4

Geometric Models

Having laid the general foundations in the previous chapters, we now study geometric processes in $\mathbb{R}^d$ and the random sets derived from them. By geometric processes we understand point processes of closed sets which are concentrated on geometrically distinguished subclasses of $\mathcal{F}'$. In particular, we consider particle processes and flat processes. **Particle processes** are point processes in the subset $\mathcal{C}'$ of nonempty compact sets. Special processes, in general more tractable, are obtained if only particles from the convex ring $\mathcal{R}$ or even the class $\mathcal{K}$ of convex bodies are admitted. A k-**flat process** is a point process in $\mathcal{F}'$ whose intensity measure is concentrated on the space $A(d, k)$ of k-dimensional flats (planes, affine subspaces) of $\mathbb{R}^d$.

We begin with the investigation of particle processes. For these we introduce, in the stationary case, intensities, grain distributions, and densities of functionals in various representations. Special cases are fiber and surface processes; they are treated after the flat processes, in the fifth section. The second section establishes a connection between particle processes and marked point processes. In particular, we introduce the **germ-grain processes**, where compact sets serve as marks. The **germ-grain models** of the third section, which are generated from germ-grain processes, are important examples of random closed sets. An especially tractable subclass are the **Boolean models**, derived from Poisson processes. In the fourth section we treat flat processes. Of particular interest are the processes arising from flat processes by intersections, either with a fixed plane or by intersecting fixed numbers of the flats in the process. Some assertions about the intensities and the directional distributions of these derived processes are obtained, mainly in the case of Poisson processes. The sixth section is concerned with a set-valued parameter, which can be attached to different processes of geometric objects. This is Matheron's 'Steiner compact set', which we call here the **associated zonoid**. It permits us to obtain, among other results, several geometric inequalities for particle or flat processes. In some cases, these can be used to characterize processes with specific extremal properties.

We remind the reader of two general assumptions that we have made. The first one (at the end of Section 3.6) is that *all point processes considered from now on are simple*, except when the opposite is explicitly stated. The second assumption (at the end of Section 3.1) is that *only point processes with locally finite intensity measures are admitted*.

4.1 Particle Processes

By a **particle process** in $\mathbb{R}^d$ we understand a point process in $\mathcal{F}' = \mathcal{F}'(\mathbb{R}^d)$ that is concentrated on the subset $\mathcal{C}'$ of nonempty compact sets, that is, the intensity measure Θ of which satisfies $\Theta(\mathcal{F}' \setminus \mathcal{C}') = 0$. In particular, a point process in $\mathcal{F}'$ whose intensity measure is concentrated on $\mathcal{R}' = \mathcal{R} \setminus \{\emptyset\}$ or $\mathcal{K}' = \mathcal{K} \setminus \{\emptyset\}$, is called a **particle process in** $\mathcal{R}$, respectively, **in** $\mathcal{K}$, in the latter case also a **process of convex particles**. The local finiteness of the intensity measure Θ of a particle process is, by Lemma 2.3.1, equivalent to

$$\Theta(\mathcal{F}_C) < \infty \qquad \text{for all } C \in \mathcal{C}. \tag{4.1}$$

The assumption (4.1) is essential for many later consequences. This is one reason for the fact that we did not define a particle process as a point process in the space $(\mathcal{C}', \delta)$ (with the Hausdorff metric); local finiteness of an intensity measure Θ in this case would only mean that $\Theta(\mathcal{F}^{C^c}) < \infty$ for $C \in \mathcal{C}$.

Nevertheless, it is convenient in the following, when we work with the set $\mathcal{C}'$, to equip it with the Hausdorff metric δ. In particular, continuity of functions on $\mathcal{C}'$ will refer to the Hausdorff metric. Although this continuity differs from continuity with respect to the topology of $\mathcal{F}$, for measurability there is no difference (see Theorem 2.4.1).

The intensity measure of a stationary particle process has a useful decomposition, obtained, roughly speaking, by factoring out the translations. For this, we need a **center function**, and we choose here the mapping

$$c : \mathcal{C}' \to \mathbb{R}^d$$

that associates with each $C \in \mathcal{C}'$ the **circumcenter** $c(C)$ of C. By definition, this is the center of the (uniquely determined) smallest ball containing C. We denote this ball by $B(C)$ and call it the **circumball** and its radius $r(C)$ the **circumradius** of C.

Lemma 4.1.1. *The mapping c is continuous on $\mathcal{C}'$.*

Proof. Let $r(C)$ denote the radius of $B(C)$, for $C \in \mathcal{C}'$. We show first that r is continuous. Let $C_i \to C$ be a convergent sequence in $\mathcal{C}'$. Every accumulation point of the sequence $(B(C_i))_{i \in \mathbb{N}}$ is a ball containing C. This implies $r(C) \leq \liminf r(C_i)$. Conversely, for given $\epsilon > 0$, almost all C_i are contained in the ball $B(C) + \epsilon B^d$, hence $\limsup r(C_i) \leq r(C) + \epsilon$. For $\epsilon \to 0$ we obtain $r(C) = \lim r(C_i)$.

Next, we show that $B(C_i) \rightarrow B(C)$. The sequence of balls $B(C_i)$ is bounded, hence we can assume without loss of generality that it converges, say $B(C_i) \rightarrow B$. The limit body B is a ball containing C, and since $r(C_i) \rightarrow r(C)$, it has radius $r(C)$. Since the circumball is unique, we have $B = B(C)$.

Finally, $B(C_i) \rightarrow B(C)$ implies $c(C_i) \rightarrow c(C)$. □

We put

$$\mathcal{C}_0 := \{C \in \mathcal{C}' : c(C) = 0\}$$

and call $\mathcal{C}_0$ the **grain space** (for particle processes). This grain space may also be considered as the set of all translation classes in $\mathcal{C}'$. The set $\mathcal{C}_0$ is closed in $\mathcal{C}'$ and hence (by Lemma 2.1.2) a Borel set in $\mathcal{F}$. Similarly, we define the subsets $\mathcal{K}_0 := \mathcal{C}_0 \cap \mathcal{K}'$, and $\mathcal{R}_0 := \mathcal{C}_0 \cap \mathcal{R}'$. For a subset $B \subset \mathbb{R}^d$ we put

$$\mathcal{C}_c(B) := \{C \in \mathcal{C}' : c(C) \in B\}.$$

The mapping

$$\Phi : \mathbb{R}^d \times \mathcal{C}_0 \rightarrow \quad \mathcal{C}'$$

$$(x, C) \mapsto x + C$$

is a homeomorphism, by Lemma 4.1.1 and Theorem 12.3.5.

Theorem 4.1.1. *Let X be a stationary particle process in $\mathbb{R}^d$ with intensity measure $\Theta \neq 0$. Then there exist a number $\gamma \in (0, \infty)$ and a probability measure $\mathbb{Q}$ on $\mathcal{C}_0$ such that*

$$\Theta = \gamma \Phi(\lambda \otimes \mathbb{Q}). \tag{4.2}$$

The number γ and the measure $\mathbb{Q}$ are uniquely determined.

Proof. Let $\widetilde{\Theta} := \Phi^{-1}(\Theta)$ be the image measure of Θ on $\mathbb{R}^d \times \mathcal{C}_0$. We first show a finiteness property. For the unit cube $C^d = [0, 1]^d$, we put $C_0^d := [0, 1)^d = C^d \setminus \partial^+ C^d$, where

$$\partial^+ C^d := \{x = (x^1, \ldots, x^d) \in C^d : \max_{1 \le i \le d} x^i = 1\}$$

is the **upper right boundary** of C^d. Let $(z_i)_{i \in \mathbb{N}}$ be an enumeration of $\mathbb{Z}^d$. We have

$$\widetilde{\Theta}(C_0^d \times \mathcal{C}_0) = \Theta(\{C \in \mathcal{C}' : c(C) \in C_0^d\})$$

$$\le \sum_{i=1}^{\infty} \Theta(\{C \in \mathcal{C}' : C \cap (C_0^d + z_i) \neq \emptyset, \ c(C) \in C_0^d\})$$

$$= \sum_{i=1}^{\infty} \Theta(\{C \in \mathcal{C}' : C \cap C_0^d \neq \emptyset, \ c(C) \in C_0^d - z_i\})$$

$$= \Theta(\{C \in \mathcal{C}' : C \cap C_0^d \neq \emptyset\}) \le \Theta(\mathcal{F}_{C^d}) < \infty,$$

due to the translation invariance of Θ (which follows from the stationarity of X) and (4.1).

Now we can copy the proof of Theorem 3.5.1, to obtain a representation

$$\widetilde{\Theta} = \gamma(\lambda \otimes \mathbb{Q})$$

with $\gamma \in (0, \infty)$ and a probability measure $\mathbb{Q}$ on $\mathcal{C}_0$. This proves (4.2). The uniqueness is trivial.   □

Note that Theorem 4.1.1 shows that for Θ-integrable functions f on $\mathcal{C}'$ we have

$$\int_{\mathcal{C}'} f \, \mathrm{d}\Theta = \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} f(C + x) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}C), \tag{4.3}$$

which will be used frequently.

We call γ the **intensity** and $\mathbb{Q}$ the **grain distribution** of the stationary particle process X. A random set with distribution $\mathbb{Q}$ is called the **typical grain** of X. If X is isotropic, then $\mathbb{Q}$ is rotation invariant (since $c(\vartheta C) = \vartheta c(C)$ for $C \in \mathcal{C}'$ and $\vartheta \in SO_d$), but the converse is generally false. Unless explicitly stated otherwise, we occasionally allow also stationary particle processes with $\Theta = 0$; in this case we define $\gamma = 0$, the grain distribution $\mathbb{Q}$ is not defined, and $\gamma\mathbb{Q}$ has to be read as the zero measure.

For later applications, it will be necessary to admit other center functions, besides the circumcenter. If c is replaced by a measurable mapping $z : \mathcal{C}' \to \mathbb{R}^d$ satisfying $z(tC) = tz(C)$ for $C \in \mathcal{C}'$ and every translation t, then again a decomposition (4.2) is obtained, with different $\mathbb{Q}$. However, isotropy of X is reflected in rotation invariance of $\mathbb{Q}$ only if z has the additional property $z(\vartheta C) = \vartheta z(C)$ for $\vartheta \in SO_d$. (An example, different from the circumcenter, with this property is provided by the Steiner point of the convex hull.) Such center functions play a role if a particle process is to be represented as a marked point process; this is explained in Section 4.2.

