5

Averaging with Invariant Measures

As soon as stochastic geometry deals with structures satisfying invariance properties with respect to some group, such as stationarity or isotropy in Euclidean spaces, there arises the need for a theory allowing averaging with respect to invariant measures. Integral geometry in the sense of Blaschke and Santaló is perfectly made for obtaining such averaging formulas. In this chapter we develop the basic tools, namely intersection formulas for fixed and moving geometric objects, where suitable geometric quantities of the intersections are integrated with respect to invariant measures. Basic facts about invariant measures on locally compact topological groups and homogeneous spaces, as far as they are needed for our purposes, are collected in the Appendix in Chapter 13.

The main purpose of Section 5.1 is the calculation of general kinematic integrals of the form

$$\int_{G_d} \varphi(K \cap gM)\, \mu(\mathrm{d}g) \tag{5.1}$$

for convex bodies K, M in $\mathbb{R}^d$. Here G_d is the motion group of $\mathbb{R}^d$, and the integration is with respect to its Haar measure μ. Such integrals are called 'kinematic', since one imagines M as moving and one averages the functional φ over all intersections of the moving set with the fixed set K. The integral (5.1) takes a simple form if the functional φ satisfies two natural assumptions, additivity and continuity. This result is known as Hadwiger's general integral geometric theorem (Theorem 5.1.2). The assumptions on φ are satisfied, in particular, by the intrinsic volumes V_j. A brief introduction to these important functionals from convex geometry is given in Section 14.2. For the intrinsic volumes, Hadwiger's general theorem reduces to the classical principal kinematic formula for convex bodies.

If the moving convex body is replaced by a moving flat, one is led to the Crofton formulas, giving explicit expressions for the integrals

$$\int_{A(d,k)} V_j(K \cap E)\, \mu_k(\mathrm{d}E),$$

where μ_k is the invariant measure on the affine Grassmannian $A(d,k)$.

The intrinsic volumes have local versions, the support measures or (generalized) curvature measures. These are introduced in Section 14.2, by means of a local Steiner formula. The purpose of the two subsequent sections is a derivation of the principal kinematic formula for curvature measures. Section 5.2 treats only integrations over the translation group, in a more general fashion with a view to later applications, and Section 5.3 then deals with the additional integrations over the rotation group. In each case, formulas for intrinsic volumes result by specialization.

Section 5.4 leaves the domain of convex or polyconvex sets and studies translative, kinematic and Crofton formulas for Hausdorff rectifiable sets and the Hausdorff measures of their intersections.

5.1 The Kinematic Formula for Additive Functionals

We make use of the homogeneous spaces and invariant measures of Euclidean geometry, as introduced in the Appendix. We assume that the reader is familiar with these, either from Chapter 13 or from other sources. We recall and collect here only the basic notation.

We denote by SO_d the group of proper (that is, orientation-preserving) rotations of $\mathbb{R}^d$. Being a compact group, it carries a unique rotation invariant (Borel) probability measure, which we denote by ν. The group of (proper) rigid motions of $\mathbb{R}^d$ is denoted by G_d. Let μ be its invariant (or Haar) measure, normalized so that

$$\mu(\{g \in G_d : gx \in B^d\}) = \kappa_d$$

for $x \in \mathbb{R}^d$. More explicitly, the mapping

$$\begin{aligned} \gamma : \mathbb{R}^d \times SO_d \to \quad & G_d \\ (x,\vartheta) \quad \mapsto \; & t_x \circ \vartheta, \end{aligned}$$

where t_x is the translation by the vector x, is a homeomorphism, and μ is the image measure of the product measure $\lambda \otimes \nu$ under γ.

The Grassmannian $G(d,q)$ of q-dimensional linear subspaces of $\mathbb{R}^d$, $q \in \{0,\dots,d\}$, is a compact homogeneous space with respect to the rotation group SO_d. It carries a unique rotation invariant probability measure, which we denote by ν_q. The affine Grassmannian $A(d,q)$, the space of q-flats in $\mathbb{R}^d$, is a locally compact homogeneous space with respect to the motion group and carries a locally finite motion invariant measure. We denote it by μ_q and normalize it so that

$$\mu_q(\{E \in A(d,q) : E \cap B^d \neq \emptyset\}) = \kappa_{d-q}.$$

More explicitly, we may choose a fixed subspace $L_q \in G(d,q)$, denote its orthogonal complement by $L_q^{\perp}$, and define mappings

$$\begin{aligned} \beta_q : SO_d &\to G(d,q) \\ \vartheta &\mapsto \vartheta L_q \end{aligned}$$

and

$$\begin{aligned} \gamma_q : L_q^\perp \times SO_d &\to \quad A(d,q) \\ (x,\vartheta) \quad &\mapsto \vartheta(L_q + x). \end{aligned} \tag{5.2}$$

These maps are continuous and surjective. Now, ν_q is the image measure of the invariant measure ν under β_q, and μ_q is the image measure of the product measure $\lambda_{L_q^\perp} \otimes \nu$ under γ_q, where $\lambda_{L_q^\perp}$ denotes the $(d-q)$-dimensional Lebesgue measure on $L_q^\perp$.

A basic task involving these invariant measures consists in the evaluation of integrals such as

$$\int_{G_d} \varphi(K \cap gM)\, \mu(\mathrm{d}g) \qquad \text{and} \qquad \int_{A(d,k)} \varphi(K \cap E)\, \mu_k(\mathrm{d}E),$$

for suitable sets K, M and functions φ. A typical simple case arises if K and M are convex bodies and $\varphi = \chi$, the Euler characteristic. Since $\chi(K) = 1$ for nonempty convex bodies K and $\chi(\emptyset) = 0$, we have

$$\int_{G_d} \chi(K \cap gM)\, \mu(\mathrm{d}g) = \mu(\{g \in G_d : K \cap gM \neq \emptyset\}),$$

which is the total invariant measure of all rigid motions g for which the body gM hits (that is, has nonempty intersection with) the body K. In order to get an idea of what an explicit computation will involve, we first consider the special case where M is a ball of radius $r > 0$. By the representation of the invariant measure μ described above, we then have

$$\begin{aligned} \int_{G_d} \chi(K \cap grB^d)\, \mu(\mathrm{d}g) &= \int_{SO_d} \int_{\mathbb{R}^d} \chi(K \cap (\vartheta rB^d + x))\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta) \\ &= V_d(K + rB^d), \end{aligned}$$

since $K \cap (\vartheta rB^d + x) \neq \emptyset$ if and only if x lies in the parallel body

$$K + rB^d = \{k + b : k \in K,\ b \in rB^d\}.$$

The **Steiner formula** of convex geometry (see (14.5)) tells us that

$$V_d(K + rB^d) = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K),$$

where $V_0, \dots, V_d$ are the **intrinsic volumes**. Once it is known (and this is not difficult to prove) that $V_d(K + rB^d)$ is a polynomial in r, the Steiner formula can serve to define the intrinsic volumes. For these functionals and their properties, as well as for a local version of the Steiner formula, we refer to Section 14.3 and the literature quoted there.

We see already from this special case, $M = rB^d$, that for the computation of the integrals $\int_{G_d} \chi(K \cap gM)\, \mu(\mathrm{d}g)$ the intrinsic volumes must play an essential role. It is a remarkable fact that no further functions are needed for the general case: the integrals

$$\int_{G_d} \chi(K \cap gM)\, \mu(\mathrm{d}g) \qquad \text{and} \qquad \int_{A(d,k)} \chi(K \cap E)\, \mu_k(\mathrm{d}E)$$

can be expressed in terms of the intrinsic volumes of K and M. These results will be obtained as special cases of formulas involving more general functions φ in the integrands. The essential property of these integrand functions, which makes explicit formulas possible, is their additivity. Generally, a function φ on $\mathcal{K}'$ with values in an abelian group is **additive** if

$$\varphi(K \cup M) + \varphi(K \cap M) = \varphi(K) + \varphi(M)$$

for all $K, M \in \mathcal{K}'$ with $K \cup M \in \mathcal{K}'$. For an additive function φ on $\mathcal{K}'$, one always extends the definition by $\varphi(\emptyset) := 0$. A reader not familiar with additive functionals on convex bodies is advised to have a look at Section 14.4. We shall make essential use of Hadwiger's characterization theorem for the intrinsic volumes, which is proved in that section.

To obtain these formulas for more general integrands, we begin with computing the integral

$$\psi(K) := \int_{A(d,k)} V_j(K \cap E)\, \mu_k(\mathrm{d}E) \tag{5.3}$$

for convex bodies $K \in \mathcal{K}'$, where V_j is the jth intrinsic volume, $j \in \{0, \dots, d\}$. (Recall that V_j is additive, and that we have defined $V_j(\emptyset) = 0$.) Equation (5.3) defines a functional ψ on $\mathcal{K}'$. Since the intrinsic volume V_j is additive, invariant under rigid motions, and continuous, it is not difficult to show that the functional ψ is additive, motion invariant and continuous (for the continuity, compare the argument used in the proof of Theorem 5.1.2 below). Therefore, Hadwiger's characterization theorem (Theorem 14.4.6) yields a representation

$$\psi(K) = \sum_{r=0}^{d} c_r V_r(K), \qquad K \in \mathcal{K}',$$

with constant coefficients $c_0, \dots, c_d$. Here only one coefficient is different from zero, due to the homogeneity property

$$\psi(\alpha K) = \alpha^{d-k+j} \psi(K)$$

for $\alpha > 0$; this property follows from the representation

$$\psi(K) = \int_{G(d,k)} \int_{L^\perp} V_j(K \cap (L+x))\, \lambda_{d-k}(\mathrm{d}x)\, \nu_k(\mathrm{d}L).$$

Since V_r is homogeneous of degree r, we see that $c_r = 0$ for $r \neq d-k+j$, hence

$$\int_{A(d,k)} V_j(K \cap E)\, \mu_k(\mathrm{d}E) = cV_{d-k+j}(K)$$

with some constant c. In order to determine this constant, we take for K the unit ball B^d. For $\epsilon \geq 0$, the Steiner formula gives

$$\sum_{j=0}^{d} \epsilon^{d-j} \kappa_{d-j} V_j(B^d) = V_d(B^d + \epsilon B^d) = (1+\epsilon)^d \kappa_d = \sum_{j=0}^{d} \epsilon^{d-j} \binom{d}{j} \kappa_d,$$

hence

$$V_j(B^d) = \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}} \qquad \text{for } j = 0, \dots, d.$$

In the following, we make use of the fact that the intrinsic volume V_j of a convex body does not depend on the dimension of the space in which the body is embedded. Choosing $L \in G(d,k)$, we obtain

$$\begin{aligned} cV_{d-k+j}(B^d) &= \int_{A(d,k)} V_j(B^d \cap E)\, \mu_k(\mathrm{d}E) \\ &= \int_{SO_d} \int_{L^\perp} V_j(B^d \cap \vartheta(L+x))\, \lambda_{d-k}(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta) \\ &= \int_{L^\perp \cap B^d} (1 - \|x\|^2)^{j/2} V_j(B^d \cap L)\, \lambda_{d-k}(\mathrm{d}x) \\ &= \binom{k}{j} \frac{\kappa_k}{\kappa_{k-j}} \int_{L^\perp \cap B^d} (1 - \|x\|^2)^{j/2}\, \lambda_{d-k}(\mathrm{d}x). \end{aligned}$$

Introducing polar coordinates, we transform the latter integral into a Beta integral and obtain

$$c = \binom{k}{j} \frac{\kappa_k \kappa_{d-k+j}}{V_{d-k+j}(B^d) \kappa_{k-j} \kappa_j} = c_j^k c_d^{d-k+j}.$$

Here we have denoted by

$$c_j^k := \frac{k! \kappa_k}{j! \kappa_j} \tag{5.4}$$

a frequently occurring constant. By using the identity

$$m! \kappa_m = 2^m \pi^{\frac{m-1}{2}} \Gamma\left(\frac{m+1}{2}\right),$$

it can also be put in the form

$$c_j^k = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)}.$$

To simplify later expressions, we also introduce the notation

$$c_{s_1,\dots,s_k}^{r_1,\dots,r_k} := \prod_{i=1}^{k} c_{s_i}^{r_i} = \prod_{i=1}^{k} \frac{r_i!\kappa_{r_i}}{s_i!\kappa_{s_i}}. \tag{5.5}$$

With these notations, we have obtained the following result.

Theorem 5.1.1. *Let $K \in \mathcal{K}'$ be a convex body. For $k \in \{1,\dots,d-1\}$ and $j \le k$ the* **Crofton formula**

$$\int_{A(d,k)} V_j(K \cap E)\,\mu_k(\mathrm{d}E) = c_{j,d}^{k,d-k+j} V_{d-k+j}(K) \tag{5.6}$$

holds.

The special case $j = 0$ of (5.6) gives

$$V_m(K) = c_{m,d-m}^{0,d} \int_{A(d,d-m)} \chi(K \cap E)\,\mu_{d-m}(\mathrm{d}E) \tag{5.7}$$

and thus provides an integral geometric interpretation of the intrinsic volumes: $V_m(K)$ is, up to a normalizing factor, the invariant measure of the set of $(d-m)$-flats intersecting K.

Using the explicit representation of the measure μ_{d-m} and the fact that the map $L \mapsto L^\perp$ transforms ν_{d-m} into ν_m, we can rewrite the representation (5.7) as

$$V_m(K) = c_{m,d-m}^{0,d} \int_{G(d,m)} \lambda_m(K|L)\,\nu_m(\mathrm{d}L), \tag{5.8}$$

where $K|L$ denotes the image of K under orthogonal projection to the subspace L. The special case $m = 1$ shows that V_1, up to a factor, is the mean width.

When we consider in the following a fixed and a moving convex body, we shall often have to exclude the touching positions. We say that the convex bodies K and M **touch** if $K \cap M \neq \emptyset$, but K and M can be separated weakly by a hyperplane. The following lemma is useful.

Lemma 5.1.1. *Let $K, M \in \mathcal{K}'$ be convex bodies, and let $(K_i)_{i\in\mathbb{N}}$, $(M_i)_{i\in\mathbb{N}}$ be sequences in $\mathcal{K}'$ with $K_i \to K$ and $M_i \to M$ for $i \to \infty$. Then the following holds:*

(a) *If $K \cap M = \emptyset$, then $K_i \cap M_i = \emptyset$ for all sufficiently large i.*

(b) *If $K \cap M \neq \emptyset$ and K and M do not touch, then $K_i \cap M_i \to K \cap M$ for $i \to \infty$.*

Proof. Assertion (a) follows immediately from the definition of convergence with respect to the Hausdorff metric.

To prove (b), let $x \in K \cap M$. We put $x_i := p(K_i \cap M_i, x)$ (the point in $K_i \cap M_i$ nearest to x, see Section 14.2) for those i for which $K_i \cap M_i \neq \emptyset$. We claim that x_i is defined for almost all i and that $x_i \to x$ for $i \to \infty$. Suppose this were false. Then there exists a ball B with center x such that $B \cap K_i \cap M_i = \emptyset$ holds for infinitely many i. For sufficiently large i we have $B \cap K_i \neq \emptyset$, since $K_i \to K$ and $x \in K$. By a standard separation theorem, for each such i there exists a hyperplane separating $B \cap K_i$ and M_i. A suitable subsequence of this sequence of hyperplanes converges to a hyperplane H; this hyperplane separates $B \cap K$ and M. Since $x \in K \cap M$, we have $x \in H$, hence H separates also K and M. This contradicts the assumption that K and M do not touch. It follows that $x_i \to x$ for $i \to \infty$.

Let $x_{i_j} \in K_{i_j} \cap M_{i_j}$ for some increasing sequence $(i_j)_{j \in \mathbb{N}}$, and assume that $x_{i_j} \to y$ for $j \to \infty$. Then $y \in K \cap M$.

The assertion $K_i \cap M_i \to K \cap M$ now follows from Theorem 12.2.2 (together with Theorem 12.3.4). □

From Hadwiger's characterization theorem, we now deduce a general kinematic formula, involving a functional on convex bodies that need not have any invariance property; crucial is the additivity of this functional. (Recall that an additive functional φ on $\mathcal{K}'$ is always extended to $\mathcal{K}$, by $\varphi(\emptyset) := 0$.)

