6

Extended Concepts of Integral Geometry

In this chapter, we derive further integral geometric formulas for convex bodies. They are related to the principal kinematic formula, either directly or indirectly. As in the latter formula, we have a fixed and a moving set, but in the two subsequent sections we do not consider intersections of both; we form sums of convex bodies or projections of convex bodies to subspaces. First we treat rotation means of Minkowski sums, which will later (Section 8.5) be applied to touching probabilities. The global version is an immediate consequence of the principal kinematic formula; the local version will be proved by techniques similar to those in Sections 5.2 and 5.3. From the formulas for rotation means of sums we deduce projection formulas.

In Section 6.3, we admit (infinite) convex cylinders as moving sets. For these, we derive a local kinematic formula, and we also obtain a formula that combines sections with projections.

Section 6.4 is devoted to a continuation of translative integral geometry. We treat iterated translative formulas, which involve a more general series of mixed measures, and consider rotation means and results of Crofton type for the mixed measures. The integral formulas for mixed measures and their global versions, the mixed functionals, also yield kinematic formulas for certain mixed volumes and for projection functions and support functions of convex bodies.

Section 6.5 provides an introduction to the integral geometry of spherically convex sets in the spherical space S^{d-1}.

6.1 Rotation Means of Minkowski Sums

In this section, we are interested in mean value formulas for the Minkowski sum of a fixed and a moving convex body. The functions to be integrated are again intrinsic volumes and curvature measures. Since $V_j(K + (\vartheta M + x))$, for example, does not depend on x, only the rotations of M are relevant, hence we shall be interested in the integral

$$\int_{SO_d} V_j(K + \vartheta M)\, \nu(\mathrm{d}\vartheta).$$

In order to illustrate the connection with the principal kinematic formula, we first prove the global version of the **rotational mean value formula**.

Theorem 6.1.1. *If $K, M \in \mathcal{K}'$ are convex bodies and if $j \in \{0, \dots, d\}$, then*

$$\int_{SO_d} V_j(K + \vartheta M)\, \nu(\mathrm{d}\vartheta) = \sum_{k=0}^{j} c_{d-j,d}^{d+k-j,d-k} V_k(K) V_{j-k}(M).$$

Proof. First we consider the case $j = d$. We have

$$\int_{SO_d} V_d(K + \vartheta M)\, \nu(\mathrm{d}\vartheta) = \int_{SO_d} \int_{\mathbb{R}^d} \mathbf{1}_{K+\vartheta M}(x)\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta).$$

The relation $x \in K+\vartheta M$ is equivalent to $K \cap (\vartheta M' + x) \neq \emptyset$, where $M' := -M$. Hence, we obtain

$$\begin{aligned}
\int_{SO_d} V_d(K + \vartheta M)\, \nu(\mathrm{d}\vartheta) &= \int_{SO_d} \int_{\mathbb{R}^d} V_0(K \cap (\vartheta M' + x))\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta) \\
&= \int_{G_d} V_0(K \cap gM')\, \mu(\mathrm{d}g) \\
&= \sum_{k=0}^{d} c_{0,d}^{k,d-k} V_k(K) V_{d-k}(M),
\end{aligned}$$

where we have used the principal kinematic formula (Theorem 5.1.3) and the fact that $V_j(M') = V_j(M)$ for $j = 0, \dots, d$.

Now we replace K by $K + \epsilon B^d$ with $\epsilon > 0$ and apply the Steiner formula (14.16), to obtain

$$\begin{aligned}
&\sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} \int_{SO_d} V_j(K + \vartheta M)\, \nu(\mathrm{d}\vartheta) \\
&= \int_{SO_d} V_d((K + \vartheta M) + \epsilon B^d)\, \nu(\mathrm{d}\vartheta) \\
&= \int_{SO_d} V_d((K + \epsilon B^d) + \vartheta M)\, \nu(\mathrm{d}\vartheta) \\
&= \sum_{m=0}^{d} c_{0,d}^{m,d-m} V_m(K + \epsilon B^d) V_{d-m}(M) \\
&= \sum_{m=0}^{d} \sum_{k=0}^{m} \epsilon^{m-k} \frac{1}{(m-k)!} c_{0,d}^{m,d-k} V_k(K) V_{d-m}(M).
\end{aligned}$$

Putting $m = d + k - j$ and changing the order of summation, we get the double sum

$$\sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} \sum_{k=0}^{j} c_{d-j,d}^{d+k-j,d-k} V_k(K) V_{j-k}(M).$$

Comparing the coefficients, we obtain the assertion for all $j \in \{0, \ldots, d\}$. $\square$

We want to extend the previous theorem to curvature measures, that is, replace the integrand $V_j(K + \vartheta M)$ by $\Phi_j(K + \vartheta M, A + \vartheta B)$. Evidently, this requires a restriction to the cases $j < d$ and to Borel sets $A \subset K$, $B \subset M$, contained in the respective bodies. Even under this assumption, $A + \vartheta B$ is in general not a Borel set, so that $\Phi_j(K + \vartheta M, A + \vartheta B)$ would not be defined. However, it will be sufficient to know the following.

Lemma 6.1.1. *Let $K, M \in \mathcal{K}'$, $A, B \in \mathcal{B}(\mathbb{R}^d)$ and $A \subset K$, $B \subset M$. For ν-almost all $\vartheta \in SO_d$ the set*

$$(A + \vartheta B) \cap \mathrm{bd}\,(K + \vartheta M)$$

is a Borel set, hence $\Phi_j(K + \vartheta M, A + \vartheta B)$ is defined for $j = 0, \ldots, d-1$.

Proof. For $x \in \mathrm{bd}\,(K + \vartheta M)$ there is a representation $x = y + z$ with $y \in K$, $z \in \vartheta M$. The points x, y, z lie in parallel supporting hyperplanes of $K + \vartheta M$, K, and ϑM, respectively; in particular, $y \in \mathrm{bd}\, K$ and $z \in \mathrm{bd}\, \vartheta M$. Suppose there is another representation $x = y_1 + z_1$ with $y_1 \in K$ and $z_1 \in \vartheta M$, then $y - y_1 = z_1 - z$, and the segments $\overline{yy_1}$, $\overline{z_1 z}$ satisfy $\overline{yy_1} \subset \mathrm{bd}\, K$ and $\overline{z_1 z} \subset \mathrm{bd}\, \vartheta M$. Hence the bodies K and ϑM contain parallel segments lying in parallel supporting hyperplanes. A theorem from the theory of convex bodies (see Schneider [695, Theorem 2.3.10]) says that for ν-almost all $\vartheta \in SO_d$ this does not occur. Hence, for these ϑ the representation $x = y + z$ with $y \in K$ and $z \in \vartheta M$ is unique for each $x \in \mathrm{bd}\,(K + \vartheta M)$. Putting

$$\pi_1(K, M, \vartheta, x) := y, \qquad \pi_2(K, M, \vartheta, x) := \vartheta^{-1} z,$$

we obtain mappings

$$\pi_1(K, M, \vartheta, \cdot): \ \mathrm{bd}\,(K + \vartheta M) \to \mathrm{bd}\, K,$$
$$\pi_2(K, M, \vartheta, \cdot): \ \mathrm{bd}\,(K + \vartheta M) \to \mathrm{bd}\, M.$$

From the compactness of the bodies K, M it follows easily that the mapping

$$\pi := \pi_1(K, M, \vartheta, \cdot) \times \pi_2(K, M, \vartheta, \cdot): \ \mathrm{bd}\,(K + \vartheta M) \to \mathrm{bd}\, K \times \mathrm{bd}\, M$$

is continuous. Hence, for Borel sets $A \subset K$, $B \subset M$ the set

$$(A + \vartheta B) \cap \mathrm{bd}\,(K + \vartheta M) = \pi^{-1}(A \times B)$$

is a Borel set, too. $\square$

In proving the local version of Theorem 6.1.1, we proceed similarly to the case of the principal kinematic formula, so we first consider polytopes. We say that two polytopes $K, M \in \mathcal{P}'$ are in **general relative position** if for any two faces F of K and G of M the linear subspaces $L(F), L(G)$ parallel to aff F, aff G, respectively, are in general position.

Theorem 6.1.2. *If $K, M \in \mathcal{K}'$ are convex bodies, $A, B \in \mathcal{B}(\mathbb{R}^d)$ are Borel sets satisfying $A \subset K$ and $B \subset M$, and if $j \in \{0, \dots, d-1\}$, then*

$$\int_{SO_d} \Phi_j(K + \vartheta M, A + \vartheta B)\, \nu(\mathrm{d}\vartheta)$$

$$= \sum_{k=0}^{j} c_{d-j,d}^{d+k-j,d-k} \Phi_k(K, A) \Phi_{j-k}(M, B). \tag{6.1}$$

Proof. The measurability of the integrand will be verified in the course of the proof. First we consider the case $j = d - 1$.

Let K, M be d-dimensional polytopes. By Lemmas 13.2.1 and 6.1.1, there is a Borel set $D_{K,M} \subset SO_d$ with $\nu(D_{K,M}) = 1$ such that K and ϑM are in general relative position and $(A + \vartheta B) \cap \mathrm{bd}\,(K + \vartheta M)$ is a Borel set if $\vartheta \in D_{K,M}$. Let $\vartheta \in D_{K,M}$. Since $K + \vartheta M$ is a polytope, we have

$$\Phi_{d-1}(K + \vartheta M, A + \vartheta B) = \sum_{F' \in \mathcal{F}_{d-1}(K+\vartheta M)} \gamma(F', K + \vartheta M) \lambda_{F'}(A + \vartheta B).$$

Because of $\vartheta \in D_{K,M}$, each facet $F' \in \mathcal{F}_{d-1}(K + \vartheta M)$ is of the form $F' = F + \vartheta G$ with $F \in \mathcal{F}_k(K)$ and $G \in \mathcal{F}_{d-1-k}(M)$, for some $k \in \{0, \dots, d-1\}$. For faces $F \in \mathcal{F}_k(K)$ and $G \in \mathcal{F}_{d-1-k}(M)$, we put $L_1 := (\mathrm{aff}\, F)^{\perp}$, $L_2 := (\mathrm{aff}\, G)^{\perp}$; then $L_1 \cap \vartheta L_2$ is of dimension one. The external angle $\gamma(F + \vartheta G, K + \vartheta M)$ is zero if $F + \vartheta G$ is not a face of $K + \vartheta M$; otherwise it is equal to $1/2$, and this happens if and only if

$$N(K, F) \cap \vartheta N(M, G) \cap S^{d-1} \neq \emptyset.$$

For arbitrary subsets $U \subset L_1$, $V \subset L_2$, the intersection $U \cap \vartheta V \cap S^{d-1}$ is either empty or one-pointed or two-pointed; we put

$$I(U, V, \vartheta) := \frac{1}{2} \mathrm{card}\,(U \cap \vartheta V \cap S^{d-1}).$$

If $F + \vartheta G$ is a face of $K + \vartheta M$, then

$$(A + \vartheta B) \cap (F + \vartheta G) = (A \cap F) + \vartheta (B \cap G).$$

Since the sum $F + \vartheta G$ is direct, we obtain

$$\lambda_{F+\vartheta G}(A + \vartheta B) = [F, \vartheta G] \lambda_F(A) \lambda_G(B).$$

This yields

$$\begin{aligned}
&\Phi_{d-1}(K+\vartheta M, A+\vartheta B) \\
&= \sum_{k=0}^{d-1} \sum_{F\in\mathcal{F}_k(K)} \sum_{\mathcal{F}_{d-1-k}(M)} \lambda_F(A)\lambda_G(B) I(N(K,F),N(M,G),\vartheta)[F,\vartheta G].
\end{aligned}$$

If faces F, G are given, we now define

$$J(U,V) := \int_{SO_d} I(U,V,\vartheta)[F,\vartheta G]\,\nu(\mathrm{d}\vartheta)$$

for arbitrary Borel sets $U \subset L_1 \cap S^{d-1},\ V \subset L_2 \cap S^{d-1}$. The measurability of the integrand is easily verified. This also yields the measurability of the integrand in (6.1) for the case where K and M are polytopes. Similarly to the proof of Theorem 5.3.1 one now proves the equality

$$J(U,V) = \alpha_{dk}\,\sigma_{d-k-1}(U)\,\sigma_k(V)$$

with a certain constant $\alpha_{dk} > 0$. This gives

$$\begin{aligned}
&\int_{SO_d} \Phi_{d-1}(K+\vartheta M, A+\vartheta B)\,\nu(\mathrm{d}\vartheta) \\
&= \sum_{k=0}^{d-1} \sum_{F\in\mathcal{F}_k(K)} \sum_{G\in\mathcal{F}_{d-k-1}(M)} \alpha'_{dk}\gamma(F,K)\gamma(G,M)\lambda_F(A)\lambda_G(B) \\
&= \sum_{k=0}^{d-1} \alpha'_{dk}\Phi_k(K,A)\Phi_{d-k-1}(M,B).
\end{aligned}$$

If $A = K$ and $B = M$, this formula must coincide with the corresponding one in Theorem 6.1.1, hence $\alpha'_{dk} = c_{1,d}^{k+1,d-k}$. Thus the proof of the case $j = d-1$ of (6.1) for d-dimensional polytopes K, M is complete.

Now let K, M be arbitrary d-dimensional convex bodies. Without loss of generality, we assume $0 \in \operatorname{int} K \cap \operatorname{int} M$. Then 0 is an inner point of $K + \vartheta M$, for all rotations $\vartheta \in SO_d$. By Lemma 6.1.1, there is a Borel set $D_{K,M} \subset SO_d$ with $\nu(D_{K,M}) = 1$ such that for $\vartheta \in D_{K,M}$ there exist the mappings

$$\begin{aligned}
\pi_1(K,M,\vartheta,\cdot) &:\ \operatorname{bd}(K+\vartheta M) \to \operatorname{bd} K, \\
\pi_2(K,M,\vartheta,\cdot) &:\ \operatorname{bd}(K+\vartheta M) \to \operatorname{bd} M
\end{aligned}$$

introduced in the proof of the lemma. Let $\vartheta \in D_{K,M}$. We extend the domain of $\pi_1(K,M,\vartheta,\cdot)$ and $\pi_2(K,M,\vartheta,\cdot)$ to all of $\mathbb{R}^d$. Since $0 \in \operatorname{int}(K+\vartheta M)$, to each $x \in \mathbb{R}^d$ there exist $\alpha \geq 0$ and $\overline{x} \in \operatorname{bd}(K+\vartheta M)$ with $x = \alpha\overline{x}$. We set

$$\pi_k(K,M,\vartheta,x) := \alpha\pi_k(K,M,\vartheta,\overline{x}) \qquad \text{for } k = 1,2.$$

Evidently the mappings $\pi_k(K,M,\vartheta,\cdot):\mathbb{R}^d\to\mathbb{R}^d$ thus defined are continuous ($k=1,2$). Let $\varphi(K,M,\vartheta,\cdot)$ be the image measure of $\Phi_{d-1}(K+\vartheta M,\cdot)$ under the map

$$\pi(K,M,\vartheta,\cdot):=\pi_1(K,M,\vartheta,\cdot)\times\pi_2(K,M,\vartheta,\cdot).$$

Then $\varphi(K,M,\vartheta,\cdot)$ is a finite Borel measure on $\mathbb{R}^d\times\mathbb{R}^d$, and for $A,B\in\mathcal{B}(\mathbb{R}^d)$ with $A\subset K$ and $B\subset M$, we have

$$\varphi(K,M,\vartheta,A\times B)=\Phi_{d-1}(K+\vartheta M,A+\vartheta B). \tag{6.2}$$

By the transformation rule for integrals,

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,y)\,\varphi(K,M,\vartheta,\mathrm{d}(x,y))\\ &=\int_{\mathbb{R}^d} f(\pi(K,M,\vartheta,z))\,\Phi_{d-1}(K+\vartheta M,\mathrm{d}z)\end{aligned} \tag{6.3}$$

for all continuous functions f on $\mathbb{R}^d\times\mathbb{R}^d$.

Now let $\vartheta\in D_{K,M}$ and let $(\vartheta_i)_{i\in\mathbb{N}}$ be a sequence in $D_{K,M}$ converging to ϑ. We show that $\varphi(K,M,\vartheta_i,\cdot)$ converges weakly to $\varphi(K,M,\vartheta,\cdot)$ if $i\to\infty$. We can choose a convex body $C\in\mathcal{K}^d$ with $K+\vartheta_i M\subset C$ for all $i\in\mathbb{N}$. Let $f\in\mathbf{C}(\mathbb{R}^d\times\mathbb{R}^d)$. The function f is uniformly continuous on $C\times C$, hence for given $\epsilon>0$ there is δ with $|f(x,y)-f(x',y')|<\epsilon$ for all $x,y,x',y'\in C$ with $\|x-x'\|+\|y-y'\|<2\delta$. It is easy to see that $\pi_k(K,M,\vartheta_i,\cdot)$ converges to $\pi_k(K,M,\vartheta,\cdot)$, uniformly on C, for $k=1,2$. We infer that $\|\pi_k(K,M,\vartheta_i,z)-\pi_k(K,M,\vartheta,z)\|<\delta$ for all $z\in C$, almost all $i\in\mathbb{N}$, and $k=1,2$. Together with (6.3) this gives

$$\begin{aligned}&\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{d}\varphi(K,M,\vartheta_i,\cdot)-\int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{d}\varphi(K,M,\vartheta,\cdot)\right|\\ &\le\int_{\mathbb{R}^d}|f(\pi(K,M,\vartheta_i,z))-f(\pi(K,M,\vartheta,z))|\,\Phi_{d-1}(K+\vartheta_i M,\mathrm{d}z)\\ &\quad+\left|\int_{\mathbb{R}^d} f(\pi(K,M,\vartheta,z))\,(\Phi_{d-1}(K+\vartheta_i M,\mathrm{d}z)-\Phi_{d-1}(K+\vartheta M,\mathrm{d}z))\right|\\ &< a\epsilon\end{aligned}$$

for almost all $i\in\mathbb{N}$, with a constant a not depending on i. Here we have used the fact that $\Phi_{d-1}(K+\vartheta_i M,\mathbb{R}^d)$ is bounded by a constant depending only on C (similarly to the proof of Theorem 5.2.3); further, the weak convergence $\Phi_{d-1}(K+\vartheta_i M,\cdot)\xrightarrow{w}\Phi_{d-1}(K+\vartheta M,\cdot)$ and the continuity of the function $f(\pi(K,M,\vartheta,\cdot))$ were applied.

The weak convergence thus established shows that for each $f\in\mathbf{C}(\mathbb{R}^d\times\mathbb{R}^d)$ the mapping

$$\vartheta\mapsto\int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{d}\varphi(K,M,\vartheta,\cdot)$$

is continuous on $D_{K,M}$. By Lemma 12.1.1 this implies the measurability of the mapping

$$\vartheta \mapsto \varphi(K, M, \vartheta, U)$$

on $D_{K,M}$, for all $U \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$. In particular, for $A, B \in \mathcal{B}(\mathbb{R}^d)$ with $A \subset K$ and $B \subset M$ we obtain from (6.2) the measurability of the map

$$\vartheta \mapsto \Phi_{d-1}(K + \vartheta M, A + \vartheta B)$$

on $D_{K,M}$ and hence the measurability ν-almost everywhere of the integrand in (6.1), if $j = d - 1$.

Putting

$$\varphi(K, M, \cdot) := \int_{SO_d} \varphi(K, M, \vartheta, \cdot)\, \nu(\mathrm{d}\vartheta),$$

we now obtain a finite measure on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f \,\mathrm{d}\varphi(K, M, \cdot) = \int_{SO_d} \int_{\mathbb{R}^d} f(\pi(K, M, \vartheta, z))\, \Phi_{d-1}(K + \vartheta M, \mathrm{d}z)\, \nu(\mathrm{d}\vartheta)$$

for $f \in \mathbf{C}(\mathbb{R}^d \times \mathbb{R}^d)$. We consider convergent sequences $K_i \to K$ and $M_i \to M$ of convex bodies K_i, M_i with $0 \in \operatorname{int} K_i \cap \operatorname{int} M_i$, and we put $D := \bigcap_{i=1}^{\infty} D_{K_i,M_i} \cap D_{K,M}$. As before, we see that for $\vartheta \in D$ the functions $f(\pi(K_i, M_i, \vartheta, \cdot))$ converge for $i \to \infty$, uniformly on every compact set, and we deduce in a similar way that

$$\int_{\mathbb{R}^d} f(\pi(K_i, M_i, \vartheta, z))\, \Phi_{d-1}(K_i + \vartheta M_i, \mathrm{d}z)$$
$$\to \int_{\mathbb{R}^d} f(\pi(K, M, \vartheta, z))\, \Phi_{d-1}(K + \vartheta M, \mathrm{d}z)$$

for $i \to \infty$. The dominated convergence theorem yields

$$\int_{SO_d} \int_{\mathbb{R}^d} f(\pi(K_i, M_i, \vartheta, z))\, \Phi_{d-1}(K_i + \vartheta M_i, \mathrm{d}z)\, \nu(\mathrm{d}\vartheta)$$
$$\to \int_{SO_d} \int_{\mathbb{R}^d} f(\pi(K, M, \vartheta, z))\, \Phi_{d-1}(K + \vartheta M, \mathrm{d}z)\, \nu(\mathrm{d}\vartheta)$$

and thus the weak convergence $\varphi(K_i, M_i, \cdot) \xrightarrow{w} \varphi(K, M, \cdot)$ for $i \to \infty$.