It must be noted that the assumed local finiteness of the intensity measure on $\mathcal{F}'$ has the consequence that not every probability measure on $\mathcal{C}_0$ can occur as the grain distribution $\mathbb{Q}$ of a stationary particle process. The following theorem clarifies this.

Theorem 4.1.2. *The probability measure $\mathbb{Q}$ on $\mathcal{C}_0$ is the grain distribution of some stationary particle process if and only if*

$$\int_{\mathcal{C}_0} V_d(C + rB^d) \, \mathbb{Q}(\mathrm{d}C) < \infty \quad \text{for some (or all) } r > 0. \tag{4.4}$$

This is equivalent to the $\mathbb{Q}$-integrability of the dth power of the circumradius, and in the case of a process of convex particles it is equivalent to the $\mathbb{Q}$-integrability of the intrinsic volumes $V_1, \ldots, V_d$.

If $\mathbb{Q}$ satisfies (4.4) and if $\gamma > 0$ is given, then there exists (up to stochastic equivalence) precisely one stationary Poisson particle process X in $\mathbb{R}^d$ with intensity γ and grain distribution $\mathbb{Q}$. The process X is isotropic if and only if $\mathbb{Q}$ is rotation invariant.

Proof. If Θ is the intensity measure of the stationary particle process X with intensity $\gamma > 0$ and grain distribution $\mathbb{Q}$, then (4.3) gives, for $K \in \mathcal{C}$,

$$\Theta(\mathcal{F}_K) = \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_{\mathcal{F}_K}(C + x) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \gamma \int_{\mathcal{C}_0} V_d(-C + K) \, \mathbb{Q}(\mathrm{d}C),$$

since

$$\mathbf{1}_{\mathcal{F}_K}(C + x) = 1 \Leftrightarrow (C + x) \cap K \neq \emptyset \Leftrightarrow x \in -C + K.$$

Hence, the local finiteness of Θ implies, in particular, that

$$\int_{\mathcal{C}_0} V_d(C + rB^d) \, \mathbb{Q}(\mathrm{d}C) < \infty \tag{4.5}$$

for all $r > 0$. If (4.5) is satisfied for one number $r > 0$, then

$$\int_{\mathcal{C}_0} V_d(C + K) \, \mathbb{Q}(\mathrm{d}C) < \infty$$

holds for all $K \in \mathcal{C}$, since K can be covered by finitely many translates of rB^d. The remaining equivalences follow from

$$V_d(C + B^d) \leq 2^d \kappa_d \max\{r(C)^d, 1\}$$

and, in the case of a process of convex particles, from the Steiner formula (14.5).

Suppose, conversely, that $\mathbb{Q}$ satisfies (4.4) and that $\gamma > 0$ is given. Then the measure Θ defined by (4.2) is locally finite and translation invariant. By Theorems 3.2.1 and 3.6.1, there exists a Poisson process, unique up to equivalence, with intensity measure Θ. It is stationary, and if $\mathbb{Q}$ is rotation invariant, it is also isotropic. □

The intuitive meaning of the intensity and the grain distribution of a stationary particle process will become clearer by the representations given below, as special cases of the next theorem. With this theorem, we turn to a refined quantitative description of particle processes, which we begin with the definition of densities for geometric functionals. For stationary particle processes, Theorem 4.1.1 opens an easy way of introducing mean values of geometric quantities.

Let $\varphi : \mathcal{C}' \to \mathbb{R}$ be a translation invariant, measurable function, and let X be a stationary particle process with intensity $\gamma > 0$ and grain distribution $\mathbb{Q}$. If φ is nonnegative or $\mathbb{Q}$-integrable, we define the φ-**density** of X by

$$\overline{\varphi}(X) := \gamma \int_{\mathcal{C}_0} \varphi \, \mathrm{d}\mathbb{Q}. \tag{4.6}$$

Remark. We emphasize that $\overline{\varphi}(X)$ is defined here as the mean value of φ with respect to the grain distribution $\mathbb{Q}$, *multiplied by the intensity* γ. That the factor γ has been included in the definition, simplifies many formulas, but must be observed when these formulas are compared with other literature.

For nonnegative φ, it is permitted in (4.6) that $\overline{\varphi}$ (and thus also the limit in Theorem 4.1.3(b)) is infinite.

The following theorem gives different representations of the φ-density, and it also justifies this name.

Theorem 4.1.3. *Let X be a stationary particle process in $\mathbb{R}^d$ with grain distribution $\mathbb{Q}$, and let $\varphi : \mathcal{C}' \to \mathbb{R}$ be a translation invariant measurable function which is nonnegative or $\mathbb{Q}$-integrable.*

(a) *For all $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda(B) < \infty$,*

$$\overline{\varphi}(X) = \frac{1}{\lambda(B)} \, \mathbb{E} \sum_{C \in X, \, c(C) \in B} \varphi(C).$$

(b) *For all $W \in \mathcal{K}$ with $V_d(W) > 0$,*

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{C \in X, \, C \subset rW} \varphi(C).$$

(c) *If*

$$\int_{\mathcal{C}_0} |\varphi(C)| V_d(C + B^d) \, \mathbb{Q}(\mathrm{d}C) < \infty,$$

then

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{C \in X, \, C \cap rW \neq \emptyset} \varphi(C)$$

for all $W \in \mathcal{K}$ with $V_d(W) > 0$.

Proof. (a) From Campbell's theorem (Theorem 3.1.2) and (4.3), together with the translation invariance of φ, we get

$$\mathbb{E} \sum_{C \in X, \, c(C) \in B} \varphi(C) = \mathbb{E} \sum_{C \in X} \mathbf{1}_B(c(C)) \varphi(C)$$

$$= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_B(c(C + x)) \varphi(C) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \gamma \lambda(B) \int_{\mathcal{C}_0} \varphi(C) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \lambda(B) \overline{\varphi}(X).$$

(b) As above, we get

$$\mathbb{E} \sum_{C \in X,\, C \subset rW} \varphi(C) = \gamma \int_{\mathcal{C}_0} \varphi(C) \lambda(\{x \in \mathbb{R}^d : C + x \subset rW\})\, \mathbb{Q}(dC).$$

We may assume that 0 is in the interior of W. Then there is a nonnegative measurable function $\rho : \mathcal{C}_0 \to \mathbb{R}$ such that $C' \subset \rho(C')W$ for all $C' \in \mathcal{C}_0$. For given $C \in \mathcal{C}_0$, suppose that $r > \rho(C)$. For $x \in (r - \rho(C))W$ we have $C + x \subset rW$. It follows that

$$\left(1 - \frac{\rho(C)}{r}\right)^d \leq \frac{\lambda(\{x \in \mathbb{R}^d : C + x \subset rW\})}{V_d(rW)} \leq 1.$$

For $r \to \infty$, the left side converges monotonically to 1, hence the monotone convergence theorem proves the assertion if φ is nonnegative. The dominated convergence theorem gives the result if φ is integrable.

(c) We have

$$\frac{1}{V_d(rW)} \mathbb{E} \sum_{C \in X,\, C \cap rW \neq \emptyset} \varphi(C) = \frac{\gamma}{V_d(W)} \int_{\mathcal{C}_0} \varphi(C) V_d\left(W - \frac{1}{r}C\right) \mathbb{Q}(dC).$$

For $C \in \mathcal{C}_0$, let B be a ball with $-C \subset B$, then $W - \frac{1}{r}C \subset W + \frac{1}{r}B$ and hence $V_d(W) \leq V_d(W - \frac{1}{r}C) \leq V_d(W + \frac{1}{r}B)$. From the continuity of the volume functional on $\mathcal{K}$ it follows that

$$V_d\left(W - \frac{1}{r}C\right) \to V_d(W) \qquad \text{for } r \to \infty.$$

Let $y_1, \dots, y_m \in -C$ be points with $(W + y_i) \cap (W + y_j) = \emptyset$ for $i \neq j$, and assume that m is maximal. For $x \in -C$ there exists i with $(W+x) \cap (W+y_i) \neq \emptyset$, thus $x \in W - W + y_i$. This shows that $-C \subset \bigcup_i (W - W + y_i)$. We may assume that $0 \in W$. For $r \geq 1$, we get $W - \frac{1}{r}C \subset \bigcup_i \left[2W - W + \frac{1}{r}y_i\right]$ and, therefore, $V_d(W - \frac{1}{r}C) \leq m V_d(2W - W)$. From $m V_d(W) \leq V_d(W - C)$ we obtain

$$V_d\left(W - \frac{1}{r}C\right) \leq b(W) V_d(W - C)$$

with a constant $b(W)$ that does not depend on C or r.

There are finitely many vectors $t_1, \dots, t_n \in \mathbb{R}^d$ such that $W \subset \bigcup_{i=1}^n (B^d + t_i)$. This yields $W - C \subset \bigcup_{i=1}^n (B^d - C + t_i)$, hence $V_d(W - C) \leq n V_d(B^d - C) = n V_d(B^d + C)$ and thus

$$\int_{\mathcal{C}_0} |\varphi(C)| V_d(W - C)\, \mathbb{Q}(dC) < \infty.$$

The assertion now follows from the dominated convergence theorem. $\square$

For additive functionals φ, further representations of the φ-density will be given in Theorem 9.2.2. The most important such functionals will be the intrinsic volumes of convex bodies.

As a special case of Theorem 4.1.3, we may choose

$$\varphi(C) := \mathbf{1}_A(C - c(C)) \qquad \text{with } A \in \mathcal{B}(\mathcal{C}_0).$$

If B is as in Theorem 4.1.3, we get

$$\gamma\mathbb{Q}(A) = \frac{1}{\lambda(B)} \, \mathbb{E} \sum_{C \in X, \, c(C) \in B} \mathbf{1}_A(C - c(C)),$$

in particular

$$\gamma = \frac{1}{\lambda(B)} \, \mathbb{E} \sum_{C \in X, \, c(C) \in B} 1. \tag{4.7}$$

Thus, the intensity γ can be interpreted as the expected number of particles per unit volume.