Theorem 5.1.2 (Hadwiger's general integral geometric theorem). *If $\varphi : \mathcal{K}' \to \mathbb{R}$ is additive and continuous, then*

$$\int_{G_d} \varphi(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=0}^{d} \varphi_{d-k}(K) V_k(M) \tag{5.9}$$

for $K, M \in \mathcal{K}'$, where the coefficients $\varphi_{d-k}(K)$ are given by

$$\varphi_{d-k}(K) = \int_{A(d,k)} \varphi(K \cap E)\, \mu_k(\mathrm{d}E).$$

Proof. The μ-integrability of the integrand in (5.9) is seen as follows. For $K, M \in \mathcal{K}'$, let $G_d(K, M)$ be the set of all motions $g \in G_d$ for which K and gM touch. It is not difficult to check that $\gamma(x, \vartheta) \in G_d(K, M)$ if and only if $x \in \mathrm{bd}\,(K - \vartheta M)$ and, hence, that $\mu(G_d(K, M)) = 0$.

Let $g \in G_d \setminus G_d(K, M)$, and let $(M_j)_{j \in \mathbb{N}}$ be a sequence in $\mathcal{K}'$ converging to M. Then $gM_j \to gM$ and hence $K \cap gM_j \to K \cap gM$, by Lemma 5.1.1, thus $\varphi(K \cap gM_j) \to \varphi(K \cap gM)$ for $j \to \infty$. It follows that the map $g \mapsto \varphi(K \cap gM)$ is continuous outside a closed set of μ-measure zero. Moreover, the continuous function φ is bounded on the compact set $\{L \in \mathcal{K}' : L \subset K\}$, and

$$\mu(\{g \in G_d : \varphi(K \cap gM) \neq 0\}) \leq \mu(\{g \in G_d : K \cap gM \neq \emptyset\}) < \infty.$$

This shows the μ-integrability of the function $g \mapsto \varphi(K \cap gM)$.

Now we fix a convex body $K \in \mathcal{K}'$ and define

$$\psi(M) := \int_{G_d} \varphi(K \cap gM)\, \mu(\mathrm{d}g) \qquad \text{for } M \in \mathcal{K}'.$$

Then $\psi : \mathcal{K}' \to \mathbb{R}$ is additive and motion invariant. The foregoing consideration together with the dominated convergence theorem shows that ψ is continuous. By Hadwiger's characterization theorem, there exist constants $\varphi_0(K), \dots, \varphi_d(K)$ such that

$$\psi(M) = \sum_{i=0}^{d} \varphi_{d-i}(K) V_i(M)$$

for all $M \in \mathcal{K}'$. We have to determine the coefficients $\varphi_{d-i}(K)$.

Let $k \in \{0, \dots, d\}$, and choose $L_k \in G(d,k)$. Let $C \subset L_k$ be a k-dimensional unit cube with center 0, and let $r > 0$. Then

$$\psi(rC) = \sum_{i=0}^{d} \varphi_{d-i}(K) V_i(rC) = \sum_{i=0}^{k} \varphi_{d-i}(K) r^i V_i(C).$$

On the other hand, using the rotation invariance of λ, we get

$$\begin{aligned}
&\psi(rC)\\
&= \int_{G_d} \varphi(K \cap grC)\, \mu(\mathrm{d}g)\\
&= \int_{SO_d} \int_{\mathbb{R}^d} \varphi(K \cap (\vartheta rC + x))\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)\\
&= \int_{SO_d} \int_{L_k^\perp} \int_{L_k} \varphi(K \cap (\vartheta rC + \vartheta x_1 + \vartheta x_2))\, \lambda_k(\mathrm{d}x_1)\, \lambda_{d-k}(\mathrm{d}x_2)\, \nu(\mathrm{d}\vartheta)\\
&= \int_{SO_d} \int_{L_k^\perp} \int_{L_k} \varphi(K \cap [\vartheta r(C + x_1) + \vartheta x_2]) r^k\, \lambda_k(\mathrm{d}x_1)\, \lambda_{d-k}(\mathrm{d}x_2)\, \nu(\mathrm{d}\vartheta).
\end{aligned}$$

Comparison gives

$$\begin{aligned}
&\varphi_{d-k}(K)\\
&= \lim_{r\to\infty} \int_{SO_d} \int_{L_k^\perp} \int_{L_k} \varphi(K \cap [\vartheta r(C + x_1) + \vartheta x_2])\, \lambda_k(\mathrm{d}x_1)\, \lambda_{d-k}(\mathrm{d}x_2)\, \nu(\mathrm{d}\vartheta).
\end{aligned}$$

For $r \to \infty$, we have

$$\varphi(K \cap [\vartheta r(C + x_1) + \vartheta x_2]) \to \begin{cases} \varphi(K \cap \vartheta(L_k + x_2)) & \text{if } 0 \in \operatorname{relint}(C + x_1),\\ 0 & \text{if } 0 \notin C + x_1. \end{cases}$$

Hence, the dominated convergence theorem yields

$$\varphi_{d-k}(K) = \int_{SO_d} \int_{L_k^\perp} \varphi(K \cap \vartheta(L_k + x_2))\lambda_k(C)\,\lambda_{d-k}(\mathrm{d}x_2)\,\nu(\mathrm{d}\vartheta)$$
$$= \int_{A(d,k)} \varphi(K \cap E)\,\mu_k(\mathrm{d}E),$$

as asserted. □

In Theorem 5.1.2 we can choose for φ, in particular, the intrinsic volume V_j. In this case, the Crofton formula (5.6) gives

$$(V_j)_{d-k}(K) = \int_{A(d,k)} V_j(K \cap E)\,\mu_k(\mathrm{d}E) = c_{j,d}^{k,d-k+j} V_{d-k+j}(K).$$

Hence, we obtain the following result.

Theorem 5.1.3. *Let $K, M \in \mathcal{K}'$ be convex bodies, and let $j \in \{0, \ldots, d\}$. Then the* **principal kinematic formula**

$$\int_{G_d} V_j(K \cap gM)\,\mu(\mathrm{d}g) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} V_k(K) V_{d-k+j}(M) \tag{5.10}$$

holds.

We note that the special case $j = 0$, or

$$\int_{G_d} \chi(K \cap gM)\,\mu(\mathrm{d}g) = \sum_{k=0}^{d} c_{0,d}^{k,d-k} V_k(K) V_{d-k}(M),$$

gives the total measure of the set of rigid motions bringing M into a hitting position with K.

Hadwiger's general formula can be iterated, that is, extended to a finite number of moving convex bodies.

Theorem 5.1.4. *Let $\varphi : \mathcal{K}' \to \mathbb{R}$ be additive and continuous, and let $K_0, K_1, \ldots, K_k \in \mathcal{K}'$, $k \geq 1$, be convex bodies. Then*

$$\int_{G_d} \cdots \int_{G_d} \varphi(K_0 \cap g_1 K_1 \cap \ldots \cap g_k K_k)\,\mu(\mathrm{d}g_1) \cdots \mu(\mathrm{d}g_k)$$
$$= \sum_{\substack{r_0, \ldots, r_k = 0 \\ r_0 + \ldots + r_k = kd}}^{d} c_{d-r_0}^{d} \varphi_{r_0}(K_0) \prod_{i=1}^{k} c_d^{r_i} V_{r_i}(K_i),$$

where the coefficients are given by (5.4).

The specialization $\varphi = V_j$ yields the following.

Theorem 5.1.5 (Iterated kinematic formula). *Let $K_0, K_1, \dots, K_k \in \mathcal{K}'$, $k \geq 1$, be convex bodies, and let $j \in \{0, \dots, d\}$. Then*

$$\int_{G_d} \cdots \int_{G_d} V_j(K_0 \cap g_1 K_1 \cap \ldots \cap g_k K_k)\, \mu(\mathrm{d}g_1) \cdots \mu(\mathrm{d}g_k)$$

$$= \sum_{\substack{m_0,\dots,m_k=j \\ m_0+\ldots+m_k=kd+j}}^{d} c_j^d \prod_{i=0}^{k} c_d^{m_i} V_{m_i}(K_i).$$

Proof. We prove Theorem 5.1.4. The proof proceeds by induction with respect to k. Theorem 5.1.2 is the case $k = 1$. Suppose that $k \geq 1$ and that the assertion of Theorem 5.1.4, and hence that of Theorem 5.1.5, has been proved for $k+1$ convex bodies. Let $K_0, \dots, K_{k+1} \in \mathcal{K}'$. Using Fubini's theorem twice, the invariance of the measure μ, and Theorem 5.1.2 followed by Theorem 5.1.5 for $k+1$ convex bodies, we obtain

$$\int_{G_d} \cdots \int_{G_d} \varphi(K_0 \cap g_1 K_1 \cap \ldots \cap g_{k+1} K_{k+1})\, \mu(\mathrm{d}g_1) \cdots \mu(\mathrm{d}g_{k+1})$$

$$= \int_{G_d} \cdots \int_{G_d} \left[\int_{G_d} \varphi(K_0 \cap g_1(K_1 \cap g_2 K_2 \cap \ldots \cap g_{k+1} K_{k+1}))\, \mu(\mathrm{d}g_1) \right]$$

$$\times\, \mu(\mathrm{d}g_2) \cdots \mu(\mathrm{d}g_{k+1})$$

$$= \int_{G_d} \cdots \int_{G_d} \sum_{j=0}^{d} \varphi_{d-j}(K_0) V_j(K_1 \cap g_2 K_2 \cap \ldots \cap g_{k+1} K_{k+1})$$

$$\times\, \mu(\mathrm{d}g_2) \cdots \mu(\mathrm{d}g_{k+1})$$

$$= \sum_{j=0}^{d} c_j^d \varphi_{d-j}(K_0) \sum_{\substack{m_0,\dots,m_k=j \\ m_0+\ldots+m_k=kd+j}}^{d} c_d^{m_0} \cdots c_d^{m_k} V_{m_0}(K_1) \cdots V_{m_k}(K_{k+1})$$

$$= \sum_{\substack{r_0,\dots,r_{k+1}=0 \\ r_0+\ldots+r_{k+1}=(k+1)d}}^{d} c_{d-r_0}^d \varphi_{r_0}(K_0) c_d^{r_1} \cdots c_d^{r_{k+1}} V_{r_1}(K_1) \cdots V_{r_{k+1}}(K_{k+1}).$$

This completes the proof. □

Remark on renormalization. The preceding formulas suggest renormalization of the intrinsic volumes, by putting

$$\widetilde{V}_j := c_d^j V_j,$$

and also of the invariant measures on the affine Grassmannians, by putting

$$\widetilde{\mu}_k := c_k^d \mu_k.$$

Then the Crofton formula (5.6) becomes

$$\int_{A(d,k)} \widetilde{V}_j(K \cap E)\, \widetilde{\mu}_k(\mathrm{d}E) = \widetilde{V}_{d-k+j}(K).$$

Hadwiger's general integral geometric theorem reads

$$\int_{G_d} \varphi(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=0}^{d} \widetilde{\varphi}_{d-k}(K)\widetilde{V}_k(M)$$

with

$$\widetilde{\varphi}_{d-k}(K) = \int_{A(d,k)} \varphi(K \cap E)\, \widetilde{\mu}_k(\mathrm{d}E).$$

In particular,

$$\widetilde{(\widetilde{V}_j)}_{d-k} = \widetilde{V}_{d-k+j}.$$

The principal kinematic formula (5.10) simplifies to

$$\int_{G_d} \widetilde{V}_j(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=j}^{d} \widetilde{V}_k(K)\widetilde{V}_{d-k+j}(M). \tag{5.11}$$

Theorem 5.1.4 becomes

$$\int_{G_d} \cdots \int_{G_d} \varphi(K_0 \cap g_1 K_1 \cap \ldots \cap g_k K_k)\, \mu(\mathrm{d}g_1) \cdots \mu(\mathrm{d}g_k)$$
$$= \sum_{\substack{r_0,\ldots,r_k=0 \\ r_0+\ldots+r_k=kd}}^{d} \widetilde{\varphi}_{r_0}(K_0) \prod_{i=1}^{k} \widetilde{V}_{r_i}(K_i),$$

and the iterated kinematic formula attains the form

$$\int_{G_d} \cdots \int_{G_d} \widetilde{V}_j(K_0 \cap g_1 K_1 \cap \ldots \cap g_k K_k)\, \mu(\mathrm{d}g_1) \cdots \mu(\mathrm{d}g_k)$$
$$= \sum_{\substack{m_0,\ldots,m_k=j \\ m_0+\ldots+m_k=kd+j}}^{d} \prod_{i=0}^{k} \widetilde{V}_{m_i}(K_i).$$

Although these simplifications increase the elegance of the formulas, we retain the original normalization of the intrinsic volumes, since a different normalization might lead to confusion in several other instances.

Remark on extension to the convex ring. All the integral geometric formulas of this section remain valid if the involved convex bodies are replaced by polyconvex sets, that is, finite unions of convex bodies, and the involved additive functionals are replaced by their additive extensions to the convex ring. The simple principle of such extensions will be explained at the end of Section 5.2.

Notes for Section 5.1

1. Integral geometry as a subject of its own was first presented in two booklets by Blaschke in 1935 and 1937; a third edition [107] appeared in 1955 (see also vol. 2 of Blaschke's Collected Works [108]). The earlier development and its connection with geometric probability are subsumed in the book by Deltheil [202]. Introductions to integral geometry, from distinctly different points of view, were given by Santaló [659], Hadwiger [307, ch. 6], Stoka [738]. The standard source on integral geometry is the monograph by Santaló [662]. It stresses the applications to geometric probability. In a similar spirit is the book by Ren [635]. The book by Voss [772] describes integral geometry as a tool for stereology and image reconstruction.

The survey by Schneider and Wieacker [720] emphasizes the relations to convex geometry, and the article by Hug and Schneider [369] surveys integral geometric intersection formulas.

Combinatorial aspects of integral geometry are in the foreground of the original approaches in the books by Ambartzumian [34] and by Klain and Rota [416].

2. The principal kinematic formula (5.10) is a central result of classical integral geometry. For easier comparison with older literature, we write the special case $j = 0$ in terms of the Euler characteristic $\chi = V_0$ and the quermassintegrals, which are defined by (14.6). It then takes the form

$$\int_{G_d} \chi(K \cap gM)\, \mu(\mathrm{d}g) = \frac{1}{\kappa_d} \sum_{k=0}^{d} \binom{d}{k} W_k(K) W_{d-k}(M). \tag{5.12}$$

Here K and M can be arbitrary polyconvex sets. Often, only (5.12) is called the **principal kinematic formula**. It goes back to Blaschke and to Santaló, under different assumptions on the sets occurring in it. Hints to the origins can be found in the work of Blaschke [106], Hadwiger [307], Santaló [662].

When comparing with this literature, one has to observe that Santaló and Hadwiger normalize the invariant measure on the rotation group so that SO_d has total measure given by

$$c_d := \frac{d!}{2} \kappa_1 \cdots \kappa_d.$$

Under this normalization, the right side of (5.12) attains the additional factor c_d, and in Santaló's work the further factor 2, since Santaló integrates also over the improper rigid motions. (The constant $\mathcal{O}_k$ that often occurs in Santaló's work is given by $(k+1)\kappa_{k+1}$.)