Obviously, the assertion of the theorem for $j = d - 1$ is equivalent to

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) g(y)\, \varphi(K, M, \mathrm{d}(x, y))$$
$$= \sum_{k=0}^{d-1} c_{1,d}^{k+1,d-k} \int_{\mathbb{R}^d} f \,\mathrm{d}\Phi_k(K, \cdot) \int_{\mathbb{R}^d} g \,\mathrm{d}\Phi_{d-k-1}(M, \cdot)$$

for all $f, g \in \mathbf{C}(\mathbb{R}^d)$. Since we have proved the assertion for d-dimensional polytopes, we can use the latter equality, where both sides depend continuously on K and M, to extend it by approximation to arbitrary d-dimensional convex bodies K, M.

The extension to convex bodies without interior points and to $j < d-1$ is now achieved by an application of the local Steiner formula of Theorem 14.2.4. We assume first that $M \in \mathcal{K}'$ still has interior points, while $K \in \mathcal{K}'$ may be arbitrary. The assertion to be proved holds for $j = d-1$ and for the bodies $K + \epsilon B^d$ and M, where $\epsilon > 0$ is arbitrary. Using Theorem 14.2.4 twice, we therefore obtain the measurability of the integrand in (6.1) and the equalities

$$\sum_{j=0}^{d-1} \epsilon^{d-1-j} \frac{1}{(d-1-j)!} c_1^{d-j} \int_{SO_d} \Phi_j(K + \vartheta M, A + \vartheta B)\, \nu(\mathrm{d}\vartheta)$$

$$= \int_{SO_d} \Phi_{d-1}(K + \epsilon B^d + \vartheta M, A + \epsilon S^{d-1} + \vartheta B)\, \nu(\mathrm{d}\vartheta)$$

$$= \sum_{r=0}^{d-1} c_{1,d}^{r+1,d-r} \Phi_r(K + \epsilon B^d, A + \epsilon S^{d-1}) \Phi_{d-r-1}(M, B)$$

$$= \sum_{r=0}^{d-1} c_{1,d}^{r+1,d-r} \sum_{k=0}^{r} \epsilon^{r-k} \frac{1}{(r-k)!} c_{d-r}^{d-k} \Phi_k(K, A) \Phi_{d-r-1}(M, B)$$

$$= \sum_{j=0}^{d-1} \epsilon^{d-1-j} \frac{1}{(d-1-j)!} c_1^{d-j} \sum_{k=0}^{j} c_{d-j,d}^{d+k-j,d-k} \Phi_k(K, A) \Phi_{j-k}(M, B).$$

Comparing the coefficients, we obtain the assertion for the bodies K and M. Analogously, M can be replaced by an arbitrary convex body. □

Notes for Section 6.1

1. Theorem 6.1.1 goes back, with a different proof, to Hadwiger [307, p. 231]. A local version of this mean value formula under Minkowski addition was first proved by Schneider [673], though not for the curvature measures Φ_j, but for the area measures Ψ_j. Weil [780] used a result of Schneider [677] on curvature measures to prove Theorem 6.1.2. A simpler proof and a generalization appear in Schneider [688].

2. The rotation formula (6.1) has an extension to support measures. It involves an operation for sets of support elements which is adapted to the Minkowski addition of convex bodies. For sets $A, B \subset \Sigma = \mathbb{R}^d \times S^{d-1}$ we define

$$A * B := \{(x + y, u) \in \Sigma : (x, u) \in A,\ (y, u) \in B\}.$$

This operation combines the behaviors of sets of boundary points and of normal vectors of convex bodies under addition, in the following way. If $A \subset \operatorname{Nor} K$ and

$B \subset \operatorname{Nor} M$, then $A * B \subset \operatorname{Nor}(K+M)$, and for $A_1, A_2 \subset \mathbb{R}^d$ and $B_1, B_2 \subset S^{d-1}$ we have

$$(A_1 \times B_1) * (A_2 \times B_2) = (A_1 + A_2) \times (B_1 \cap B_2).$$

The following result holds for convex bodies $K, M \in \mathcal{K}'$, Borel sets $A \subset \operatorname{Nor} K$, $B \subset \operatorname{Nor} M$, and for $j = 0, \dots, d-1$:

$$\int_{SO_d} \Xi_j(K + \vartheta M, A * \vartheta B)\, \nu(\mathrm{d}\vartheta) = \sum_{k=0}^{j} c_{d-j,d}^{d+k-j,d-k} \Xi_k(K, A) \Xi_{j-k}(M, B). \tag{6.4}$$

Special cases are (6.1) and the formula

$$\int_{SO_d} \Psi_j(K + \vartheta M, A \cap \vartheta B)\, \nu(\mathrm{d}\vartheta) = \sum_{k=0}^{j} c_{d-j,d}^{d+k-j,d-k} \Psi_k(K, A) \Psi_{j-k}(M, B) \tag{6.5}$$

for Borel sets $A, B \subset S^{d-1}$.

Formula (6.4) was proved by Schneider [688]; the special case (6.5) was obtained earlier by Schneider [672].

3. For special pairs of convex bodies $K, M \in \mathcal{K}'$, a counterpart to Theorem 6.1.1 holds with the sum $K + \vartheta M$ replaced by the Minkowski difference $K \ominus \vartheta M$. One says that M **rolls freely** in K if for each rotation $\vartheta \in SO_d$ and each point $x \in \operatorname{bd} K$ there is a vector t such that $x \in \vartheta M + t \subset K$, equivalently, if each rotation image ϑM is a summand of K (see Schneider [695, p. 150]). If M rolls freely in K, then

$$\int_{SO_d} V_d(K \ominus \vartheta M)\, \nu(\mathrm{d}\vartheta) = \sum_{k=0}^{d} (-1)^{d-k} c_{d,0}^{d-k,k} V_k(K) V_{d-k}(M). \tag{6.6}$$

In fact, Theorem 6.1.1 together with (14.20) yields

$$\begin{aligned}
\sum_{k=0}^{d} c_{d,0}^{d-k,k} V_k(K) V_{d-k}(M) \epsilon^{d-k} &= \int_{SO_d} V_d(K + \epsilon\vartheta M)\, \nu(\mathrm{d}\vartheta) \\
= \sum_{k=0}^{d} \epsilon^{d-k} \binom{d}{k} &\int_{SO_d} V(K[k], \vartheta M[d-k])\, \nu(\mathrm{d}\vartheta).
\end{aligned}$$

Comparing the coefficients, we obtain

$$\binom{d}{k} \int_{SO_d} V(K[k], \vartheta M[d-k])\, \nu(\mathrm{d}\vartheta) = c_{d,0}^{d-k,k} V_k(K) V_{d-k}(M) \tag{6.7}$$

for $k = 0, \dots, d$. Since $(K \ominus M) + M = K$, the symmetry and linearity properties of mixed volumes imply

$$\begin{aligned}
&V_d(K \ominus M) = V(K \ominus M, \dots, K \ominus M) \\
&= V(K \ominus M, \dots, K \ominus M, K) - V(K \ominus M, \dots, K \ominus M, M) \\
&= \dots = \sum_{k=0}^{d} (-1)^{d-k} \binom{d}{k} V(K[k], M[d-k]),
\end{aligned}$$

and similarly for ϑM instead of M. Together with (6.7) this yields (6.6).

4. Containment measures. While the principal kinematic formula expresses the hitting measure $\mu(\{g \in G_d : gM \cap K \neq \emptyset\})$ of two convex bodies $K, M \in \mathcal{K}'$ in terms of intrinsic volumes, there is in general no simple expression for the **containment measure** (also called **inclusion measure**)

$$I(M,K) := \mu(\{g \in G_d : gM \subset K\}).$$

An exception is the case of the previous note: if M rolls freely in K, then $\{t \in \mathbb{R}^d : \vartheta M + t \subset K\} = K \ominus \vartheta M$, hence $I(M,K) = \int_{SO_d} V_d(K \ominus \vartheta M)\, \nu(\mathrm{d}\vartheta)$.

For results on containment measures, in particular for the case where M is a segment, we refer to Santaló [664], Ren [635], Zhang [832], the survey by Zhang and Zhou [834], and the literature quoted there.

6.2 Projection Formulas

A further familiar operation for convex bodies is the projection to a subspace. For a subspace $L \in G(d,q)$, recall that $A|L$ is the image of the set $A \subset \mathbb{R}^d$ under the orthogonal projection to L. From the results of the last sections we shall now derive **projection formulas**.

Theorem 6.2.1. *If $K \in \mathcal{K}'$ is a convex body, $A \in \mathcal{B}(\mathbb{R}^d)$ is a Borel set satisfying $A \subset K$, and if $q \in \{1, \dots, d-1\}$, $j \in \{0, \dots, q-1\}$, then*

$$\int_{G(d,q)} \Phi_j(K|L, A|L)\, \nu_q(\mathrm{d}L) = c_{d,q-j}^{q,d-j} \Phi_j(K,A). \tag{6.8}$$

Proof. Let $L_q \in G(d,q)$ be fixed. By the definition of ν_q,

$$\int_{G(d,q)} \Phi_j(K|L, A|L)\, \nu_q(\mathrm{d}L) = \int_{SO_d} \Phi_j(K|\vartheta L_q, A|\vartheta L_q)\, \nu(\mathrm{d}\vartheta).$$

Let M be a unit cube in $L_q^\perp$ and $B := \operatorname{relint} M$, then

$$\Phi_k(M,B) = \begin{cases} 1 \text{ for } k = d-q, \\ 0 \text{ for } k \neq d-q. \end{cases} \tag{6.9}$$

Let $\vartheta \in SO_d$ be chosen in such a way that K and ϑM do not contain parallel segments in parallel supporting hyperplanes. We consider the local parallel set

$$U_\epsilon(K,A) := \{x \in \mathbb{R}^d : \|x - p(K,x)\| \leq \epsilon,\ p(K,x) \in A\}.$$

For $\epsilon > 0$ we have

$$\begin{aligned} &U_\epsilon(K + \vartheta M, (A + \vartheta B) \cap \mathrm{bd}\,(K + \vartheta M)) \\ &= \{z \in U_\epsilon(K, A') : z - p(K,z) \in \vartheta L_q\} + \vartheta B \end{aligned}$$

with $A' := \{a \in A : a|\vartheta L_q \in \operatorname{relbd}(K|\vartheta L_q)\}$. In fact, if $y = p(K + \vartheta M, x)$ and $y \in (A + \vartheta B) \cap \operatorname{bd}(K + \vartheta M)$, then $y = a + \vartheta b$ with $a \in A,\ b \in B$. There is a supporting hyperplane H to $K + \vartheta M$ through y. Since ϑb lies in a supporting hyperplane of ϑM parallel to H and since $b \in \operatorname{relint} M$, we obtain $\vartheta L_q^{\perp} + y \subset H$ and thus $a \in A'$. The argument can be reversed.

Trivially we have

$$(A|\vartheta L_q) \cap \operatorname{relbd}(K|\vartheta L_q) = A'|\vartheta L_q. \tag{6.10}$$

This set is a Borel set, since the orthogonal projection to ϑL_q, restricted to the points that are projected to $\operatorname{relbd}(K|\vartheta L_q)$, is a homeomorphism, by the choice of ϑ. Fubini's theorem now gives

$$\lambda(U_\epsilon(K + \vartheta M, (A + \vartheta B) \cap \operatorname{bd}(K + \vartheta M))) = \lambda_q(U_\epsilon^{(q)}(K|\vartheta L_q, A'|\vartheta L_q)),$$

where $U_\epsilon^{(q)}$ is a local parallel set in ϑL_q. Using the local Steiner formula (14.12) and (6.10), we obtain

$$\sum_{i=0}^{d-1} \epsilon^{d-i}\kappa_{d-i}\Phi_i(K + \vartheta M, A + \vartheta B) = \sum_{j=0}^{q-1} \epsilon^{q-j}\kappa_{q-j}\Phi_j(K|\vartheta L_q, A|\vartheta L_q),$$

hence

$$\Phi_j(K|\vartheta L_q, A|\vartheta L_q) = \Phi_{d-q+j}(K + \vartheta M, A + \vartheta B)$$

for $j = 0, \dots, q-1$ and

$$\Phi_i(K + \vartheta M, A + \vartheta B) = 0$$

for $i = 0, \dots, d-q-1$. This entails the measurability, up to a set of ν-measure zero, of the mapping $\vartheta \mapsto \Phi_j(K|\vartheta L_q, A|\vartheta L_q)$, and from Theorem 6.1.2 and (6.9) we then obtain

$$\begin{aligned}
&\int_{SO_d} \Phi_j(K|\vartheta L_q, A|\vartheta L_q)\,\nu(\mathrm{d}\vartheta) \\
&= \int_{SO_d} \Phi_{d-q+j}(K + \vartheta M, A + \vartheta B)\,\nu(\mathrm{d}\vartheta) \\
&= \sum_{r=0}^{d+j-q} c_{d,q-j}^{q+r-j,d-r}\Phi_r(K, A)\Phi_{d-q+j-r}(M, B) \\
&= c_{d,q-j}^{q,d-j}\Phi_j(K, A),
\end{aligned}$$

as asserted. □

Theorem 6.2.1 implies a projection formula for the intrinsic volumes $V_j,\ j = 0, \dots, q-1$. This formula holds for V_q, too.

Theorem 6.2.2. *If $K \in \mathcal{K}'$ and $q \in \{1, \dots, d-1\}$, $j \in \{0, \dots, q\}$, then*

$$\int_{G(d,q)} V_j(K|L)\, \nu_q(\mathrm{d}L) = c_{d,q-j}^{q,d-j} V_j(K).$$

Proof. Only the case $j = q$ needs to be proved. Fix $L_q \in G(d,q)$ and let B^{d-q} be the unit ball in $L_q^\perp$. For $\epsilon > 0$, Theorem 6.1.1 gives

$$\int_{SO_d} V_d(\vartheta K + \epsilon B^{d-q})\, \nu(\mathrm{d}\vartheta) = \sum_{k=0}^{d} c_{d,0}^{k,d-k} V_k(K) V_{d-k}(B^{d-q}) \epsilon^{d-k}.$$

The coefficient of ϵ^{d-q} is $c_{d,0}^{q,d-q} V_q(K) \kappa_{d-q}$. On the other hand, Fubini's theorem gives

$$V_d(\vartheta K + \epsilon B^{d-q}) = \int_{L_q} V_{d-q}((\vartheta K \cap (L_q^\perp + x)) + \epsilon B^{d-q})\, \lambda_q(\mathrm{d}x).$$

Applying the Steiner formula (14.5) in $L_q^\perp + x$, we obtain on the right side a polynomial in ϵ, where the coefficient of ϵ^{d-q} is equal to

$$\int_{L_q} \kappa_{d-q} V_0(\vartheta K \cap (L_q^\perp + x))\, \lambda_q(\mathrm{d}x) = \kappa_{d-q} V_q(\vartheta K | L_q) = \kappa_{d-q} V_q(K | \vartheta^{-1} L_q).$$

Integrating over SO_d (observing the invariance of ν) and comparing the coefficients, we obtain the assertion. □

Choosing $q = j$ in Theorem 6.2.2, we get

$$V_j(K) = c_{j,d-j}^{0,d} \int_{G(d,j)} V_j(K|L)\, \nu_j(\mathrm{d}L), \tag{6.11}$$

which is known as **Kubota's formula**. The special case $j = d-1$ of (6.11) yields the representation

$$\begin{aligned} S(K) = 2V_{d-1}(K) &= \frac{d\kappa_d}{\kappa_{d-1}} \int_{G(d,d-1)} V_{d-1}(K|L)\, \nu_{d-1}(\mathrm{d}L) \\ &= \frac{1}{\kappa_{d-1}} \int_{S^{d-1}} V_{d-1}(K|u^\perp)\, \sigma(\mathrm{d}u) \end{aligned} \tag{6.12}$$

for the surface area $S(K)$ of K. The latter equation is called **Cauchy's surface area formula**. For $j = 1$, (6.11) reduces to

$$V_1(K) = \frac{d\kappa_d}{2\kappa_{d-1}} \int_{G(d,1)} V_1(K|L)\, \nu_1(\mathrm{d}L).$$

Since $V_1(K|L)$ is the width of K in direction L, the integral

$$\int_{G(d,1)} V_1(K|L)\,\nu_1(\mathrm{d}L)$$

is the mean width, $b(K)$, of K. Hence we obtain formula (14.7),

$$V_1(K) = \frac{d\kappa_d}{2\kappa_{d-1}}\, b(K).$$

Notes for Section 6.2

1. The projection formulas of Theorem 6.2.2 are classical results of the integral geometry of convex bodies; a special case was already known to Cauchy. Local versions are found in Schneider [673] and Weil [780]. The reduction to the rotation formula for sums, which is used in the proof of Theorem 6.2.1, was noted in Schneider [688].

2. The projection formula (6.8) has an extension to support measures. For a set $A \subset \Sigma$ and a linear subspace of $\mathbb{R}^d$ we define

$$A|L := \{(x|L, u) : (x,u) \in A,\ u \in L\}.$$

Let $K \in \mathcal{K}'$ be a convex body and $A \subset \operatorname{Nor} K$ a Borel set. For $q \in \{1, \dots, d-1\}$, $j \in \{0, \dots, q-1\}$, the formula

$$\int_{G(d,q)} \Xi'_j(K|L, A|L)\,\nu_q(\mathrm{d}L) = c^{q,d-j}_{d,q-j}\,\Xi_j(K, A)$$

holds, where Ξ'_j denotes the jth support measure with respect to L. In a different, but equivalent formulation, this is Theorem 4.5.10 in Schneider [695].

3. An extension of the projection formula (6.8) to polyconvex sets was treated in Schneider [693]; here suitable multiplicities of tangential projections have to be taken into account.

6.3 Cylinders and Thick Sections

As we have seen, the Crofton formulas can be deduced from the principal kinematic formula, and the Cauchy–Kubota formulas are consequences of the rotation formulas for Minkowski sums. This shows that integral geometric formulas for convex bodies on one side and for affine subspaces on the other side are closely connected. This connection will become even more evident when we now consider cylinders and prove a common generalization of the principal kinematic formula and the Crofton formula.

By a (convex) **cylinder** C in $\mathbb{R}^d$ we understand a set of the form $C = M + L$ with $L \in G(d,q)$, $q \in \{0, \dots, d-1\}$, and $M \in \mathcal{K}'$, $M \subset L^\perp$. The linear subspace L is called the **direction space** of the cylinder C, and M is its **base**. Also the images gC of C under $g \in G_d$ are called cylinders, but C will always be of the standard form as described (with fixed $L \in G(d,q)$).

Since C is a closed convex set, the curvature measures $\Phi_0(C,\cdot),\dots,\Phi_d(C,\cdot)$ are well defined. They are finite on bounded Borel sets and have a special form. In the following, we identify $\mathbb{R}^d$ with $L^\perp \times L$. We denote by λ_{d-q} and λ_q, respectively, the Lebesgue measures in $L^\perp$ and L.

Lemma 6.3.1. *The curvature measures of the cylinder C satisfy*

$$\Phi_j(C,\cdot) = \begin{cases} \Phi_{j-q}(M,\cdot) \otimes \lambda_q & \text{for } q \le j \le d, \\ 0 & \text{for } 0 \le j < q. \end{cases}$$

Proof. We can assume that the base M is a polytope; the general case then follows by approximation, due to the weak continuity of the mapping $C \mapsto \Phi_j(C \cap K,\cdot)$, for each $K \in \mathcal{K}'$. Since C is polyhedral in that case, the representation (14.13) of the curvature measures of polytopes gives

$$\Phi_j(C,\cdot) = \sum_{F \in \mathcal{F}_j(C)} \gamma(F,C)\lambda_F.$$

Since $C = M + L$, we have $\mathcal{F}_j(C) = \emptyset$ for $j < q$, thus $\Phi_j(C,\cdot) = 0$. For $j \ge q$,

$$\mathcal{F}_j(C) = \{F + L : F \in \mathcal{F}_{j-q}(M)\},$$

hence in this case we get

$$\Phi_j(C,\cdot) = \sum_{F \in \mathcal{F}_{j-q}(M)} \gamma(F+L, M+L)\lambda_{F+L}.$$

Together with $\gamma(F+L, M+L) = \gamma(F,M)$ and $\lambda_{F+L} = \lambda_F \otimes \lambda_q$, this yields

$$\Phi_j(C,\cdot) = \left(\sum_{F \in \mathcal{F}_{j-q}(M)} \gamma(F,M)\lambda_F \right) \otimes \lambda_q = \Phi_{j-q}(M,\cdot) \otimes \lambda_q,$$

as stated. □

In analogy to the principal kinematic formula and the Crofton formula, we now consider intersections of a fixed convex body and a moving cylinder. The principal kinematic formula involves an integration over the motion group. Although the motion group has infinite invariant measure, the integrals remain finite, since for $K, M \in \mathcal{K}$ the relation $K \cap gM \ne \emptyset$ holds only for the motions g from a suitable compact set. However, for a convex body K with inner points and for a cylinder C with $q > 0$, the set of rigid motions g with $K \cap gC \ne \emptyset$ has infinite measure. In the case of the Crofton formula, which concerns the case $\dim M = 0$, the integration was therefore with respect to the invariant measure μ_q on the space $A(d,q)$ of q-flats. In a similar way, we can interpret the set of cylinders congruent to C as a homogeneous space, on which we can introduce an invariant measure. Implicitly, this has been done in the following theorem where, though, we work directly with a suitable representation of this invariant measure.