Further we obtain, with W is as in Theorem 4.1.3,

$$\gamma\mathbb{Q}(A) = \lim_{r \to \infty} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{C \in X, \, C \subset rW} \mathbf{1}_A(C - c(C)) \tag{4.8}$$

$$= \lim_{r \to \infty} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{C \in X, \, C \cap rW \neq \emptyset} \mathbf{1}_A(C - c(C)). \tag{4.9}$$

If X is a stationary particle process in $\mathbb{R}^d$, then the points $c(C)$, $C \in X$, almost surely generate an ordinary point process X^0 in $\mathbb{R}^d$ (the necessary finiteness condition follows from the proof of Theorem 4.1.1). The corresponding assertion is also true in the non-stationary case if the particles are convex; however, for non-convex particles, the measure $\sum_{C \in X} \delta_{c(C)}$ is not necessarily locally finite a.s. For a stationary (simple) particle process X, the point process X^0 need no longer be simple. We shall see, however, that for a stationary Poisson particle process the point process X^0 is always simple (and thus a stationary Poisson process, too).

In the case of a stationary particle process X the intensity γ is, according to (4.7), also the intensity of the stationary point process X^0. One might interpret this as a construction of X, starting from the ordinary point process X^0 and attaching random compact sets with distribution $\mathbb{Q}$ to the points (regarding multiplicities). However, this idea is not precise, since the random $\mathcal{C}_0$-sets corresponding to different points of X^0 need not be independent. The following example may be instructive.

In $\mathbb{R}^2$, let s_h be a horizontal and s_v a vertical unit segment, both with center 0, and consider the system of segments $s_h + z$, $s_v + z'$, $z, z' \in \mathbb{Z}^2$, where z has even and z' has odd sum of coordinates. Applying to this system the translation by a random vector, uniformly distributed in $[0, 1]^2$, we obtain a stationary particle process for which $\mathbb{Q}$ is concentrated on the set $\{s_h, s_v\}$ (associating probability $1/2$ to either segment). In this case, the particle, s_h

or s_v, attached to one point of the ordinary point process X^0, completely determines all the other particles of the realization.

The situation is different for Poisson processes.

Theorem 4.1.4. *Let X be a stationary Poisson particle process in $\mathbb{R}^d$ with intensity $\gamma > 0$ and grain distribution $\mathbb{Q}$. Then the ordinary point process X^0 is a (stationary) Poisson process, and the following holds.*

To every $B \in \mathcal{C}$ with $\lambda(B) > 0$ and every $k \in \mathbb{N}$ there exist random points $\xi_1, \ldots, \xi_k$ in B with distribution $\lambda \llcorner B / \lambda(B)$ and random closed sets $Z_1, \ldots, Z_k$ in $\mathcal{C}_0$ with distribution $\mathbb{Q}$ such that $\xi_1, \ldots, \xi_k, Z_1, \ldots, Z_k$ are independent and

$$\mathbb{P}(X \llcorner \mathcal{C}_c(B) \in \cdot \mid X(\mathcal{C}_c(B)) = k) = \mathbb{P}\left(\sum_{i=1}^{k} \delta_{\xi_i + Z_i} \in \cdot\right).$$

In other words, a stationary Poisson process X in $\mathcal{C}'$ can be generated (and also simulated) by taking an ordinary stationary Poisson process X^0 with intensity γ and adding to every $x \in X^0$ independently a random closed set Z_x with distribution $\mathbb{Q}$,

$$X = \sum_{x \in X^0} \delta_{x + Z_x}.$$

Proof. Since

$$\mathbb{P}(X^0(A) = k) = \mathbb{P}(X(\mathcal{C}_c(A)) = k), \qquad A \in \mathcal{B}(\mathbb{R}^d),$$

X^0 is a stationary Poisson process in $\mathbb{R}^d$ with intensity measure

$$\Theta_c(A) = \Theta(\mathcal{C}_c(A)) = \gamma\lambda(A), \qquad A \in \mathcal{B}(\mathbb{R}^d);$$

here we have applied Theorem 4.1.1 to the set $\mathcal{C}_c(A)$. In particular, if B is compact and $\lambda(B) > 0$, then $0 < \Theta(\mathcal{C}_c(B)) < \infty$, hence we can apply Theorem 3.2.2(b) and obtain independent random closed sets $\widetilde{Z}_1, \ldots, \widetilde{Z}_k$ with

$$\mathbb{P}(X \llcorner \mathcal{C}_c(B) \in \cdot \mid X(\mathcal{C}_c(B)) = k) = \mathbb{P}\left(\sum_{i=1}^{k} \delta_{\widetilde{Z}_i} \in \cdot\right);$$

here each $\widetilde{Z}_i$ has distribution

$$\mathbb{P}_{\widetilde{Z}_i} = \frac{\Theta \llcorner \mathcal{C}_c(B)}{\Theta(\mathcal{C}_c(B))}.$$

From $\Theta \llcorner \mathcal{C}_c(B) = \Phi((\lambda \llcorner B) \otimes \gamma\mathbb{Q})$ and $\Theta(\mathcal{C}_c(B)) = \gamma\lambda(B)$ it follows that

$$\mathbb{P}_{\widetilde{Z}_i} = \Phi\left(\frac{\lambda \llcorner B}{\lambda(B)} \otimes \mathbb{Q}\right).$$

Hence, if we define Z_i, ξ_i by $\Phi^{-1} \circ \widetilde{Z}_i =: (\xi_i, Z_i)$, we obtain independent random variables Z_i (random closed sets with distribution $\mathbb{Q}$) and ξ_i (random points with distribution $\lambda \llcorner B / \lambda(B)$), and we have

$$\widetilde{Z}_i = \Phi(\xi_i, Z_i) = \xi_i + Z_i,$$

therefore also

$$\mathbb{P}\left(\sum_{i=1}^{k} \delta_{\widetilde{Z}_i} \in \cdot\right) = \mathbb{P}\left(\sum_{i=1}^{k} \delta_{\xi_i + Z_i} \in \cdot\right).$$

This completes the proof.   □

Marked Particle Processes

In Chapter 10 it will be useful to have limit relations, analogous to those of Theorem 4.1.3, for marked particle processes. By a **marked particle process** we understand a simple point process in $\mathcal{C}' \times M$, where M denotes the mark space, as in Section 3.5. For the intensity measure Θ we assume, corresponding to (3.12), that

$$\Theta(C \times M) < \infty \qquad \text{for all } C \in \mathcal{C}(\mathcal{F}').$$

Stationarity again means invariance of the distribution $\mathbb{P}_X$ under translations, where these, as in the case of marked point processes, affect only the first component. Let X be a stationary marked particle process with intensity measure $\Theta \neq 0$. Using the mapping

$$\Phi : \mathbb{R}^d \times \mathcal{C}_0 \times M \to \quad \mathcal{C}' \times M$$
$$(x, C, m) \quad \mapsto (x + C, m),$$

we obtain, in analogy to Theorem 4.1.1, a decomposition

$$\Theta = \gamma \Phi(\lambda \otimes \mathbb{Q})$$

where $\mathbb{Q}$ is now a probability measure on $\mathcal{C}_0 \times M$; it is called the **grain-mark distribution** of X. With this decomposition, Campbell's theorem and the analog of (4.3) read as follows. For every nonnegative measurable function f on $\mathcal{C}' \times M$,

$$\mathbb{E} \sum_{(C,m) \in X} f(C, m) = \int_{\mathcal{C}' \times M} f \, \mathrm{d}\Theta$$

$$= \gamma \int_{\mathcal{C}_0 \times M} \int_{\mathbb{R}^d} f(C + x, m) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}(C, m)).$$

Theorem 4.1.5. *Let X be a stationary marked particle process in $\mathbb{R}^d$ with grain-mark distribution $\mathbb{Q}$, and let $\varphi : \mathcal{C}' \times M \to \mathbb{R}$ be a measurable function*

which is translation invariant in the first variable and either nonnegative or $\mathbb{Q}$-integrable. Then the φ-density defined by

$$\overline{\varphi}(X) := \gamma \int_{\mathcal{C}_0 \times M} \varphi \, \mathrm{d}\mathbb{Q}$$

has the representations

$$\overline{\varphi}(X) = \frac{1}{\lambda(B)} \, \mathbb{E} \sum_{(C,m) \in X, \, c(C) \in B} \varphi(C, m) \qquad (4.10)$$

for $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda(B) < \infty$,

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{(C,m) \in X, \, C \subset rW} \varphi(C, m) \qquad (4.11)$$

for $W \in \mathcal{K}$ with $V_d(W) > 0$, and

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{(C,m) \in X, \, C \cap rW \neq \emptyset} \varphi(C, m), \qquad (4.12)$$

if, in addition,

$$\int_{\mathcal{C}_0 \times M} |\varphi(C, m)| V_d(C + B^d) \, \mathbb{Q}(\mathrm{d}(C, m)) < \infty$$

is satisfied.

The proof is obtained by the obvious modification of that of Theorem 4.1.3.

Note for Section 4.1

Theorem 4.1.1 shows, as did also Theorems 3.3.1 and 3.5.1, how the assumption of stationarity leads to a decomposition of the intensity measure, where the Lebesgue measure appears as a factor. Such a factorization of measures with partial invariance properties on (local) product spaces, where a Haar measure appears as one factor, was raised by Ambartzumian [35] to a basic principle of stochastic geometry.

4.2 Germ-grain Processes

We recall the convention, agreed upon in Section 3.1, that simple counting measures are often identified with their supports, so that, for example, for $\eta \in \mathsf{N}_s(\mathcal{C}')$ the notations $\eta(\{C\}) = 1$ and $C \in \eta$ are employed synonymously.