If K is a convex body whose boundary is a regular (twice continuously differentiable) hypersurface, then

$$dW_i(K) = \int_{\mathrm{bd}\, K} H_{i-1}\, \mathrm{d}S =: M_{i-1}(\mathrm{bd}\, K) \qquad \text{for } i = 1, \dots, d,$$

where H_{i-1} denotes the $(i-1)$th normalized elementary symmetric function of the principal curvatures of $\mathrm{bd}\, K$ (and M_{i-1} is the notation used by Santaló). With this interpretation of the functionals W_i as curvature integrals, equation (5.12) holds also if K and M are non-convex domains of $\mathbb{R}^d$ with boundary hypersurfaces of class C^2. This is the differential-geometric version of the principal kinematic formula. It goes back to Chern and Yien [171] and was proved with greater care by Chern [169]; see also Santaló [662, pp. 262 ff]. There (p. 269) one also finds a differential-geometric version of the formula

$$\int_{G_d} W_i(K \cap gM)\, \mu(\mathrm{d}g) = \frac{1}{\kappa_d} \sum_{k=d-i}^{d} C_{ki} W_{i+k-d}(K) W_{d-k}(M)$$

with

$$C_{ki} := \binom{i}{d-k} \frac{\kappa_k \kappa_i \kappa_{2d-k-i}}{\kappa_{d-k}\kappa_{d-i}\kappa_{k+i-d}}$$

($i = 1, \dots, d$). For elements of the convex ring, this is formula (5.10), rewritten in terms of the quermassintegrals W_i. Further kinematic intersection formulas in a differential-geometric version, valid for lower-dimensional compact differentiable submanifolds without boundary, are due to Chern [170]; see also Chapter V in the book by Sulanke and Wintgen [749]. This book, in contrast to [662], also provides the technical foundations that are required for using the elegant machinery of differential forms in the derivation of integral geometric formulas.

Also for the Crofton formulas, there are differential-geometric versions; one finds them in the quoted books of Sulanke and Wintgen and of Santaló.

A common generalization of kinematic formulas for convex bodies and for smooth compact submanifolds is Federer's extension to sets of positive reach; see Note 1 for Section 5.3.

3. The approach to integral geometric formulas for convex bodies that uses the axiomatic characterization of the intrinsic volumes, goes in principle back to W. Blaschke. It came into full force only when Hadwiger had proved his characterization theorem (Theorem 14.4.6). Hadwiger's general integral geometric theorem (Theorem 5.1.2) was proved in this way in [306, 307]; no other proof is currently known.

4. Hadwiger's general integral geometric theorem provides a kinematic formula for arbitrary additive continuous functions on convex bodies. For integrations over the translation group, an analogous result can be proved for simply additive functions. Let φ be a continuous real function on $\mathcal{K}'$ which is a simple valuation, that is, additive and satisfying $\varphi(K) = 0$ for convex bodies of dimension less than d. Then

$$\int_{\mathbb{R}^d} \varphi(K \cap (M + x))\, \lambda(\mathrm{d}x) = \varphi(K) V_d(M) + \int_{S^{d-1}} f_{K,\varphi}(u)\, S_{d-1}(M, \mathrm{d}u)$$

for convex bodies $K, M \in \mathcal{K}'$, where the function $f_{K,\varphi} : S^{d-1} \to \mathbb{R}$ is given by

$$f_{K,\varphi}(u) = \int_{-h(K,-u)}^{h(K,u)} \varphi(K \cap H^-(u,\tau))\, \mathrm{d}\tau - \varphi(K) h(K,u).$$

Here $h(K,\cdot)$ is the support function of K and $H^-(u,\tau)$ is the closed halfspace $\{x \in \mathbb{R}^d : \langle x, u \rangle \le \tau\}$; the measure $S_{d-1}(M,\cdot)$ is the surface area measure of M (see Section 14.2). This formula was proved by Schneider [708].

5. Alesker's work on valuations (see [22] for a survey) has also shed new light on kinematic formulas and their generalizations. Part of his work extends Hadwiger's characterization theorem and its integral geometric applications. Let G be a compact subgroup of the orthogonal group O_d acting transitively on the unit sphere S^{d-1}. Let $\mathbf{Val}^G$ denote the vector space of continuous, translation invariant and G-invariant real valuations on $\mathcal{K}$. Alesker [18] has shown that $\mathbf{Val}^G$ has finite dimension. In [19], he provided explicit bases for the case of $G = U(n)$ (where $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$), thus establishing a unitary counterpart to Hadwiger's characterization theorem (with a much deeper proof, though). For the case of $SU(2)$, see Alesker [21]. Further, Alesker [20] has introduced a multiplication for continuous, translation invariant valuations. With this, $\mathbf{Val}^G$ becomes a graded algebra over $\mathbb{R}$, satisfying the Poincaré duality. The structure of this algebra was determined for the case $G = SO_d$ by Alesker [20], and for $G = U(d)$ by Fu [239].

Applications to kinematic formulas, where the role of the rotation group in the classical case is now played by a group G as above, were investigated by Alesker [19], Fu [239], Bernig and Fu [95, 96], Bernig [92].

6. Iterations of the principal kinematic formula, as in Theorem 5.1.5, were used, for instance, by Streit [747].

7. As explained in the remark on renormalization, the intrinsic volumes and the invariant measures on the Grassmannians can be renormalized so that the principal kinematic formula takes the particularly simple form (5.11), where all the coefficients of the bilinear expression are equal to one. Some authors have elaborated upon this fact from a structural point of view; see Nijenhuis [585] and Fu [239].

Questions of normalization with desirable properties are also an issue in the book by Klain and Rota [416].

5.2 Translative Integral Formulas

Our next major aim is an extension of the principal kinematic formula (5.10) to the curvature measures Φ_m, which are introduced in Section 14.2. Thus, we want to compute the integral

$$\int_{G_d} \Phi_j(K \cap gM, A \cap gB)\, \mu(\mathrm{d}g)$$
$$= \int_{SO_d} \int_{\mathbb{R}^d} \Phi_j(K \cap (\vartheta M + x), A \cap (\vartheta B + x))\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$

for convex bodies K, M and Borel sets $A, B \in \mathcal{B}(\mathbb{R}^d)$ (recall that $\Phi_j(\emptyset, \cdot) = 0$, by definition). The result will be stated in Theorem 5.3.2. In this section we study only the inner integral

$$\int_{\mathbb{R}^d} \Phi_j(K \cap (M + x), A \cap (B + x))\, \lambda(\mathrm{d}x). \tag{5.13}$$

Integrals of this type can be considered to extend over the translation group of $\mathbb{R}^d$ and are therefore known as **translative integrals**. The computation of (5.13) is the first step towards a direct proof of the kinematic formula for curvature measures, but is also of independent interest, in view of later applications to non-isotropic stochastic models.

The integral (5.13) is easily computed for $j = d$. This is a special case of a simple but often useful integral geometric formula with respect to the translation group, which can be obtained without much effort (and was already used in Sections 4.5 and 4.6). It is quite general and is an immediate consequence of the translation and inversion invariance of the Lebesgue measure.

Theorem 5.2.1. *If α is a σ-finite measure on $\mathbb{R}^d$ and if $A, B \in \mathcal{B}(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \alpha(A \cap (B+t))\, \lambda(\mathrm{d}t) = \alpha(A)\lambda(B).$$

Proof. Fubini's theorem gives

$$\begin{aligned}\int_{\mathbb{R}^d} \alpha(A \cap (B+t))\, \lambda(\mathrm{d}t) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_A(x)\mathbf{1}_{B+t}(x)\, \alpha(\mathrm{d}x)\, \lambda(\mathrm{d}t)\\ &= \int_{\mathbb{R}^d} \mathbf{1}_A(x) \int_{\mathbb{R}^d} \mathbf{1}_{-B+x}(t)\, \lambda(\mathrm{d}t)\, \alpha(\mathrm{d}x)\\ &= \int_{\mathbb{R}^d} \mathbf{1}_A(x)\lambda(-B+x)\, \alpha(\mathrm{d}x)\\ &= \alpha(A)\lambda(B),\end{aligned}$$

as asserted. ☐

The formulas

$$\int_{\mathbb{R}^d} V_d(K \cap (M+x))\, \lambda(\mathrm{d}x) = V_d(K)V_d(M) \tag{5.14}$$

and

$$\int_{\mathbb{R}^d} V_{d-1}(K \cap (M+x))\, \lambda(\mathrm{d}x) = V_{d-1}(K)V_d(M) + V_d(K)V_{d-1}(M) \tag{5.15}$$

for convex bodies $K, M \in \mathcal{K}'$ are special cases of Theorem 5.2.1. The second is obtained by applying the theorem twice, taking for the measure α the $(d-1)$-dimensional Hausdorff measure, restricted to the boundary of one of the bodies. We do not carry this out here, since we shall give a detailed proof of the much more general Theorem 5.2.3.

The corresponding translative formulas for the intrinsic volumes V_j with $j < d-1$ are no longer so simple as (5.14) and (5.15). This is already seen from the case $j = 0$. Since $V_0(K \cap (M+x)) = 1$ is equivalent to $K \cap (M+x) \neq \emptyset$ and hence to $x \in K - M$, we have

$$\int_{\mathbb{R}^d} V_0(K \cap (M+x))\, \lambda(\mathrm{d}x) = V_d(K - M).$$

Similarly to the case of the Steiner formula, the volume of a sum $K + \epsilon L$, $\epsilon \geq 0$, for convex bodies $K, L \in \mathcal{K}'$, can be expanded into a polynomial in ϵ (see (14.17)). In this way, one obtains

$$\int_{\mathbb{R}^d} V_0(K \cap (M + x))\, \lambda(\mathrm{d}x) = \sum_{k=0}^{d} \binom{d}{k} V(\underbrace{K, \ldots, K}_{k}, \underbrace{-M, \ldots, -M}_{d-k}), \quad (5.16)$$

where V on the right side denotes a mixed volume. We see that the result involves functionals that depend on K and M simultaneously. It is in general not possible to separate the roles of K and M, as was the case with the principal kinematic formula (5.10). There, the resulting bilinear form owes its existence to the further integration over the rotation group. The occurrence of simultaneous functionals is typical for translative integral geometry.

Our proof of translative and kinematic formulas for curvature measures is prepared by a measurability lemma. We recall from the proof of Theorem 5.1.2 that the set $G_d(K, M) = \{g \in G_d : K \text{ and } M \text{ touch}\}$ satisfies $\mu(G_d(K, M)) = 0$.

Lemma 5.2.1. *Let $K, M \in \mathcal{K}'$ and $A, B \in \mathcal{B}(\mathbb{R}^d)$, let $j \in \{0, \ldots, d\}$. The mapping*

$$x \mapsto \Phi_j(K \cap (M + x), A \cap (B + x)), \qquad x \in \mathbb{R}^d,$$

is measurable on $\mathbb{R}^d \setminus \mathrm{bd}\,(K - M)$, where $\lambda(\mathrm{bd}\,(K - M)) = 0$.

The mapping

$$g \mapsto \Phi_j(K \cap gM, A \cap gB), \qquad g \in G_d,$$

is measurable on $G_d \setminus G_d(K, M)$, where $\mu(G_d(K, M)) = 0$.

Proof. It suffices to prove the second assertion, since the proof of the first one is analogous. For fixed $(x, \vartheta) \in \mathbb{R}^d \times SO_d$, we define

$$\begin{aligned} T_{x,\vartheta} : \mathbb{R}^d &\to \quad \mathbb{R}^d \times \mathbb{R}^d \\ y &\mapsto (y, \vartheta^{-1}(y - x)) \end{aligned}$$

and the image measure

$$\varphi^{(j)}(x, \vartheta, K, M, \cdot) := T_{x,\vartheta}(\Phi_j(K \cap (\vartheta M + x), \cdot)).$$

Then $\varphi^{(j)}(x, \vartheta, K, M, \cdot)$ is a finite measure on $\mathbb{R}^d \times \mathbb{R}^d$, and

$$\varphi^{(j)}(x, \vartheta, K, M, A \times B) = \Phi_j(K \cap (\vartheta M + x), A \cap (\vartheta B + x))$$

for $A, B \in \mathcal{B}(\mathbb{R}^d)$. By the transformation formula for integrals,

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}^d} f(y, z)\, \varphi^{(j)}(x, \vartheta, K, M, \mathrm{d}(y, z)) \\ &= \int_{\mathbb{R}^d} f(y, \vartheta^{-1}(y - x))\, \Phi_j(K \cap (\vartheta M + x), \mathrm{d}y) \end{aligned}$$

for $f \in \mathbf{C}(\mathbb{R}^d \times \mathbb{R}^d)$. By Lemma 5.1.1, the mapping $(x, \vartheta) \mapsto K \cap (\vartheta M + x)$ is continuous outside the set $\gamma^{-1}(G_d(K, M))$, hence, by Theorem 14.2.2(c) the mapping $(x, \vartheta) \mapsto \Phi_j(K \cap (\vartheta M + x), \cdot)$ is continuous (with respect to the weak topology) on $(\mathbb{R}^d \times SO_d) \setminus \gamma^{-1}(G_d(K, M))$. For $f \in \mathbf{C}(\mathbb{R}^d \times \mathbb{R}^d)$ and for $(x_i, \vartheta_i) \to (x_0, \vartheta_0) \notin \gamma^{-1}(G_d(K, M))$ we deduce that

$$\int_{\mathbb{R}^d} f(y, \vartheta_i^{-1}(y - x_i))\, \Phi_j(K \cap (\vartheta_i M + x_i), \mathrm{d}y)$$
$$\to \int_{\mathbb{R}^d} f(y, \vartheta_0^{-1}(y - x_0))\, \Phi_j(K \cap (\vartheta_0 M + x_0), \mathrm{d}y)$$

(since $\Phi_j(K \cap (\vartheta_i M + x_i), \cdot)$ vanishes outside a suitable compact set independent of i, and since f is uniformly continuous on any compact set). Therefore, the mapping

$$(x, \vartheta) \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} f(y, z)\, \varphi^{(j)}(x, \vartheta, K, M, \mathrm{d}(y, z))$$

is continuous on $\mathbb{R}^d \setminus \gamma^{-1}(G_d(K, M))$. As shown in Lemma 12.1.1, this implies the measurability of the mapping

$$(x, \vartheta) \mapsto \varphi^{(j)}(x, \vartheta, K, M, A \times B) = \Phi_j(K \cap (\vartheta M + x), A \cap (\vartheta B + x))$$

on $\mathbb{R}^d \setminus \gamma^{-1}(G_d(K, M))$, for arbitrary $A, B \in \mathcal{B}(\mathbb{R}^d)$. $\square$

In the following, we shall have to use the subspace determinant $[L_1, \dots, L_k]$, which is introduced in Section 14.1. We extend its definition as follows. If $A_1, \dots, A_k \subset \mathbb{R}^d$ are nonempty subsets, we denote by $L(A_i)$ the linear subspace which is a translate of the affine hull of A_i, and we write

$$[A_1, \dots, A_k] := [L(A_1), \dots, L(A_k)]$$

if the latter is defined.

First we investigate a translative formula for polytopes. For the external angles, we refer to (14.10). For polytopes $K, M \in \mathcal{P}'$ and for faces F of K and G of M we define a **common external angle** by

$$\gamma(F, G; K, M) := \gamma(F \cap (G + x), K \cap (M + x)),$$

where $x \in \mathbb{R}^d$ is chosen so that

$$\operatorname{relint} F \cap \operatorname{relint}(G + x) \neq \emptyset.$$

Obviously, this definition does not depend on the special choice of x.