Theorem 6.3.1 (Local kinematic formula for cylinders). *Suppose that* $q \in \{0, \dots, d-1\}$ *and* $j \in \{0, \dots, d\}$*. Let* $K \in \mathcal{K}'$ *be a convex body, let* C *be a cylinder with direction space* $L \in G(d,q)$ *and base* M*, and let* $A, B \in \mathcal{B}(\mathbb{R}^d)$ *be Borel sets with* $B \subset L^\perp$*. Then*

$$\int_{SO_d} \int_{L^\perp} \Phi_j(K \cap \vartheta(C+x), A \cap \vartheta(B+L+x))\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$
$$= \sum_{k=j}^{N(d,j,q)} c_{j,d}^{k,d-k+j} \Phi_k(K,A) \Phi_{d-k+j-q}(M,B)$$

with $N(d,j,q) := \min\{d, d+j-q\}$.

Proof. First we note that

$$\int_{SO_d} \int_{L^\perp} \Phi_j(K \cap \vartheta(C+x), A \cap \vartheta(B+L+x))\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$
$$= \int_{SO_d} \int_{L^\perp} \Phi_j(\vartheta K \cap (C+x), \vartheta A \cap (B+L+x))\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta).$$

Since $\{\vartheta K : \vartheta \in SO_d\}$ is bounded, there exists a compact set $B' \subset L$ (with $\lambda_q(B') > 0$) such that

$$\Phi_j(\vartheta K \cap (C+x), \vartheta A \cap (B+L+x))$$
$$= \Phi_j(\vartheta K \cap (C+x), \vartheta A \cap (B+B'+x))$$

for all $x \in L^\perp$ and all $\vartheta \in SO_d$. From Theorem 5.3.2 and Lemma 6.3.1 we get

$$\int_{SO_d} \int_{L^\perp} \int_L \Phi_j(\vartheta K \cap (C+x), \vartheta A \cap (B+B'+x+y))$$
$$\times \lambda_q(\mathrm{d}y)\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$
$$= \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \Phi_k(K,A) \Phi_{d-k+j}(C, B+B')$$
$$= \sum_{k=j}^{N(d,j,q)} c_{j,d}^{k,d-k+j} \Phi_k(K,A) \Phi_{d-k+j-q}(M,B) \lambda_q(B').$$

On the other hand, with $K' := \vartheta K - x$ and $A' := \vartheta A - x$ we have

$$\int_L \Phi_j(\vartheta K \cap (C+x), \vartheta A \cap (B+B'+x+y))\, \lambda_q(\mathrm{d}y)$$
$$= \int_L \Phi_j(K' \cap C, A' \cap (B+B'+y))\, \lambda_q(\mathrm{d}y)$$

$$
\begin{aligned}
&= \int_L \int_{\mathbb{R}^d} \mathbf{1}_{A'}(u)\mathbf{1}_{B+B'}(u-y)\, \Phi_j(K' \cap C, \mathrm{d}u)\, \lambda_q(\mathrm{d}y) \\
&= \int_{\mathbb{R}^d} \int_L \mathbf{1}_{B+B'}(u-y)\, \lambda_q(\mathrm{d}y) \mathbf{1}_{A'}(u)\, \Phi_j(K' \cap C, \mathrm{d}u) \\
&= \int_{\mathbb{R}^d} \mathbf{1}_{B+L}(u) \lambda_q(B') \mathbf{1}_{A'}(u)\, \Phi_j(K' \cap C, \mathrm{d}u) \\
&= \Phi_j(K' \cap C, A' \cap (B+L)) \lambda_q(B') \\
&= \Phi_j(\vartheta K \cap (C+x), \vartheta A \cap (B+L+x)) \lambda_q(B').
\end{aligned}
$$

Dividing by $\lambda_q(B')$, we obtain the assertion. □

As special cases, Theorem 6.3.1 contains both, the principal kinematic formula (Theorem 5.3.2) and the Crofton formula (Theorem 5.3.3). The former is obtained for $q = 0$ (thus $L = \{0\}$), and the latter for $M = B = \{0\}$ and $j \leq q$, since then

$$
\Phi_{d-k+j-q}(M, B) = \begin{cases} 1 & \text{for } d-k+j-q = 0, \\ 0 & \text{else.} \end{cases}
$$

The global version of Theorem 6.3.1 (that is, $A = B = \mathbb{R}^d$) results in a cylinder formula for intrinsic volumes.

Corollary 6.3.1 (Principal kinematic formula for cylinders). *Let $q \in \{0, \ldots, d-1\}$ and $j \in \{0, \ldots, d\}$. If $K \in \mathcal{K}'$ is a convex body and C is a cylinder with direction space $L \in G(d, q)$ and base M, then*

$$
\begin{aligned}
&\int_{SO_d} \int_{L^\perp} V_j(K \cap \vartheta(C+x))\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta) \\
&\quad = \sum_{k=j}^{N(d,j,q)} c_{j,d}^{k,d-k+j} V_k(K) V_{d-k+j-q}(M).
\end{aligned}
$$

Especially for cylinders, there is a further operation besides section and projection – combining section and projection. Namely, for K and C as above, the intersection $K \cap \vartheta(C+x)$ can be projected orthogonally to the direction space ϑL of $\vartheta(C+x)$. In a special case, such a combination appears in certain applications. For example, microscopical sections, as they are treated in stereology by means of integral geometric methods, have a non-zero thickness. Therefore, a microscopical section is not an intersection with a plane, but with a cylinder $C = M + L$, where L is a plane and $M \subset L^\perp$ is a segment. Only the projection $(K \cap C)|L$ is observable. For such projections of sections with cylinders we state a general integral geometric formula, restricted to the global case.

Theorem 6.3.2 (Projected thick sections). *Let $q \in \{0, \dots, d-1\}$ and $j \in \{0, \dots, q\}$. If $K \in \mathcal{K}'$ is a convex body and C is a cylinder with direction space $L \in G(d,q)$ and base M, then*

$$\int_{SO_d} \int_{L^\perp} V_j((K \cap \vartheta(C+x))|\vartheta L)\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$
$$= \sum_{k=j}^{d+j-q} c_{j,q-j,d}^{k,d-k,q} V_k(K) V_{d-k+j-q}(M).$$

Proof. First, the double integral in the assertion is again written in the form

$$I_j := \int_{SO_d} \int_{L^\perp} V_j(((\vartheta K + x) \cap (M+L))|L)\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta).$$

Since

$$((\vartheta K + x) \cap (M+L))|L = (\vartheta K - M + x) \cap L,$$

we get

$$I_j = \int_{SO_d} \int_{L^\perp} V_j((\vartheta K - M + x) \cap L)\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta).$$

We put $\vartheta K - M =: C$ and $B^d \cap L =: B^q$ and let $\epsilon > 0$. Using Fubini's theorem, the Steiner formula (14.5) and the invariance properties of the Lebesgue measure, we obtain

$$\begin{aligned} V_d(C + \epsilon B^q) &= \int_{L^\perp} V_q((C + \epsilon B^q) \cap (L+y))\, \lambda_{d-q}(\mathrm{d}y) \\ &= \int_{L^\perp} V_q((C \cap (L+y)) + \epsilon B^q)\, \lambda_{d-q}(\mathrm{d}y) \\ &= \int_{L^\perp} \sum_{j=0}^{q} \epsilon^{q-j} \kappa_{q-j} V_j(C \cap (L+y))\, \lambda_{d-q}(\mathrm{d}y) \\ &= \sum_{j=0}^{q} \epsilon^{q-j} \kappa_{q-j} \int_{L^\perp} V_j((C+x) \cap L)\, \lambda_{d-q}(\mathrm{d}x). \end{aligned}$$

Inserting $C = \vartheta K - M$ and integrating over SO_d, we get

$$\begin{aligned} \sum_{j=0}^{q} \epsilon^{q-j} \kappa_{q-j} I_j &= \int_{SO_d} V_d(\vartheta K - M + \epsilon B^q)\, \nu(\mathrm{d}\vartheta) \\ &= \sum_{k=0}^{d} c_{0,d}^{d-k,k} V_k(K) V_{d-k}(-M + \epsilon B^q) \\ &= \sum_{k=0}^{d} c_{0,d}^{d-k,k} V_k(K) \sum_{r=0}^{d-k} V_r(-M) V_{d-k-r}(B^q) \epsilon^{d-k-r} \end{aligned}$$

$$= \sum_{j=0}^{q} \sum_{k=j}^{d+j-q} c_{0,d}^{d-k,k} V_k(K) V_{d-k+j-q}(M) V_{q-j}(B^q) \epsilon^{q-j}.$$

Here we have used Theorem 6.1.1 and Lemma 14.2.1. Since

$$V_{q-j}(B^q) = \binom{q}{j} \frac{\kappa_q}{\kappa_j},$$

a comparison of the coefficients yields the assertion. □

Notes for Section 6.3

1. Kinematic formulas for cylinders were treated by Santaló [662, p. 270 ff]. The local kinematic formula for a fixed convex body and a moving cylinder (Theorem 6.3.1) was proved in Schneider [680].

2. Theorem 6.3.2 and its proof are taken from Schneider [681].

6.4 Translative Integral Geometry, Continued

Our proof of the local principal kinematic formula, Theorem 5.3.2, was preceded by a translative version, Theorem 5.2.3. This translative formula involves a series of mixed measures $\Phi_k^{(j)}(K, M; \cdot)$, which are measures on the product space $\mathbb{R}^d \times \mathbb{R}^d$, depending homogeneously (of degrees k and $d-k+j$, respectively) and additively on the convex bodies K and M. In the following, we continue the investigation of translative formulas and consider iterations, rotation means and Crofton-type results for the mixed measures. In contrast to the iterated kinematic formula of Theorem 5.1.5, the iteration of the translative formula of Theorem 5.2.3 involves new functions at each iteration step. Altogether, a series of **mixed measures** is required, which depend on an increasing number of convex bodies. The total mixed measures define **mixed functionals**, which generalize the intrinsic volumes of one body and the mixed volumes of two convex bodies. We start with the definition of the mixed measures.

In order to simplify the presentation within this section, we frequently abbreviate the translate $A + x$ of a set A by A^x.

For polytopes $P_1, \dots, P_k$ and faces F_i of P_i $(i = 1, \dots, k)$ with

$$\sum_{i=1}^{k} \dim F_i \geq (k-1)d,$$

we define the **common external angle** $\gamma(F_1, \dots, F_k; P_1, \dots, P_k)$ by

$$\gamma(F_1, \dots, F_k; P_1, \dots, P_k) := \gamma(F_1 \cap F_2^{x_2} \cap \dots \cap F_k^{x_k}, P_1 \cap P_2^{x_2} \cap \dots \cap P_k^{x_k}),$$

where $x_2, \dots, x_k \in \mathbb{R}^d$ are chosen so that the sets $F_1, F_2^{x_2}, \dots, F_k^{x_k}$ have relatively interior points in common. The common external angle does not depend on the choice of $x_2, \dots, x_k$.

Definition 6.4.1. *Let*

$$\begin{gathered} k \in \mathbb{N}, \quad j \in \{0, \dots, d\}, \quad m_1, \dots, m_k \in \{j, \dots, d\}, \\ j = \sum_{i=1}^k m_i - (k-1)d. \end{gathered} \tag{6.13}$$

For polytopes $K_1, \dots, K_k \in \mathcal{P}'$, the **mixed measure** $\Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; \cdot)$ *is the measure on $(\mathbb{R}^d)^k$ defined by*

$$\begin{aligned} &\Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; A_1 \times \dots \times A_k) \\ &:= \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \gamma(F_1, \dots, F_k; K_1, \dots, K_k) \\ &\quad \times [F_1, \dots, F_k] \lambda_{F_1}(A_1) \cdots \lambda_{F_k}(A_k) \end{aligned} \tag{6.14}$$

for $A_1, \dots, A_k \in \mathcal{B}(\mathbb{R}^d)$.

Convention. Integers k, j and k-tuples $(m_1, \dots, m_k)$ occurring in this section are always assumed to satisfy (6.13).

Obviously, the case $k = 1$ of (6.14) reduces to the representations of the curvature measures of polytopes given by (14.13), thus

$$\Phi^{(j)}_j(K; \cdot) = \Phi_j(K, \cdot).$$

The case $k = 2$ reduces to the mixed measures introduced in Theorem 5.2.2.

First we collect the essential properties of the mixed measures and state the iterated translative formula.

Theorem 6.4.1. *The mixed measure $\Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; \cdot)$ depends continuously on the polytopes $K_1, \dots, K_k$ (in the weak topology). It has a (unique) continuous extension to arbitrary convex bodies $K_1, \dots, K_k \in \mathcal{K}'$. The extended measures have the following properties, valid for all $K_1, \dots, K_k \in \mathcal{K}'$ and $A_1, \dots, A_k \in \mathcal{B}(\mathbb{R}^d)$.*

(a) **Symmetry:**

$$\begin{aligned} &\Phi^{(j)}_{m_{i_1},\dots,m_{i_k}}(K_{i_1}, \dots, K_{i_k}; A_{i_1} \times \dots \times A_{i_k}) \\ &= \Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; A_1 \times \dots \times A_k) \end{aligned}$$

for every permutation $(i_1, \dots, i_k)$ of $(1, \dots, k)$.

(b) **Decomposability:**

$$\begin{aligned}&\Phi^{(j)}_{m_1,\dots,m_{k-1},d}(K_1,\dots,K_{k-1},K_k;\cdot)\\&=\Phi^{(j)}_{m_1,\dots,m_{k-1}}(K_1,\dots,K_{k-1};\cdot)\otimes(\lambda\llcorner K_k).\end{aligned}$$

For $m_1,\dots,m_k<d$, *the measure* $\Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\cdot)$ *is concentrated on* $\mathrm{bd}\,K_1\times\ldots\times\mathrm{bd}\,K_k$.

(c) **Homogeneity:**

$$\begin{aligned}&\Phi^{(j)}_{m_1,\dots,m_k}(\alpha K_1,K_2,\dots,K_k;\alpha A_1\times A_2\times\ldots\times A_k)\\&=\alpha^{m_1}\Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;A_1\times\ldots\times A_k)\end{aligned}$$

for $\alpha\geq 0$.

(d) **Additivity:** *The measure* $\Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\cdot)$ *is additive in each of its arguments* $K_1,\dots,K_k$.

(e) **Local determination:** *The measure* $\Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\cdot)$ *is locally determined, that is, for an open set* $U\subset(\mathbb{R}^d)^k$ *and for* $M_1,\dots,M_k\in\mathcal{K}'$ *with* $K_1\times\ldots\times K_k\cap U=M_1\times\ldots\times M_k\cap U$, *we have*

$$\Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\cdot)=\Phi^{(j)}_{m_1,\dots,m_k}(M_1,\dots,M_k;\cdot)$$

on U.

The following **iterated translative formula** *holds:*

$$\begin{aligned}&\int_{(\mathbb{R}^d)^{k-1}}\Phi_j(K_1\cap K_2^{x_2}\cap\ldots\cap K_k^{x_k},A_1\cap A_2^{x_2}\cap\ldots\cap A_k^{x_k})\,\lambda^{k-1}(\mathrm{d}(x_2,\dots,x_k))\\&=\sum_{\substack{m_1,\dots,m_k=j\\m_1+\ldots+m_k=(k-1)d+j}}^{d}\Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;A_1\times\ldots\times A_k).\end{aligned}\tag{6.15}$$

Proof. Concerning (6.15), the measurability of the integrand on the left side follows from the obvious extension of Lemma 5.2.1. We now show first that (6.15) holds for polytopes $K_1,\dots,K_k$, by using induction on k. For $k=1$, (6.15) is trivial, and for $k=2$ it reduces to Theorem 5.2.2. For $k\geq 3$, the induction hypothesis, Theorem 5.2.2 and Lemma 14.1.1 yield

$$\begin{aligned}&\int_{(\mathbb{R}^d)^{k-1}}\Phi_j(K_1\cap K_2^{x_2}\cap\ldots\cap K_k^{x_k},A_1\cap A_2^{x_2}\cap\ldots\cap A_k^{x_k})\,\lambda^{k-1}(\mathrm{d}(x_2,\dots,x_k))\\&=\sum_{\substack{m_1,\dots,m_{k-2},m=j\\m_1+\ldots+m_{k-2}+m=(k-2)d+j}}^{d}\int_{\mathbb{R}^d}\Phi^{(j)}_{m_1,\dots,m_{k-2},m}(K_1,\dots,K_{k-2},K_{k-1}\cap K_k^{x};\end{aligned}$$

$$A_1 \times \ldots \times A_{k-2} \times (A_{k-1} \cap A_k^x))\, \lambda(\mathrm{d}x)$$

$$= \sum_{\substack{m_1,\ldots,m_{k-2},m=j \\ m_1+\ldots+m_{k-2}+m=(k-2)d+j}}^{d} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_{k-2} \in \mathcal{F}_{m_{k-2}}(K_{k-2})}$$

$$\int_{\mathbb{R}^d} \sum_{F \in \mathcal{F}_m(K_{k-1} \cap K_k^x)} \gamma(F_1,\ldots,F_{k-2},F;K_1,\ldots,K_{k-2},K_{k-1} \cap K_k^x)$$

$$\times [F_1,\ldots,F_{k-2},F] \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-2}}(A_{k-2}) \lambda_F(A_{k-1} \cap A_k^x)\, \lambda(\mathrm{d}x)$$

$$= \sum_{\substack{m_1,\ldots,m_k=j \\ m_1+\ldots+m_k=(k-1)d+j}}^{d} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-2}}(A_{k-2})$$

$$\times \int_{\mathbb{R}^d} \gamma(F_1,\ldots,F_{k-2},F_{k-1} \cap F_k^x;K_1,\ldots,K_{k-2},K_{k-1} \cap K_k^x)$$

$$\times [F_1,\ldots,F_{k-2},F_{k-1} \cap F_k^x] \lambda_{F_{k-1} \cap F_k^x}(A_{k-1} \cap A_k^x)\, \lambda(\mathrm{d}x)$$

$$= \sum_{\substack{m_1,\ldots,m_k=j \\ m_1+\ldots+m_k=(k-1)d+j}}^{d} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \gamma(F_1,\ldots,F_k;K_1,\ldots,K_k)$$

$$\times [F_1,\ldots,F_{k-2},L(F_{k-1}) \cap L(F_k)] \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-2}}(A_{k-2})$$

$$\times \int_{\mathbb{R}^d} \lambda_{F_{k-1} \cap F_k^x}(A_{k-1} \cap A_k^x)\, \lambda(\mathrm{d}x)$$

$$= \sum_{\substack{m_1,\ldots,m_k=j \\ m_1+\ldots+m_k=(k-1)d+j}}^{d} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \gamma(F_1,\ldots,F_k;K_1,\ldots,K_k)$$

$$\times [F_1,\ldots,F_{k-2},L(F_{k-1}) \cap L(F_k)][F_{k-1},F_k]$$

$$\times \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-2}}(A_{k-2}) \lambda_{F_{k-1}}(A_{k-1}) \lambda_{F_k}(A_k)$$

$$= \sum_{\substack{m_1,\ldots,m_k=j \\ m_1+\ldots+m_k=(k-1)d+j}}^{d} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \gamma(F_1,\ldots,F_k;K_1,\ldots,K_k)$$

$$\times [F_1,\ldots,F_k] \lambda_{F_1}(A_1) \cdots \lambda_{F_k}(A_k)$$

$$= \sum_{\substack{m_1,\ldots,m_k=j \\ m_1+\ldots+m_k=(k-1)d+j}}^{d} \Phi^{(j)}_{m_1,\ldots,m_k}(K_1,\ldots,K_k;A_1 \times \ldots \times A_k).$$

The integral formula (6.15) is thus established for polytopes.

