3

1. Theorem 7.3.1 appears, with a sketched proof, in Miles [521], equation (70). It was proved in a different way by Affentranger [9]. The proof given here goes back (for $d=3$) to Møller [553].

2. Theorem 7.3.2 is taken from Calka [149].