Two faces F and G of a polytope are said to be in **special position** if the linear subspaces $L(F)$ and $L(G)$ parallel to F and G are in special position, that is, satisfy

$$L(F) \cap L(G) \neq \{0\} \quad \text{and} \quad \mathrm{lin}\,(L(F) \cup L(G)) \neq \mathbb{R}^d.$$

For a face F of a polytope, the measure λ_F is defined by

$$\lambda_F(A) := \lambda_{\dim F}(A \cap F) \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^d).$$

Theorem 5.2.2. *If $K, M \in \mathcal{P}'$ are polytopes, $A, B \in \mathcal{B}(\mathbb{R}^d)$ are Borel sets and $j \in \{0, \dots, d\}$, then*

$$\int_{\mathbb{R}^d} \Phi_j(K \cap (M + x), A \cap (B + x))\, \lambda(\mathrm{d}x) = \sum_{k=j}^{d} \Phi_k^{(j)}(K, M; A \times B)$$

with finite measures $\Phi_k^{(j)}(K, M; \cdot)$ on $\mathbb{R}^d \times \mathbb{R}^d$, which are defined by

$$\Phi_k^{(j)}(K, M; \cdot) := \sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{d-k+j}(M)} \gamma(F, G; K, M)[F, G] \lambda_F \otimes \lambda_G$$

$(k = j, \dots, d)$. In particular,

$$\Phi_j^{(j)}(K, M; A \times B) = \Phi_j(K, A) \Phi_d(M, B),$$

$$\Phi_d^{(j)}(K, M; A \times B) = \Phi_d(K, A) \Phi_j(M, B).$$

Proof. Let

$$I := \int_{\mathbb{R}^d} \Phi_j(K \cap (M + x), A \cap (B + x))\, \lambda(\mathrm{d}x).$$

By Lemma 5.2.1, this is well defined. The representation (14.13) gives

$$I = \int_{\mathbb{R}^d} \sum_{F' \in \mathcal{F}_j(K \cap (M+x))} \gamma(F', K \cap (M + x)) \lambda_{F'}(A \cap (B + x))\, \lambda(\mathrm{d}x).$$

The faces $F' \in \mathcal{F}_j(K \cap (M + x))$ are precisely the j-dimensional sets of the form $F' = F \cap (G + x)$, where $F \in \mathcal{F}_k(K)$ and $G \in \mathcal{F}_i(M)$ for suitable $k, i \in \{j, \dots, d\}$. For the computation of the integral I, only those vectors x are relevant that together with $F \cap (G + x) \neq \emptyset$ for a pair F, G satisfy $\mathrm{relint}\, F \cap \mathrm{relint}\,(G + x) \neq \emptyset$, since the remaining vectors x form a set of Lebesgue measure zero. Moreover, the pairs F, G for which $k + i < d$ or which are in special position, do not contribute to the integral, since for these we have

$$\lambda(\{x \in \mathbb{R}^d : F \cap (G + x) \neq \emptyset\}) = \lambda(F - G) = 0.$$

In the other cases, $\dim F' = \dim F + \dim G - d$, hence $k + i = d + j$. Thus, we obtain

$$
\begin{aligned}
I = & \sum_{k=j}^{d} \sum_{F\in\mathcal{F}_k(K)} \sum_{G\in\mathcal{F}_{d-k+j}(M)} \\
& \int_{\mathbb{R}^d} \gamma(F\cap(G+x), K\cap(M+x))\lambda_{F\cap(G+x)}(A\cap(B+x))\,\lambda(\mathrm{d}x) \\
= & \sum_{k=j}^{d} \sum_{F\in\mathcal{F}_k(K)} \sum_{G\in\mathcal{F}_{d-k+j}(M)} \gamma(F,G;K,M)J(F,G)
\end{aligned}
$$

with

$$J(F,G) := \int_{\mathbb{R}^d} \lambda_{F\cap(G+x)}(A\cap(B+x))\,\lambda(\mathrm{d}x).$$

For the computation of $J(F,G)$ we suppose, without loss of generality, that

$$0 \in L_1 := \operatorname{aff} F \cap \operatorname{aff} G.$$

Let

$$L_2 := L_1^\perp \cap \operatorname{aff} F, \qquad L_3 := L_1^\perp \cap \operatorname{aff} G,$$

and let $\lambda_j, \lambda_{k-j}, \lambda_{d-k}$ be the Lebesgue measure on L_1, L_2, L_3, respectively. Then $\mathbb{R}^d = L_1 \oplus L_2 \oplus L_3$, and $x \in \mathbb{R}^d$ can uniquely be written in the form $x = x_1 + x_2 + x_3$ with $x_i \in L_i$ for $i = 1,2,3$. Writing $A' := A\cap F$, $B' := B\cap G$, we have

$$
\begin{aligned}
J(F,G) = [F,G] \int_{L_3}\int_{L_2}\int_{L_1} & \lambda_{F\cap(G+x_1+x_2+x_3)}(A'\cap(B'+x_1+x_2+x_3)) \\
& \times \lambda_j(\mathrm{d}x_1)\,\lambda_{k-j}(\mathrm{d}x_2)\,\lambda_{d-k}(\mathrm{d}x_3).
\end{aligned}
$$

Since

$$(A'\cap(B'+x_1+x_2+x_3)) - x_2 = (A'-x_2)\cap(B'+x_1+x_3) \subset L_1,$$

we obtain

$$
\begin{aligned}
& \int_{L_1} \lambda_{F\cap(G+x_1+x_2+x_3)}(A'\cap(B'+x_1+x_2+x_3))\,\lambda_j(\mathrm{d}x_1) \\
& = \int_{L_1} \lambda_j((A'-x_2)\cap(B'+x_3+x_1))\,\lambda_j(\mathrm{d}x_1) \\
& = \lambda_j((A'-x_2)\cap L_1)\lambda_j((B'+x_3)\cap L_1),
\end{aligned}
$$

by Theorem 5.2.1. Fubini's theorem yields

$$\int_{L_2} \lambda_j((A'-x_2)\cap L_1)\,\lambda_{k-j}(\mathrm{d}x_2) = \lambda_j \otimes \lambda_{k-j}(A') = \lambda_F(A)$$

and

$$\int_{L_3} \lambda_j((B' + x_3) \cap L_1)\, \lambda_{d-k}(\mathrm{d}x_3) = \lambda_j \otimes \lambda_{d-k}(B') = \lambda_G(B).$$

Altogether this gives

$$J(F,G) = [F,G]\lambda_F(A)\lambda_G(B),$$

and thus the representation of the measure $\Phi_k^{(j)}(K,M;\cdot)$ as stated in the theorem.

In the special case $k = j$ we have

$$\begin{aligned}
&\sum_{F\in\mathcal{F}_k(K)} \sum_{G\in\mathcal{F}_{d-k+j}(M)} \gamma(F,G;K,M)[F,G]\lambda_F \otimes \lambda_G \\
&= \sum_{F\in\mathcal{F}_j(K)} \gamma(F,M;K,M)[F,M]\lambda_F \otimes \lambda_M \\
&= \sum_{F\in\mathcal{F}_j(K)} \gamma(F,K)\lambda_F \otimes \lambda_M \\
&= \Phi_j(K,\cdot) \otimes \Phi_d(M,\cdot).
\end{aligned}$$

Similarly, for $k = d$ we obtain the measure $\Phi_d(K,\cdot) \otimes \Phi_j(M,\cdot)$. □

Corollary 5.2.1. *If $K, M \in \mathcal{P}'$ are polytopes and $j \in \{0,\dots,d\}$, then*

$$\begin{aligned}
&\int_{\mathbb{R}^d} V_j(K \cap (M+x))\, \lambda(\mathrm{d}x) \\
&= V_j(K)V_d(M) + \sum_{k=j+1}^{d-1} V_k^{(j)}(K,M) + V_d(K)V_j(M),
\end{aligned}$$

where

$$V_k^{(j)}(K,M) := \sum_{F\in\mathcal{F}_k(K)} \sum_{G\in\mathcal{F}_{d-k+j}(M)} \gamma(F,G;K,M)[F,G]V_k(F)V_{d-k+j}(G).$$

Theorem 5.2.2 and Corollary 5.2.1 will now be extended, by means of approximation, to arbitrary convex bodies $K, M \in \mathcal{K}'$. In contrast to the case of polytopes, for the measures $\Phi_k^{(j)}(K,M;\cdot)$ and functionals $V_k^{(j)}(K,M)$ that occur, no simple explicit representations are known in the general case.

Theorem 5.2.3. *For convex bodies $K, M \in \mathcal{K}'$ and for $j \in \{0,\dots,d\}$, there exist finite measures $\Phi_{j+1}^{(j)}(K,M;\cdot),\dots,\Phi_{d-1}^{(j)}(K,M;\cdot)$ on $\mathbb{R}^d \times \mathbb{R}^d$, concentrated on $\mathrm{bd}\,K \times \mathrm{bd}\,M$, such that*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Phi_j(K \cap (M+x), A \cap (B+x))\, \lambda(\mathrm{d}x) \qquad (5.17) \\
&= \Phi_j(K,A)\Phi_d(M,B) + \sum_{k=j+1}^{d-1} \Phi_k^{(j)}(K,M;A\times B) + \Phi_d(K,A)\Phi_j(M,B)
\end{aligned}$$

for all $A, B \in \mathcal{B}(\mathbb{R}^d)$. In particular,

$$\int_{\mathbb{R}^d} V_j(K \cap (M+x))\, \lambda(\mathrm{d}x) = V_j(K)V_d(M) + \sum_{k=j+1}^{d-1} V_k^{(j)}(K,M) + V_d(K)V_j(M)$$

with $V_k^{(j)}(K,M) := \Phi_k^{(j)}(K,M;\mathbb{R}^d \times \mathbb{R}^d)$.

The measure $\Phi_k^{(j)}(K,M;\cdot)$ depends continuously on $K, M \in \mathcal{K}'$ and is homogeneous of degree k in K and of degree $d-k+j$ in M. It is additive in each of its first two arguments. For polytopes K, M, the measure $\Phi_k^{(j)}(K,M;\cdot)$ coincides with the one appearing in Theorem 5.2.2.

Proof. As was already verified in the proof of Theorem 5.2.2, the integrand on the left side of (5.17) is measurable for λ-almost all x, hence the integral in (5.17) is well defined. We now first remark that equality (5.17) is equivalent to

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x, x-y)\, \Phi_j(K \cap (M+y), \mathrm{d}x)\, \lambda(\mathrm{d}y)\\ &= \sum_{k=j}^{d} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y)\, \Phi_k^{(j)}(K,M;\mathrm{d}(x,y))\end{aligned} \tag{5.18}$$

for all continuous functions f on $\mathbb{R}^d \times \mathbb{R}^d$, provided that the measures $\Phi_k^{(j)}(K,M;\cdot)$ exist; here we have written

$$\Phi_j^{(j)}(K,M;\cdot) := \Phi_j(K,\cdot) \otimes \Phi_d(M,\cdot),$$
$$\Phi_d^{(j)}(K,M;\cdot) := \Phi_d(K,\cdot) \otimes \Phi_j(M,\cdot).$$

In fact, if (5.17) holds, then (5.18) is true for $f = \mathbf{1}_{A\times B}$, hence (5.18) follows for elementary functions and then, by a standard argument, for integrable functions. If (5.18) holds, then (5.17) is obtained for compact sets A, B, since $\mathbf{1}_{A\times B}$ is in this case the limit of a decreasing sequence of continuous functions, and for arbitrary Borel sets it then follows since both sides, as functions of A and B, are measures.

By Theorem 5.2.2, formulas (5.17) and (5.18) are valid if K and M are polytopes.

For convex bodies $K, M \in \mathcal{K}'$ and for a continuous function f on $\mathbb{R}^d \times \mathbb{R}^d$ we now define

$$J(f,K,M) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x, x-y)\, \Phi_j(K \cap (M+y), \mathrm{d}x)\, \lambda(\mathrm{d}y).$$

We show that $J(f,K,M)$ depends continuously on K and M. For this, let $K_i \to K$, $M_i \to M$ be convergent sequences in $\mathcal{K}'$. From Lemma 5.1.1 and Theorem 14.2.2(c) we infer the weak convergence

$$\varPhi_j(K_i \cap (M_i + y), \cdot) \xrightarrow{w} \varPhi_j(K \cap (M + y), \cdot)$$

and, therefore, the pointwise convergence

$$\int_{\mathbb{R}^d} f(x, x - y)\, \varPhi_j(K_i \cap (M_i + y), \mathrm{d}x) \to \int_{\mathbb{R}^d} f(x, x - y)\, \varPhi_j(K \cap (M + y), \mathrm{d}x)$$

for $i \to \infty$, for all $y \notin \mathrm{bd}\,(K - M)$. From this we deduce that

$$\begin{aligned} &\lim_{i\to\infty} J(f, K_i, M_i) \\ &= \int_{\mathbb{R}^d} \left(\lim_{i\to\infty} \int_{\mathbb{R}^d} f(x, x - y)\, \varPhi_j(K_i \cap (M_i + y), \mathrm{d}x) \right) \lambda(\mathrm{d}y) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, x - y)\, \varPhi_j(K \cap (M + y), \mathrm{d}x)\, \lambda(\mathrm{d}y) \\ &= J(f, K, M). \end{aligned}$$

Here we have applied the dominated convergence theorem. This is legitimate, since we can find a λ-integrable function of y dominating

$$\left| \int_{\mathbb{R}^d} f(x, x - y)\, \varPhi_j(K_i \cap (M_i + y), \mathrm{d}x) \right| \tag{5.19}$$

for all i. To see this, we choose a ball rB^d, $r > 0$, containing all K_i, M_i and hence also K and M, and denote by $\|f\|_r$ the maximum of the continuous function f on $rB^d \times rB^d$. Then

$$\left| \int_{\mathbb{R}^d} f(x, x - y)\, \varPhi_j(K_i \cap (M_i + y), \mathrm{d}x) \right| \le \|f\|_r V_j(K_i \cap (M_i + y)).$$

The monotonicity of the intrinsic volumes gives

$$V_j(K_i \cap (M_i + y)) \le V_j(K_i)\mathbf{1}_{K_i - M_i}(y),$$

and this yields the required function dominating (5.19).

For $r, s > 0$ we now define a continuous mapping $D_{r,s}$ from $\mathbb{R}^d \times \mathbb{R}^d$ into itself by

$$D_{r,s}(x, y) := \left(\frac{x}{r}, \frac{y}{s} \right) \qquad \text{for } x, y \in \mathbb{R}^d.$$

If K and M are polytopes, (5.18) gives

$$\begin{aligned} D_{r,s}J(f, K, M) &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f\left(\frac{x}{r}, \frac{x - y}{s} \right) \varPhi_j(K \cap (M + y), \mathrm{d}x)\, \lambda(\mathrm{d}y) \\ &= \sum_{k=j}^{d} \int_{\mathbb{R}^d \times \mathbb{R}^d} f\left(\frac{x}{r}, \frac{y}{s} \right) \varPhi_k^{(j)}(K, M; \mathrm{d}(x, y)) \\ &= \sum_{k=j}^{d} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y)\, D_{r,s}\left(\varPhi_k^{(j)}(K, M; \cdot) \right) (\mathrm{d}(x, y)). \end{aligned}$$

For the polytopes rK and sM, the image measure $D_{r,s}\left(\Phi_k^{(j)}(rK, sM; \cdot)\right)$ can be determined by means of the formula in Theorem 5.2.2; this yields

$$D_{r,s}\left(\Phi_k^{(j)}(rK, sM; \cdot)\right) = r^k s^{d-k+j} \Phi_k^{(j)}(K, M; \cdot).$$

For given convex bodies K, M we now choose polytopes K_i, M_i ($i \in \mathbb{N}$) so that $K_i \to K$ and $M_i \to M$ for $i \to \infty$. Then it follows that

$$D_{r,s} J(f, rK_i, sM_i) \to D_{r,s} J(f, rK, sM)$$

for every continuous function f on $\mathbb{R}^d \times \mathbb{R}^d$ and all $r, s > 0$. As we have just seen,

$$\begin{aligned} &D_{r,s} J(f, rK_i, sM_i) \\ &= \sum_{k=j}^{d} r^k s^{d-k+j} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y)\, \Phi_k^{(j)}(K_i, M_i; \mathrm{d}(x, y)). \end{aligned} \tag{5.20}$$

We deduce the convergence of the coefficients

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y)\, \Phi_k^{(j)}(K_i, M_i; \mathrm{d}(x, y))$$

in the polynomial (5.20) and thus the weak convergence of the measures

$$\Phi_k^{(j)}(K_i, M_i; \cdot), \qquad k = j, \dots, d,$$

for $i \to \infty$. The limits, denoted by $\Phi_k^{(j)}(K, M; \cdot)$, $k = j, \dots, d$, are again finite measures, satisfying

$$D_{r,s} J(f, rK, sM) = \sum_{k=j}^{d} r^k s^{d-k+j} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y)\, \Phi_k^{(j)}(K, M; \mathrm{d}(x, y)), \tag{5.21}$$

from which we see that they are independent of the approximating sequences $(K_i)_{i \in \mathbb{N}}$, $(M_i)_{i \in \mathbb{N}}$. For $r = s = 1$ we obtain (5.18).