We now extend (6.15), and thus the mixed measures, to arbitrary convex bodies, employing approximation by polytopes. For this purpose, we first remark that (6.15), for all Borel sets $A_1,\ldots,A_k$, is equivalent to

$$\int_{(\mathbb{R}^d)^{k-1}} \int_{\mathbb{R}^d} f(x_1, x_1 - x_2, \dots, x_1 - x_k)\, \Phi_j(K_1 \cap K_2^{x_2} \cap \dots \cap K_k^{x_k}, \mathrm{d}x_1)$$
$$\times \lambda^{k-1}(\mathrm{d}(x_2, \dots, x_k)) \tag{6.16}$$
$$= \sum_{\substack{m_1, \dots, m_k = j \\ m_1 + \dots + m_k = (k-1)d+j}}^{d} \int_{(\mathbb{R}^d)^k} f(x_1, \dots, x_k)\, \Phi^{(j)}_{m_1, \dots, m_k}(K_1, \dots, K_k; \mathrm{d}(x_1, \dots, x_k))$$

for all continuous functions f on $(\mathbb{R}^d)^k$ (provided that the mixed measures exist). For $k = 2$, this equivalence is explained at the beginning of the proof of Theorem 5.2.3; the general case follows analogously. Hence, (6.15) and (6.16) are valid if $K_1, \dots, K_k$ are polytopes. As in the proof of Theorem 5.2.3, we consider the functional

$$J(f, K_1, \dots, K_k)$$
$$:= \int_{(\mathbb{R}^d)^{k-1}} \int_{\mathbb{R}^d} f(x_1, x_1 - x_2, \dots, x_1 - x_k)\, \Phi_j(K_1 \cap K_2^{x_2} \cap \dots \cap K_k^{x_k}, \mathrm{d}x_1)$$
$$\times \lambda^{k-1}(\mathrm{d}(x_2, \dots, x_k))$$

and obtain that J depends continuously on $K_1, \dots, K_k$. For $r_1, \dots, r_k > 0$, we define a continuous mapping $D_{r_1, \dots, r_k}$ from $(\mathbb{R}^d)^k$ into itself by

$$D_{r_1, \dots, r_k}(x_1, \dots, x_k) := \left(\frac{x_1}{r_1}, \dots, \frac{x_k}{r_k} \right) \qquad \text{for } x_1, \dots, x_k \in \mathbb{R}^d.$$

For polytopes $K_1, \dots, K_k$, relation (6.16) and the definition of the mixed measures imply

$$D_{r_1, \dots, r_k} J(f, r_1 K_1, \dots, r_k K_k)$$
$$:= \int_{(\mathbb{R}^d)^{k-1}} \int_{\mathbb{R}^d} f\left(\frac{x_1}{r_1}, \frac{x_1 - x_2}{r_2}, \dots, \frac{x_1 - x_k}{r_k} \right)$$
$$\times \Phi_j(r_1 K_1 \cap (r_2 K_2)^{x_2} \cap \dots \cap (r_k K_k)^{x_k}, \mathrm{d}x_1)\, \lambda^{k-1}(\mathrm{d}(x_2, \dots, x_k))$$
$$= \sum_{\substack{m_1, \dots, m_k = j \\ m_1 + \dots + m_k = (k-1)d+j}}^{d}$$
$$\int_{(\mathbb{R}^d)^k} f\left(\frac{x_1}{r_1}, \dots, \frac{x_k}{r_k} \right) \Phi^{(j)}_{m_1, \dots, m_k}(r_1 K_1, \dots, r_2 K_k; \mathrm{d}(x_1, \dots, x_k))$$
$$= \sum_{\substack{m_1, \dots, m_k = j \\ m_1 + \dots + m_k = (k-1)d+j}}^{d} r_1^{m_1} \cdots r_k^{m_k}$$
$$\times \int_{(\mathbb{R}^d)^k} f(x_1, \dots, x_k)\, \Phi^{(j)}_{m_1, \dots, m_k}(K_1, \dots, K_k; \mathrm{d}(x_1, \dots, x_k)).$$

For arbitrary convex bodies $K_1, \dots, K_k$, we choose sequences of polytopes $K_{1i}, \dots K_{ki}$ ($i \in \mathbb{N}$) such that $K_{1i} \to K_1, \dots, K_{ki} \to K_k$ for $i \to \infty$. Then

$$D_{r_1,\dots,r_k} J(f, r_1 K_{1i}, \dots, r_k K_{ki}) \to D_{r_1,\dots,r_k} J(f, r_1 K_1, \dots, r_k K_k)$$

for every continuous function f on $(\mathbb{R}^d)^k$ and all $r_1, \dots, r_k > 0$. From the polynomial expansion just established, we deduce the convergence of the coefficients

$$\int_{(\mathbb{R}^d)^k} f(x_1, \dots, x_k)\, \Phi^{(j)}_{m_1,\dots,m_k}(K_{1i}, \dots, K_{ki}; \mathrm{d}(x_1, \dots, x_k))$$

and thus the weak convergence of the measures

$$\Phi^{(j)}_{m_1,\dots,m_k}(K_{1i}, \dots, K_{ki}; \cdot)$$

for $i \to \infty$. The limits, denoted by $\Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; \cdot)$, are again finite measures, satisfying

$$\begin{aligned} &D_{r_1,\dots,r_k} J(f, r_1 K_1, \dots, r_k K_k) \\ &= \sum_{\substack{m_1,\dots,m_k=j \\ m_1+\dots+m_k=(k-1)d+j}}^{d} r_1^{m_1} \cdots r_k^{m_k} \\ &\quad \times \int_{(\mathbb{R}^d)^k} f(x_1, \dots, x_k)\, \Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; \mathrm{d}(x_1, \dots, x_k)), \end{aligned} \tag{6.17}$$

from which we see that they are independent of the approximating sequences $(K_{1i})_{i\in\mathbb{N}}, \dots, (K_{ki})_{i\in\mathbb{N}}$. For $r_1 = \dots = r_k = 1$, we obtain (6.16).

Thus, mixed measures for arbitrary bodies $K_1, \dots, K_k$ are defined which fulfill (6.15). Moreover, properties (a), (b) and (c), which follow for polytopes $K_1, \dots, K_k$ from the definition, transfer to general convex bodies by means of approximation and an application of (6.17). Also, (6.17) shows that $\Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; \cdot)$ depends continuously on the bodies $K_1, \dots, K_k$, and (d) can be deduced from the corresponding additivity property of curvature measures, similarly to the proof of Theorem 5.2.3. To prove (e), suppose that its assumptions are satisfied. Without loss of generality, we may assume that $U = U_1 \times \dots \times U_k$ with open sets $U_1, \dots, U_k \subset \mathbb{R}^d$. Then, for $r_1, \dots, r_k > 0$ and $x_2, \dots, x_k \in \mathbb{R}^d$, the set

$$r_1 K_1 \cap (r_2 K_2)^{x_2} \cap \dots \cap (r_k K_k)^{x_k} \cap r_1 U_1 \cap (r_2 U_2)^{x_2} \cap \dots \cap (r_k U_k)^{x_k}$$

remains the same if K_i is replaced by M_i, $i = 1, \dots, k$. Since, by Theorem 14.2.3, the curvature measures are locally determined, the value

$$\Phi_j(r_1 K_1 \cap (r_2 K_2)^{x_2} \cap \dots \cap (r_k K_k)^{x_k}, r_1 A_1 \cap (r_2 A_2)^{x_2} \cap \dots \cap (r_k A_k)^{x_k}),$$

for Borel sets $A_i \subset U_i$, does not change if K_i is replaced by M_i. Let f be a continuous function on $(\mathbb{R}^d)^k$ with support in U. Then the case $r_1 = \ldots = r_k = 1$ shows that the left side of (6.16) does not change if K_i is replaced by M_i. More generally, we obtain

$$D_{r_1,\ldots,r_k} J(f, r_1 K_1, \ldots, r_k K_k) = D_{r_1,\ldots,r_k} J(f, r_1 M_1, \ldots, r_k M_k).$$

Therefore, the right side of (6.17) does not change if K_i is replaced by M_i. This yields the assertion. □

For the total mixed measures, we introduce the notation

$$V^{(j)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k) := \Phi^{(j)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k; (\mathbb{R}^d)^k),$$

and we call these the **mixed functionals**. In particular,

$$V^{(j)}_j(K) = V_j(K),$$

and the case $k = 2$ reduces to the mixed functionals introduced in Theorem 5.2.3. If $K_1, \ldots, K_k$ are polytopes, then

$$\begin{aligned} &V^{(j)}_{m_1,\ldots,m_k}(K_1, \ldots, K_k) \\ &= \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_k \in \mathcal{F}_{m_k}(K_k)} \gamma(F_1, \ldots, F_k; K_1, \ldots, K_k) \\ &\quad \times [F_1, \ldots, F_k] V_{m_1}(F_1) \cdots V_{m_k}(F_k). \end{aligned} \tag{6.18}$$

Results on mixed measures contain results on mixed functionals as special cases. In the sequel, we therefore concentrate on mixed measures and mention mixed functionals only when their behavior deviates from that of mixed measures.

In view of the decomposability property (b) we can, in large parts of the following, concentrate on the case $k \le d$.

Since mixed measures are locally determined, we can extend them to unbounded closed convex sets $K_1, \ldots, K_k$. We shall use this extension, in particular, for linear or affine subspaces and for closed halfspaces. The representation (6.14) remains valid for polyhedral sets. It is important to note that also the integral geometric formulas for mixed measures obtained in this section extend in the same way. In fact, any unbounded convex set K_i in such a formula, with bounded corresponding Borel set A_i, can be replaced by the intersection of K_i with a cube (say) that contains A_i in its interior. This replacement does not affect the values of the involved mixed measures.

The next theorem collects some of the integral geometric formulas that hold for mixed measures.

Theorem 6.4.2. *For $k \in \mathbb{N}$, convex bodies $K_1, \ldots, K_k \in \mathcal{K}'$ and Borel sets $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^d)$, the mixed measures satisfy the* **translative formula**

$$\int_{\mathbb{R}^d} \Phi^{(j)}_{m_1,\ldots,m_{k-2},m}(K_1,\ldots,K_{k-2},K_{k-1}\cap K_k^x;$$
$$A_1\times\ldots\times A_{k-2}\times(A_{k-1}\cap A_k^x))\,\lambda(\mathrm{d}x)$$
$$= \sum_{\substack{m_{k-1},m_k=m\\ m_{k-1}+m_k=d+m}}^{d} \Phi^{(j)}_{m_1,\ldots,m_k}(K_1,\ldots,K_k;A_1\times\ldots\times A_k), \tag{6.19}$$

the **rotation formula**

$$\int_{SO_d} \Phi^{(j)}_{m_1,\ldots,m_{k-1},m}(K_1,\ldots,K_{k-1},\vartheta K_k;$$
$$A_1\times\ldots\times A_{k-1}\times\vartheta A_k)\,\nu(\mathrm{d}\vartheta) \tag{6.20}$$
$$= c_{d,j}^{m,d-m+j}\Phi^{(d-m+j)}_{m_1,\ldots,m_{k-1}}(K_1,\ldots,K_{k-1};A_1\times\ldots\times A_{k-1})\Phi_m(K_k,A_k),$$

and the **principal kinematic formula**

$$\int_{G_d} \Phi^{(j)}_{m_1,\ldots,m_{k-2},m}(K_1,\ldots,K_{k-2},K_{k-1}\cap gK_k;$$
$$A_1\times\ldots\times A_{k-2}\times(A_{k-1}\cap gA_k))\,\mu(\mathrm{d}g)$$
$$= \sum_{r=m}^{d} c_{d,j}^{d-r+m,j-m+r}\Phi^{(j-m+r)}_{m_1,\ldots,m_{k-2},r}(K_1,\ldots,K_{k-1};A_1\times\ldots\times A_{k-1})$$
$$\times\,\Phi_{d-r+m}(K_k,A_k). \tag{6.21}$$

Proof. It is sufficient to prove the results for polytopes; the general case then follows by approximation, using arguments similar to those of the previous proof.

For polytopes, (6.19) was obtained during the proof of (6.15). In the case of (6.20), we use the definition of the mixed measures, Lemma 14.1.1, and Theorem 5.3.1 to get

$$\int_{SO_d} \Phi^{(j)}_{m_1,\ldots,m_{k-1},m}(K_1,\ldots,K_{k-1},\vartheta K_k;A_1\times\ldots\times A_{k-1}\times\vartheta A_k)\,\nu(\mathrm{d}\vartheta)$$
$$= \sum_{F_1\in\mathcal{F}_{m_1}(K_1)}\cdots\sum_{F_{k-1}\in\mathcal{F}_{m_{k-1}}(K_{k-1})}\sum_{F\in\mathcal{F}_m(K_k)} \lambda_{F_1}(A_1)\cdots\lambda_{F_{k-1}}(A_{k-1})\lambda_F(A_k)$$
$$\times\int_{SO_d}\gamma(F_1,\ldots,F_{k-1},\vartheta F;K_1,\ldots,K_{k-1},\vartheta K_k)[F_1,\ldots,F_{k-1},\vartheta F]\,\nu(\mathrm{d}\vartheta)$$
$$= \sum_{F_1\in\mathcal{F}_{m_1}(K_1)}\cdots\sum_{F_{k-1}\in\mathcal{F}_{m_{k-1}}(K_{k-1})}\sum_{F\in\mathcal{F}_m(K_k)} \lambda_{F_1}(A_1)\cdots\lambda_{F_{k-1}}(A_{k-1})\lambda_F(A_k)$$
$$\times[F_1,\ldots,F_{k-1}]\int_{SO_d}\gamma(G,\vartheta F;M,\vartheta K_k)[G,\vartheta F]\,\nu(\mathrm{d}\vartheta)$$

$$= c_{d,j}^{m,d-m+j} \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_{k-1} \in \mathcal{F}_{m_{k-1}}(K_{k-1})} [F_1, \ldots, F_{k-1}]$$

$$\times \gamma(G, M) \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-1}}(A_{k-1}) \sum_{F \in \mathcal{F}_m(K_k)} \gamma(F, K_k) \lambda_F(A_k)$$

$$= c_{d,j}^{m,d-m+j} \Phi_{m_1,\ldots,m_{k-1}}^{(d-m+j)}(K_1, \ldots, K_{k-1}; A_1 \times \ldots \times A_{k-1}) \Phi_m(K_k, A_k),$$

where $G := F_1 \cap F_2^{x_2} \cap \ldots \cap F_{k-1}^{x_{k-1}}$, $M := K_1 \cap K_2^{x_2} \cap \ldots \cap K_{k-1}^{x_{k-1}}$ and $x_2, \ldots, x_{k-1}$ are suitably chosen vectors.

Finally, (6.21) follows immediately by combining (6.19) and (6.20). □

Further formulas can be obtained by iteration. In particular, there is an iterated translative formula for mixed measures.

We present next a Crofton formula for the mixed measures, first in a translative version and then in its kinematic form.

Theorem 6.4.3. *For convex bodies $K_1, \ldots, K_k \in \mathcal{K}'$, Borel sets $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^d)$, a subspace $L \in G(d, q)$ with $q \in \{m, \ldots, d-1\}$, and any Borel set $A_L \subset L$ with $\lambda_q(A_L) = 1$, the mixed measures satisfy the* **translative Crofton formula**

$$\int_{L^\perp} \Phi_{m_1,\ldots,m_{k-1},m}^{(j)}(K_1, \ldots, K_{k-1}, K_k \cap L^x;$$
$$A_1 \times \ldots \times A_{k-1} \times (A_k \cap L^x))\, \lambda_{d-q}(\mathrm{d}x)$$
$$= \Phi_{m_1,\ldots,m_{k-1},d-q+m,q}^{(j)}(K_1, \ldots, K_k, L; A_1 \times \ldots \times A_k \times A_L) \quad (6.22)$$

and the **Crofton formula**

$$\int_{A(d,q)} \Phi_{m_1,\ldots,m_{k-1},m}^{(j)}(K_1, \ldots, K_{k-1}, K_k \cap E;$$
$$A_1 \times \ldots \times A_{k-1} \times (A_k \cap E))\, \mu_q(\mathrm{d}E)$$
$$= c_{d,j}^{q,d-q+j} \Phi_{m_1,\ldots,m_{k-1},d-q+m}^{(d-q+j)}(K_1, \ldots, K_k; A_1 \times \ldots \times A_k). \quad (6.23)$$

Proof. For the proof of (6.22) we may again concentrate on the case of polytopes. Moreover, we can assume that the faces of the polytope K_k and the subspace L are in general position. This implies that the m-dimensional faces of $K_k \cap L^x$, for $x \in L^\perp$, are intersections $F \cap L^x$ of $(d-q+m)$-dimensional faces F of K_k, at least for those x for which the sets intersect at relatively interior points.

Using this observation, we can proceed, in large parts, similarly to the proof of (6.15) and get

$$\int_{L^\perp} \Phi_{m_1,\ldots,m_{k-1},m}^{(j)}(K_1, \ldots, K_{k-1}, K_k \cap L^x;$$

$$A_1 \times \ldots \times A_{k-1} \times (A_k \cap L^x))\, \lambda_{d-q}(\mathrm{d}x)$$

$$= \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_{k-1} \in \mathcal{F}_{m_{k-1}}(K_{k-1})} \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-1}}(A_{k-1})$$

$$\times \int_{L^\perp} \sum_{F \in \mathcal{F}_m(K_k \cap L^x)} \gamma(F_1, \ldots, F_{k-1}, F; K_1, \ldots, K_{k-1}, K_k \cap L^x)$$

$$\times [F_1, \ldots, F_{k-1}, F] \lambda_F(A_k \cap L^x)\, \lambda_{d-q}(\mathrm{d}x)$$

$$= \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_{k-1} \in \mathcal{F}_{m_{k-1}}(K_{k-1})} \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-1}}(A_{k-1})$$

$$\times \sum_{F \in \mathcal{F}_{d-q+m}(K_k)} \int_{L^\perp} \gamma(F_1, \ldots, F_{k-1}, F \cap L^x; K_1, \ldots, K_{k-1}, K_k \cap L^x)$$

$$\times [F_1, \ldots, F_{k-1}, F \cap L^x] \lambda_{F \cap L^x}(A_k \cap L^x)\, \lambda_{d-q}(\mathrm{d}x)$$

$$= \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_{k-1} \in \mathcal{F}_{m_{k-1}}(K_{k-1})} \sum_{F \in \mathcal{F}_{d-q+m}(K_k)}$$

$$\gamma(F_1, \ldots, F_{k-1}, F, L; K_1, \ldots, K_{k-1}, K_k, L)[F_1, \ldots, F_{k-1}, L(F) \cap L]$$

$$\times \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-1}}(A_{k-1}) \int_{L^\perp} \lambda_{F \cap L^x}(A_k \cap L^x)\, \lambda_{d-q}(\mathrm{d}x)$$

$$= \sum_{F_1 \in \mathcal{F}_{m_1}(K_1)} \cdots \sum_{F_{k-1} \in \mathcal{F}_{m_{k-1}}(K_{k-1})} \sum_{F \in \mathcal{F}_{d-q+m}(K_k)}$$

$$\gamma(F_1, \ldots, F_{k-1}, F, L; K_1, \ldots, K_{k-1}, K_k, L)[F_1, \ldots, F_{k-1}, L(F) \cap L]$$

$$\times [F, L] \lambda_{F_1}(A_1) \cdots \lambda_{F_{k-1}}(A_{k-1}) \lambda_F(A_k)$$

$$= \Phi^{(j)}_{m_1, \ldots, m_{k-1}, d-q+m, q}(K_1, \ldots, K_k, L; A_1 \times \ldots \times A_k \times A_L).$$

The Crofton formula (6.23) is a direct consequence of (6.22) and the rotation formula (6.20). □

Remark on extension to the convex ring. Since the mixed measure $\Phi^{(j)}_{m_1,\ldots,m_k}$ is additive and weakly continuous in each of its first k arguments, it has a unique additive extension to the convex ring. As in Section 5.1, the integral geometric formulas for mixed measures obtained so far in this section remain valid if the involved convex bodies are replaced by polyconvex sets. The arguments explained at the end of Section 5.2 can easily be adapted to the present situation.

By specializing some of the integral geometric formulas, we obtain useful information about mixed measures and functionals. For that, we assume one of the bodies to be the unit ball $B^d \subset \mathbb{R}^d$. If we put $K_k = B^d$ and $A_k = \mathbb{R}^d$ in (6.20) and insert the value of $V_m(B^d)$ given by (14.8), then we obtain

$$\Phi^{(j)}_{m_1,\dots,m_{k-1},m}(K_1,\dots,K_{k-1},B^d;A_1\times\dots\times A_{k-1}\times\mathbb{R}^d)$$
$$=\frac{1}{m!}c^{m,d-m+j}_{j,d-m}\Phi^{(d-m+j)}_{m_1,\dots,m_{k-1}}(K_1,\dots,K_{k-1};A_1\times\dots\times A_{k-1}). \quad (6.24)$$

The following result is a consequence of (6.24).