In the previous section, the decomposition of the intensity measure of a stationary particle process X in $\mathbb{R}^d$ was based on a representation of the

particles $C \in X$ in the form $C = C_0 + x$ with $x := c(C)$ and $C_0 := C - c(C)$. Of course, the formulation of Theorem 4.1.1 is strongly reminiscent of the corresponding Theorem 3.5.1 on marked point processes. One can, in fact, identify the stationary particle process X with the marked point process

$$\Phi^{-1}(X) = \{(x, C) \in \mathbb{R}^d \times \mathcal{C}_0 : C + x \in X\}.$$

Here the mark space is $\mathcal{C}_0$, and the grain distribution $\mathbb{Q}$ becomes the mark distribution. Before we make use of this connection and apply the results of Section 3.5 on Palm distributions of marked point processes to particle processes, we want to clarify the role that the choice of the circumcenter as a center of the particles plays here.

Generally, we understand by a **center function** a measurable map $z : \mathcal{C}' \to \mathbb{R}^d$ which is compatible with translations, that is, satisfies

$$z(C + x) = z(C) + x \qquad \text{for all } x \in \mathbb{R}^d.$$

Examples of center functions, besides the circumcenter c, are the center of gravity, if only particles with positive Lebesgue measure are considered, or the Steiner point of the convex hull. These center functions are also equivariant under rotations (that is, $z(\vartheta C) = \vartheta z(C)$ for $\vartheta \in SO_d$). The following examples do not have this property. For $C \in \mathcal{C}'(\mathbb{R}^2)$ (the definition can be extended to $d \geq 2$), the **lower tangent point** of C is defined by $\widetilde{z}(C) = (z^1, z^2)$ with

$$z^2 := \min\{x^2 : (x^1, x^2) \in C\}$$
$$z^1 := \min\{x^1 : (x^1, z^2) \in C\},$$

where x^1, x^2 are the coordinates of x. Further, the **left lower corner** of C is defined by

$$z'(C) := \left(\min_{x \in C} x^1, \min_{x \in C} x^2 \right)$$

(in general, $z'(C) \notin C$). These center functions are applied in certain estimation procedures. As in the example of the center of gravity, it is sometimes convenient to allow center functions that are defined only on measurable subclasses of $\mathcal{C}'$ which are closed under translations.

If X is a particle process and z is a center function, then

$$X^z := \sum_{C \in X} \delta_{z(C)}$$

is a random counting measure on $\mathbb{R}^d$ which, however, need neither be simple nor locally finite. In the stationary case, local finiteness is ensured. Thus, the following connection between stationary particle processes and marked point processes can be established.

Theorem 4.2.1. *Let X be a stationary particle process in $\mathbb{R}^d$, and let z be a center function. Then X^z is a stationary point process in $\mathbb{R}^d$, and*

$$X_z := \sum_{C \in X} \delta_{(z(C), C - z(C))}$$

is a stationary marked point process with mark space $\mathcal{C}'$. The intensities of X, X^z and X_z are the same. The mark distribution $\mathbb{Q}_z$ of X_z is the image of the grain distribution $\mathbb{Q}$ of X under the mapping $C \mapsto C - z(C)$.

In particular, the grain distribution $\mathbb{Q}$ is the mark distribution of X_c.

Proof. To show the measurability of X_z, it suffices by Lemma 3.1.5 to verify that $\{X_z(A) = k\}$ is measurable for all $A \in \mathcal{B}(\mathbb{R}^d \times \mathcal{C}')$ and all $k \in \mathbb{N}_0$. Let A and k be given. The function

$$\begin{aligned} \varphi : \mathcal{C}' &\to & \mathbb{R}^d \times \mathcal{C}' \\ C &\mapsto (z(C), C - z(C)) \end{aligned}$$

is measurable, since z is measurable. Hence, $\{X_z(A) = k\} = \{X(\varphi^{-1}(A)) = k\}$ is measurable.

As in the proof of Theorem 4.1.1 (where we replace $\mathcal{C}_0$ by $\mathcal{C}'$ and c by z) we obtain

$$\mathbb{E} X_z(C_0^d \times \mathcal{C}') \leq \Theta(\mathcal{F}_{C^d}) < \infty,$$

where Θ is the intensity measure of X. This gives $\mathbb{E} X_z(C \times \mathcal{C}') < \infty$ for every compact set $C \in \mathcal{C}$; thus the measure X_z is a.s. locally finite. Hence, X^z is a point process with locally finite intensity measure, and X_z satisfies (3.12) and is, therefore, a marked point process in $\mathbb{R}^d$.

For $t \in \mathbb{R}^d$, the definition of the operation of the translation group on $\mathbb{R}^d \times \mathcal{C}'$ and the compatibility of z with translations give

$$\begin{aligned} X_z + t &= \sum_{C \in X} \delta_{(z(C)+t, C - z(C))} \\ &= \sum_{C \in X} \delta_{(z(C+t), C + t - z(C+t))} \\ &= \sum_{C \in X+t} \delta_{(z(C), C - z(C))} \\ &= (X + t)_z. \end{aligned}$$

Since X and $X + t$ have the same distribution, the same holds for X_z and $X_z + t$, which means that X_z is stationary. From this it follows that also X^z is stationary.

Let $\mathbb{Q}_z$ be the mark distribution of X_z and let γ_z be its intensity. Denoting by γ and $\mathbb{Q}$ the intensity and the grain distribution of X, from Theorems 3.5.1, 3.1.2 and 4.1.1 we get, for $B \in \mathcal{B}(\mathbb{R}^d)$ and $A \in \mathcal{B}(\mathcal{C}')$,

$$\gamma_z \lambda \otimes \mathbb{Q}_z (B \times A) = \mathbb{E} X_z (B \times A)$$

$$= \mathbb{E} \sum_{C \in X} \mathbf{1}_B (z(C)) \mathbf{1}_A (C - z(C))$$

$$= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_B (z(C+x)) \mathbf{1}_A (C + x - z(C+x)) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \gamma \lambda(B) \int_{\mathcal{C}_0} \mathbf{1}_A (C - z(C)) \, \mathbb{Q}(dC)$$

$$= \gamma (\lambda \otimes f_z(\mathbb{Q}))(B \times A)$$

with $f_z : \mathcal{C}_0 \to \mathcal{C}'$ defined by $f_z(C) = C - z(C)$. The special case $B = C^d$ and $A = \mathcal{C}'$ gives $\gamma = \gamma_z$, which is also the intensity of X^z. Now it follows that $\mathbb{Q}_z$ is the image measure of $\mathbb{Q}$ under the mapping f_z. □

Some properties of the mark distribution, for example rotation invariance, depend essentially on the choice of the center function. If X is isotropic and z is equivariant under rotations, then the mark distribution, too, is rotation invariant.

By Theorem 4.2.1, to every stationary particle process X there corresponds a whole family of marked point processes with mark space $\mathcal{C}'$; every center function z generates an element X_z of this family. The special choice $z = c$ yields the canonical model X_c of X with $X^z = X^0$, which we mostly use in the following. If X is a stationary Poisson process, a corresponding assertion holds for X_z.

Theorem 4.2.2. *Let X be a stationary Poisson particle process in $\mathbb{R}^d$, and let z be a center function. Then X_z is an independently marked stationary Poisson process.*

Proof. We define

$$\varphi : \mathcal{C}' \to \qquad \mathbb{R}^d \times \mathcal{C}_0$$
$$C \mapsto (z(C), C - z(C)).$$

As we have seen in the proof of Theorem 4.2.1,

$$\{X_z(A) = k\} = \{X(\varphi^{-1}(A)) = k\}$$

holds for Borel sets $A \in \mathcal{B}(\mathbb{R}^d \times \mathcal{C}_0)$ and $k \in \mathbb{N}_0$. Hence, X_z is a stationary Poisson process in $\mathbb{R}^d \times \mathcal{C}_0$. The assertion now follows from Theorem 3.5.8. □

We return to the general situation and now consider, conversely, a marked point process $\widetilde{X}$ with mark space $\mathcal{C}'$. Then

$$X := \sum_{(x,C) \in \widetilde{X}} \delta_{x+C} \tag{4.13}$$

defines a particle process X, if the local finiteness of the counting measures on the right side (and of the intensity measure) is guaranteed. In this case, we call $\widetilde{X}$ a **germ-grain process**. The intuitive idea behind this is that the points x of the pairs $(x, C) \in \widetilde{X}$ are the 'germs' and the compact sets $x + C$ are the 'grains'. The process (4.13) is called the **particle process generated by $\widetilde{X}$**. If, in particular, $\widetilde{X}$ is stationary, then in analogy to Theorem 4.1.2 one finds that for the local finiteness of the intensity measure of X the condition

$$\int_{\mathcal{C}'} V_d(C + B^d)\, \mathbb{Q}(dC) < \infty \tag{4.14}$$

on the mark distribution $\mathbb{Q}$ of $\widetilde{X}$ is necessary and sufficient. In the stationary case, the mark distribution $\mathbb{Q}$ is also called the **distribution of the typical grain**, and every random closed set Z_0 with distribution $\mathbb{Q}$ is called the **typical grain** (or **primary grain**) of $\widetilde{X}$.

If $\widetilde{X}$ is an independently marked point process with mark space $\mathcal{C}'$, then even without stationarity it is possible to work with the mark distribution $\mathbb{Q}$, as in Section 3.5. Also in this case, a random closed set Z_0 with distribution $\mathbb{Q}$ is called the **typical grain** of $\widetilde{X}$. The finiteness condition (4.1) for the intensity measure Θ of the particle process X generated by (4.13) can now be rewritten in the following way. For $C \in \mathcal{C}'$ we have

$$\Theta(\mathcal{F}_C) = \int_{\mathbb{R}^d} \int_{\mathcal{C}'} \mathbf{1}_{\mathcal{F}_C}(x + K)\, \mathbb{Q}(dK)\, \vartheta(dx)$$

$$= \int_{\mathbb{R}^d} T_{Z_0}(C - x)\, \vartheta(dx).$$

Here, ϑ is the intensity measure of the unmarked process X^0, and T_{Z_0} is the capacity functional of the typical grain Z_0 of $\widetilde{X}$. Hence, for the particle process X, (4.1) is equivalent to

$$\int_{\mathbb{R}^d} T_{Z_0}(C - x)\, \vartheta(dx) < \infty \qquad \text{for } C \in \mathcal{C}'. \tag{4.15}$$

If $\widetilde{X}$ is stationary, this is again equivalent to (4.14). An independently marked point process $\widetilde{X}$ satisfying (4.15) is called an **independent germ-grain process**.