From the polynomial expansion (5.21), we also deduce that $\Phi_k^{(j)}(K, M; \cdot)$ depends continuously on K and M. That $\Phi_{j+1}^{(j)}(K, M; \cdot), \dots, \Phi_{d-1}^{(j)}(K, M; \cdot)$ are concentrated on $\mathrm{bd}\, K \times \mathrm{bd}\, M$ is a consequence of Theorem 5.2.2, if K and M are polytopes, and for general convex bodies K, M it is obtained by approximation with polytopes. The stated homogeneity properties are obvious for polytopes, and for the general case they follow by approximation. The additivity of $\Phi_k^{(j)}$ in any of its first two arguments follows immediately from the expansion (5.17), if one uses the additivity of Φ_j in its first argument and then compares summands of equal degrees of homogeneity. □

We supplement the definition by $\Phi_k^{(j)}(K, M; \cdot) = 0$ if $K = \emptyset$ or $M = \emptyset$. In Section 6.4 we shall extend $\Phi_m^{(j)}(K, M; \cdot)$ to more than two convex bodies; these functions are then called mixed measures.

Additive Extension

The integral geometric formulas obtained so far are not restricted to convex bodies, but can be extended to sets of the convex ring $\mathcal{R}$, by means of additivity. First we note that the curvature measure Φ_j, as a function of its first argument, has an additive extension to $\mathcal{R}$. This follows from Groemer's extension theorem (Theorem 14.4.2), since Φ_j is additive on $\mathcal{K}'$ and is continuous as a map from $\mathcal{K}'$ into the vector space of finite signed measures on $\mathbb{R}^d$ with the weak topology. The extension is denoted by the same symbol. In a similar way, the function $\Phi_k^{(j)}$ can be extended. First we fix a convex body $M \in \mathcal{K}'$. By the same argument as just used, $\Phi_k^{(j)}(\cdot, M; \cdot)$ as a function of its first argument has an additive extension to the convex ring $\mathcal{R}$; we denote it by the same symbol. Next, we fix a polyconvex set $K \in \mathcal{R}$. We choose a representation $K = K_1 \cup \ldots \cup K_m$ with convex bodies $K_i \in \mathcal{K}'$. From the representation

$$\Phi_k^{(j)}(K, \cdot; \cdot) = \sum_{v \in S(m)} (-1)^{|v|-1} \Phi_k^{(j)}(K_v, \cdot; \cdot)$$

it follows that $\Phi_k^{(j)}(K, \cdot; \cdot)$, as a function of its second argument, is additive and continuous, hence it has an additive extension to $\mathcal{R}$. In this way, $\Phi_k^{(j)}(K, M; \cdot)$ is defined for all $K, M \in \mathcal{R}$ and is additive in each of its first two arguments.

Now both sides of the formula (5.17) make sense for arbitrary polyconvex sets $K, M \in \mathcal{R}$. Suppose, first, that M is convex. As a function of K, both sides are additive, and they are equal if K is convex. By the inclusion–exclusion principle, two additive functions coinciding on $\mathcal{K}'$ also coincide on $\mathcal{R}$. Thus, (5.17) remains true if K is a polyconvex set. In the same way, M can be replaced by a polyconvex set.

Theorem 5.2.4. *The translative formula* (5.17) *holds for polyconvex sets* $K, M \in \mathcal{R}$.

The investigation of translative integral geometry will be continued in Section 6.4.

The Notes for this section are included in those for Section 5.3.

5.3 The Principal Kinematic Formula for Curvature Measures

As mentioned at the beginning of the previous section, our aim is the derivation of a formula for the kinematic integral

$$\int_{G_d} \Phi_j(K \cap gM, A \cap gB)\, \mu(\mathrm{d}g)$$
$$= \int_{SO_d} \int_{\mathbb{R}^d} \Phi_j(K \cap (\vartheta M + x), A \cap (\vartheta B + x))\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta). \qquad (5.22)$$

The measurability of the integrand was proved in Lemma 5.2.1. If K and M are polytopes, we can apply Theorem 5.2.2 and obtain for the right side of (5.22) the expression

$$\Phi_j(K,A)\Phi_d(M,B) + \sum_{k=j+1}^{d-1} \sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{d-k+j}(M)} \lambda_F(A)\lambda_G(B)$$
$$\times \int_{SO_d} \gamma(F, \vartheta G; K, \vartheta M)[F, \vartheta G]\, \nu(\mathrm{d}\vartheta) + \Phi_d(K,A)\Phi_j(M,B). \qquad (5.23)$$

The integral over the rotation group occurring here is evaluated in the next theorem.

Theorem 5.3.1. *If $K, M \in \mathcal{P}'$ are polytopes, $j \in \{0, \dots, d-2\}$, $k \in \{j+1, \dots, d-1\}$, $F \in \mathcal{F}_k(K)$ and $G \in \mathcal{F}_{d-k+j}(M)$, then*

$$\int_{SO_d} \gamma(F, \vartheta G; K, \vartheta M)[F, \vartheta G]\, \nu(\mathrm{d}\vartheta) = c_{j,d}^{k,d-k+j} \gamma(F,K)\gamma(G,M), \qquad (5.24)$$

where the constant is given by (5.5).

Proof. Let $\vartheta \in SO_d$ be a rotation for which F and ϑG are not in special position. By Lemma 13.2.1, only such rotations need to be considered for the computation of the integral in (5.24). By definition,

$$\gamma(F, \vartheta G; K, \vartheta M) = \gamma(F \cap (\vartheta G + x), K \cap (\vartheta M + x))$$

with a suitable vector $x \in \mathbb{R}^d$. Denoting by $N(P,F)$ the normal cone of P at a relatively interior point of F, we see from the definition of the external angle that

$$\gamma(F, \vartheta G; K, \vartheta M) = \frac{\sigma_{d-1-j}(N(K \cap (\vartheta M + x), F \cap (\vartheta G + x)) \cap S^{d-1})}{\sigma_{d-1-j}(L \cap S^{d-1})},$$

where $L \in G(d, d-j)$ is the subspace totally orthogonal to $F \cap (\vartheta G + x)$. Since

$$N(K \cap (\vartheta M + x), F \cap (\vartheta G + x)) = N(K,F) + \vartheta N(M,G)$$

(see Schneider [695, Theorem 2.2.1]), we have to consider the integral

$$\int_{SO_d} \sigma_{d-j-1}((N(K,F) + \vartheta N(M,G)) \cap S^{d-1})[F, \vartheta G]\, \nu(\mathrm{d}\vartheta).$$

More generally, we denote by L_1 and L_2 the orthogonal spaces of F and G, respectively. Noting that $[F, \vartheta G] = [L_1^\perp, \vartheta L_2^\perp] = [L_1, \vartheta L_2]$, we define

$$I(A,B) := \int_{SO_d} \sigma_{d-j-1}((\breve{A} + \vartheta \breve{B}) \cap S^{d-1})[L_1, \vartheta L_2]\, \nu(\mathrm{d}\vartheta)$$

for arbitrary Borel sets $A \subset L_1 \cap S^{d-1}$ and $B \subset L_2 \cap S^{d-1}$, where

$$\breve{A} := \{\alpha x : x \in A,\, \alpha \geq 0\}$$

denotes the cone generated by A. Concerning the measurability of the integrand, we observe the following. The function $\vartheta \mapsto [L_1, \vartheta L_2]$ is continuous. Let U be the set of all rotations $\vartheta \in SO_d$ for which L_1 and ϑL_2 are not in special position; then $\nu(SO_d \setminus U) = 0$ by Lemma 13.2.1. Since $\dim L_1 + \dim L_2 = d - j \leq d$, the sum $L_1 + \vartheta L_2$ is direct if $\vartheta \in U$, hence $\breve{A} + \vartheta \breve{B}$ is a Borel set. For $\vartheta \in U$, all sets $\breve{A} + \vartheta \breve{B}$ are images of a fixed one under linear transformations of $\mathbb{R}^d$. Using this fact, it is not difficult to show that the map

$$\vartheta \mapsto \sigma_{d-j-1}((\breve{A} + \vartheta \breve{B}) \cap S^{d-1})$$

is measurable on U.

For fixed $B \in \mathcal{B}(L_2 \cap S^{d-1})$ we now set

$$\omega(A) := I(A,B) \qquad \text{for } A \in \mathcal{B}(L_1 \cap S^{d-1}).$$

If $\bigcup_{i=1}^\infty A_i$ is a disjoint union of sets $A_i \in \mathcal{B}(L_1 \cap S^{d-1})$, then

$$\left(\bigcup_{i=1}^\infty \breve{A}_i + \vartheta \breve{B} \right) \cap S^{d-1} = \bigcup_{i=1}^\infty \left((\breve{A}_i + \vartheta \breve{B}) \cap S^{d-1} \right)$$

for $\vartheta \in U$, and this is again a disjoint union up to a set of σ_{d-j-1}-measure zero. It follows that

$$\sigma_{d-j-1}\left(\left(\bigcup_{i=1}^\infty \breve{A}_i + \vartheta \breve{B} \right) \cap S^{d-1} \right) = \sum_{i=1}^\infty \sigma_{d-j-1}\left((\breve{A}_i + \vartheta \breve{B}) \cap S^{d-1} \right)$$

for $\vartheta \in U$, hence

$$\omega\left(\bigcup_{i=1}^\infty A_i \right) = \sum_{i=1}^\infty \omega(A_i)$$

by the monotone convergence theorem. Thus ω is a finite measure on $L_1 \cap S^{d-1}$. Let $\rho \in SO_d$ be a rotation mapping L_1 into itself and fixing every point of $L_1^\perp$. Then

$$\rho \breve{A} + \vartheta \breve{B} = \rho(\breve{A} + \rho^{-1} \vartheta \breve{B})$$

and

$$[L_1, \vartheta L_2] = [\rho L_1, \vartheta L_2] = [L_1, \rho^{-1} \vartheta L_2],$$

hence

$$\begin{aligned}\omega(\rho A) &= \int_{SO_d} \sigma_{d-j-1}((\rho\breve{A} + \vartheta\breve{B}) \cap S^{d-1})[L_1, \vartheta L_2]\,\nu(\mathrm{d}\vartheta) \\ &= \int_{SO_d} \sigma_{d-j-1}((\breve{A} + \rho^{-1}\vartheta\breve{B}) \cap S^{d-1})[L_1, \rho^{-1}\vartheta L_2]\,\nu(\mathrm{d}\vartheta) \\ &= \omega(A).\end{aligned}$$

By the uniqueness of spherical Lebesgue measure (a special case of Theorem 13.1.3), ω is a constant multiple of σ_{d-k-1} on $L_1 \cap S^{d-1}$. Similarly we obtain for fixed $A \in \mathcal{B}(L_1 \cap S^{d-1})$ that $I(A, \cdot)$ is a constant multiple of σ_{k-j-1} on $L_2 \cap S^{d-1}$. Altogether this yields a representation

$$I(A, B) = \alpha(L_1, L_2)\sigma_{d-k-1}(A)\sigma_{k-j-1}(B)$$

for all $A \in \mathcal{B}(L_1 \cap S^{d-1})$, $B \in \mathcal{B}(L_2 \cap S^{d-1})$, where $\alpha(L_1, L_2)$ is a constant that depends only on L_1 and L_2. The choice $A = L_1 \cap S^{d-1}$ and $B = L_2 \cap S^{d-1}$, together with the invariance properties of the functional I resulting from its definition, shows that $\alpha(L_1, L_2)$ does, in fact, depend only on the dimensions d, j, k.

In particular, this gives

$$I(N(K, F) \cap S^{d-1}, N(M, G) \cap S^{d-1}) = \alpha_{djk}\gamma(F, K)\gamma(G, M)$$

with a constant $\alpha_{djk} > 0$ and thus the assertion of Theorem 5.3.1, up to the determination of α_{djk}. We insert (5.24) into (5.22), using (5.23) for the right side. If we choose for K an r-polytope with $r \in \{j + 1, \dots, d - 1\}$, for M a $(d - r + j)$-polytope, and $A = B = \mathbb{R}^d$, then the result must coincide with formula (5.10). From this we conclude that $\alpha_{djr} = c_{j,d}^{r,d-r+j}$. This completes the proof. ☐

Corollary 5.3.1. *If $K, M \in \mathcal{K}'$ are convex bodies, $A, B \in \mathcal{B}(\mathbb{R}^d)$ are Borel sets and if $j \in \{0, \dots, d - 2\}$, $k \in \{j + 1, \dots, d - 1\}$, then*

$$\int_{SO_d} \Phi_k^{(j)}(K, \vartheta M; A \times \vartheta B)\,\nu(\mathrm{d}\vartheta) = c_{j,d}^{k,d-k+j}\Phi_k(K, A)\Phi_{d-k+j}(M, B). \quad (5.25)$$

Proof. If K, M are polytopes, the definition of $\Phi_k^{(j)}(K, M; \cdot)$ and formula (5.24) show that

$$\begin{aligned}&\int_{SO_d} \Phi_k^{(j)}(K, \vartheta M; A \times \vartheta B)\,\nu(\mathrm{d}\vartheta) \\ &= \sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{d-k+j}(M)} \lambda_F(A)\lambda_G(B) \int_{SO_d} \gamma(F, \vartheta G; K, \vartheta M)[F, \vartheta G]\,\nu(\mathrm{d}\vartheta)\end{aligned}$$

$$= c_{j,d}^{k,d-k+j} \left(\sum_{F \in \mathcal{F}_k(K)} \lambda_F(A) \gamma(F,K) \right) \left(\sum_{G \in \mathcal{F}_{d-k+j}(M)} \lambda_G(B) \gamma(G,M) \right)$$
$$= c_{j,d}^{k,d-k+j} \Phi_k(K,A) \Phi_{d-k+j}(M,B).$$

Approximation by polytopes yields (5.25) for general convex bodies $K, M \in \mathcal{K}'$. For this, we first have to verify the measurability of the integrand. It is obtained from the weak continuity of the measures $\Phi_k^{(j)}(K, \vartheta M; \cdot)$, established in Theorem 5.2.3, and from the identity

$$\int_{SO_d} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, \vartheta^{-1} y) \, \Phi_k^{(j)}(K, \vartheta M; \mathrm{d}(x,y)) \, \nu(\mathrm{d}\vartheta)$$
$$= c_{j,d}^{k,d-k+j} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,y) \, \Phi_k(K, \mathrm{d}x) \, \Phi_{d-k+j}(M, \mathrm{d}y),$$

valid for all $f \in \mathbf{C}(\mathbb{R}^d \times \mathbb{R}^d)$. The latter identity is equivalent to (5.25); this is seen as in the proof of Theorem 5.2.3. Also the final limit procedure is analogous to that in the proof of Theorem 5.2.3; one uses the weak continuity of the involved measures (Theorems 14.2.2 and 5.2.3) and the dominated convergence theorem. □

We can now state the main result of this section.

Theorem 5.3.2 (Local principal kinematic formula). *If $K, M \in \mathcal{R}$ are polyconvex sets, $A, B \in \mathcal{B}(\mathbb{R}^d)$ are Borel sets and if $j \in \{0, \dots, d\}$, then*

$$\int_{G_d} \Phi_j(K \cap gM, A \cap gB) \, \mu(\mathrm{d}g) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \Phi_k(K,A) \Phi_{d-k+j}(M,B).$$

Proof. The extension to polyconvex sets K, M was explained at the end of Section 5.2. □

Remark. The **extended convex ring** $\mathcal{S}$ is defined as the system of all subsets of $\mathbb{R}^d$ that intersect every convex body in a union of finitely many convex bodies. Since the curvature measures are locally determined (Theorem 14.2.3), they can be extended to sets of the extended convex ring, as long as the involved Borel sets remain bounded. Hence, Theorem 5.3.2 can be extended in the same sense.