Theorem 6.4.4. *For $K_1,\dots,K_k\in\mathcal{K}'$ and $A_1,\dots,A_k\in\mathcal{B}(\mathbb{R}^d)$, the mixed measures satisfy the* **reduction property**

$$\Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;A_1\times\dots\times A_k)$$
$$=\frac{1}{\kappa_{d-j}}\Phi^{(0)}_{m_1,\dots,m_k,d-j}(K_1,\dots,K_k,B^d;A_1\times\dots\times A_k\times\mathbb{R}^d)$$
$$=\left(\frac{2}{\kappa_{d-1}}\right)^j\frac{1}{j!\kappa_j}\,\Phi^{(0)}_{m_1,\dots,m_k,d-1,\dots,d-1}(K_1,\dots,K_k,\underbrace{B^d,\dots,B^d}_{j};$$
$$A_1\times\dots\times A_k\times(\mathbb{R}^d)^j).$$

Proof. The first equation is obtained from (6.24) if m,j,k are replaced by $d-j,0,k$, respectively. For the second, we put $m=d-1$ and replace $k-1$ by k and j by $j-1$ in (6.24). This gives

$$\Phi^{(j-1)}_{m_1,\dots,m_k,d-1}(K_1,\dots,K_k,B^d;A_1\times\dots\times A_k\times\mathbb{R}^d)$$
$$=\frac{1}{(d-1)!}\,c^{j,d-1}_{j-1,1}\,\Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;A_1\times\dots\times A_k).$$

The assertion is now obtained by j-fold iteration. □

It follows from this result that all mixed measures can be reduced to the (series of) measures $\Phi^{(0)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\cdot)$, where $k\in\{1,\dots,d\}$ and $m_1,\dots,m_k\in\{1,\dots,d-1\}$ satisfy $m_1+\dots+m_k=(k-1)d$.

As a consequence of the Crofton formula, we note a connection between the mixed functionals $V^{(j)}_{k,d-k+j}(K,M)$ of two convex bodies K,M and mixed volumes. It already follows from (5.16) and Corollary 5.2.1 that

$$V^{(0)}_{k,d-k}(K,M)=\binom{d}{k}V(K[k],-M[d-k]). \quad (6.25)$$

Combining (6.25) with the Crofton formula (6.23), we immediately get a representation of the mixed functionals $V^{(j)}_{k,d-k+j}(K,M)$ as Crofton-type integrals of mixed volumes.

Theorem 6.4.5. *Let $K,M\in\mathcal{K}'$. If $j\in\{0,\dots,d-2\}$ and $k\in\{j+1,\dots,d-1\}$, then*

$$V^{(j)}_{k,d-k+j}(K,M) \quad (6.26)$$
$$=\binom{d}{k-j}c^{d,0}_{j,d-j}\int_{A(d,d-j)}V((K\cap E)[k-j],-M[d-k+j])\,\mu_{d-j}(\mathrm{d}E).$$

Our next aim is the derivation of a translative integral formula for support functions. It can be deduced from a translative formula for special mixed measures. As in (4.42), we replace the support function $h(K,\cdot)$ by its centered version $h^*(K,\cdot)$. Here, a continuous function f on S^{d-1} is **centered** if

$$\int_{S^{d-1}} f(u)u\,\sigma(\mathrm{d}u) = 0.$$

The centered support function of K is invariant under translations of K. The following lemma establishes a connection between the centered support function and a special mixed measure. We use the notation

$$u^+ := \{x \in \mathbb{R}^d : \langle x, u\rangle \geq 0\}$$

for the closed halfspace with 0 in the boundary and inner normal vector $u \in S^{d-1}$.

Lemma 6.4.1. *Let $P \in \mathcal{P}'$ be a polytope, and let $u \in S^{d-1}$. Then*

$$h^*(P,u) = \sum_{F\in\mathcal{F}_1(P)} \gamma(F,u^\perp;P,u^+)[F,u^\perp]\lambda_1(F). \tag{6.27}$$

Let $K \in \mathcal{K}'$, and let $A_{u^\perp} \subset u^\perp$ be a Borel set with $\lambda_{d-1}(A_{u^\perp}) = 1$. Then

$$h^*(K,u) = \Phi^{(0)}_{1,d-1}(K,u^+;\mathbb{R}^d \times A_{u^\perp}). \tag{6.28}$$

Proof. For a vertex e of P, we do not distinguish between the vector e and the corresponding 0-face $\{e\}$. We use the relations

$$\sum_{e\in\mathcal{F}_0(P)} \gamma(e,P) = \chi(P) = 1,$$

which is obvious, and

$$\sum_{e\in\mathcal{F}_0(P)} \gamma(e,P)e = s(K),$$

which is given by (14.29). Writing

$$u_t^+ := \{x \in \mathbb{R}^d : \langle x,u\rangle \geq t\}, \qquad u_t^\perp := \{x \in \mathbb{R}^d : \langle x,u\rangle = t\}$$

for $u \in S^{d-1}$ and $t \in \mathbb{R}$, and choosing a number c with $P \subset u_c^+$, we get

$$\begin{aligned}
&h(P,u) - c\\
&= \int_c^\infty \chi(P\cap u_t^+)\,\mathrm{d}t\\
&= \int_c^\infty \sum_{e\in\mathcal{F}_0(P\cap u_t^+)} \gamma(e,P\cap u_t^+)\,\mathrm{d}t
\end{aligned}$$

$$= \int_c^\infty \sum_{e\in\mathcal{F}_0(P)} \gamma(e,P)\mathbf{1}\{\langle e,u\rangle \geq t\}\,\mathrm{d}t$$

$$+ \int_c^\infty \sum_{F\in\mathcal{F}_1(P)} \gamma(F,u^\perp;P,u^+)\chi(F\cap u_t^\perp)\,\mathrm{d}t$$

$$= \sum_{e\in\mathcal{F}_0(P)} \gamma(e,P)(\langle e,u\rangle - c) + \sum_{F\in\mathcal{F}_1(P)} \gamma(F,u^\perp;P,u^+)[F,u^\perp]\lambda_1(F)$$

$$= \langle s(P),u\rangle - c + \sum_{F\in\mathcal{F}_1(P)} \gamma(F,u^\perp;P,u^+)[F,u^\perp]\lambda_1(F).$$

This proves (6.27), and (6.28) for polytopes follows from (6.14) (extended to polyhedral sets).

For a polytope P, the definition (6.14) implies

$$\Phi^{(0)}_{1,d-1}(P,u^+;\cdot) = \left(\sum_{F\in\mathcal{F}_1(P)} \gamma(F,u^\perp;P,u^+)[F,u^\perp]\,\lambda_F\right)\otimes\lambda_{u^\perp}.$$

Using the weak continuity of the mixed measures, we conclude by approximation that also $\Phi^{(0)}_{1,d-1}(K,u^+;\cdot)$ for $K\in\mathcal{K}'$ is a product measure with $\lambda_{u^\perp}$ as second factor. Now (6.28) follows by approximation and continuity. □

We state a translative formula for centered support functions. Here we restrict ourselves to the case of two convex bodies; the extension to $k\geq 2$ bodies presents no additional difficulties.

Theorem 6.4.6. *For convex bodies $K,M\in\mathcal{K}'$, there exist continuous functions $h_1^*(K,M;\cdot),\dots,h_d^*(K,M;\cdot)$ on S^{d-1} such that*

$$\int_{\mathbb{R}^d} h^*(K\cap M^x,\cdot)\,\lambda(\mathrm{d}x) = \sum_{k=1}^d h_k^*(K,M;\cdot), \tag{6.29}$$

where $h_1^*(K,M;\cdot) = h^*(K,\cdot)V_d(M)$ and $h_d^*(K,M;\cdot) = V_d(K)h^*(M,\cdot)$. The function $h_k^*(K,M;\cdot)$ is centered and symmetric, in the sense that

$$h_k^*(K,M;\cdot) = h_{d+1-k}^*(M,K;\cdot),$$

it depends continuously on $K,M\in\mathcal{K}'$ and is homogeneous of degree k in K and of degree $d+1-k$ in M. Moreover, it is additive in each of its arguments K and M.

For polytopes K,M, we have

$$h_k^*(K,M;u) \tag{6.30}$$
$$= \sum_{F\in\mathcal{F}_k(K)}\sum_{G\in\mathcal{F}_{d+1-k}(M)} \gamma(F,G,u^\perp;K,M,u^+)[F,G,u^\perp]\lambda_k(F)\lambda_{d-k+1}(G).$$

Proof. Let $K, M \in \mathcal{K}'$, let $u \in S^{d-1}$. By (6.28),

$$h^*(K \cap M^x, u) = \Phi^{(0)}_{d-1,1}(u^+, K \cap M^x; A_{u^\perp} \times \mathbb{R}^d).$$

The translative formula (6.19) with $k = 3$, $K_1 = u^+$, $K_2 = K$, $K_3 = M$, $A_1 = A_{u^\perp}$, $A_2 = A_3 = \mathbb{R}^d$, $j = 0$, $m_1 = d-1$ and $m = 1$ gives (6.29) with

$$h^*_k(K, M; u) := \Phi^{(0)}_{d-1,k,d+1-k}(u^+, K, M; A_{u^\perp} \times \mathbb{R}^d \times \mathbb{R}^d) \tag{6.31}$$

for $k = 1, \dots, d$. The representation (6.30) in the case of polytopes follows from (6.14).

From (6.31) and the known properties of mixed measures we immediately obtain the assertions about symmetry, continuity in K, M, homogeneity, and additivity of $h^*_k(K, M; u)$. By the homogeneity property, we have

$$\int_{\mathbb{R}^d} h^*(rK \cap M^x, \cdot)\, \lambda(\mathrm{d}x) = \sum_{k=1}^{d} r^k h^*_k(K, M; \cdot)$$

for all $r \geq 0$. Inserting $r = 1, \dots, d$ and solving the resulting system of equations, we get a representation

$$h^*_k(K, M; \cdot) = \sum_{n=1}^{d} a_{kn} \int_{\mathbb{R}^d} h^*(nK \cap M^x, \cdot)\, \lambda(\mathrm{d}x)$$

with coefficients a_{kn} independent of K and M. From this representation, we see that $h^*_k(K, M; \cdot)$ is a continuous function. Inserting u, multiplying by u, integrating over S^{d-1}, and using Fubini's theorem, we also see that $h^*(K, M; \cdot)$ is centered. $\square$

We next derive a kinematic formula for support functions. For fixed $u \in \mathbb{R}^d$, the function $K \mapsto h^*(K, u)$ satisfies the assumptions of Theorem 5.1.2, hence we get

$$\int_{G_d} h^*(K \cap gM, \cdot)\, \mu(\mathrm{d}g) = \sum_{k=1}^{d} \left(\int_{A(d,k)} h^*(K \cap E, \cdot)\, \mu_k(\mathrm{d}E) \right) V_k(M) \tag{6.32}$$

(observing that $h^*(\{x\}, \cdot) = 0$). The coefficient of $V_k(M)$ is evidently a support function. We define, for $k \in \{1, \dots, d-1\}$, the kth **mean section body** $M_k(K)$ of a convex body $K \in \mathcal{K}'$ by

$$h(M_k(K), \cdot) := \int_{A(d,k)} h^*(K \cap E, \cdot)\, \mu_k(\mathrm{d}E).$$

We complement the definition by setting $h(M_d(K), \cdot) := h^*(K, \cdot)$, that is, $M_d(K) = K - s(K)$, and $h(M_0(K), \cdot) := 0$, thus $M_0(K) = \{0\}$. To obtain a connection with mixed measures, we choose $M = B^d$ in (6.32). Applying (6.29), (6.31) and Theorem 6.4.4, we get, for $u \in \mathbb{R}^d$,

$$\begin{aligned} h(M_k(K),u)V_k(B^d) &= \Phi^{(0)}_{d+1-k,k,d-1}(K,B^d,u^+;\mathbb{R}^d \times \mathbb{R}^d \times A_{u^\perp}) \\ &= \kappa_k \Phi^{(d-k)}_{d+1-k,d-1}(K,u^+;\mathbb{R}^d \times A_{u^\perp}). \qquad (6.33) \end{aligned}$$

If K is a polytope, an explicit form of the latter expression is obtained from (6.14). We collect the obtained results in the following theorem.

Theorem 6.4.7. *If $K, M \in \mathcal{K}'$, then*

$$\int_{G_d} h^*(K \cap gM, \cdot)\, \mu(\mathrm{d}g) = \sum_{k=1}^{d} h(M_k(K), \cdot)V_k(M),$$

where

$$h(M_k(K),u) = c^{d-k,k}_{d,0} \Phi^{(d-k)}_{d+1-k,d-1}(K,u^+;\mathbb{R}^d \times A_{u^\perp}).$$

If K is a polytope, then

$$h(M_k(K),u) = c^{d-k,k}_{d,0} \sum_{F \in \mathcal{F}_{d+1-k}(K)} \gamma(F,u^\perp)[F,u^\perp]\lambda_{d+1-k}(F). \qquad (6.34)$$

Finally, we use some of the obtained information on mixed measures to derive a kinematic and a Crofton formula for projection functions, that is, volumes of projections of convex bodies. Let $j \in \{1,\dots,d-1\}$ and $L \in G(d,j)$. For $K \in \mathcal{K}'$, the j-dimensional volume of the orthogonal projection $K|L$ defines the jth **projection function** $L \mapsto V_j(K|L)$. From (14.19) and (6.25) we have

$$V_j(K|L) = \binom{d}{j} V(K[j], B_{L^\perp}[d-j]) = V^{(0)}_{j,d-j}(K, B_{L^\perp})$$

where $B_{L^\perp} \subset L^\perp$ is a ball with $\lambda_{d-j}(B_{L^\perp}) = 1$. Therefore, (6.21) and (6.23) yield, for $K, M \in \mathcal{K}'$,

$$\int_{G_d} V_j((K \cap gM)|L)\, \mu(\mathrm{d}g) = \sum_{i=j}^{d} c^{d-i+j,i-j}_{d,0} V^{(i-j)}_{i,d-j}(K,B_{L^\perp})V_{d-i+j}(M),$$

$$\int_{A(d,d-i+j)} V_j((K \cap E)|L)\, \mu_{d-i+j}(\mathrm{d}E) = c^{d-i+j,i-j}_{d,0} V^{(i-j)}_{i,d-j}(K,B_{L^\perp}).$$

(Of course, if one of the two results is known, the other one can also be deduced from Theorem 5.1.2.) The mixed functionals appearing here can be expressed as Radon transforms of the projection function. The **Radon transform** R_{ij} : $\mathbf{C}(G(d,i)) \to \mathbf{C}(G(d,j))$ is defined by

$$(R_{ij}f)(L) := \int_{G(L,i)} f(M)\, \nu_i^L(\mathrm{d}M), \qquad L \in G(d,j). \tag{6.35}$$

In the following, we assume $i \in \{j+1, \dots, d\}$.

Using the symmetry of the mixed functionals together with (6.26) and denoting by $c_1, c_2, \dots$ constants depending only on d, i, j, we get

$$\begin{aligned} V_{i,d-j}^{(i-j)}(K, B_{L^\perp}) &= V_{d-j,i}^{(i-j)}(B_{L^\perp}, K) \\ &= c_1 \int_{A(d,d-i+j)} V((B_{L^\perp} \cap E)[d-i], K[i])\, \mu_{d-i+j}(\mathrm{d}E). \end{aligned}$$

The integrand, as a function of E, depends only on $E \cap L^\perp$. Therefore, we can use the integral geometric identity

$$\int_{A(d,d-i+j)} f(E \cap L^\perp)\, \mu_{d-i+j}(\mathrm{d}E) = c_2 \int_{A(L^\perp,d-i)} f(F)\, \mu_{d-i}^{L^\perp}(\mathrm{d}F),$$

which holds for all nonnegative measurable functions f on $A(L^\perp, d-i)$. Here $A(L^\perp, d-i)$ is the space of $(d-i)$-flats contained in $L^\perp$, and $\mu_{d-i}^{L^\perp}$ is its invariant measure (see Section 13.2). To prove the identity, we note that its left side, applied to indicator functions of Borel sets in $A(L^\perp, d-i)$, defines a measure on $A(L^\perp, d-i)$, which is locally finite and invariant under rigid motions of $L^\perp$ into itself and hence is a multiple of the invariant measure $\mu_{d-i}^{L^\perp}$. Thus, we obtain

$$\begin{aligned} &V_{i,d-j}^{(i-j)}(K, B_{L^\perp}) \\ &= c_3 \int_{A(L^\perp,d-i)} V((B_{L^\perp} \cap F)[d-i], K[i])\, \mu_{d-i}^{L^\perp}(\mathrm{d}F) \\ &= c_3 \int_{G(L^\perp,d-i)} \int_{H^\perp \cap L^\perp} V((B_{L^\perp} \cap H^x)[d-i], K[i])\, \lambda_{i-j}(\mathrm{d}x)\, \nu_{d-i}^{L^\perp}(\mathrm{d}H). \end{aligned}$$

Here we have

$$\int_{H^\perp \cap L^\perp} V((B_{L^\perp} \cap H^x)[d-i], K[i])\, \lambda_{i-j}(\mathrm{d}x) = c_4 V_i(K|H^\perp),$$

which follows from (14.19), since $B_{L^\perp} \cap H^x$, if not empty, is homothetic to $B_{L^\perp} \cap H$, with homothety factor depending only on $\|x\|$. We deduce that

$$\begin{aligned} V_{i,d-j}^{(i-j)}(K, B_{L^\perp}) &= c_5 \int_{G(L^\perp,d-i)} V_i(K|H^\perp)\, \nu_{d-i}^{L^\perp}(\mathrm{d}H) \\ &= c_5 \int_{G(L,i)} V_i(K|M)\, \nu_i^L(\mathrm{d}M) \\ &= c_5 (R_{ij} V_i(K|\cdot))(L). \end{aligned}$$

Since (6.24) implies

$$V_{i,d-j}^{(i-j)}(B^d, B_{L^\perp}) = \kappa_i c_{d-i,i-j}^{d-j,0},$$

we conclude that $c_5 = c_{d-i,i-j}^{d-j,0}$. We have obtained the following result.

Theorem 6.4.8. *Let $K, M \in \mathcal{K}'$. If $j \in \{1, \ldots, d-1\}$ and $L \in G(d,j)$, then the* **principal kinematic formula for projection functions***,*

$$\int_{G_d} V_j((M \cap gK)|L)\, \mu(\mathrm{d}g) = \sum_{i=j}^{d} c_{d,d-i}^{d-i+j,d-j} (R_{ij} V_i(K|\cdot))(L) V_{d-i+j}(M),$$

and the **Crofton formula for projection functions***,*

$$\int_{A(d,d-i+j)} V_j((K \cap E)|L)\, \mu_{d-i+j}(\mathrm{d}E) = c_{d,d-i}^{d-i+j,d-j} (R_{ij} V_i(K|\cdot))(L),$$

hold.

Notes for Section 6.4

1. An iterated translative integral formula in the plane was first derived by Miles [529].

The iterated translation formula for curvature measures was proved in Weil [792] and applied to non-isotropic Poisson particle processes and Boolean models. Shorter surveys are given in [790] and [791]. The presentation in this section follows closely the one in Weil [800].

2. Extensions of translative integral formulas to sets of positive reach have been studied by Rataj and Zähle, using methods of geometric measure theory. First, a translative formula for support measures of sets with positive reach was proved in [617]. An iterated version was obtained by Rataj [613]. Various extensions and supplements were provided by Rataj [614], Hug [355], Zähle [831], Rataj and Zähle [618], [619]. Translative Crofton formulas for support measures were treated by Rataj [615].

The iterated translative integral formula for support measures can be written in the form

$$\begin{aligned}
&\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} h(x, x - x_2, \ldots, x - x_k, u) \\
&\times \Xi_j(K_1 \cap K_2^{x_2} \cap \ldots \cap K_k^{x_k}, \mathrm{d}(x,u))\, \lambda(\mathrm{d}x_2) \cdots \lambda(\mathrm{d}x_k) \\
&= \sum_{\substack{m_1,\ldots,m_k=j \\ m_1+\ldots+m_k=(k-1)d+j}}^{d} \int_{(\mathbb{R}^d)^{k+1}} h(x_1, \ldots, x_k, u) \\
&\quad \times \Xi_{m_1,\ldots,m_k}^{(j)}(K_1, \ldots, K_k; \mathrm{d}(x_1, \ldots, x_k, u))
\end{aligned}$$

for nonnegative measurable functions h on $(\mathbb{R}^d)^{k+1}$, with certain **mixed support measures** $\Xi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\cdot)$ on $(\mathbb{R}^d)^{k+1}$. In Rataj's version for sets of positive reach, the mixed support measures are expressed as currents evaluated at specially chosen differential forms. For closed convex sets, Hug [355] has a more general version for relative support measures, as well as more explicit expressions for the mixed support measures, which imply, in particular, representations of special mixed measures by Goodey and Weil [280].

3. In Schneider [702], the mixed functionals $V^{(0)}_{m_1,\dots,m_k}(K_1,\dots,K_k)$ of convex bodies are embedded in a wider theory, together with the mixed volumes. For polytopes, more general representations of type (6.18) (for $j=0$) are obtained.