If, for an independent germ-grain process $\widetilde{X}$, the germ process X^0 is a Poisson process, and thus $\widetilde{X}$, according to Theorem 3.5.7, is a Poisson process in $\mathbb{R}^d \times \mathcal{C}'$, with intensity measure $\vartheta \otimes \mathbb{Q}$, then the generated particle process X is the image $\sigma(\widetilde{X})$ of $\widetilde{X}$ under the mapping

$$\sigma : \mathbb{R}^d \times \mathcal{C}' \to \quad \mathcal{C}'$$
$$(x, C) \mapsto x + C$$

and hence is also a Poisson process (with intensity measure $\Theta = \sigma(\vartheta \otimes \mathbb{Q})$). Since ϑ has no atoms, the same holds true for Θ.

We continue these considerations in the next section, where we treat random closed sets which arise from independent germ-grain processes by taking union sets of the generated particle processes.

Now we apply the results of Section 3.5 to particle processes.

Theorem 4.2.3. *Let X be a stationary particle process in $\mathbb{R}^d$ with intensity $\gamma > 0$, and let z be a center function. Then there is a (uniquely determined) probability measure $\mathbb{P}^0$ on $\mathcal{C}' \times \mathsf{N}_s(\mathcal{C}')$ such that*

$$\gamma \mathbb{P}^0(A) = \mathbb{E} \sum_{C \in X} \mathbf{1}_B(z(C)) \mathbf{1}_A(C - z(C), X - z(C))$$

for all $A \in \mathcal{B}(\mathcal{C}') \otimes \mathcal{N}_s(\mathcal{C}')$ and all $B \in \mathcal{B}(\mathbb{R}^d)$ with $\lambda(B) = 1$.

If $f : \mathbb{R}^d \times \mathcal{C}' \times \mathsf{N}_s(\mathcal{C}') \to \mathbb{R}$ is a nonnegative measurable function, then $\sum_{C \in X} f(z(C), C - z(C), X)$ is measurable and

$$\mathbb{E} \sum_{C \in X} f(z(C), C - z(C), X)$$
$$= \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}' \times \mathsf{N}_s(\mathcal{C}')} f(x, C, \eta + x) \, \mathbb{P}^0(\mathrm{d}(C, \eta)) \, \lambda(\mathrm{d}x).$$

Proof. We apply Theorem 3.5.2 to the marked point process X_z with mark space $\mathcal{C}'$. More precisely, if $A \in \mathcal{B}(\mathcal{C}') \otimes \mathcal{N}_s(\mathcal{C}')$ is given, we apply it to the set $\widetilde{A} := (\mathrm{id} \times \psi)(A)$, where $\psi : \mathsf{N}_s(\mathcal{C}') \to \mathsf{N}_s(\mathbb{R}^d \times \mathcal{C}')$ is defined by

$$\psi(\eta) := \sum \delta_{(z(C_i), C_i - z(C_i))}, \qquad \text{if } \eta = \sum \delta_{C_i}.$$

If the measure that results from Theorem 3.5.2 is denoted by $\widetilde{\mathbb{P}}^0$, then $\mathbb{P}^0(A) = \widetilde{\mathbb{P}}^0(\widetilde{A})$ yields the required measure. The second part of Theorem 4.2.3 follows from Theorem 3.5.3. $\qquad\square$

Theorem 4.2.4. *Let X be a stationary particle process in $\mathbb{R}^d$ with intensity $\gamma > 0$, let z be a center function. Let $\mathcal{C}_{z,0} := \{C \in \mathcal{C}' : z(C) = 0\}$ denote the mark space of the marked point process X_z, and let $\mathbb{Q}$ be the mark distribution of X_z. Then there exists a ($\mathbb{Q}$-a.s. uniquely determined) regular family $(\mathbb{P}^{0,C})_{C \in \mathcal{C}_{z,0}}$ of probability measures on $\mathcal{N}_s(\mathcal{C}')$ with*

$$\mathbb{P}^0(B \times A) = \int_B \mathbb{P}^{0,C}(A) \, \mathbb{Q}(\mathrm{d}C)$$

for $B \in \mathcal{B}(\mathcal{C}_{z,0})$ and $A \in \mathcal{N}_s(\mathcal{C}')$.

If $f : \mathbb{R}^d \times \mathcal{C}_{z,0} \times \mathsf{N}_s(\mathcal{C}') \to \mathbb{R}$ is a nonnegative measurable function, then $\sum_{C \in X} f(z(C), C - z(C), X)$ is measurable, and

$$\mathbb{E} \sum_{C \in X} f(z(C), C - z(C), X)$$

$$= \gamma \int_{\mathbb{R}^d} \int_{\mathcal{C}_{z,0}} \int_{\mathsf{N}_s(\mathcal{C}')} f(x, C, \eta + x) \, \mathbb{P}^{0,C}(\mathrm{d}\eta) \, \mathbb{Q}(\mathrm{d}C) \, \lambda(\mathrm{d}x).$$

Proof. Let $\widetilde{\mathbb{P}}^0$ be the measure obtained in the proof of Theorem 4.2.3. By Theorem 3.5.4, there exists a ($\mathbb{Q}$-a.s. uniquely determined) regular family $(\widetilde{\mathbb{P}}^{0,C})_{C \in \mathcal{C}_{z,0}}$ of probability measures on $\mathsf{N}_s(\mathbb{R}^d \times \mathcal{C}_{z,0})$ with

$$\widetilde{\mathbb{P}}^0(B \times A) = \int_B \widetilde{\mathbb{P}}^{0,C}(A) \, \mathbb{Q}(\mathrm{d}C)$$

for all $B \in \mathcal{B}(\mathcal{C}_{z,0})$ and $A \in \mathcal{N}_s(\mathbb{R}^d \times \mathcal{C}_{z,0})$. Defining $\mathbb{P}^{0,C}$ as the image measure of $\widetilde{\mathbb{P}}^{0,C}$ under the mapping

$$\begin{aligned} \mathsf{N}(\mathbb{R}^d \times \mathcal{C}_{z,0}) &\to \quad \mathsf{N}_s(\mathcal{C}'), \\ \widetilde{\eta} &\mapsto \sum_{(x,C) \in \widetilde{\eta}} \delta_{x+C} \end{aligned}$$

we obtain the assertion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we consider sections with a fixed k-dimensional plane $S \in G(d, k)$. Let $\widetilde{X}$ be a stationary (but otherwise arbitrary) germ-grain process in $\mathbb{R}^d$. With it, we can associate in a natural way the section process

$$\widetilde{X} \cap S := \sum_{(x,C) \in \widetilde{X}, \, (x+C) \cap S \neq \emptyset} \delta_{(x_S, (x^S + C) \cap S)},$$

where $x = x_S + x^S$ with $x_S \in S$ and $x^S \in S^\perp$ is the orthogonal decomposition. Thus, the germs of $\widetilde{X} \cap S$ arise by orthogonally projecting to S those germs of $\widetilde{X}$ for which the corresponding grain has nonempty intersection with S. Observe that

$$(x + C) \cap S = x_S + [(x^S + C) \cap S]. \tag{4.16}$$

If we assume in addition that $\widetilde{X} \cap S$ is simple and that the condition corresponding to (3.12) is satisfied, then $\widetilde{X} \cap S$ is a germ-grain process in the space S (which we can identify with $\mathbb{R}^k$) with mark space $\mathcal{C}'(S)$; the marked process $\widetilde{X} \cap S$ is stationary in S. For the particle process generated by $\widetilde{X}$,

$$X := \sum_{(x,C) \in \widetilde{X}} \delta_{x+C},$$

the section process $X \cap S$ was already defined in Section 3.6. Because of (4.16), $X \cap S$ coincides with the particle process generated by $\widetilde{X} \cap S$.

Suppose now that γ is the intensity and $\mathbb{Q}$ is the mark distribution of $\widetilde{X}$, and let $\gamma_{\widetilde{X} \cap S}$, $\mathbb{Q}_{\widetilde{X} \cap S}$ be the corresponding parameters for the section process

$\widetilde{X} \cap S$. Then, for $B \in \mathcal{B}(S)$ and $A \in \mathcal{B}(\mathcal{C}'(S))$, by Theorem 3.5.1 and the Campbell theorem we have

$$\gamma_{\widetilde{X} \cap S} \lambda_S(B) \mathbb{Q}_{\widetilde{X} \cap S}(A)$$

$$= \mathbb{E} \sum_{(x,C) \in \widetilde{X}} \mathbf{1}_B(X_S) \mathbf{1}_A((x^S + C) \cap S)$$

$$= \gamma \int_{\mathcal{C}'} \int_{\mathbb{R}^d} \mathbf{1}_B(x_S) \mathbf{1}_A((x^S + C) \cap S) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \gamma \int_{\mathcal{C}'} \int_{S^\perp} \int_S \mathbf{1}_B(y) \mathbf{1}_A((z + C) \cap S) \, \lambda_S(\mathrm{d}y) \, \lambda_{S^\perp}(\mathrm{d}z) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \gamma \lambda_S(B) \int_{\mathcal{C}'} \int_{S^\perp} \mathbf{1}_A((z + C) \cap S) \, \lambda_{S^\perp}(\mathrm{d}z) \, \mathbb{Q}(\mathrm{d}C).$$

Hence, setting

$$M_S(A) := \int_{\mathcal{C}'} \int_{S^\perp} \mathbf{1}_A((z + C) \cap S) \, \lambda_{S^\perp}(\mathrm{d}z) \, \mathbb{Q}(\mathrm{d}C),$$

we obtain

$$\gamma_{\widetilde{X} \cap S} = \gamma M_S(\mathcal{C}'(S)) = \gamma \int_{\mathcal{C}'} \lambda_{S^\perp}(C|S^\perp) \, \mathbb{Q}(\mathrm{d}C), \qquad (4.17)$$

where $C|S^\perp$ is the image of C under the orthogonal projection to $S^\perp$, and

$$\mathbb{Q}_{\widetilde{X} \cap S}(A) = M_S(A)/M_S(\mathcal{C}'(S)), \qquad (4.18)$$

if $M_S(\mathcal{C}'(S)) \neq 0$. Thus, the mark distribution of the section process depends only on the mark distribution of the original process. More explicit results for $\gamma_{\widetilde{X} \cap S}$ can be obtained for stationary and isotropic processes of convex particles (a general result of this type is Theorem 9.4.8).