In the principal kinematic formula, curvature measures of the intersection of a fixed and a moving set from the convex ring are integrated over all rigid motions. Here the moving compact set can be replaced by a moving affine subspace, and the integration can be carried out with respect to the corresponding invariant measure. One can derive such formulas directly from the principal kinematic formula.

Theorem 5.3.3 (Local Crofton formula). *If $K \in \mathcal{R}^d$ is a polyconvex set, $A \in \mathcal{B}(\mathbb{R}^d)$ is a Borel set and $q \in \{0, \dots, d\}$, $j \in \{0, \dots, q\}$, then*

$$\int_{A(d,q)} \Phi_j(K \cap E, A \cap E)\, \mu_q(\mathrm{d}E) = c_{j,d}^{q,d-q+j} \Phi_{d-q+j}(K, A).$$

Proof. We may assume that $K \in \mathcal{K}'$; the extension to $K \in \mathcal{R}$ is then achieved as explained at the end of Section 5.2. We fix $L_q \in G(d, q)$ and use the map γ_q defined by (5.2), then $\mu_q = \gamma_q(\lambda_{d-q} \otimes \nu)$. Let C be a q-dimensional unit cube in L_q. Since $L_q \in \mathcal{S}$, C is bounded and A can be replaced by the bounded set $A \cap K$, the remark after the proof of Theorem 5.3.2 shows that

$$J := \int_{G_d} \Phi_j(L_q \cap gK, C \cap gA)\, \mu(\mathrm{d}g) = \sum_{k=j}^{d} c_{j,d}^{k,d-k+j} \Phi_k(L_q, C) \Phi_{d-k+j}(K, A).$$

Now

$$\Phi_k(L_q, C) = \begin{cases} \lambda_q(C) & \text{for } k = q, \\ 0 & \text{for } k \neq q, \end{cases}$$

hence

$$J = c_{j,d}^{q,d-q+j} \Phi_{d-q+j}(K, A).$$

On the other hand,

$$\begin{aligned} J &= \int_{SO_d} \int_{\mathbb{R}^d} \Phi_j(L_q \cap (\vartheta K + x), C \cap (\vartheta A + x))\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta) \\ &= \int_{SO_d} \int_{L_q^\perp} \int_{L_q} \Phi_j(L_q \cap (\vartheta K + x_1 + x_2), C \cap (\vartheta A + x_1 + x_2))\, \lambda_q(\mathrm{d}x_2) \\ &\quad \times \lambda_{d-q}(\mathrm{d}x_1)\, \nu(\mathrm{d}\vartheta). \end{aligned}$$

To compute the inner integral, we put

$$\Phi_j(L_q \cap (\vartheta K + x_1), \cdot) =: \phi, \qquad \vartheta A + x_1 =: A';$$

then

$$\begin{aligned} &\int_{L_q} \Phi_j(L_q \cap (\vartheta K + x_1 + x_2), C \cap (\vartheta A + x_1 + x_2))\, \lambda_q(\mathrm{d}x_2) \\ &= \int_{L_q} \phi((C - x_2) \cap A')\, \lambda_q(\mathrm{d}x_2) \\ &= \phi(A') \lambda_q(C) \\ &= \Phi_j(L_q \cap (\vartheta K + x_1), L_q \cap (\vartheta A + x_1)), \end{aligned}$$

where Theorem 5.2.1 was used. This yields

$$
\begin{aligned}
J &= \int_{SO_d} \int_{L_q^\perp} \Phi_j(L_q \cap (\vartheta K + x_1), L_q \cap (\vartheta A + x_1)) \, \lambda_{d-q}(\mathrm{d}x_1) \, \nu(\mathrm{d}\vartheta) \\
&= \int_{SO_d} \int_{L_q^\perp} \Phi_j(K \cap \vartheta(L_q + x), A \cap \vartheta(L_q + x)) \, \lambda_{d-q}(\mathrm{d}x) \, \nu(\mathrm{d}\vartheta) \\
&= \int_{A(d,q)} \Phi_j(K \cap E, A \cap E) \, \mu_q(\mathrm{d}E),
\end{aligned}
$$

where we have used the motion covariance of the curvature measures and the inversion invariance of λ_{d-q} and ν. The two representations obtained for J prove the assertion. □

The case $j = 0$,

$$
\Phi_{d-q}(K, A) = c_{q,d-q}^{d,0} \int_{A(d,q)} \Phi_0(K \cap E, A \cap E) \, \mu_q(\mathrm{d}E),
$$

gives an interpretation of the measure $\Phi_{d-q}(K, A)$: up to a numerical factor, it is the mean value of $\Phi_0(K \cap E, A \cap E)$, where the mean is taken over the intersections with q-flats. The Gaussian curvature measure Φ_0 has a simple intuitive interpretation, as mentioned in Section 14.2.

Notes for Sections 5.2 and 5.3

1. A general local principal kinematic formula, which coincides with Theorem 5.3.2 in the case of convex bodies, was first obtained by Federer [228]. He proved it for sets of positive reach and for their curvature measures, which he introduced for this purpose. The generality of the admissible point sets requires deeper techniques from geometric measure theory. Using such techniques, in particular Martina Zähle studied new approaches to curvature measures and to integral geometric formulas valid for them; see Zähle [824, 825, 826, 827], Rother and Zähle [649].

There have been several successful attempts to define curvature measures and to obtain kinematic and Crofton formulas in very general situations, where strong singularities are permitted. We refer here to Fu [236, 237, 238], Bröcker and Kuppe [122], Bernig and Bröcker [94, 93], Rataj and Zähle [620, 621].

In contrast to this trend to deep generalizations, it has been our aim in this book to follow an approach to local integral geometric formulas for convex bodies and sets of the convex ring that needs only elementary measure-theoretic and geometric arguments, and which (we hope) is more in the spirit of the integral geometry of Blaschke and Hadwiger. Different approaches of this kind are also found in Schneider [676, 680].

In deriving the Crofton formula (Theorem 5.3.3) from the local principal kinematic formula, we followed Federer [228].

2. In order to extend the curvature measures additively to the convex ring, we have referred here to Groemer's extension theorem. For the support measures, and thus for the curvature measures, a more explicit construction of an additive extension to polyconvex sets is found in Schneider [679] and in Section 4.4 of [695]. It is based on an extension of the local Steiner formula for polyconvex sets, with the Lebesgue measure replaced by the integral of the multiplicity function that arises from additive extension of the indicator function of a parallel set. See also Note 3 of Section 14.4.

3. A more general version of Theorem 5.2.1 is Theorem 13.1.4. We refer to Note 2 of Section 13.1 for some references.

Translative integral geometry was first investigated by Blaschke [106] and Berwald and Varga [98]; see Schneider and Weil [715] for further references. From the latter paper, we essentially took the proofs of Theorems 5.2.2 and 5.3.1, and thus of the local principal kinematic formula, Theorem 5.3.2. A first version of Theorem 5.2.3 appeared in Weil [786]. A better understanding of the mixed measures $\Phi_k^{(j)}(K,M;\cdot)$ of Theorem 5.2.3 is desirable. Results concerning the total measures $\Phi_k^{(j)}(K,M;\mathbb{R}^d\times\mathbb{R}^d) =: V_k^{(j)}(K,M)$ were found by Goodey and Weil [277], Weil [790, 791, 800].

For further information on translative integral geometry, we refer to the Notes for Section 6.4.

4. Kinematic formulas for support measures. The curvature measures, for which we have proved the local principal kinematic formula and the Crofton formula, are specializations of the support measures Ξ_m introduced in Section 14.2. There are also versions of these formulas for support measures. They require that the intersection of Borel sets in $\mathbb{R}^d$ be replaced by a suitable law of composition for subsets of $\Sigma = \mathbb{R}^d \times S^{d-1}$, which is adapted to intersections of convex bodies. For $A, B \subset \Sigma$, let

$$A \wedge B := \{(x,u) \in \Sigma: \text{ there are } u_1, u_2 \in S^{d-1} \text{ with}$$
$$(x,u_1) \in A,\ (x,u_2) \in B,\ u \in \operatorname{pos}\{u_1,u_2\}\},$$

where $\operatorname{pos}\{u_1,u_2\} := \{\lambda_1 u_1 + \lambda_2 u_2 : \lambda_1, \lambda_2 \geq 0\}$ is the positive hull of $\{u_1,u_2\}$. Now for convex bodies $K, M \in \mathcal{K}'$, Borel sets $A \subset \operatorname{Nor} K$ and $B \subset \operatorname{Nor} M$ (where Nor denotes the generalized normal bundle, see Section 14.2), and for $j \in \{0,\dots,d-2\}$, the formula

$$\int_{G_d} \Xi_j(K \cap gM, A \wedge gB)\,\mu(\mathrm{d}g) = \sum_{k=j+1}^{d-1} c_{j,d}^{k,d-k+j}\,\Xi_k(K,A)\Xi_{d-k+j}(M,B) \quad (5.26)$$

holds (for $j = d-1$, both sides would give 0).

For a q-flat $E \in A(d,q)$, $q \in \{1,\dots,d-1\}$, one defines

$$A \wedge E := \{(x,u) \in \Sigma: \text{ there are } u_1, u_2 \in S^{d-1} \text{ with}$$
$$(x,u_1) \in A,\ x \in E,\ u_2 \in E^\perp,\ u \in \operatorname{pos}\{u_1,u_2\}\},$$

where $E^\perp$ is the linear subspace totally orthogonal to E. Then the local Crofton formula has the following extension. Let $K \in \mathcal{K}'$ be a convex body, $k \in \{1,\dots,d-1\}$, $j \in \{0,\dots,k-1\}$, and let $A \subset \operatorname{Nor} K$ be a Borel set. Then

$$\int_{A(d,k)} \Xi_j(K \cap E, A \wedge E)\,\mu_k(\mathrm{d}E) = c_{j,d}^{k,d-k+j}\,\Xi_{d-k+j}(K,A).$$

These results are due to Glasauer [266], under an additional assumption in the case of (5.26). A common boundary point x of the convex bodies K, M is said to be 'exceptional' if the linear hulls of the normal cones of K and M at x have a non-zero intersection. Glasauer assumed that the set of rigid motions g for which K and gM have some exceptional common boundary point, is of Haar measure zero. He conjectured that this assumption is always satisfied. This was proved by Schneider [700]. An alternative proof of a more general result appears in Zähle [831].

5. A local version of Hadwiger's general integral geometric theorem. The local principal kinematic formula together with the local Crofton formula (Theorems 5.3.2 and 5.3.3) can be extended in the same way as the principal kinematic formula and the Crofton formula (Theorems 5.1.3 and 5.1.1) are extended by Hadwiger's general integral geometric theorem. This abstract version of (5.17) reads as follows.

Theorem. *Let $\Lambda : \mathcal{K}' \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$ be a mapping with the following properties:*

(a) *$\Lambda(K, \cdot)$ is a finite positive measure concentrated on K, for all $K \in \mathcal{K}'$.*

(b) *The map $K \mapsto \Lambda(K, \cdot)$ is additive and weakly continuous.*

(c) *If $K, M \in \mathcal{K}'$, $A \subset \mathbb{R}^d$ is open and $K \cap A = M \cap A$, then $\Lambda(K, B) = \Lambda(M, B)$ for all Borel sets $B \subset A$.*

Then, for $K, M \in \mathcal{K}'$, $A, B \in \mathcal{B}(\mathbb{R}^d)$ and $j \in \{0, \dots, d\}$, the formula

$$\int_{G_d} \Lambda(K \cap gM, A \cap gB)\, \mu(\mathrm{d}g) = \sum_{k=0}^{d} \Lambda_{d-k}(K, A) \Phi_k(M, B)$$

(with $\Lambda(\emptyset, \cdot) := 0$) holds, where

$$\Lambda_{d-k}(K, B) := \int_{A(d,k)} \Lambda(K \cap E, B)\, \mu_k(\mathrm{d}E).$$

This was proved by Schneider [696]. An analog in spherical space and a simpler proof in Euclidean space were given by Glasauer [264]. Examples of mappings Λ satisfying the above properties are the relative curvature measures introduced in Schneider [696]. Also (5.26) admits an abstract generalization in the spirit of Hadwiger's general integral geometric theorem; see Glasauer [268], Theorem 7.

6. Tensor valuations. The intrinsic volumes and their local versions arise from the notion of volume, through the Steiner formula. Replacement of the volume by vectorial or higher rank tensorial moments leads to tensor-valued valuations on convex bodies and raises the question whether their properties and their role in integral geometry extend those of the intrinsic volumes. To explain this, we denote by $\mathbb{T}^p$ the vector space of symmetric tensors of rank p over $\mathbb{R}^d$ (we identify $\mathbb{R}^d$ with its dual space, using the scalar product, so that no distinction between covariant and contravariant tensors is necessary). If $p \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we write x^p for the p-fold tensor product $x \otimes \cdots \otimes x$, and we put $x^0 := 1$. For symmetric tensors a and b, their symmetric product is denoted by ab. For $K \in \mathcal{K}'$ and $p \in \mathbb{N}_0$, let

$$\Psi_p(K) := \frac{1}{p!} \int_K x^p \, \lambda(\mathrm{d}x).$$

The Steiner formula extends to a polynomial expansion

$$\Psi_p(K + \epsilon B^d) = \sum_{k=0}^{d+p} \epsilon^{d+p-k} \kappa_{d+p-k} V_k^{(p)}(K) \tag{5.27}$$

for $\epsilon > 0$, with $V_k^{(p)}(K) \in \mathbb{T}^p$. Each function $V_k^{(p)} : \mathcal{K}' \to \mathbb{T}^p$ is additive, continuous and isometry covariant, which means that $V_k^{(p)}(\vartheta K) = \vartheta V_k^{(p)}(K)$ for every rotation $\vartheta \in SO_d$ and that $V_k^{(p)}(K+t)$ is a (tensor) polynomial in $t \in \mathbb{R}^d$ of degree p. The known facts in the case $p = 0$ suggest the following questions: (a) Is an additive, continuous, isometry covariant function $f : \mathcal{K}' \to \mathbb{T}^p$ necessarily a linear combination of $V_0^{(p)}, \dots, V_{d+p}^{(p)}$? (b) Do the coefficients $V_k^{(p)}$ satisfy kinematic and Crofton formulas? For $p = 0$, positive answers were given in this chapter. For $p = 1$, both questions were answered affirmatively by Hadwiger and Schneider [312]. For $p > 1$, however, the situation is different. One has to consider more general tensor valuations, defined by

$$\Phi_{m,r,s}(K) := \frac{1}{r!s!} \frac{\omega_{d-m}}{\omega_{d-m+s}} \int_\Sigma x^r u^s \, \Xi_m(K, \mathrm{d}(x,u))$$

for $K \in \mathcal{K}'$ and integers $r, s \geq 0$, $0 \leq m \leq d-1$ (the factors before the integral turn out to be convenient). They were introduced (via a polytopal approach) by McMullen [472]. Besides these tensor functions $\Phi_{m,r,s} : \mathcal{K}' \to \mathbb{T}^{r+s}$, one also needs the metric tensor $G \in \mathbb{T}^2$ of $\mathbb{R}^d$. The functions $G^q \Phi_{m,r,s}$ and $G^q \Psi_p$ ($q \in \mathbb{N}_0$) are called **basic tensor valuations**. Answering a question posed by McMullen [472], Alesker [17], based on his earlier work in [16], proved the following extension of Hadwiger's characterization theorem:

Theorem. *If $p \in \mathbb{N}_0$ and if $f : \mathcal{K}' \to \mathbb{T}^p$ is an additive, continuous, isometry covariant function, then f is a linear combination of the functions $G^q \Phi_{m,r,s}$ (with $2q + r + s = p$) and the functions $G^q \Psi_r$ (with $2q + r = p$).*

McMullen [472] had already discovered a set of nontrivial linear relations between the basic tensor valuations. Therefore, Alesker's result yielded a generating system, but not a basis or the dimension of the vector space of continuous, isometry covariant tensor valuations of fixed rank. This remaining problem was settled by Hug, Schneider and Schuster [374], who proved that the relations between the basic tensor valuations discovered by McMullen are essentially the only ones.