4. The reduction property in Theorem 6.4.4 can be generalized to lower-dimensional balls. The following result was proved in Weil [800]. It also provides a kind of **exchangeability**, since the role of the subspace L and the unit ball B_L in L can be exchanged. Let $K_1,\dots,K_k\in\mathcal{K}'$ and $A_1,\dots,A_k\in\mathcal{B}(\mathbb{R}^d)$. For $q\in\{j,\dots,d\}$ and $m\in\{j,\dots,q\}$, let $L\in G(d,q)$ and let B_L be the unit ball in L. Then

$$\begin{aligned}
&\Phi^{(j)}_{m_1,\dots,m_k,m}(K_1,\dots,K_k,B_L;A_1\times\ldots\times A_k\times L)\\
&=\frac{1}{m!\kappa_q}c^{m,q-m+j}_{j,q-m}\Phi^{(q-m+j)}_{m_1,\dots,m_k,q}(K_1,\dots,K_k,L;A_1\times\ldots\times A_k\times B_L)\\
&=\frac{1}{m!\kappa_q}c^{m,q-m+j}_{j,q-m}\Phi^{(q-m+j)}_{m_1,\dots,m_k,q}(K_1,\dots,K_k,B_L;A_1\times\ldots\times A_k\times L). \qquad (6.36)
\end{aligned}$$

Replacing m and j by $q-j$ and 0, we obtain

$$\begin{aligned}
&\Phi^{(j)}_{m_1,\dots,m_k,q}(K_1,\dots,K_k,L;A_1\times\ldots\times A_k\times B_L)\\
&=\Phi^{(j)}_{m_1,\dots,m_k,q}(K_1,\dots,K_k,B_L;A_1\times\ldots\times A_k\times L)\\
&=\frac{\kappa_q}{\kappa_{q-j}}\Phi^{(0)}_{m_1,\dots,m_k,q-j}(K_1,\dots,K_k,B_L;A_1\times\ldots\times A_k\times L). \qquad (6.37)
\end{aligned}$$

For $q=d$ (and using the reduction property of mixed measures), formula (6.36) reduces to (6.24) and (6.37) yields the first formula in Theorem 6.4.4.

For the mixed functionals, (6.37) implies

$$V^{(j)}_{m_1,\dots,m_k,q}(K_1,\dots,K_k,B_L)=\frac{\kappa_q}{\kappa_{q-j}}V^{(0)}_{m_1,\dots,m_k,q-j}(K_1,\dots,K_k,B_L). \qquad (6.38)$$

5. Translative Crofton formula for mixed volumes. Combining (6.25) with the translative Crofton formula in Theorem 6.4.3, we obtain a translative integral formula for mixed volumes of convex bodies K,M (see Weil [800]).

Let $j\in\{1,\dots,d-1\}$, $q\in\{j,\dots,d-1\}$ and $L\in G(d,q)$, then

$$\int_{L^\perp}V((K\cap L^x)[j],M[d-j])\,\lambda_{d-q}(\mathrm{d}x)=\frac{1}{\binom{d}{j}\kappa_q}V^{(0)}_{d-q+j,d-j,q}(K,-M,B_L).$$

For $M=B^d$, and using (6.25) and (6.38), a translative Crofton formula for intrinsic volumes results,

$$\int_{L^\perp}V_j(K\cap L^x)\,\lambda_{d-q}(\mathrm{d}x)=\frac{\binom{d}{q-j}}{\kappa_{q-j}}V(K[d-q+j],B_L[q-j]), \qquad (6.39)$$

which was first proved in Schneider [681].

6. Formulas for halfspaces. Crofton-type formulas, where the moving k-flat is replaced by a moving halfspace, were also discussed in Weil [800]. Let $C_{u^\perp}$ be a unit cube in $u^\perp$.

The following formula is the analog of Theorem 6.4.3 (for $q = d - 1$) and is proved in the same way:

$$\int_{-\infty}^{\infty} \Phi^{(j)}_{m_1,\dots,m_{k-1},m}(K_1,\dots,K_{k-1},K\cap(u^+)^{ru};$$
$$A_1\times\dots\times A_{k-1}\times(A\cap(u^\perp)^{ru}))\,\mathrm{d}r$$
$$=\Phi^{(j)}_{m_1,\dots,m_{k-1},m+1,d-1}(K_1,\dots,K_{k-1},K,u^+;$$
$$A_1\times\dots\times A_{k-1}\times A\times C_{u^\perp}). \tag{6.40}$$

Combining Theorem 6.4.3 and (6.40), a quite general result for halfspaces of the form $L^{u,+} := L\cap u^+, L\in G(d,q), u\in S^{d-1}\cap L$, with bounding flat $L^u := L\cap u^\perp$, can be deduced:

$$\Phi^{(j)}_{m_1,\dots,m_k,q-1}(K_1,\dots,K_k,L^{u,+};A_1\times\dots\times A_k\times C_{L^u})$$
$$=\Phi^{(j)}_{m_1,\dots,m_k,q,d-1}(K_1,\dots,K_k,L,u^+;A_1\times\dots\times A_k\times C_L\times C_{u^\perp}).$$

7. Mean section body. The kth mean section body $M_k(K)$ of a convex body K was introduced and investigated in Goodey and Weil [279]. In particular, the representation (6.34) was proved there.

A special case of (6.26) is worth mentioning. If $k = j+1$, then

$$V^{(j)}_{j+1,d-1}(K,M) = dc^{d,0}_{j,d-j}\int_{A(d,d-j)} V((K\cap E)[1],-M[d-1])\,\mu_{d-j}(\mathrm{d}E).$$

The linearity properties of the mixed volume imply that the latter integral equals the mixed volume $V(M_{d-j}(K)[1],-M[d-1])$, thus

$$V^{(j)}_{j+1,d-1}(K,M) = dc^{d,0}_{j,d-j}V(M_{d-j}(K)[1],-M[d-1]).$$

8. Spherical integral representations. For mixed volumes $V(K[1],M[d-1])$ of two convex bodies K,M the spherical integral representation (14.23), namely

$$V(K[1],M[d-1]) = \frac{1}{d}\int_{S^{d-1}} h^*(K,u)\,S(M,\mathrm{d}u),$$

is classical. It involves the support function $h(K,\cdot)$ of K (here replaced by its centered version $h^*(K,\cdot)$) and the surface area measure $S(M,\cdot) := S_{d-1}(M,\cdot)$ of M. Since

$$V(K[1],M[d-1]) = \frac{1}{d}V^{(0)}_{1,d-1}(K,-M),$$

one can ask for extensions to mixed functionals of more than two bodies. The following result is obtained in Weil [800] for convex bodies $K_1,\dots,K_k$, $M_1,\dots,M_i$:

$$\begin{aligned}&\Phi^{(j)}_{m_1,\dots,m_k,d-1,\dots,d-1}(K_1,\dots,K_k,-M_1,\dots,-M_i;A_1\times\dots\times A_k\times(\mathbb{R}^d)^i)\\&=\int_{S^{d-1}}\cdots\int_{S^{d-1}}\Phi^{(j)}_{m_1,\dots,m_k,d-1,\dots,d-1}(K_1,\dots,K_k,u_1^+,\dots,u_i^+;\\&\quad A_1\times\dots\times A_k\times C_{u_1^\perp}\times\dots\times C_{u_i^\perp})\,S(M_1,\mathrm{d}u_1)\cdots S(M_i,\mathrm{d}u_i).\end{aligned}\tag{6.41}$$

For $k=1$, (6.41) again implies the formulas (6.28) and (6.33). The latter representation was proved by Goodey and Weil [281] (in correction of an erroneous statement from [795]).

If there are no bodies K_j in (6.41), then a formula for $V^{(j)}_{d-1,\dots,d-1}(M_1,\dots,M_{d-j})$ results. Since

$$\begin{aligned}&\Phi^{(j)}_{d-1,\dots,d-1}(u_1^+,\dots,u_{d-j}^+;C_{u_1^\perp}\times\dots\times C_{u_{d-j}^\perp})\\&=\frac{1}{(d-j)\kappa_{d-j}}\sigma_{d-j-1}(\mathrm{co}(u_1,\dots,u_{d-j}))|\det(u_1,\dots,u_{d-j})|,\end{aligned}$$

where co denotes the spherical convex hull, we obtain

$$\begin{aligned}&V^{(j)}_{d-1,\dots,d-1}(M_1,\dots,M_{d-j})\\&=\frac{1}{(d-j)\kappa_{d-j}}\int_{S^{d-1}}\cdots\int_{S^{d-1}}\sigma_{d-j-1}(\mathrm{co}(u_1,\dots,u_{d-j}))\\&\quad\times|\det(u_1,\dots,u_{d-j})|\,S(M_1,\mathrm{d}u_1)\cdots S(M_{d-j},\mathrm{d}u_{d-j}).\end{aligned}\tag{6.42}$$

From the case $j=d-2$ of (6.42) (or from (6.33)), we get a representation of $h(M_2(K),\cdot)$ due to Goodey and Weil [279],

$$h(M_2(K),u)=\frac{1}{2\pi}c^{2,d-2}_{d,0}\int_{S^{d-1}}\alpha(u,v)\,\sin\alpha(u,v)\,S(-K,\mathrm{d}v).$$

Here, $\alpha(u,v)\in[0,\pi]$ denotes the (smaller) angle between $u,v\in S^{d-1}$.

9. Centrally symmetric bodies. For smooth centrally symmetric bodies, representations of mixed measures in terms of the projection generating measures are possible. If M is a generalized zonoid (a centrally symmetric body, for which (14.33) holds with a signed measure ρ), the signed measure $\rho_{(j)}$ introduced by (14.36) exists and satisfies (14.38). For convex bodies $K_1,\dots,K_k\in\mathcal{K}$ and generalized zonoids $M_1,\dots,M_i$, the following was shown in Weil [800]:

$$\begin{aligned}&\Phi^{(j)}_{m_1,\dots,m_k,r_1,\dots,r_i}(K_1,\dots,K_k,M_1,\dots,M_i;A_1\times\dots\times A_k\times(\mathbb{R}^d)^i)\\&=2^{\sum_{j=1}^i r_j}\int_{G(d,r_1)}\cdots\int_{G(d,r_i)}\Phi^{(j)}_{m_1,\dots,m_k,r_1,\dots,r_i}(K_1,\dots,K_k,L_1,\dots,L_i;\\&\quad A_1\times\dots\times A_k\times C_{L_1}\times\dots\times C_{L_i})\,\rho_{(r_1)}(M_1,\mathrm{d}L_1)\cdots\rho_{(r_i)}(M_i,\mathrm{d}L_i).\end{aligned}\tag{6.43}$$

For $k=0$, (6.43) yields Theorem 10.1 in Weil [792]:

$$\begin{aligned}&V^{(j)}_{r_1,\dots,r_i}(M_1,\dots,M_i)\\&=\frac{2^{(i-1)d+j}}{r_1!\cdots r_i!}\int_{G(d,r_1)}\cdots\int_{G(d,r_i)}[L_1,\dots,L_i]\,\rho_{(r_1)}(M_1,\mathrm{d}L_1)\cdots\rho_{(r_i)}(M_i,\mathrm{d}L_i).\end{aligned}\tag{6.44}$$

In the case $r_1 = \ldots = r_i = d-1$, (6.44) implies an iterated variant of (6.40) for centrally symmetric bodies $M_1, \ldots, M_i$ where the halfspaces are replaced by their bounding hyperplanes (see Weil [800], for details). As special cases, for centrally symmetric convex bodies K, M, the following formulas result:

$$V^{(j)}_{j+1,d-1}(K,M) = \frac{1}{2} \int_{S^{d-1}} \Phi^{(j)}_{j+1,d-1}(K, u^\perp; \mathbb{R}^d \times C_{u^\perp})\, S(M, \mathrm{d}u),$$

$$h(M_{d-j}(K), u) = \frac{c^{j,d-j}_{d,0}}{2} \Phi^{(j)}_{j+1,d-1}(K, u^\perp; \mathbb{R}^d \times C_{u^\perp}).$$

10. Support functions. The translative formula for support functions of Theorem 6.4.6 and the kinematic formula of Theorem 6.4.7 were proved by Weil [795]. The approach to (6.29) that is presented here comes from Schneider [708].

Since the left side of (6.29) defines a support function and the summands on the right side have different degrees of homogeneity, one may conjecture that the mixed functions $h^*_k(K, M; \cdot)$ are support functions, too. This was indeed proved by Goodey and Weil [281]. A simpler approach and an extension to mixed functions of more than two convex bodies are found in Schneider [709].

The case of two convex bodies can also be formulated as follows. If $K, M \in \mathcal{K}'$, then the translative integral

$$\int_{\mathbb{R}^d} h^*(K \cap M^x, \cdot)\, \lambda(\mathrm{d}x)$$

defines the support function of a convex body $T(K, M)$, called the **translation mixture** of K and M. There exists a polynomial expansion

$$T(rK, sM) = \sum_{k=1}^{d} r^k s^{d+1-k} T_k(K, M)$$

with convex bodies $T_k(K, M)$, called the **mixed bodies** of K and M. For the case of polytopes K, M, the vertices and edges of $T_k(K, M)$ were explicitly determined in [709].

Applications of the integral geometric formulas for support functions to stochastic geometry appear in Weil [793, 798].

11. Projection functions. Kinematic and Crofton formulas for projection functions were first studied by Goodey and Weil [278]. Theorem 6.4.8 in its present form appears in Goodey, Schneider and Weil [275].

6.5 Spherical Integral Geometry

Large parts of integral geometry in Euclidean spaces can be extended, in a suitable way, to spaces of constant curvature. In this section, we treat basic facts of the integral geometry of convex bodies in spherical space, since this is of some relevance for stochastic geometry. The approach will be similar to the Euclidean case: for (spherically) convex bodies we introduce generalized curvature measures via a Steiner formula, and integral geometric intersection formulas involving curvature measures are proved for polytopes, using characterization theorems, and then extended to general convex bodies. Our presentation owes much to the work of Glasauer [264, 265], which we follow in several aspects and details. We shall be rather brief at points where the procedure is in an obvious way similar to the Euclidean case. On the other hand, some geometric facts of spherical geometry are proved here instead of deferring them to the Appendix, since they are needed only in this section.

The spherical space to be considered is the unit sphere S^{d-1} of $\mathbb{R}^d$. The usual metric in S^{d-1} is denoted by d_s, thus $d_s(x,y) = \arccos\langle x,y\rangle$ for $x,y \in S^{d-1}$. It induces the trace topology from $\mathbb{R}^d$ on S^{d-1}, and topological notions in S^{d-1} refer to this topology. For points $x,y \in S^{d-1}$ with $d_s(x,y) < \pi$, the set $[x,y] := S^{d-1} \cap \operatorname{pos}\{x,y\}$ is the unique **spherical segment** joining x and y. A **spherically convex body** in S^{d-1} is the intersection of S^{d-1} with a closed convex cone different from $\{0\}$ in $\mathbb{R}^d$. In the present section, we shall mostly say 'convex' instead of 'spherically convex'. The convex bodies in S^{d-1} are precisely the nonempty closed subsets that contain with any two points of spherical distance less than π also the spherical segment joining them. The set of all convex bodies in S^{d-1} is denoted by $\mathcal{K}_s$. It is equipped with the Hausdorff metric induced by the metric d_s. Note that S^{d-1} is an isolated point of $\mathcal{K}_s$. For $K \in \mathcal{K}_s$, we denote by $\breve{K} := \operatorname{pos} K$ the cone with $K = S^{d-1} \cap \breve{K}$. The correspondence $K \leftrightarrow \breve{K}$ is quite useful for the study of spherically convex bodies. The **dimension** of $K \in \mathcal{K}_s$ is defined as $\dim K := \dim \breve{K} - 1$. The **relative interior** of K, denoted by $\operatorname{relint} K$, is the interior of K relative to $S^{d-1} \cap \operatorname{lin} K$.

A distinguished subset of $\mathcal{K}_s$ is the set $\mathcal{S}_k$ of k-dimensional great subspheres, which are the intersections of S^{d-1} with $(k+1)$-dimensional linear subspaces of $\mathbb{R}^d$, $k = 0, \dots, d-1$. We write $\mathcal{S}_\bullet := \bigcup_{k=0}^{d-1} \mathcal{S}_k$, and we often say 'subsphere' instead of 'great subsphere'. A set $K \in \mathcal{K}_s$ is called a **proper convex body** if it is contained in an open hemisphere, equivalently, if the cone $\breve{K}$ is pointed (does not contain a line). We write $\mathcal{K}_s^p$ for the set of all proper convex bodies.

Let $K, M \in \mathcal{K}_s$. We denote by

$$K \vee M := S^{d-1} \cap \operatorname{pos}(K \cup M)$$

the **spherically convex hull** of K and M. For $K \vee \{x\}$ we write $K \vee x$, and $x \vee y := [x,y]$ if x and y are not antipodal. The set

$$K^* := \{x \in S^{d-1} : \langle x,y\rangle \le 0 \text{ for all } y \in K\}$$

is the **polar body** of K; thus K^* is the intersection of S^{d-1} with the dual cone of $\breve{K}$. It is again in $\mathcal{K}_s$. Further, $(K^*)^* = K$ and

$$(K \vee M)^* = K^* \cap M^*, \qquad (K \cap M)^* = K^* \vee M^*. \tag{6.45}$$

The polar body of K can also be represented as

$$K^* := \{x \in S^{d-1} : d_s(K, x) \geq \pi/2\}.$$

If $S \in \mathcal{S}_k$ for $k \in \{0, \dots, d-2\}$, then $S^* = S^{d-1} \cap (\operatorname{lin} S)^\perp \in \mathcal{S}_{d-k-2}$.

Let $K \in \mathcal{K}_s$ and $x \in S^{d-1}$. If $0 \leq d_s(K, x) < \pi/2$, there is a unique point in K that is nearest to x; we denote it by $p_s(K, x)$. This defines the **nearest-point map** or **metric projection** $p_s(K, \cdot)$. If $x \notin K$, we define $u_s(K, x) = u(\breve{K}, x)$ (see Section 14.2 for the latter); note that $u_s(K, x) = p_s(K^*, x)$. For $x \in \operatorname{bd} K$,

$$N_s(K, x) := \{y \in K^* : \langle x, y \rangle = 0\}$$

is the set of outer (unit) normal vectors to K at x. Note that $\operatorname{pos} N_s(K, x) = N(\breve{K}, x)$ is the normal cone of $\breve{K}$ at x (as introduced in Section 14.2), but $N_s(K, x)$ consists of unit vectors. A pair (x, u) with $x \in \operatorname{bd} K$ and $u \in N_s(K, x)$ is called a **support element** of K. It is easy to see that

$$(x, u) \text{ is a support element of } K \quad \Leftrightarrow \quad (u, x) \text{ is a support element of } K^*.$$

The set of all support elements of K, denoted by $\operatorname{Nor} K$, is a closed subset of the product space $\Sigma_s := S^{d-1} \times S^{d-1}$.

A convex body $P \in \mathcal{K}_s$ is a (spherical) **polytope** if the cone $\breve{P}$ is polyhedral, that is, an intersection of finitely many closed halfspaces with 0 in the boundary. The set of polytopes in S^{d-1} is denoted by $\mathcal{P}_s$. Let $P \in \mathcal{P}_s$. A **k-face** of P is a set $F = S^{d-1} \cap \breve{F}$, where $\breve{F}$ is a $(k+1)$-face of $\breve{P}$, $k \in \{0, \dots, d-1\}$. The set of all k-faces of P is denoted by $\mathcal{F}_k(P)$, and we write $\mathcal{F}_\bullet(P) := \bigcup_{k=0}^{d-1} \mathcal{F}_k(P)$.

Let F be a k-face of P. The set $N_s(P, x)$ is the same for all $x \in \operatorname{relint} F$ and is denoted by $N_s(P, F)$. The **internal angle** $\beta(0, \breve{F})$ of the cone $\breve{F}$ at 0 is defined by

$$\beta(0, \breve{F}) := \frac{\sigma_k(F)}{\omega_{k+1}},$$

and the **external angle** of P at F by

$$\gamma(F, P) := \gamma(\breve{F}, \breve{P}) := \frac{\sigma_{d-k-2}(N_s(P, F))}{\omega_{d-k-1}}.$$

The proof of a local Steiner formula for spherical polytopes will rest on the following lemma. It is a spherical counterpart to the representation of Lebesgue measure in $\mathbb{R}^d$ as the product of the Lebesgue measures on a subspace and its orthogonal complement.