Notes for Section 4.2

1. Generalized center functions. As a generalization of the notion of center function z, a **generalized center function** z maps each particle C in a particle collection η to a point which may depend not only on C but also on the other particles in η. Formally, z is a measurable mapping from $\mathcal{C}' \circ \mathsf{N}_s(\mathcal{C}') := \{(C, \eta) \in \mathcal{C}' \times \mathsf{N}_s(\mathcal{C}') : C \in \eta\}$ to $\mathbb{R}^d$, which is compatible with translations,

$$\mathsf{z}(C + x, \eta + x) = \mathsf{z}(C, \eta) + x \qquad \text{for all } x \in \mathbb{R}^d.$$

Every center function z defines a generalized center function $\langle z \rangle$ by $\langle z \rangle(C, \eta) := z(C)$. In generalization of Theorem 4.2.1, the following holds (see Schneider and Weil [717, Satz 4.3.1]).

Let X be a stationary particle process in $\mathbb{R}^d$, and let z be a generalized center function. Then

$$X^{\mathsf{z}} := \sum_{C \in X} \delta_{\mathsf{z}(C,X)}$$

is a stationary point process in $\mathbb{R}^d$, and

$$X_{\mathsf{z}} := \sum_{C \in X} \delta_{(\mathsf{z}(C,X), C - \mathsf{z}(C,X))}$$

is a stationary marked point process with mark space $\mathcal{C}'$. The intensities of X, X^{z} and X_{z} are the same.

Generalized center functions occur, for example, in connection with Voronoi mosaics. If A is a locally finite set in $\mathbb{R}^d$ such that the Voronoi cells $C(x, A)$, $x \in A$, are all bounded (see Section 10.2), then, for the corresponding Voronoi mosaic $\mathsf{m} := \{C(x, A) : x \in A\}$, the mapping $\mathsf{z} : (C, \eta) \mapsto x$, if $\eta = \mathsf{m}$ and $C = C(x, A)$, (and $\mathsf{z}(C, \eta) := c(C)$ otherwise) is a generalized center function.

2. The section formulas (4.17) and (4.18) can be found in Stoyan [740].

3. The interpretation of germ-grain processes as processes of points around which grains have grown randomly already indicates a temporal aspect which can, and has been, pursued further. Motivated by applications to the growth of crystals, tumor cells and other growing structures, various spatio-temporal models have been developed. Starting with the realization of a spatial point process X, one can, for example, let balls grow around the points of X with constant or random speed, in a dependent way or independently, at the same time or at different, random times. The growth can be stopped or modified, according to different rules, when the growing balls touch, or the growing balls can overlap, penetrate or get deformed to form a tessellation of space. Finally, also the underlying point process X may vary in time, for example as a birth-and-death process. Examples of this kind are the Stienen model (compare Note 9 to Section 10.2), the lilypond model (lilypond growth protocol, see Daley and Last [193], Heveling and Last [340]), the dead leaves model (see Serra [729, pp. 508–511], Cowan and Tsang [184]), the Johnson–Mehl tessellation model (see Møller [552]) and the general class of crystallization processes investigated by Capasso and co-workers [157], [158], [159], [515].

Some of the mentioned spatio-temporal models produce random systems of non-overlapping balls, others can be modified to do so, for example by thinning. There are current efforts by statisticians and physicists to generate random packings of balls with high volume density (Torquato [759], Stoyan and Schlather [745], Döge et al. [205]).

4.3 Germ-grain Models, Boolean Models

In Theorem 3.6.2 we have seen that for a point process X in $\mathcal{F}'$ the union set

$$Z_X := \bigcup_{F \in X} F$$

is a random closed set, and in Theorem 3.6.3 we have characterized those Z_X for which X has Poisson distributed counting variables. Now we study the random closed sets arising as the union sets of particle processes. We shall

be particularly interested in the random closed sets resulting from special germ-grain processes.

It is easy to see that a given random closed set Z can always be represented as the union set of a particle process X. In the following, we describe a construction which has the advantage that invariance properties of Z carry over to X. If Z is a random $\mathcal{S}$-set, then it is even possible to choose the particles of X as convex bodies. However, in order to ensure in this case the local finiteness of the intensity measure of X, we need an integrability assumption on the random $\mathcal{S}$-set Z. To formulate it, we define

$$N(K) := \min \left\{ m \in \mathbb{N} : K = \bigcup_{i=1}^{m} K_i \text{ with } K_i \in \mathcal{K} \right\} \qquad \text{for } K \in \mathcal{R}',$$

and $N(\emptyset) := 0$.

Lemma 4.3.1. *The function $N : \mathcal{R} \to \mathbb{N}_0$ is measurable.*

Proof. By Theorem 2.4.1, it is sufficient to show that N is semicontinuous with respect to the Hausdorff metric. Let $M_j, M \in \mathcal{R}$ be sets with $M_j \to M$ (in the Hausdorff metric) as $j \to \infty$. We assert that

$$N(M) \leq \liminf N(M_j). \tag{4.19}$$

Suppose this were false. Going over to a subsequence, we can assume that there exists a number $m \in \mathbb{N}$ with

$$N(M_j) = m < N(M) \qquad \text{for } j \in \mathbb{N},$$

thus

$$M_j = \bigcup_{i=1}^{m} K_j^{(i)} \qquad \text{with } K_j^{(i)} \in \mathcal{K}'.$$

Since the sets M_j and hence also the sets $K_j^{(i)}$ are uniformly bounded, there exists a subsequence $(j_k)_{k \in \mathbb{N}}$ such that

$$K_{j_k}^{(i)} \to K^{(i)} \qquad \text{as } k \to \infty, \qquad i = 1, \ldots, m,$$

with $K^{(i)} \in \mathcal{K}'$. Theorem 12.3.5 gives

$$M_{j_k} = \bigcup_{i=1}^{m} K_{j_k}^{(i)} \to \bigcup_{i=1}^{m} K^{(i)},$$

thus $M = \bigcup_{i=1}^{m} K^{(i)}$, and hence $N(M) \leq m$, a contradiction. This completes the proof of (4.19) and thus of the lemma. $\qquad\square$

Now we can prove the announced representation result.

Theorem 4.3.1. *To every random closed set Z in $\mathbb{R}^d$ there exists a simple particle process X with $Z = Z_X$ and such that $X \overset{\mathcal{D}}{=} gX$ for all rigid motions $g \in G_d$ for which $Z \overset{\mathcal{D}}{=} gZ$. In particular, X is stationary (isotropic) if Z is stationary (isotropic).*

If Z is a random $\mathcal{S}$-set with $\mathbb{E}N(Z \cap C) < \infty$ for all $C \in \mathcal{C}'$, then X can in addition be chosen so that all particles are convex.

Proof. The decomposition of a random closed set Z into compact particles is easier to achieve than that of a random $\mathcal{S}$-set into convex bodies. Therefore, we restrict ourselves in the proof to this more difficult situation. The decomposition into compact particles can be done in a similar way if the mapping ψ used below is replaced by $\widetilde{\psi} : C \mapsto \delta_C$, $C \in \mathcal{C}'$ (and $\widetilde{\psi}(\emptyset) = 0$).

From the proof of Theorem 14.4.4 we get the existence of a measurable map $\psi : \mathcal{R} \to \mathsf{N}_s(\mathcal{K}')$ with

$$\psi(C) = \sum_{i=1}^{N(C)} \delta_{K_i}, \qquad C = \bigcup_{i=1}^{N(C)} K_i$$

for $C \neq \emptyset$, and $\psi(\emptyset) = 0$.

As before, we denote by C_0^d the half-open unit cube. With an enumeration $(z_k)_{k\in\mathbb{N}}$ of $\mathbb{Z}^d$ and with $C_{0k}^d := C_0^d + z_k$, we put

$$\Psi(Z) := \sum_{k=1}^{\infty} \left[\psi \left(\mathrm{cl}\left(Z \cap C_{0k}^d \right) - z_k \right) + z_k \right]. \tag{4.20}$$

Then $\Psi(Z)$ is a simple point process in $\mathcal{K}'$ with a locally finite intensity measure. In fact, we have

$$\mathbb{E}\,\Psi(Z)(\mathcal{F}_C) \leq p(C)\mathbb{E}N(Z \cap w(C)) < \infty$$

for $C \in \mathcal{C}'$, where $p(C)$ denotes the number of cubes $C^d + z_k$, $k \in \mathbb{N}$, with $C \cap (C^d + z_k) \neq \emptyset$, and $w(C)$ is the union of these cubes.

Obviously, we have $Z = \bigcup_{K\in\Psi(Z)} K$ and $t_{-z}\Psi(t_z Z) = \Psi(Z)$ for all $z \in \mathbb{Z}^d$ (to achieve this invariance, and the simplicity, Ψ has been defined by (4.20)). To obtain the stronger invariance properties as required, the construction has yet to be modified. For a motion $g \in G_d$, we define $\Psi_g(Z) := g\Psi(g^{-1}Z)$. We put

$$G_d^0 := \left\{ g = \vartheta t_x \in G_d : \vartheta \in SO_d,\, x \in C_0^d \right\}$$

and denote by μ^0 the Haar measure on the motion group G_d, restricted to the relatively compact set G_d^0 and normalized to a probability measure. Let ξ be a random motion, independent of Z and with distribution μ^0. We define

$$X := \Psi_\xi(Z).$$

As shown above, X is a point process in $\mathcal{K}'$ with $Z = Z_X$.