By Alesker's result, the coefficients $V_k^{(p)}$ appearing in the Steiner polynomial (5.27) are linear combinations of basic tensor valuations. Question (b) above should, therefore, be modified, asking whether the functions $\Phi_{m,r,s}$ and Ψ_p satisfy kinematic and Crofton formulas. Unlike in the cases of rank zero or one, the characterization theorem does not seem useful for obtaining integral geometric formulas, due to the linear relations between the basic tensor valuations; hence, direct computations are required. It is sufficient to derive Crofton formulas, since then Hadwiger's general integral geometric theorem, which in the case of tensor functions can be applied coordinate-wise, immediately yields kinematic formulas. For dimension two and rank one or two, kinematic formulas were already obtained by Müller [567] (except for $\Phi_{0,1,1}$, in our notation), who took up a suggestion of Blaschke. An investigation for all dimensions and ranks was begun by Schneider [701] and continued by Schneider and Schuster [713]. This led, in particular, to a complete set of Crofton and kinematic formulas in two and three dimensions. The higher-dimensional case turned out to be intricate; it was settled by Hug, Schneider and Schuster [375].

7. Non-intersecting sets: distances. All the integral geometric results considered up to now in this chapter concern the intersection of a fixed and a moving set. For convex sets, there are also kinematic formulas involving relations between non-intersecting sets. One possibility consists in taking distances into account. The distance $d(K, L)$ of a compact set $K \subset \mathbb{R}^d$ and a closed set $L \subset \mathbb{R}^d$, $K, L \neq \emptyset$, is defined by

$$d(K, L) := \min\{\|x - y\| : x \in K,\ y \in L\}.$$

Let $f : [0, \infty) \to [0, \infty)$ be a measurable function satisfying $f(0) = 0$ and

$$m_k(f) := \frac{1}{k!} \int_0^\infty f(r) r^k \, \mathrm{d}r < \infty \qquad \text{for } k = 0, \ldots, d-1.$$

Then, for convex bodies $K, M \in \mathcal{K}'$, the kinematic formula

$$\int_{G_d} f(d(K, gM)) \, \mu(\mathrm{d}g) = \sum_{j=0}^{d-1} \sum_{k=0}^{d-j-1} c_{d,0}^{d-j,d-k} m_{d-j-1-k}(f) V_j(K) V_k(M)$$

holds. This can be generalized in various directions. To give one example, suppose that for the convex bodies K, M with $K \cap M = \emptyset$ there is a unique pair $x \in K$, $y \in M$ with $\|x - y\| = d(K, M)$. Then one can define $p(K, M) := x$. One can show that $p(K, gM)$ exists for μ-almost all $g \in G_d$ with $K \cap gM = \emptyset$. If $f : (0, \infty) \times \operatorname{bd} K \times \operatorname{bd} M \to \mathbb{R}$ is a measurable function for which the integral

$$\int_{K \cap gM = \emptyset} f(d(K, gM), p(K, gM), g^{-1} p(gM, K)) \, \mu(\mathrm{d}g)$$

is finite, then this integral can be expressed in terms of integrals of curvature measures of K and M. Similarly, one can treat kinematic integrals involving functions of the unit vector pointing from K to gM. Further, the moving convex body can be replaced by a moving flat.

For the special case where M is one-pointed, a related formula is given by Theorem 14.3.3.

Contributions to this area are due to Hadwiger [309, 310], Bokowski, Hadwiger and Wills [111], Schneider [675], Groemer [291], Weil [779, 780, 782]. We refer also to Section 4 of the survey article by Schneider and Wieacker [720].

Translative formulas for non-intersecting convex bodies in suitable general position have been studied by Kiderlen and Weil [409]; the results involve mixed curvature measures. Hug, Last and Weil [358] give a quite general translative formula, allowing also non-Euclidean distances and using relative support measures (a special case is Theorem 14.3.3). A corresponding version for flats is contained in Hug, Last and Weil [360].

8. Non-intersecting sets: convex hulls. Glasauer [267] found a new type of kinematic formulas, involving the convex hull of a fixed and a moving convex body. Since convex hulls with a freely moving convex body are not uniformly bounded, the results can only be of the type of weighted limits. Let $K \vee M$ denote the convex hull of $K \cup M$. A typical result of Glasauer concerns the mixed volumes with fixed convex bodies $K_{j+1}, \ldots, K_d$ and states that

$$\lim_{r \to \infty} \frac{1}{r^{d+1}} \frac{(d+1)\kappa_d}{\kappa_{d-1}} \int_{\{g \in G_d :\, gM \subset rB^d\}} V(K \vee gM[j], K_{j+1}, \ldots, K_d) \, \mu(\mathrm{d}g)$$

$$= \sum_{k=0}^{j-1} V(K[k], B^d[j-k], K_{j+1}, \ldots, K_d) V(M[j-k-1], B^d[d-j+k+1]).$$

This is a special case of Theorem 3 of Glasauer [267]. He has considerably more general results, for not necessarily invariant measures, and with mixed area measures instead of mixed volumes. For $K_{j+1} = \ldots = K_d = B^d$, the formula reduces to one for intrinsic volumes. For this result, there is also a local version, which is 'dual' to formula (5.26). It involves a law of composition for subsets of Σ which is adapted to the convex hull operation for pairs of convex bodies. For $A, B \subset \Sigma$, let

$$\begin{aligned} A \vee B := \ & \{(x,u) \in \Sigma : \text{ there are } x_1, x_2 \in \mathbb{R}^d \text{ with} \\ & \langle x_1 - x_2, u\rangle = 0,\ (x_1,u) \in A,\ (x_2,u) \in B,\ x \in \operatorname{conv}\{x_1,x_2\}\}. \end{aligned}$$

Now suppose that $K, M \in \mathcal{K}'$, $A \subset \operatorname{Nor} K$ and $B \subset \operatorname{Nor} M$ are Borel sets, and $j \in \{0, \ldots, d-1\}$. Then Glasauer [268] proved (with different notation) that

$$\begin{aligned} & \lim_{r\to\infty} \frac{1}{r^{d+1}} \int_{\{g \in G_d :\, gM \subset rB^d\}} \Xi_j(K \vee gM, A \vee gB)\, \mu(\mathrm{d}g) \\ & = \sum_{k=0}^{j-1} \beta_{djk} \Xi_k(K, A) \Xi_{j-k-1}(M, B), \end{aligned}$$

with explicit constants β_{djk}. The proof requires the following regularity result. A common supporting hyperplane H of the convex bodies K, M (leaving K and M on the same side) is said to be exceptional if the affine hulls of the sets $H \cap K$ and $H \cap M$ have a nonempty intersection or contain parallel lines. Then the set of all rigid motions g for which K and gM have some exceptional common supporting hyperplane is of Haar measure zero. This was conjectured by Glasauer and proved by Schneider [700].

9. Dual quermassintegrals. The principal kinematic formula for convex bodies involves the intrinsic volumes, which belong to the Brunn–Minkowski theory. There are analogs in the dual Brunn–Minkowski theory. This analogy becomes clearer in terms of the quermassintegrals $W_0, \ldots, W_n$ (see (14.6)). Equivalent to (5.8) is the formula

$$W_{d-i}(K) = \frac{\kappa_d}{\kappa_i} \int_{G(d,i)} \lambda_i(K|L)\, \nu_i(\mathrm{d}L)$$

for $i = 0, \ldots, d$. In terms of the quermassintegrals, the principal kinematic formula has the form (5.12). Let $K \subset \mathbb{R}^d$ be a star body (a compact set, star-shaped with respect to 0, with continuous radial function). The **dual quermassintegrals** $\widetilde{W}_0, \ldots, \widetilde{W}_d$ are defined by

$$\widetilde{W}_{d-i}(K) = \frac{\kappa_d}{\kappa_i} \int_{G(d,i)} \lambda_i(K \cap L)\, \nu_i(\mathrm{d}L).$$

For $g \in G_d$, let N_g denote the segment joining 0 and $g0$. Zhang [833] has proved the kinematic formula

$$\int_{G_d} \chi(K \cap gM \cap N_g)\, \mu(\mathrm{d}g) = \frac{1}{\kappa_d} \sum_{i=0}^{d} \binom{d}{i} \widetilde{W}_i(K) \widetilde{W}_{d-i}(M)$$

for star bodies $K, M \subset \mathbb{R}^d$, which is formally very similar to (5.12).

10. Striking combinatorial analogs of the kinematic formula in the context of finite lattices were found by Klain [415]; see also Klain and Rota [416, p. 29].

11. Kinematic formulas for boundaries of convex bodies. Let $K, M \subset \mathbb{R}^d$ be convex bodies with nonempty interiors, and let $\partial K, \partial L$ denote their boundaries. The following two kinematic formulas, involving intersections of two convex surfaces or of a convex surface and a convex body, were conjectured by Firey (see Problem 18 in the collection of Gruber and Schneider [298]):

$$\int_{G_d} \chi(\partial K \cap g\partial M)\,\mu(\mathrm{d}g) = \frac{1+(-1)^d}{\kappa_d} \sum_{k=0}^{d-1} \binom{d}{k} (1-(-1)^k) W_{d-k}(K) W_k(M), \quad (5.28)$$

$$\int_{G_d} \chi(\partial K \cap gM)\,\mu(\mathrm{d}g) = \frac{1}{\kappa_d} \sum_{k=0}^{d-1} \binom{d}{k} (1-(-1)^{d-k}) W_{d-k}(K) W_k(M). \quad (5.29)$$

For polytopes, these formulas can easily be verified. However, there is no simple approximation argument to extend the results to general convex bodies. In Hug and Schätzle [368], Firey's conjecture was confirmed by proving the following more general translative versions of (5.28) and (5.29):

$$\int_{\mathbb{R}^d} \chi(\partial K \cap (\partial M + x))\,\lambda(\mathrm{d}x)$$
$$= (1+(-1)^d) \sum_{k=0}^{d-1} \binom{d}{k} \left(V_k(K, -M) + (-1)^{k-1} V_k(K, M) \right),$$

where $V_k(K, L)$ denotes the mixed volume of k copies of K and $d-k$ copies of L, and

$$\int_{\mathbb{R}^d} \chi(\partial K \cap (M + x))\,\lambda(\mathrm{d}x) = \sum_{k=0}^{d-1} \binom{d}{k} \left(V_k(K, -M) + (-1)^{d-k-1} V_k(K, M) \right).$$

From these formulas, (5.28) and (5.29) are obtained if one replaces M by ϑM, integrates over all $\vartheta \in SO_d$ with respect to the invariant measure, and then applies [695, formula (5.3.25)]. In fact, Firey's original question was already answered implicitly by a result of Fu [238], which, however, does not cover the translative case.

In Hug, Mani–Levitska and Schätzle [363], these integral geometric results are extended further, to lower-dimensional sets. Furthermore, iterated formulas are established concerning intersections of several convex bodies, which then are applied to obtain formulas of stochastic geometry. Defining intrinsic volumes for intersections of convex surfaces in a suitable way by a Crofton type expression, integral formulas for such functionals are also derived.

12. Further information on kinematic and Crofton formulas is contained in the survey article by Hug and Schneider [369].

13. A Gaussian kinematic formula. Taylor [754] obtains an analog of the Steiner formula and Weyl tube formula, with Lebesgue measure replaced by Gaussian measure. This is then applied in an analog of the principal kinematic formula, expressing the expected Euler characteristic of excursion sets for certain random fields. For the geometry of random fields, see Adler [1] and Adler and Taylor [2].

5.4 Intersection Formulas for Submanifolds

The integral geometric formulas considered so far all refer to intersections of a fixed and a moving set, and these sets, with the exception of Theorem 5.2.1, were either convex bodies or affine subspaces. Certain applications to stochastic geometry or stereology, dealing with fiber or surface processes, require intersection formulas for submanifolds of various dimensions and for Hausdorff measures of their intersections. In the present section we describe such results. The technical requirements for such a treatment depend on the generality of the notion of k-dimensional surface that is used. For the most elementary notion, polyhedral surfaces, the results stated below are easily obtained from the results previously established and by the methods used in this book. However, already smooth surfaces would require different methods. The more general k-surfaces, for which the results will be formulated, need notions and techniques from geometric measure theory. Since this is outside the scope of this book, we present only the results and give references to complete proofs (including the measurability considerations omitted here).

Some notions from geometric measure theory, which are used in the following, are collected in Section 14.5 of the Appendix. In this section, we do not aim at the greatest generality, but prefer simpler formulations which are sufficient for our applications.

Let $k \in \{0, \dots, d\}$. Recall that a subset $M \subset \mathbb{R}^d$ is **k-rectifiable** if it is the image of some bounded subset of $\mathbb{R}^k$ under some Lipschitz map. The set M is **countably k-rectifiable** if it is the union of countably many k-rectifiable sets. By a **k-surface** we understand, in this section, a countably k-rectifiable Borel set M with $\mathcal{H}^k(M) < \infty$, where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

A trivial case of the translative integrals we want to consider is obtained if we take a k-dimensional convex body K and a $(d-k)$-dimensional convex body M. In that case, we immediately get

$$\int_{\mathbb{R}^d} \mathcal{H}^0(K \cap (M+t))\, \lambda(\mathrm{d}t) = [K, M]\mathcal{H}^k(K)\mathcal{H}^{d-k}(M).$$

The first theorem of this section is a generalization of this simple formula to k-surfaces.

Let M be a k-surface. Then there exist k-dimensional C^1-submanifolds $N_1, N_2, \dots$ such that $\mathcal{H}^k(M \setminus \bigcup_{i\in\mathbb{N}} N_i) = 0$. Let $T_x N_i$ denote the tangent space of N_i at $x \in N_i$ (considered as a subspace of $\mathbb{R}^d$). For a Borel set $A \in \mathcal{B}(G(d,k))$, we define

$$\tau_M(A) := \mathcal{H}^k\left(\bigcup_{i\in\mathbb{N}} \{x \in M \cap N_i : T_x N_i \in A\}\right).$$

This defines a finite measure τ_M on $G(d,k)$, which depends only on M.

Theorem 5.4.1. *Let $n \in \{1, \dots, d-1\}$, and let M_i be a k_i-surface, for $i = 0, \dots, n$, with $k := k_0 + \dots + k_n \geq nd$. Then*

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathcal{H}^{k-nd}(M_0 \cap (M_1 + t_1) \cap \dots \cap (M_n + t_n)) \, \lambda(\mathrm{d}t_1) \cdots \lambda(\mathrm{d}t_n)$$

$$= \int_{G(d,k_n)} \cdots \int_{G(d,k_0)} [L_0, \dots, L_n] \, \tau_{M_0}(\mathrm{d}L_0) \cdots \tau_{M_n}(\mathrm{d}L_n).$$

For the proof, we refer to Wieacker [816]. He has a more general result, for $(\mathcal{H}^{k_i}, k_i)$-rectifiable subsets M_i, but in that case, an additional assumption on the product $M_0 \times \dots \times M_n$ is required.

Our first conclusion from Theorem 5.4.1 is a kinematic formula for two surfaces.

Theorem 5.4.2. *Let $k_0, k_1 \in \{1, \dots, d-1\}$ be numbers with $k_0 + k_1 \geq d$, let M_i be a k_i-surface, for $i = 0, 1$. Then*

$$\int_{G_d} \mathcal{H}^{k_0+k_1-d}(M_0 \cap gM_1) \, \mu(\mathrm{d}g) = c_{k_0+k_1-d,d}^{k_0,k_1} \mathcal{H}^{k_0}(M_0)\mathcal{H}^{k_1}(M_1).$$

This theorem holds, more generally, for $\mathcal{H}^{k_i}$-rectifiable subsets M_i, $i = 0, 1$, if it is assumed that $M_0 \times M_1$ is $(\mathcal{H}^{k_0+k_1}, k_0 + k_1)$-rectifiable; see Zähle [823]. We also refer to this paper for the necessary measurability considerations and the proof that $M_0 \cap gM_1$ is $(\mathcal{H}^{k_0+k_1-d}, k_0 + k_1 - d)$-rectifiable for μ-almost all $g \in G_d$. Here we show only how the formula of Theorem 5.4.2 follows from that of Theorem 5.4.1.