Lemma 6.5.1. *Let $S \in \mathcal{S}_k$, where $k \in \{0, \dots, d-2\}$, and let $f : S^{d-1} \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\int_{S^{d-1}} f \, \mathrm{d}\sigma = \int_S \int_{S^* \vee v} \sin^k(d_s(S^*, u)) f(u) \, \sigma_{d-k-1}(\mathrm{d}u) \, \sigma_k(\mathrm{d}v).$$

Proof. We extend f to $\mathbb{R}^d$ by $\bar{f}(x) := \|x\|^{-d+1} f(x/\|x\|)$ for $0 < \|x\| < 1$, and $\bar{f}(x) = 0$ otherwise. Let $L := \operatorname{lin} S$. Using spherical coordinates, we obtain

$$\begin{aligned}\int_{S^{d-1}} f \, \mathrm{d}\sigma &= \int_{S^{d-1}} \int_0^1 (f(x)/t^{d-1}) t^{d-1} \, \mathrm{d}t \, \sigma(\mathrm{d}x) \\ &= \int_{\mathbb{R}^d} \bar{f} \, \mathrm{d}\lambda \\ &= \int_L \int_{L^\perp} \bar{f}(x+y) \, \lambda_{L^\perp}(\mathrm{d}y) \, \lambda_L(\mathrm{d}x) \\ &= \int_S \left[\int_0^1 t^k \int_{L^\perp} \bar{f}(tv+y) \, \lambda_{L^\perp}(\mathrm{d}y) \, \mathrm{d}t \right] \sigma_k(\mathrm{d}v).\end{aligned}$$

Recall that $d(\cdot,\cdot)$ denotes the Euclidean distance. With $tv + y =: w = \tau u$, $\|u\| = 1$, we have $t = d(L^\perp, w) = \tau \sin(d_s(S^*, u))$. Hence, the integral in brackets is equal to

$$\begin{aligned}&\int_{\operatorname{pos}(L^\perp \cup \{v\})} d(L^\perp, w)^k \bar{f}(w) \, \lambda_{d-k}(\mathrm{d}w) \\ &= \int_{S^* \vee v} \int_0^1 \tau^{d-k-1} \tau^k \sin^k(d_s(S^*, u)) \bar{f}(\tau u) \, \mathrm{d}\tau \, \sigma_{d-k-1}(\mathrm{d}u) \\ &= \int_{S^* \vee v} \sin^k(d_s(S^*, u)) f(u) \, \sigma_{d-k-1}(\mathrm{d}u).\end{aligned}$$

This yields the assertion. □

For $K \in \mathcal{K}_s$, the **local parallel set** of K, determined by a Borel set $A \subset \Sigma_s$ and a number $0 < \epsilon < \pi/2$, is defined by

$$M_\epsilon(K, A) := \{x \in S^{d-1} : d_s(K, x) \leq \epsilon, \, (p_s(K, x), u_s(K, x)) \in A\}.$$

Theorem 6.5.1 (Local spherical Steiner formula). *For $K \in \mathcal{K}_s$, there exist uniquely determined finite measures $\Theta_0(K, \cdot), \ldots, \Theta_{d-2}(K, \cdot)$ on Σ_s such that the following holds. If $A \in \mathcal{B}(\Sigma_s)$ and $0 < \epsilon < \pi/2$, then*

$$\sigma(M_\epsilon(K, A)) = \sum_{m=0}^{d-2} g_{d,m}(\epsilon) \Theta_m(K, A)$$

with

$$g_{d,m}(\epsilon) := \omega_{m+1} \omega_{d-m-1} \int_0^\epsilon \cos^m \varphi \sin^{d-m-2} \varphi \, \mathrm{d}\varphi, \qquad 0 \leq \epsilon \leq \pi/2.$$

If $P \in \mathcal{P}_s$, then

$$\Theta_m(P, A) = \frac{1}{\omega_{m+1} \omega_{d-m-1}} \sum_{F \in \mathcal{F}_m(P)} \int_F \int_{N_s(P,F)} \mathbf{1}_A(x, u) \, \sigma_{d-m-2}(\mathrm{d}u) \, \sigma_m(\mathrm{d}x).$$

Proof. First let $P \in \mathcal{P}_s$, let $m \in \{0, \ldots, d-2\}$ and $F \in \mathcal{F}_m(P)$. Put $S := S^{d-1} \cap \text{lin}\, F$ and $U := p_s(P, \cdot)^{-1}(\text{relint}\, F)$. Let $f \geq 0$ be a measurable function on S^{d-1}. We apply Lemma 6.5.1, first to S^{d-1} and its subsphere S, then to the sphere $S^* \vee \{-x, x\}$, where $x \in S$, and its subsphere S^*. This gives

$$\begin{aligned}
\int_U f \, \mathrm{d}\sigma &= \int_S \int_{S^* \vee x} \mathbf{1}_U(u) f(u) \sin^m(d_s(S^*, u))\, \sigma_{d-m-1}(\mathrm{d}u)\, \sigma_m(\mathrm{d}x) \\
&= \int_S \int_{S^*} \int_{\{-x,x\} \vee v} \mathbf{1}_{U \cap (S^* \vee x)}(z) f(z) \sin^m(d_s(S^*, z)) \\
&\quad \times \sin^{d-m-2}(d_s(\{-x, x\}, z))\, \sigma_1(\mathrm{d}z)\, \sigma_{d-m-2}(\mathrm{d}v)\, \sigma_m(\mathrm{d}x) \\
&= \int_S \int_{S^*} \int_{[x,v]} \mathbf{1}_U(z) f(z) \cos^m(d_s(S, z)) \sin^{d-m-2}(d_s(S, z))\, \sigma_1(\mathrm{d}z) \\
&\quad \times \sigma_{d-m-2}(\mathrm{d}v)\, \sigma_m(\mathrm{d}x) \\
&= \int_F \int_{N_s(P,F)} \int_0^{\pi/2} f(x \cos\varphi + v \sin\varphi) \cos^m \varphi \sin^{d-m-2} \varphi \, \mathrm{d}\varphi \\
&\quad \times \sigma_{d-m-2}(\mathrm{d}v)\, \sigma_m(\mathrm{d}x).
\end{aligned}$$

The choice $f = \mathbf{1}_{M_\epsilon(P,A)}$ yields

$$\begin{aligned}
&\sigma(M_\epsilon(P, A) \cap p_s(P, \cdot)^{-1}(\text{relint}\, F)) \\
&= \int_F \int_{N_s(P,F)} \mathbf{1}_A(x, v)\, \sigma_{d-m-2}(\mathrm{d}v)\, \sigma_m(\mathrm{d}x) \int_0^\epsilon \cos^m \varphi \sin^{d-m-2} \varphi \, \mathrm{d}\varphi.
\end{aligned}$$

Similarly to Euclidean space, $\text{bd}\, P = \bigcup_{m=0}^{d-2} \bigcup_{F \in \mathcal{F}_m(P)} \text{relint}\, F$ is a disjoint union, hence we get

$$\begin{aligned}
\sigma(M_\epsilon(P, A)) &= \sum_{m=0}^{d-2} \sum_{F \in \mathcal{F}_m(P)} \int_F \int_{N_s(P,F)} \mathbf{1}_A(x, v)\, \sigma_{d-m-2}(\mathrm{d}v)\, \sigma_m(\mathrm{d}x) \\
&\quad \times \int_0^\epsilon \cos^m \varphi \sin^{d-m-2} \varphi \, \mathrm{d}\varphi \\
&= \sum_{m=0}^{d-2} g_{d,m}(\epsilon) \Theta_m(P, A),
\end{aligned}$$

if $\Theta_m(P, A)$ is defined as shown in the theorem. We observe that the functions $g_{d,0}, \ldots, g_{d,d-2}$ are linearly independent on $(0, \pi/2)$. The remaining parts of the proof (measurability, extension to general convex bodies, uniqueness) are so similar to the Euclidean case (which is treated in Schneider [695, sect. 4.1, 4.2]) that we omit them. □

We add a remark to the Steiner formula. Applying it with $K = S \in \mathcal{S}_i$ for $i \in \{0, \ldots, d-2\}$ and $A = \Sigma_s$, and observing that $\lim_{\epsilon \to \pi/2} \sigma(M_\epsilon(S, \Sigma_s)) = \omega_d$ and

$$\Theta_m(S, \Sigma_s) = \delta_{im} \qquad \text{for } S \in \mathcal{S}_i \tag{6.46}$$

(where δ_{im} denotes the Kronecker symbol), we find that $g_{d,i}(\pi/2) = \omega_d$ (which can, of course, also be obtained from the definition). If P is a polytope and $F \in \mathcal{F}_m(P)$, then

$$\operatorname{cl} \bigcup_{0<\epsilon<\pi/2} M_\epsilon(P, \Sigma_s) \cap p_s(P, \cdot)^{-1}(\operatorname{relint} F) = F \vee N_s(P, F),$$

hence

$$\begin{aligned}\frac{\sigma(F \vee N_s(P, F))}{\omega_d} &= \frac{\sigma_m(F)}{\omega_{m+1}} \frac{\sigma_{d-m-2}(N_s(P, F))}{\omega_{d-m-1}} \\ &= \frac{1}{\omega_{m+1}} \gamma(F, P)\sigma_m(F).\end{aligned} \tag{6.47}$$

The polytopes $F \vee N_s(P, F)$, $F \in \mathcal{F}_m(P)$, $m = 0, \dots, d-2$, together with P and P^*, tile the sphere S^{d-1}, that is, they cover it and have pairwise no common interior points. It follows that

$$\sum_{m=0}^{d-2} \frac{1}{\omega_{m+1}} \sum_{F \in \mathcal{F}_m(P)} \gamma(F, P)\sigma_m(F) + \frac{1}{\omega_d}\sigma(P) + \frac{1}{\omega_d}\sigma(P^*) = 1, \tag{6.48}$$

a fact which will later become important.

We call $\Theta_m(K, \cdot)$ the mth **support measure** or **generalized curvature measure** of K. The chosen normalization has a simplifying effect in later formulas. The following theorem collects the main properties of the support measures.

Theorem 6.5.2. *For $m = 0, \dots, d-2$, the mapping $\Theta_m : \mathcal{K}_s \times \mathcal{B}(\Sigma_s) \to \mathbb{R}$ has the following properties:*

(a) **Rotation covariance:** *$\Theta_m(\vartheta K, \vartheta A) = \Theta_m(K, A)$ for $\vartheta \in SO_d$, where $\vartheta A := \{(\vartheta x, \vartheta u) : (x, u) \in A\}$,*
(b) **Weak continuity:** *$K_j \to K$ (in the Hausdorff metric on $\mathcal{K}_s$) implies $\Theta_m(K_j, \cdot) \xrightarrow{w} \Theta_m(K, \cdot)$,*
(c) *$\Theta_m(\cdot, A)$ is* **additive***, for each fixed $A \in \mathcal{B}(\Sigma_s)$,*
(d) *$\Theta_m(\cdot, A)$ is* **measurable***, for each fixed $A \in \mathcal{B}(\Sigma_s)$.*

This theorem is analogous to Theorem 14.2.2, whose proof can be found in Schneider [695]. In the spherical case, the proof is very similar, so that we omit it here.

There is no Euclidean counterpart to the following nice behavior of the support measures under polarity.

Theorem 6.5.3. *If $K \in \mathcal{K}_s$ and $A \in \mathcal{B}(\Sigma_s)$, then*

$$\Theta_m(K, A) = \Theta_{d-m-2}(K^*, A^{-1})$$

for $m \in \{0, \dots, d-2\}$, where $A^{-1} := \{(u, x) \in \Sigma_s : (x, u) \in A\}$.

Proof. By the weak continuity of the support measures and the continuity of the polarity $K \mapsto K^*$ (which is easy to see), it suffices to prove this for the case where K is a polytope P. The assertion then follows from the explicit representation of $\Theta_m(P, A)$ given in Theorem 6.5.1 and the fact that $F \in \mathcal{F}_m(P)$ and $N_s(P, F) =: G$ implies $G \in \mathcal{F}_{d-m-2}(P^*)$ and $N_s(P^*, G) = F$; the latter is again easy to see. □

As a marginal measure of the mth support measure, we obtain the mth **curvature measure**, by

$$\phi_m(K, A) := \Theta_m(K, A \times S^{d-1}), \qquad A \in \mathcal{B}(S^{d-1}).$$

We supplement the definition by

$$\phi_{d-1}(K, A) := \frac{1}{\omega_d}\sigma(K \cap A), \qquad A \in \mathcal{B}(S^{d-1}).$$

Theorem 6.5.2 (together with properties of the spherical Lebesgue measure) implies that the curvature measure ϕ_m, $m \in \{0, \dots, d-1\}$, is rotation covariant, in the sense that $\phi_m(\vartheta K, \vartheta A) = \phi(K, A)$ for $K \in \mathcal{K}_s$ and $A \in \mathcal{B}(S^{d-1})$, weakly continuous, and additive and measurable in its first argument. Further, it follows easily from the definition that $\phi_m(K, \cdot)$ is concentrated on K and that $\phi_m(K, \cdot)$ is **locally determined**, in the sense that $K_1, K_2 \in \mathcal{K}_s$ and $K_1 \cap B = K_2 \cap B$ for an open set $B \subset S^{d-1}$ implies $\phi(K_1, A) = \phi(K_2, A)$ for all $A \in \mathcal{B}(B)$. These properties can be defined similarly for mappings $\psi : \mathcal{K}_s \times \mathcal{B}(S^{d-1}) \to \mathbb{R}$ and play a role in the following characterization theorem.

Theorem 6.5.4. *Let $\psi : \mathcal{P}_s \times \mathcal{B}(S^{d-1}) \to \mathbb{R}$ be a mapping which is rotation covariant, locally determined, additive in its first argument, and such that $\psi(P, \cdot)$ is a finite measure concentrated on P, for all $P \in \mathcal{P}_s$. Then there are constants $c_0, \dots, c_{d-1} \geq 0$ such that*

$$\psi(P, \cdot) = \sum_{m=0}^{d-1} c_m \phi_m(P, \cdot)$$

for all $P \in \mathcal{P}_s$.

Proof. Let $k \in \{0, \dots, d-2\}$. First let $S_k \in \mathcal{S}_k$, and let $\mathcal{P}(S_k^*)$ be the set of all polytopes contained in S_k^*. Let $Q \in \mathcal{P}(S_k^*) \cup \{\emptyset\}$. For $Q = \emptyset$, we define $Q^* := S^{d-1}$. The mapping $A \mapsto \psi(S_k \vee Q, A)$, $A \in \mathcal{B}(S_k)$, is a finite measure which is invariant under all rotations that map S_k into itself and fix S_k^* pointwise. By the uniqueness of the spherical Lebesgue measure, there exists a constant $c(S_k, Q) \geq 0$ with

$$\psi(S_k \vee Q, A) = c(S_k, Q)\sigma_k(A), \qquad A \in \mathcal{B}(S_k). \tag{6.49}$$

We write Q^o for the polar body of Q with respect to S_k^* as surrounding sphere, thus $Q^o = Q^* \cap S_k^*$; in particular, $Q^o = S_k^*$ if $Q = \emptyset$ (one has to keep in mind that Q^o depends on S_k^*). Choosing $A = S_k$, we put

$$f(Q) := c(S_k, Q^o) = \frac{1}{\omega_{k+1}} \psi(S_k \vee Q^o, S_k) \qquad \text{for } Q \in \mathcal{P}(S_k^*) \cup \{\emptyset\}.$$

The function f is nonnegative and invariant under the rotations of S_k^* into itself. If $Q_1 \cup Q_2$ is convex, then

$$S_k \vee (Q_1 \cup Q_2)^o = (S_k \vee Q_1^o) \cap (S_k \vee Q_2^o),$$
$$S_k \vee (Q_1 \cap Q_2)^o = (S_k \vee Q_1^o) \cup (S_k \vee Q_2^o).$$

From the additivity of ψ in its first argument it follows that f is additive. Let $Q \in \mathcal{P}(S_k^*)$ be a polytope with $\dim Q < d - k - 2$. Then there exists $S_0 \in \mathcal{S}_0$ with $S_0 \subset Q^o$. Putting $S_{k+1} := S_k \vee S_0$, we have $S_{k+1} \in \mathcal{S}_{k+1}$ and

$$S_k \vee Q^o = S_{k+1} \vee (Q^* \cap S_{k+1}^*).$$

Therefore, using (6.49) with k replaced by $k+1$,

$$\begin{aligned} \omega_{k+1} f(Q) &= \psi(S_k \vee Q^o, S_k) = \psi(S_{k+1} \vee (Q^* \cap S_{k+1}^*), S_k) \\ &= c(S_{k+1}, Q^* \cap S_{k+1}^*) \sigma_{k+1}(S_k) = 0. \end{aligned}$$

This shows that the mapping f satisfies the assumptions of Theorem 14.4.7, with S^{d-1} replaced by S_k^*. It follows that

$$f(Q) = c(S_k) \sigma_{d-k-2}(Q)$$

with a constant $c(S_k) \geq 0$. By the rotation covariance of ψ, this constant depends only on k; we put $c(S_k) =: b_k$. Thus we have $c(S_k, Q^o) = b_k \sigma_{d-k-2}(Q)$ and hence $c(S_k, Q) = b_k \sigma_{d-k-2}(Q^o)$. Altogether, we arrive at

$$\psi(S_k \vee Q, A) = b_k \sigma_{d-k-2}(Q^o) \sigma_k(A)$$

for $Q \in \mathcal{P}(S_k^*)$ and $A \in \mathcal{B}(S_k)$.

Now let $P \in \mathcal{P}_s$, $F \in \mathcal{F}_k(P)$ for some $k \in \{0, \dots, d-2\}$, and $A \in \mathcal{B}(S^{d-1})$ with $A \subset \operatorname{relint} F$. With $S_k := S^{d-1} \cap \operatorname{lin} F$ and $Q := (S_k \vee P) \cap S_k^*$ we have $N_s(P, F) = P^* \cap S_k^* = Q^* \cap S_k^* = Q^o$. A sufficiently small open neighborhood B of A satisfies $P \cap B = (S_k \vee Q) \cap B$. Since ψ is locally determined, we get

$$\begin{aligned} \psi(P, A) &= \psi(S_k \vee Q, A) = b_k \sigma_{d-k-2}(Q^o) \sigma_k(A) \\ &= b_k \sigma_{d-k-2}(N_s(P, F)) \sigma_k(A). \end{aligned}$$

Finally, let $P \in \mathcal{P}_s$ and $A \in \mathcal{B}(S^{d-1})$. Then

$$A = (A \setminus P) \cup (A \cap \operatorname{int} P) \cup \bigcup_{k=0}^{d-2} \bigcup_{F \in \mathcal{F}_k(P)} A \cap \operatorname{relint} F$$

is a disjoint union. Since $\psi(P,\cdot)$ is concentrated on P, we get

$$\begin{aligned}
&\psi(P,A)\\
&= \psi(P, A \cap \operatorname{int} P) + \sum_{k=0}^{d-2} \sum_{F \in \mathcal{F}_k(P)} b_k \sigma_{d-k-2}(N_s(P,F))\sigma_k(A \cap \operatorname{relint} F)\\
&= \psi(P, A \cap \operatorname{int} P) + \sum_{k=0}^{d-2} c_k \phi_k(P,A),
\end{aligned}$$

with $c_k := b_k\omega_{k+1}\omega_{d-k-1}$. Since ψ is locally determined, we have $\psi(P, A \cap \operatorname{int} P) = \psi(S^{d-1}, A \cap \operatorname{int} P)$. Here $\psi(S^{d-1},\cdot)$ is a rotation invariant finite measure and hence proportional to σ, thus $\psi(P, A \cap \operatorname{int} P) = b_{d-1}\sigma(A \cap \operatorname{int} P) = c_{d-1}\phi_{d-1}(P,A)$ with a constant $c_{d-1} \geq 0$. This completes the proof. $\square$

Before applying Theorem 6.5.4 to the proof of a kinematic formula for curvature measures, we consider the total curvature measures. We write

$$v_m(K) := \phi_m(K, S^{d-1}), \qquad m = 0, \dots, d-1.$$

The functional v_m is called the mth **(spherical) intrinsic volume**. For a polytope P, we obtain from Theorem 6.5.1 the representation

$$v_m(P) = \frac{1}{\omega_{m+1}} \sum_{F \in \mathcal{F}_m(P)} \gamma(F,P)\sigma_m(F) = \sum_{G \in \mathcal{F}_{m+1}(\breve{P})} \beta(0,G)\gamma(G,\breve{P}). \quad (6.50)$$

The duality relation of Theorem 6.5.3 gives

$$v_m(K) = v_{d-m-2}(K^*), \qquad m = 0, \dots, d-2. \quad (6.51)$$

It is consistent with this to supplement the definition by

$$v_{-1}(K) := v_{d-1}(K^*). \quad (6.52)$$

With this definition, the intrinsic volumes satisfy two linear relations, which also have no counterpart in Euclidean space.

Theorem 6.5.5. *For $K \in \mathcal{K}_s$,*

$$\sum_{i=-1}^{d-1} v_i(K) = 1, \quad (6.53)$$

and if $K \in \mathcal{K}_s \setminus \mathcal{S}_\bullet$, then

$$\sum_{i=-1}^{d-1} (-1)^i v_i(K) = 0, \quad (6.54)$$

hence also

$$\sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{2i}(K) = \frac{1}{2}. \quad (6.55)$$

Proof. Relation (6.53) is just (6.48). For the proof of (6.54), it is convenient to consider first a pointed polyhedral cone $C \subset \mathbb{R}^d$ with interior points and its dual cone

$$C^* := \{y \in \mathbb{R}^d : \langle x, y\rangle \le 0 \text{ for all } X \in C\}.$$

If $F \in \mathcal{F}_m(C)$ for some $m \in \{0, \dots, d\}$, then we define $\widehat{F} := N(C, F)$ (normal cone of C at F) and observe that $\widehat{F} \in \mathcal{F}_{d-m}(C^*)$. The cone $F + (-\widehat{F})$ has dimension d. We denote by U the union of all faces of dimensions less than $d-1$ of all the cones $F + (-\widehat{F})$, $F \in \mathcal{F}_\bullet(C)$, and assert that

$$\eta(x) := \sum_{F\in\mathcal{F}_\bullet(C)} (-1)^{\dim F} \mathbf{1}_{F+(-\widehat{F})}(x) = 0 \qquad \text{for all } x \in \mathbb{R}^d \setminus U. \tag{6.56}$$

Since C and C^* can be separated weakly by a hyperplane, there is a point $y \in \mathbb{R}^d$ with $y \notin C \cup (-C^*)$, hence with $\eta(y) = 0$. Let $x \in \mathbb{R}^d \setminus U$. The point x can be joined to y by a polygonal path in $\mathbb{R}^d \setminus U$. Hence, it suffices to show that η is constant along this path, and for this it is sufficient to show that η does not change when entering some $(d-1)$-face of a cone $F + (-\widehat{F})$.