Now suppose that $g_0 \in G_d$ is a motion with $Z \overset{\mathcal{D}}{=} g_0 Z$. Then

$$g_0 X = g_0 \xi \Psi \left(\xi^{-1} g_0^{-1} g_0 Z \right) = \Psi_{g_0 \xi}(g_0 Z),$$

hence, for all $A \in \mathcal{N}_s(\mathcal{K}')$,

$$\mathbb{P}(g_0 X \in A) = \int_{G_d^0} \mathbb{P}(\Psi_{g_0 g}(Z) \in A) \, \mu^0(\mathrm{d}g),$$

by the independence of Z and ξ and the g_0-invariance of Z. For $g_0 = \vartheta_0 t_{x_0}$, $g = \vartheta t_x$ we have

$$g_0 g = \vartheta_0 \vartheta t_{x + \vartheta^{-1} x_0},$$

hence (using the decomposition (13.8) of the invariant measure on G_d)

$$\mathbb{P}(g_0 X \in A) = \int_{SO_d} \int_{C_0^d} \mathbb{P}\left(\Psi_{\vartheta_0 \vartheta t_{x + \vartheta^{-1} x_0}}(Z) \in A \right) \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta).$$

If $\vartheta \in SO_d$ is fixed, to each $x \in C_0^d$ there exists a unique representation

$$x + \vartheta^{-1} x_0 = y(x) + z(x)$$

with $y(x) \in C_0^d$ and $z(x) \in \mathbb{Z}^d$. If x varies in C_0^d, the norm of $z(x)$ remains bounded, hence $z(x)$ attains only finitely many values $z_1, \ldots, z_r \in \mathbb{Z}^d$. For $D_i := \{x \in C_0^d : z(x) = z_i\}$, $i = 1, \ldots, r$, we then have

$$C_0^d = \bigcup_{i=1}^r D_i,$$

and the sets D_i are pairwise disjoint. Consider the mapping $\varphi : x \mapsto y(x)$ on C_0^d. On each D_i, it is a translation. For $x, x' \in C_0^d$ with $y(x) = y(x')$ we have $x - x' = z(x) - z(x') \in \mathbb{Z}^d$, hence $x = x'$. Thus φ is injective. This map is also surjective, since to each $y \in C_0^d$ there exists a decomposition $y - \vartheta^{-1} x_0 = x - z$, $x \in C_0^d$, $z \in \mathbb{Z}^d$. This gives $x + \vartheta^{-1} x_0 = y + z$, hence $y = y(x)$. Thus φ is a bijection onto C_0^d, which leaves λ invariant. Therefore, we obtain

$$\int_{C_0^d} \mathbb{P}\left(\Psi_{\vartheta_0 \vartheta t_{x + \vartheta^{-1} x_0}}(Z) \in A \right) \lambda(\mathrm{d}x)$$

$$= \int_{C_0^d} \mathbb{P}\left(\Psi_{\vartheta_0 \vartheta t_{y(x) + z(x)}}(Z) \in A \right) \lambda(\mathrm{d}x)$$

$$= \int_{C_0^d} \mathbb{P}\left(\vartheta_0 \vartheta \Psi_{t_{y(x) + z(x)}}(\vartheta^{-1} \vartheta_0^{-1} Z) \in A \right) \lambda(\mathrm{d}x)$$

$$= \int_{C_0^d} \mathbb{P}\left(\vartheta_0 \vartheta \Psi_{t_{y(x)}}(\vartheta^{-1} \vartheta_0^{-1} Z) \in A \right) \lambda(\mathrm{d}x)$$

$$= \int_{C_0^d} \mathbb{P}(\vartheta_0 \vartheta \Psi_{t_x}(\vartheta^{-1} \vartheta_0^{-1} Z) \in A) \lambda(\mathrm{d}x)$$

$$= \int_{C_0^d} \mathbb{P}(\Psi_{\vartheta_0 \vartheta t_x}(Z) \in A) \lambda(\mathrm{d}x).$$

The rotation invariance of the measure ν now yields

$$\mathbb{P}(g_0 X \in A) = \int_{SO_d} \int_{C_0^d} \mathbb{P}(\Psi_{\vartheta t_x}(Z) \in A)\, \lambda(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta)$$

$$= \int_{G_d} \mathbb{P}(\Psi_g(Z) \in A)\, \mu^0(\mathrm{d}g)$$

$$= \mathbb{P}(X \in A)$$

and thus $g_0 X \overset{\mathcal{D}}{=} X$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Due to this theorem, in particular every stationary random closed set Z is the union set of a stationary germ-grain process $\widetilde{X}$. Moreover, in the case of a random $\mathcal{S}$-set satisfying the finiteness condition of Theorem 4.3.1 the grains are convex. Here, the union set $Z_{\widetilde{X}}$ of a germ-grain process $\widetilde{X}$ is defined as the union set

$$Z_{\widetilde{X}} := \bigcup_{(x,C)\in\widetilde{X}} (x + C)$$

of the particle process induced by $\widetilde{X}$ according to (4.13).

To obtain more accessible models, we now consider random sets $Z = Z_{\widetilde{X}}$ arising from an independent germ-grain process. Such a random closed set is called a **germ-grain model**. If the mark distribution of $\widetilde{X}$ is concentrated on $\mathcal{K}'$, we call $Z_{\widetilde{X}}$ a **germ-grain model with convex grains**. For a germ-grain model Z, the capacity functional T_Z can be expressed in terms of the process X^0 of germs and the capacity functional of the typical grain Z_0. In fact, for $C \in \mathcal{C}$ we have

$$T_Z(C) = 1 - \mathbb{E}\prod_{x\in X^0}(1 - T_{Z_0}(C - x)). \tag{4.21}$$

To prove this, we choose a suitable representation

$$\widetilde{X} = \sum_{i=1}^{\tau}\delta_{(\xi_i, Z_i)}, \qquad \tau = \widetilde{X}(\mathbb{R}^d \times \mathcal{C}'),$$

and then argue as follows.

$$1 - T_Z(C) = \mathbb{P}\left(\bigcup_{i=1}^{\tau}(\xi_i + Z_i) \cap C = \emptyset\right)$$

$$= \mathbb{P}(\xi_i \notin C - Z_i,\ i = 1,\ldots,\tau)$$

$$= \mathbb{P}\left(\prod_{i=1}^{\tau}(1 - \mathbf{1}_{C-Z_i}(\xi_i)) = 1\right)$$

$$= \mathbb{E}\prod_{i=1}^{\tau}(1 - \mathbf{1}_{C-Z_i}(\xi_i))$$

$$= \mathbb{E} \prod_{i=1}^{\tau} \left(1 - \int_{\mathcal{C}'} \mathbf{1}_{C-K}(\xi_i) \, \mathbb{Q}(\mathrm{d}K) \right)$$

$$= \mathbb{E} \prod_{x \in X^0} (1 - T_{Z_0}(C - x)).$$

Particularly accessible are those germ-grain models Z for which the process of germs is a Poisson process. They are called **Boolean models (Boolean models with convex grains** if the generating germ-grain process $\widetilde{X}$ has convex grains). These random closed sets are the most tractable ones for applications. The Boolean model

$$Z := \bigcup_{(x,C) \in \widetilde{X}} (x + C) \tag{4.22}$$

is (up to stochastic equivalence) determined by the intensity measure ϑ of the Poisson process of germs and by the distribution $\mathbb{Q}$ of the typical grain. For that reason, we also write $Z =: Z(\vartheta, \mathbb{Q})$.

Formulas for Boolean models are the subject of Section 9.1.

Now let Z be a stationary Boolean model, that is, a stationary random closed set that is generated, according to (4.22), by an independent germ-grain process $\widetilde{X}$ with a Poisson germ process X^0. Then the generated particle process

$$X := \sum_{(x,C) \in \widetilde{X}} \delta_{x+C},$$

is a Poisson process, too, as remarked before Theorem 4.2.3. Hence, Z is the union set of a (by Theorem 3.6.4) stationary Poisson particle process X. From Theorem 4.2.2 we obtain also a converse, and thus the following result.

Theorem 4.3.2. *The stationary Boolean models are precisely the union sets of stationary Poisson particle processes.*

If Z is a stationary Boolean model and X is a generating Poisson particle process, then the intensity measure of X is uniquely determined by the distribution of Z, by Theorem 3.6.3 and Lemma 2.3.1. It is translation invariant, and by Theorem 4.1.1 the intensity γ (assumed positive) and the grain distribution $\mathbb{Q}$ of X are uniquely determined. By Theorem 4.1.2, also the corresponding particle process is uniquely determined. However, this does not hold for the corresponding marked processes. If, besides $Z = Z(\gamma\lambda, \mathbb{Q})$ one also has $Z = Z(\gamma\lambda, \mathbb{Q}')$, then $\mathbb{Q}'$ is in general distinct from the grain distribution $\mathbb{Q}$ of X. Yet, it is true that $\mathbb{Q}$ is the image of $\mathbb{Q}'$ under the mapping $\pi_c : C \mapsto C - c(C)$. This follows from Theorem 4.2.1. Thus, the generation of a stationary Boolean model by an independent germ-grain process with a Poisson germ process can be achieved in different ways. It is, however, always possible to choose the 'canonical' generating process X_c.

If a stationary Boolean model is intersected with a plane S, then in S one obtains again a Boolean model, stationary with respect to S. In fact, for a Poisson particle process X, the section process $X \cap S$, too, is a Poisson process, as follows immediately from (3.2).

For a point process of lower-dimensional sets, it may be possible to obtain certain quantities of X from studying the union set Z. In general this will be difficult, due to overlappings. In particular, for Poisson processes in $\mathcal{C}$, $\mathcal{R}$ or $\mathcal{K}$ and with full-dimensional particles, such overlappings occur with positive probability, due to Theorem 4.1.4. On the other hand, for a stationary Boolean model Z, which is the union set of a stationary Poisson particle process X, this process X is already uniquely determined, as just remarked. Therefore, all characteristic parameters of X, for example the intensity, must be obtainable from quantities of Z. We shall study this phenomenon in greater detail in Section 9.1.

Notes for Section 4.3

1. Theorem 4.3.1 is due to Weil and Wieacker [805].

2. General germ-grain models were introduced by Hanisch [319] and further studied by Heinrich [324] and others. In Hanisch [319] one finds, for example, formula (4.21).