Proof. Making use of the obvious facts that $\tau_{\vartheta M} = \vartheta(\tau_M)$ for $\vartheta \in SO_d$ and that the integral $\int_{SO_d} [L_0, \vartheta L_1] \, \nu(\mathrm{d}\vartheta)$ is invariant under rotations of L_0, we obtain

$$\int_{G_d} \mathcal{H}^{k_0+k_1-d}(M_0 \cap gM_1) \, \mu(\mathrm{d}g)$$

$$= \int_{SO_d} \int_{\mathbb{R}^d} \mathcal{H}^{k_0+k_1-d}(M_0 \cap (\vartheta M_1 + t)) \, \lambda(\mathrm{d}t) \, \nu(\mathrm{d}\vartheta)$$

$$= \int_{SO_d} \int_{G(d,k_0)} \int_{G(d,k_1)} [L_0, L_1] \, \tau_{\vartheta M_1}(\mathrm{d}L_1) \, \tau_{M_0}(\mathrm{d}L_0) \, \nu(\mathrm{d}\vartheta)$$

$$= \int_{SO_d} \int_{G(d,k_0)} \int_{G(d,k_1)} [L_0, \vartheta L_1] \, \tau_{M_1}(\mathrm{d}L_1) \, \tau_{M_0}(\mathrm{d}L_0) \, \nu(\mathrm{d}\vartheta)$$

$$= c \, \mathcal{H}^{k_0}(M_0)\mathcal{H}^{k_1}(M_1)$$

with a constant c. Its value is obtained from the principal kinematic formula (5.10), if we choose for M_i a convex body of dimension k_i ($i = 0, 1$) and observe that then $V_j(M_0 \cap gM_1) = \mathcal{H}^{k_0+k_1-d}(K \cap gM)$, $V_k(M_0) = 0$ for $k > k_0$ and $V_{k_0+k_1-k}(M_1) = 0$ for $k < k_0$. □

From this kinematic formula, we can deduce a Crofton formula.

Theorem 5.4.3. *Let $k, q \in \{1, \dots, d-1\}$ be numbers with $k+q \geq d$, let M be a k-surface. Then*

$$\int_{A(d,q)} \mathcal{H}^{k+q-d}(M \cap E)\, \mu_q(\mathrm{d}E) = c_{k+q-d,d}^{k,q}\, \mathcal{H}^k(M).$$

Proof. The proof is similar to that of Theorem 5.3.3, but simpler. Choose $L_q \in G(d,q)$ and a q-dimensional unit cube $C \subset L_q$. By Theorem 5.4.2,

$$J := \int_{G_d} \mathcal{H}^{k+q-d}(C \cap gM)\, \mu(\mathrm{d}g) = c_{k+q-d,d}^{k,q}\, \mathcal{H}^k(M).$$

On the other hand,

$$\begin{aligned} J &= \int_{SO_d} \int_{\mathbb{R}^d} \mathcal{H}^{k+q-d}(C \cap (\vartheta M + t))\, \lambda(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta) \\ &= \int_{SO_d} \int_{L_q^\perp} \int_{L_q} \mathcal{H}^{k+q-d}(C \cap (\vartheta M + t_1 + t_2))\, \lambda_q(\mathrm{d}t_2)\, \lambda_{d-q}(\mathrm{d}t_1)\, \nu(\mathrm{d}\vartheta) \end{aligned}$$

and

$$\begin{aligned} &\int_{L_q} \mathcal{H}^{k+q-d}(C \cap (\vartheta M + t_1 + t_2))\, \lambda_q(\mathrm{d}t_2) \\ &= \int_{L_q} \mathcal{H}^{k+q-d}((C - t_2) \cap (\vartheta M + t_1))\, \lambda_q(\mathrm{d}t_2) \\ &= \mathcal{H}^{k+q-d}(L_q \cap (\vartheta M + t_1)) \end{aligned}$$

by Theorem 5.2.1 (the σ-finiteness condition is satisfied for almost all ϑ). This gives

$$\begin{aligned} J &= \int_{SO_d} \int_{L_q^\perp} \mathcal{H}^{k+q-d}(L_q \cap (\vartheta M + t_1))\, \lambda_{d-q}(\mathrm{d}t_1)\, \nu(\mathrm{d}\vartheta) \\ &= \int_{SO_d} \mathcal{H}^{k+q-d}(M \cap \vartheta(L_q + t))\, \lambda_{d-q}(\mathrm{d}t)\, \nu(\mathrm{d}\vartheta) \\ &= \int_{A(d,q)} \mathcal{H}^{k+q-d}(M \cap E)\, \mu_q(\mathrm{d}E), \end{aligned}$$

which completes the proof. ☐

Finally, we consider the special case of Theorem 5.4.1 where each k_i is equal to $d-1$. If M is a $(d-1)$-surface, it is convenient to replace the measure τ_M by the even measure σ_M on the unit sphere which for $A \in \mathcal{B}(S^{d-1})$ without antipodal points is defined by

$$\sigma_M(A) := \frac{1}{2}\tau_M(\{u^\perp : u \in A\}).$$

We define an auxiliary convex body Π_M, a zonoid, by its support function

$$h(\Pi_M, u) := \frac{1}{2}\int_{S^{d-1}} |\langle u, v\rangle|\, \sigma_M(\mathrm{d}v), \qquad u \in S^{d-1}. \tag{5.30}$$

If $K \in \mathcal{K}'$ is a convex body, then $\sigma_{\mathrm{bd}\, K} = (1/2)[S_{d-1}(K, \cdot) + S_{d-1}(-K, \cdot)]$ and, therefore,

$$\Pi_{\mathrm{bd}\, K} = \Pi_K,$$

where Π_K is the projection body of K, introduced in (14.40), (14.41).

If $m \in \{2, \dots, d\}$ and M_i is a $(d-1)$-surface ($i = 1, \dots, m$), then the formula of Theorem 5.4.1 can be written as follows (recall the definition of ∇_m before Theorem 4.4.8).

$$\begin{aligned}
&\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathcal{H}^{d-m}(M_1 \cap (M_2 + t_2) \cap \ldots \cap (M_m + t_m))\, \lambda(\mathrm{d}t_2) \cdots \lambda(\mathrm{d}t_m)\\
&= \int_{S^{d-1}} \cdots \int_{S^{d-1}} \nabla_m(u_1, \dots, u_m)\, \sigma_{M_1}(\mathrm{d}u_1) \cdots \sigma_{M_m}(\mathrm{d}u_m)\\
&= \frac{d!}{(d-m)!\kappa_{d-m}} V(\Pi_{M_1}, \dots, \Pi_{M_m}, B^d, \dots, B^d),
\end{aligned}$$

where the right side is a mixed volume. The last equality follows from (14.34), observing the factor $1/2$ in (5.30).

We state the last result as a theorem.

Theorem 5.4.4. *Let $m \in \{2, \dots, d\}$, and let M_i be a $(d-1)$-surface, for $i = 1, \dots, m$. Then*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathcal{H}^{d-m}(M_1 \cap (M_2 + t_2) \cap \ldots \cap (M_m + t_m))\, \lambda(\mathrm{d}t_2) \cdots \lambda(\mathrm{d}t_m)\\
&= \frac{d!}{(d-m)!\kappa_{d-m}} V(\Pi_{M_1}, \dots, \Pi_{M_m}, B^d, \dots, B^d).
\end{aligned}$$

By a **convex hypersurface** we understand any $(d-1)$-surface of the form $F = B \cap \mathrm{bd}\, K$, where $K \in \mathcal{K}$ is a convex body with interior points and $B \in \mathcal{B}(\mathbb{R}^d)$ is a Borel set. For such a convex hypersurface, the body defined by (5.30) can be represented by

$$h(\Pi_F, u) := \frac{1}{2}\int_F |\langle u, n_K(x)\rangle|\, \mathcal{H}^{d-1}(\mathrm{d}x), \qquad u \in S^{d-1}. \tag{5.31}$$

Here $n_K(x)$ denotes the outer unit normal vector of K at $x \in \mathrm{bd}\, K$; it is uniquely determined $\mathcal{H}^{d-1}$-almost everywhere on $\mathrm{bd}\, K$. The integral in (5.31) depends only on F and not on K.

Notes for Section 5.4

1. The techniques of geometric measure theory that are needed for the general versions of the results of this section are found in the book by Federer [229].

The first general versions of Theorems 5.4.2 and 5.4.3 are due to Federer [227].

Applications to random processes of Hausdorff rectifiable closed sets were investigated by Zähle [822].

2. The translative formulas of Theorems 5.4.1 and 5.4.4 appear in Wieacker [816]. He has extended the approach considerably and has studied various applications to stochastic geometry; see [817, 818].

3. Crofton formulas in Minkowski spaces and projective Finsler spaces. The particular case of Theorem 5.4.3, where M is a k-surface and $q = d - k$ is the complementary dimension, reduces to

$$\int_{A(d,d-k)} \operatorname{card}(M \cap E)\, \mu_{d-k}(\mathrm{d}E) = \alpha_{dk} \mathcal{H}^k(M) \tag{5.32}$$

with a constant α_{dk}. This formula provides a beautiful interpretation of the k-dimensional area $\mathcal{H}^k(M)$: it is, up to a normalizing factor, the invariant measure of the $(d-k)$-flats hitting M, weighted by the number of hits. This motivates the following reverse question. If some other notion of k-dimensional area, denoted by vol_k, is given, does there exist a measure (or a signed measure) η_{d-k} on $A(d, d-k)$ such that

$$\int_{A(d,d-k)} \operatorname{card}(M \cap E)\, \eta_{d-k}(\mathrm{d}E) = \mathrm{vol}_k(M) \tag{5.33}$$

holds for all k-surfaces M (or at least for a nontrivial subclass, such as polyhedral surfaces)? This question has been studied in various degrees of generality, in particular, for Minkowski spaces and for projective Finsler spaces. A **Minkowski space** is a finite-dimensional real normed vector space. A **Finsler metric** on an open convex subset C of $\mathbb{R}^d$ is (here) a continuous function $F : C \times \mathbb{R}^d \to [0, \infty)$ such that $F(x, \cdot)$ is a norm on $\mathbb{R}^d$ for each $x \in C$. In the following, the pair (C, F) is called a **Finsler space**, and it is called smooth if F is of class C^∞ on $C \times \mathbb{R}^d \setminus \{0\}$ and the unit sphere of the norm $F(x, \cdot)$ is quadratically convex (has positive curvatures), for each $x \in C$. In a Finsler space, there is a canonical notion of curve length (and an induced metric), denoted by vol_1. The Finsler space (C, F) is called **projective** if line segments are shortest curves connecting their endpoints. The classical examples of projective Finsler spaces are Minkowski spaces and the Hilbert geometries in bounded open convex sets.

In a Finsler space, for $k > 1$ there are many different possibilities of defining a reasonable notion of k-dimensional area, but no canonical one (see Álvarez and Thompson [31] for a survey). Two such notions are particularly natural and important from a geometric point of view. These are the **Busemann k-area**, which is defined by the k-dimensional Hausdorff measure coming from the induced metric, and the **Holmes–Thompson k-area**, which is defined via the symplectic volume. For a more detailed introduction, we refer to Schneider [712, pp. 165–177].

For the existence of Crofton type formulas, it has turned out that the Holmes–Thompson area is the right area notion to be used. Let vol_k denote the k-dimensional Holmes–Thompson area. It was observed, with different degrees of generality, by Busemann [143], El–Ekhtiar [216] and Schneider and Wieacker [721] that for vol_{d-1}

in a Minkowski space there always exists a translation invariant measure η_1 on $A(d,1)$ so that (5.33) holds. In order that (5.33) hold for vol_1 with a translation invariant measure η_{d-1}, it is necessary and sufficient that the Minkowski space be hypermetric. (A metric space (S,δ) is **hypermetric** if $\sum_{i,j=1}^{k} \delta(p_i,p_j)N_iN_j \le 0$ holds for $k \ge 2$, all $p_1,\dots,p_k \in S$ and all integers $N_1,\dots,N_k$ with $\sum_{i=1}^{k} N_i = 1$.) This is equivalent to the condition that the unit ball of the dual Minkowski space is a zonoid. If this assumption is satisfied, then there are translation invariant measures η_j on $A(d,j)$ such that the general Crofton type formula

$$\int_{A(d,j)} \mathrm{vol}_{k+j-d}(M \cap E)\, \eta_j(\mathrm{d}E) = \alpha_{nkj} \mathrm{vol}_k(M) \tag{5.34}$$

holds for all $k \in \{1,\dots,d\}$, $j \in \{d-k,\dots,d-1\}$ and for all k-surfaces M. This was proved by Schneider and Wieacker [721, Th. 7.3]. For general (not necessarily smooth) hypermetric projective Finsler spaces, the existence of measures η_{d-k} so that (5.33) holds at least for k-dimensional compact convex sets M was established by Schneider [705] (for $k = d-1$, the assumption 'hypermetric' can be deleted). The proof yields merely the existence; an explicit construction for the line measure η_1 in polytopal Hilbert geometries is described in Schneider [711].

For smooth projective Finsler spaces, general investigations on Crofton densities have been undertaken by Gelfand and Smirnov [255] and by Álvarez, Gelfand and Smirnov [30], in part related to Hilbert's fourth problem and to symplectic geometry. Subsequent work by Álvarez and Fernandes [26, 27, 28, 29] and the thesis of Fernandes [231] use double fibrations and the Gelfand transform as a unifying approach to integral geometric intersection formulas and obtain, in particular, Crofton type formulas (with signed measures) for Holmes–Thompson areas of smooth submanifolds in smooth projective Finsler spaces. The first of these papers makes use of the symplectic structure on the space of geodesics of a projective Finsler space. Later it turned out that the methods applied by Schneider and Wieacker [721] for the case of hypermetric Minkowski spaces (where they yield measures η_j) can be adapted to the case of smooth projective Finsler spaces (where they yield signed measures). In this way, a very general version of the Crofton formula (5.34) was obtained, namely for $k = 1,\dots,d$, $j = d-k,\dots,d-1$ and for Holmes–Thompson areas of $(\mathcal{H}^k,k)$-rectifiable Borel sets M in smooth projective Finsler spaces (where the local unit spheres need not be quadratically convex); see Schneider [706].

The special role that the Holmes–Thompson area plays in connection with Crofton type formulas can be illuminated from other sides. Following Busemann, one can define a general notion of Minkowskian $(d-1)$-area by a few natural axioms. It was shown by Schneider [698] that there exist Minkowski spaces for which, among all Minkowskian $(d-1)$-areas, only the multiples of the Holmes–Thompson area allow a Crofton formula (5.33) for $k = d-1$ with a translation invariant measure η_1. For the Busemann area, the picture is not clear. Let us say that, for a Minkowski space $S = (\mathbb{R}^d, \|\cdot\|)$, the Busemann area is **integral geometric** if (5.33) holds for S and for the Busemann $(d-1)$-area with a translation invariant measure η_1, and at least for all $(d-1)$-dimensional compact convex sets M. The following was shown by Schneider [703], for $d \ge 3$. Every neighborhood (in the sense of the Banach–Mazur distance) of the Euclidean space ℓ_2^d contains Minkowski spaces for which the Busemann area is not integral geometric, as well as spaces (different from ℓ_2^d) for which the Busemann area is integral geometric. If d is sufficiently large, then a full neighborhood of the Minkowski space ℓ_∞^d consists of Minkowski spaces for which the Busemann area is not integral geometric. We conjecture that it is generically true (that is, for a dense open subset of the space of all d-dimensional Minkowski spaces) that the Busemann area is not integral geometric. In the preceding counterexamples, non-smoothness properties of the unit ball of the Minkowski space play a role. On the other hand, Álvarez and Berck [25] have constructed smooth projective Finsler spaces in which there is no Crofton formula for the Busemann area, not even with a signed measure.

Additional and more detailed information can be found in the survey of Schneider [710].