Let H be a $(d-1)$-face of some cone $F + (-\widehat{F})$, $F \in \mathcal{F}_\bullet(C)$. Being a facet of the direct sum $F + (-\widehat{F})$, H is the direct sum, $H = F_1 + (-G)$, of a face $F_1 \in \mathcal{F}_k(C)$, for some $k \in \{0, \dots, d-1\}$, and a face $-G \in \mathcal{F}_{d-1-k}(-\widehat{F})$. There is a face F_2 of C with $G = \widehat{F}_2$. From $\dim F_1 + \dim G = d-1$ and $\dim G = d - \dim F_2$ it follows that $\dim F_2 = k+1$. From $F_1 \subset F$ and $\widehat{F}_2 \subset \widehat{F}$, hence $F_1 \subset F \subset F_2$, it follows that either $F = F_1$ or $F = F_2$.

Since $F_1 \subset H$, $F_2 \not\subset H$, F_1 is a face of F_2 and $\dim F_1 = \dim F_2 - 1$, it follows that the cone F_2 lies in one of the closed halfspaces bounded by $\operatorname{lin} H$. Similarly, $\widehat{F}_1$ lies in one of the closed halfspaces bounded by $\operatorname{lin} H$. Let u be the unit normal vector of $\operatorname{lin} H$ pointing into the halfspace not containing F_2; note that $u \in (\operatorname{lin} \widehat{F}_2)^\perp = \operatorname{lin} F_2$. There exists $x \in F_1$ with $x - u \in F_2 \setminus F_1$. Since $x - u \in C$, but $x - u \notin F_1$, there exists $y \in \widehat{F}_1$ with $\langle x-u, y\rangle < 0$, hence with $\langle u, y\rangle > 0$. Thus, $\widehat{F}_1$ lies in the halfspace not containing F_2. Since $F_1 \subset H$ and $\widehat{F}_2 \subset H$, we conclude that $F_1 + (-\widehat{F}_1)$ and $F_2 + (-\widehat{F}_2)$ lie on the same side of the hyperplane $\operatorname{lin} H$.

As a consequence, when entering the facet H from $\operatorname{int}(F + (-\widehat{F})) \setminus U$, the changes in the contributions to the function η coming from $F_1 + (-\widehat{F}_1)$ and from $F_2 + (-\widehat{F}_2)$ cancel each other. Should part of the facet H also belong to some other cone $G + (-\widehat{G})$, $G \in \mathcal{F}_\bullet(C)$, the same argument applies. In this way, relation (6.56) is proved.

If now $P \in \mathcal{K}_s^p$ is a $(d-1)$-dimensional polytope, we can apply relation (6.56) to the cone $C = \breve{P}$ and obtain

$$\mathbf{1}_{-P^*}(x) + \sum_{i=0}^{d-1} (-1)^{i+1} \sum_{F\in\mathcal{F}_i(P)} \mathbf{1}_{F\vee(-N_s(P,F))}(x) = 0$$

for σ-almost all $x \in S^{d-1}$. Integrating this relation over the unit sphere and using the reflection invariance of σ, we obtain (6.54), in view of (6.47) and (6.50). The extension to general convex bodies which are not subspheres follows by approximation. Relation (6.55) follows from (6.53) and (6.54). $\square$

The **spherical convex ring** $\mathcal{R}_s$ is defined as the system of all finite unions of spherically convex bodies in S^{d-1}, including the empty set $\emptyset$. Groemer's extension theorem 14.4.2, with the obvious adaptation to S^{d-1}, shows that every continuous additive functional on $\mathcal{K}_s^p$ with values in a topological vector space has a continuous extension to $\mathcal{R}_s$. The function χ defined by $\chi(K) := 1$ for $K \in \mathcal{K}_s^p$ is additive and hence has such an extension, which is also denoted by χ and called the **Euler characteristic**. Since the intrinsic volumes, too, have additive extensions to $\mathcal{R}_s$, relation (6.55) generalizes to

$$2 \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{2i}(K) = \chi(K)$$

for $K \in \mathcal{R}_s$. This is a version of the **spherical Gauss–Bonnet theorem**. (Note that, in contrast to the Euclidean case, v_0 is *not* the Euler characteristic.)

Returning to the curvature measures, we prove an integral geometric intersection formula.

Theorem 6.5.6 (Spherical kinematic formula). *If $K, M \in \mathcal{R}_s$ and $A, B \in \mathcal{B}(S^{d-1})$, then*

$$\int_{SO_d} \phi_j(K \cap \vartheta M, A \cap \vartheta B)\, \nu(\mathrm{d}\vartheta) = \sum_{k=j}^{d-1} \phi_k(K, A)\phi_{d-1-k+j}(M, B) \qquad (6.57)$$

for $j = 0, \dots, d-1$.

Proof. For convex bodies $K, M \in \mathcal{K}_s$ we define

$$T(K, M) := \{\vartheta \in SO_d : K \text{ and } M \text{ touch}\},$$

where K and M are said to **touch** if $K \cap M \neq \emptyset$ but the cones $\breve{K}, \breve{M}$ can be separated weakly by a hyperplane. Similarly to the proof of Lemma 5.2.1 one shows that the mapping

$$\vartheta \mapsto \phi_j(K \cap \vartheta M, A \cap \vartheta B), \qquad \vartheta \in SO_d,$$

is measurable on $SO_d \setminus T(K, M)$ and hence coincides almost everywhere on SO_d with a measurable mapping if $\nu(T(K, M)) = 0$. The proof of the latter fact is not so straightforward as that for the Euclidean counterpart (see the beginning of Theorem 5.1.2). For polytopes P, Q, the relation $\nu(T(P, Q)) = 0$ is easily deduced from Lemma 13.2.1. Therefore, we first prove the kinematic formula for polytopes. This is used to prove $\nu(T(K,M)) = 0$ for general convex bodies, which then allows us to extend the kinematic formula to this case.

Let $j \in \{0,\dots,d-1\}$. The left side of (6.57) is well defined if K, M are polytopes. We fix $Q \in \mathcal{P}_s$ and an open set $B \subset S^{d-1}$ and put

$$\psi(P,A) := \int_{SO_d} \phi_j(P \cap \vartheta Q, A \cap \vartheta B)\,\nu(\mathrm{d}\vartheta),$$

for $P \in \mathcal{P}_s$ and $A \in \mathcal{B}(S^{d-1})$. It is easy to check that ψ satisfies the assumptions of Theorem 6.5.4, hence there exist constants $c_0(Q,B),\dots,c_{d-1}(Q,B) \geq 0$ such that

$$\int_{SO_d} \phi_j(P \cap \vartheta Q, A \cap \vartheta B)\,\nu(\mathrm{d}\vartheta) = \sum_{k=0}^{d-1} c_k(Q,B)\phi_k(P,A)$$

for all $P \in \mathcal{P}_s$ and all Borel sets $A \subset S^{d-1}$. Since, by (6.46), $\phi_m(S_k, S^{d-1}) = \delta_{km}$ for $S_k \in \mathcal{S}_k$, we obtain

$$c_k(Q,B) = \int_{SO_d} \phi_j(S_k \cap \vartheta Q, \vartheta B)\,\nu(\mathrm{d}\vartheta) \tag{6.58}$$

for $k = 0,\dots,d-1$. Admitting arbitrary Borel sets B in (6.58), we can again apply Theorem 6.5.4 and deduce that

$$c_k(Q,B) = \int_{SO_d} \phi_j(S_k \cap \vartheta Q, \vartheta B)\,\nu(\mathrm{d}\vartheta) = \sum_{i=0}^{d-1} b_{ik}\phi_i(Q,B)$$

with constants $b_{ik} \geq 0$. Here we choose $Q = S_m \in \mathcal{S}_m$ and $B = S^{d-1}$. It follows from Lemma 13.2.1 that either $S_k \cap \vartheta S_m = \emptyset$ for ν-almost all ϑ or $S_k \cap \vartheta S_m \in \mathcal{S}_{k+m-d+1}$ for ν-almost all ϑ, hence

$$b_{mk} = \begin{cases} 1, \text{ if } m = d-1-k+j, \\ 0 \ \text{ else.} \end{cases}$$

Thus we get $c_k(Q,B) = 0$ for $k = 0,\dots,j-1$ and $c_k(Q,B) = \phi_{d-1-k+j}(Q,B)$ for $k = j,\dots,d-1$. We conclude that

$$\int_{SO_d} \phi_j(P \cap \vartheta Q, A \cap \vartheta B)\,\nu(\mathrm{d}\vartheta) = \sum_{k=j}^{d-1} \phi_k(P,A)\phi_{d-1-k+j}(Q,B) \tag{6.59}$$

for $P, Q \in \mathcal{P}_s$, $A \in \mathcal{B}(S^{d-1})$ and open sets B. Since both sides define measures if B varies, (6.59) holds for arbitrary Borel sets B.

We want to replace P in (6.59) by a general convex body K. Since subspheres are polytopes, we may assume that K is not a subsphere. We can choose polytopes $P_1, P_2 \in \mathcal{P}_s$ which are not subspheres and satisfy $P_1 \subset K \subset P_2$. Then

$$T(K,Q) \subset (\{\vartheta \in SO_d : P_2 \cap \vartheta Q \neq \emptyset\} \setminus \{\vartheta \in SO_d : P_1 \cap \vartheta Q \neq \emptyset\}) \cup T(P_1, Q).$$

Since P_r is not a subsphere ($r = 1, 2$), it is easy to check that $P_r \cap \vartheta Q$ is not a subsphere for almost all ϑ, hence for almost all ϑ with $P_r \cap \vartheta Q \neq \emptyset$ we have

$$2 \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{2i}(P_r \cap \vartheta Q) = 1$$

by (6.55). Now formula (6.59) with $A = B = \mathbb{R}^d$ gives

$$\begin{aligned} \nu(\{\vartheta \in SO_d : P_r \cap \vartheta Q \neq \emptyset\}) &= \int_{SO_d} 2 \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} v_{2i}(P_r \cap \vartheta Q)\, \nu(\mathrm{d}\vartheta) \\ &= 2 \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{k=2i}^{d-1} v_k(P_r) v_{d-1-k+2i}(Q) \end{aligned}$$

for $r = 1, 2$ and hence

$$\nu(T(K,Q)) \leq 2 \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{k=2i}^{d-1} [v_k(P_2) - v_k(P_1)] v_{d-1-k+2i}(Q).$$

Since P_1 and P_2 can be chosen arbitrarily close to K in the Hausdorff metric and since the spherical intrinsic volumes are continuous (by Theorem 6.5.2(b)), we conclude that $\nu(T(K,Q)) = 0$.

Now we can conclude, as before, that $\vartheta \mapsto \phi_j(K \cap \vartheta Q, A \cap \vartheta B)$ coincides almost everywhere on SO_d with a measurable function, hence the left side of our next assertion,

$$\int_{SO_d} \phi_j(K \cap \vartheta Q, A \cap \vartheta B)\, \nu(\mathrm{d}\vartheta) = \sum_{k=j}^{d-1} \phi_k(K, A) \phi_{d-1-k+j}(Q, B), \quad (6.60)$$

is well defined. To prove (6.60), we proceed similarly to the Euclidean case, see Theorem 5.2.3. Assertion (6.60) is equivalent to

$$\begin{aligned} &\int_{SO_d} \int_{S^{d-1}} f(x) g(\vartheta^{-1} x)\, \phi_j(K \cap \vartheta Q, \mathrm{d}x)\, \nu(\mathrm{d}\vartheta) \\ &= \sum_{k=j}^{d-1} \int_{S^{d-1}} f \,\mathrm{d}\phi_k(K, \cdot) \int_{S^{d-1}} g \,\mathrm{d}\phi_{d-1-k+j}(Q, \cdot) \end{aligned}$$

for all continuous functions $f, g : S^{d-1} \to \mathbb{R}$. The proof of this relation and hence of (6.60) is now completed by approximating K by polytopes, using the weak continuity of the curvature measures and the bounded convergence theorem.

Due to the inversion invariance of the measure ν, the result (6.60) can be written in the form

$$\int_{SO_d} \phi_j(Q \cap \vartheta K, B \cap \vartheta A)\, \nu(\mathrm{d}\vartheta) = \sum_{k=j}^{d-1} \phi_k(Q, B)\phi_{d-1-k+j}(K, A),$$

valid for polytopes Q and convex bodies K. As before, the polytope Q can now similarly be replaced by a general convex body.

The final extension to the spherical convex ring $\mathcal{R}_s$ is also similar to the Euclidean case. □

As a particular case of the spherical kinematic formula, we note its global version

$$\int_{SO_d} v_j(K \cap \vartheta M)\, \nu(\mathrm{d}\vartheta) = \sum_{k=j}^{d-1} v_k(K) v_{d-1-k+j}(M), \tag{6.61}$$

for $K, M \in \mathcal{R}_s$ and $j = 0, \dots, d-1$. Due to the duality relations (6.45), (6.51), (6.52), we obtain a dual kinematic formula for convex bodies by applying (6.61) to polar bodies. The result is

$$\int_{SO_d} v_j(K \vee \vartheta M)\, \nu(\mathrm{d}\vartheta) = \sum_{k=-1}^{j} v_k(K) v_{j-k-1}(M)$$

for $K, M \in \mathcal{K}_s$ and $j = -1, \dots, d-2$.

In contrast to the Euclidean case, where V_0 is the Euler characteristic, we must use (6.55) to obtain the integral

$$\int_{SO_d} \chi(K \cap \vartheta M)\, \nu(\mathrm{d}\vartheta) = 2 \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{i=2k}^{d-1} v_i(K) v_{d-1-i+2k}(M)$$

for $K, M \in \mathcal{R}_s$.

The case of (6.57) where one of the sets is a subsphere deserves special attention. To have a concise notation, for $q \in \{0, \dots, d-1\}$ we choose $S_0 \in \mathcal{S}_q$ and denote by τ_q the image measure of ν under the map $\vartheta \mapsto \vartheta S_0$ from SO_d to $\mathcal{S}_q$. Thus, τ_q is the uniquely determined rotation invariant probability measure on $\mathcal{S}_q$. With this notation, we get the **spherical Crofton formula**

$$\int_{\mathcal{S}_q} \phi_j(K \cap S, A \cap S)\, \tau_q(\mathrm{d}S) = \phi_{d-1-q+j}(K, A)$$

for $K \in \mathcal{R}_s$, $A \in \mathcal{B}(S^{d-1})$, $q \in \{0, \dots, d-1\}$ and $j \in \{0, \dots, q\}$.

Defining

$$U_j(K) := \frac{1}{2} \int_{\mathcal{S}_{d-1-j}} \chi(K \cap S)\, \tau_{d-1-j}(\mathrm{d}S) \tag{6.62}$$

for $K \in \mathcal{R}_s$ and $j \in \{0, \dots, d-1\}$, we further obtain

$$U_j(K) = \sum_{k=0}^{\lfloor \frac{d-1-j}{2} \rfloor} v_{j+2k}(K) \tag{6.63}$$

If K in (6.62) is a convex body and not a subsphere, then for almost all $S \in \mathcal{S}_{d-1-j}$ the intersection $K \cap S$ is not a subsphere, hence $\chi(K \cap S) = 1$ if $K \cap S \neq \emptyset$. Therefore, $2U_j(K)$ is the total invariant measure of the set of all $(d-1-j)$-dimensional subspheres hitting the convex body K.

Notes for Section 6.5

1. A general source for integral geometry in spaces of constant curvature, from the differential geometric viewpoint, is the book by Santaló [662] and the literature quoted there, in particular Santaló [658].

Steiner formulas in spaces of constant curvature, in a differential geometric setting, were studied by Herglotz [336], Allendoerfer [23], Santaló [657]. A very general local version, for sets of positive reach, is due to Kohlmann [422]. The approach followed here is taken from Glasauer [264]. For differential geometric proofs of the spherical Gauss–Bonnet formula, we refer to Allendoerfer and Weil [24], Santaló [660, 661].

2. Since the linear relations between intrinsic volumes in Theorem 6.5.5 are special cases of the Steiner formula and the Gauss–Bonnet formula in spherical space, they appeared first, with differential geometric proofs, in the relevant literature quoted above. For spherical polytopes, in an equivalent version for polyhedral cones in $\mathbb{R}^d$, McMullen [469] has given interesting new proofs, more combinatorial in nature. The proof of (6.54) given here is based on a note by McMullen [470], which expands his remark at the beginning of §3 in [469].

3. The proof of Theorem 6.5.6, the spherical kinematic formula for curvature measures, is modeled after the proof given for its Euclidean counterpart by Schneider [676, Th. (6.1)]. The presentation given here follows the one by Glasauer [264]. This work contains many more results of spherical integral geometry, among them an abstract version of the kinematic formula and a version for support measures.

4. For spherical polytopes, or rather their spanned polyhedral cones, the functionals U_j were studied by Grünbaum [300], under the name of **Grassmann angles**.

5. In contrast to the Euclidean case, the spherical intrinsic volumes are in general not monotone under set inclusion. We restrict ourselves here to the set $\mathcal{K}_s \setminus \mathcal{S}_\bullet$. Clearly v_{d-1} is increasing under set inclusion, and so is v_{d-2}, being equal to U_{d-2}. By duality, v_{-1} and v_0 are decreasing. For $j \in \{1, \dots, d-3\}$, however, the functional v_j is not monotone. This follows, for example, by considering spherical balls B_r with spherical radius r, $0 \leq r \leq \pi/2$. From the Steiner formula one sees that

$$v_j(B_r) = \frac{\omega_d}{\omega_{j+1}\omega_{d-1-j}} \binom{d-2}{j} \cos^{d-2-j} r \sin^j r,$$

which is not a monotone function of r.

On the other hand, the functionals $U_0, \dots, U_{d-1}$ are increasing, as follows immediately from their definition. They may as well be considered as spherical analogs of the Euclidean intrinsic volumes, sharing with them the integral geometric interpretation as total measures of intersecting flats, respectively subspheres, of a suitable dimension. There is still another series of functionals which can be considered as counterparts to the Euclidean intrinsic volumes. Let $q \in \{0, \dots, d-1\}$ and $S \in \mathcal{S}_q$. The **spherical projection** of $K \in \mathcal{K}_s$ to S is defined by $K|S := S \cap (K \vee S^*)$. Then the function defined by

$$W_j(K) := \frac{1}{\omega_{j+1}} \int_{\mathcal{S}_j} \sigma_j(K|S)\, \tau_j(\mathrm{d}S)$$

for $K \in \mathcal{K}_s$ can be expressed in terms of intrinsic volumes. The relation

$$W_j(K) = \sum_{k=j}^{d-1} v_k(K)$$

was proved by Glasauer [264]. Clearly, also W_j is increasing. Thus, in spherical space, there are three series of functionals which, with some reason, can be considered as counterparts to the Euclidean intrinsic volumes. All functionals v_j, U_j, W_j are nonnegative, additive, continuous, and rotation invariant. The U_j and W_j are linear combinations of the v_j with nonnegative coefficients, and they are increasing under set inclusion.

It is a longstanding (and repeatedly asked) open question whether Hadwiger's characterization theorem 14.4.6 has a spherical counterpart. For example, if a function $\varphi : \mathcal{K}_s \to \mathbb{R}$ is additive, continuous and rotation invariant, must it be of the form $\varphi = \sum_{i=0}^{d-1} c_i v_i$ with constant coefficients $c_0, \dots, c_{d-1}$? An affirmative answer to the question posed in Note 6 of Section 14.4 would be an essential step towards a solution. As a variant, one might ask whether a function $\varphi : \mathcal{K}_s \setminus \mathcal{S}_\bullet \to \mathbb{R}$ which is additive, rotation invariant and increasing must be a nonnegative linear combination of the functions U_j or W_j.

6. Motivated by the Euclidean case, one may ask for inequalities existing between the functionals v_j, U_j, W_j. For example, among all convex bodies $K \in \mathcal{K}_s^p$ of given positive volume $v_{d-1}(K)$, which ones are extremal for one of the functionals? Only the following nontrivial cases seem to be known. The minimum of v_{d-2} is attained if and only if K is a ball (the classical isoperimetric problem in spherical space). The maximum of $v_{-1}(K) = v_{d-1}(K^*)$ (and, because of $U_1(K) = \frac{1}{2} - v_{d-1}(K^*)$; also the minimum of $U_1(K)$) is attained if and only if K is a ball. The latter result, which can be considered as a spherical counterpart to the Blaschke–Santaló inequality, was proved by Gao, Hug and Schneider [243].