3. The Boolean model was, after a few precursors (see Cressie [185, p. 753]), at first mainly studied by the Fontainebleau school; this is reflected in the books of Matheron [462] and Serra [729]. The book by Hall [317] contains a detailed discussion of qualitative and quantitative properties of the Boolean model and more general germ-grain models, in particular with a view to covering and connectivity properties. Meesters and Roy [509] study Boolean models in the framework of percolation theory. Statistical methods for Boolean models are treated by Molchanov [546].

4. Quermass-interaction models. Starting with a Poisson process Y of convex particles with a finite intensity measure Θ and the corresponding Boolean model Z, Kendall, van Lieshout and Baddeley [398] defined quermass-interaction processes Y' and their union sets Z'. The particle process Y' is supposed to be absolutely continuous to Y with density

$$p(\mathbf{y}) = \alpha \beta^{n(\mathbf{y})} \exp\left[- \sum_{j=0}^{d} \gamma_j V_j(U(\mathbf{y})) \right].$$

Here, $\mathbf{y} = \{K_1, \ldots, K_n\}$ is a (finite) realization of Y with $n(\mathbf{y}) = n$, $\beta > 0$ and $\gamma_j \in \mathbb{R}$ are model parameters, α is a normalizing constant and $U(\mathbf{y}) = \bigcup_{i=1}^{n} K_i$.

The particles of Y' are no longer independent, in general, but satisfy a Markov property. The main question discussed in [398] is whether Y' is stable in the sense of Ruelle, a property which implies integrability of the density p. The results mostly concern the planar case with particles being disks or convex polygons.

4.4 Processes of Flats

In this section, we study processes of flats. A **process** of k-**flats**, or k-**flat process**, in $\mathbb{R}^d$ is a point process in the space $A(d,k)$ of k-flats (k-dimensional planes) in $\mathbb{R}^d$, where $k \in \{1, \ldots, d-1\}$, and thus a point process in $\mathcal{F}'$ with intensity measure concentrated on $A(d,k)$. For $k=1$, we also speak of a **line process**, and for $k = d-1$, of a **hyperplane process**.

For stationary k-flat processes, there is again a decomposition of the intensity measure. The proof is not quite as simple as for the analogous results in Theorems 3.3.1 and 3.5.1, since $A(d,k)$ can, for $k < d-1$, only locally be represented as a product space with a Euclidean factor.

The Grassmannian $G(d,k)$ of k-dimensional linear subspaces of $\mathbb{R}^d$ is a subset of $A(d,k)$ and is closed in $\mathcal{F}$ and $\mathcal{F}'$. For $L \in G(d,k)$, recall that λ_L is the k-dimensional Lebesgue measure on L. The (continuous) mapping

$$\pi_0 : \bigcup_{k=1}^{d-1} A(d,k) \to \bigcup_{k=1}^{d-1} G(d,k)$$

associates with every plane its translate through 0.

Theorem 4.4.1. *Let Θ be a locally finite, translation invariant measure on $A(d,k)$. Then there exists a uniquely determined finite measure Θ_0 on $G(d,k)$ such that*

$$\Theta(A) = \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_A(L+x)\, \lambda_{L^\perp}(\mathrm{d}x)\, \Theta_0(\mathrm{d}L) \tag{4.23}$$

for every Borel set $A \in \mathcal{B}(A(d,k))$.

Proof. Let $U \in G(d, d-k)$, and define

$$G_U := \{L \in G(d,k) : \dim(L \cap U) = 0\}$$

and $A_U := \{L + x : L \in G_U,\, x \in U\}$. The mapping

$$\varphi : G_U \times U \to A_U$$
$$(L, x) \mapsto L + x$$

is a homeomorphism. Let $A \subset G_U$ be a Borel set. For Borel sets $B \subset U$, let

$$\eta(B) := \Theta(\varphi(A \times B)).$$

Then η is a locally finite, translation invariant measure on U and thus a multiple of the Lebesgue measure λ_U. Denoting the factor by $\rho(A)$, we have

$$\Theta(\varphi(A \times B)) = \rho(A)\lambda_U(B).$$

Evidently, ρ is a finite measure on G_U. Thus,

$$\varphi^{-1}(\Theta)(A \times B) = (\rho \otimes \lambda_U)(A \times B),$$

which gives $\varphi^{-1}(\Theta) = \rho \otimes \lambda_U$ and, therefore, $\Theta \llcorner A_U = \varphi(\rho \otimes \lambda_U)$. Hence, for every nonnegative measurable function f on $A(d,k)$ we have

$$\int_{A_U} f \, \mathrm{d}\Theta = \int_{G_U} \int_U f(L + x) \, \lambda_U(\mathrm{d}x) \, \rho(\mathrm{d}L).$$

For given $L \in G_U$, let $\Pi_L : U \to L^\perp$ denote the orthogonal projection to the orthogonal complement of L. It is bijective, since $L \in G_U$. Therefore, $\Pi_L(\lambda_U) = a(L)\lambda_{L^\perp}$, with a factor $a(L) > 0$ that depends only on L. Further, $f(L + x) = f(L + \Pi_L(x))$. This yields

$$\int_U f(L + x) \, \lambda_U(\mathrm{d}x) = a(L) \int_{L^\perp} f(L + x) \, \lambda_{L^\perp}(\mathrm{d}x).$$

Defining a measure Θ_U on G_U by $a(L)\rho(\mathrm{d}L) =: \Theta_U(\mathrm{d}L)$, we have

$$\int_{A_U} f \, \mathrm{d}\Theta = \int_{G_U} \int_{L^\perp} fL + x) \, \lambda_{L^\perp}(\mathrm{d}x) \, \Theta_U(\mathrm{d}L).$$

We interpret Θ_U as a measure on all of $G(d,k)$, with $\Theta_U(G(d,k) \setminus G_U) = 0$, and then have

$$\int_{A_U} f \, \mathrm{d}\Theta = \int_{G(d,k)} \int_{L^\perp} f(L + x) \, \lambda_{L^\perp}(\mathrm{d}x) \, \Theta_U(\mathrm{d}L).$$

Every set G_U, $U \in G(d, d-k)$, is open in $G(d,k)$, hence there are finitely many subspaces $U_1, \ldots, U_m \in G(d, d-k)$ with $G(d,k) = \bigcup_{i=1}^m G_{U_i}$. The sets A_{U_i}, $i = 1, \ldots, m$, cover $A(d,k)$ and are invariant under translations. The translation invariant Borel sets defined by $A_k := A_{U_k} \setminus (A_1 \cup \ldots \cup A_{k-1})$, $k = 1, \ldots, m$, form a disjoint covering of $A(d,k)$. Since $\Theta \llcorner A_i$ is translation invariant, the measure $\Theta_i := (\Theta \llcorner A_i)_{U_i}$, defined as above, satisfies

$$\int_{A_i} f \, \mathrm{d}\Theta = \int_{G(d,k)} \int_{L^\perp} f(L + x) \, \lambda_{L^\perp}(\mathrm{d}x) \, \Theta_i(\mathrm{d}L).$$

Therefore, the measure $\Theta_0 := \Theta_1 + \ldots + \Theta_m$ satisfies (4.23).

From (4.23) we obtain, for $A \in \mathcal{B}(G(d,k))$,

$$\Theta_0(A) = \frac{1}{\kappa_{d-k}} \Theta\left(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A)\right). \tag{4.24}$$

From (4.24) it is obvious that Θ_0 is finite and uniquely determined. $\qquad \square$

Applying the preceding theorem to intensity measures, we immediately obtain the following result.

Theorem 4.4.2. *Let X be a stationary k-flat process in $\mathbb{R}^d$ with intensity measure $\Theta \neq 0$. Then there are a number $\gamma \in (0, \infty)$ and a probability measure $\mathbb{Q}$ on $G(d, k)$ with*

$$\int_{A(d,k)} f \, \mathrm{d}\Theta = \gamma \int_{G(d,k)} \int_{L^\perp} f(L + x) \, \lambda_{L^\perp}(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}L) \qquad (4.25)$$

for all nonnegative measurable functions f on $A(d, k)$. Here γ and $\mathbb{Q}$ are uniquely determined by Θ.

We call γ the **intensity** and $\mathbb{Q}$ the **directional distribution** of the stationary flat process X. If X is moreover isotropic, then $\mathbb{Q}$ is rotation invariant, as follows from the uniqueness. By Theorem 13.2.11, there is only one normalized rotation invariant measure on $G(d, k)$, the Haar measure ν_k.

Occasionally (for example, when sections are considered) we have to allow flat processes with $\Theta = 0$; for these we define $\gamma = 0$.

The interpretation of γ and $\mathbb{Q}$ is clear from (4.25), since for $A \in \mathcal{B}(G(d, k))$ this gives

$$\gamma \mathbb{Q}(A) = \frac{1}{\kappa_{d-k}} \mathbb{E}\, X\left(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A)\right), \qquad (4.26)$$

in particular

$$\gamma = \frac{1}{\kappa_{d-k}} \, \mathbb{E}\, X\left(\mathcal{F}_{B^d}\right) \qquad (4.27)$$

and

$$\mathbb{Q}(A) = \frac{\mathbb{E}\, X\left(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A)\right)}{\mathbb{E}\, X(\mathcal{F}_{B^d})}. \qquad (4.28)$$

The representation (4.28) explains why the measure $\mathbb{Q}$ is called the directional distribution of X.

For a further interpretation of the intensity γ, we need a measurability result.

Lemma 4.4.1. *Let X be a point process in $A(d, k)$. Then*

$$\sum_{E \in X} \lambda_E(A) \qquad (4.29)$$

is measurable for all $A \in \mathcal{B}(\mathbb{R}^d)$.

Proof. It is sufficient to consider the case $A \subset mB^d$, $m \in \mathbb{N}$. First let A be compact, and assume that $E_i \to E$ in $G(d, k)$. Then there exist rotations $g_i \in SO_d$, $i \in \mathbb{N}$, converging to the identity for $i \to \infty$, and such that $g_i^{-1}E = E_i$. Using the representation

$$\lambda_{E_i}(A) = \int_E \mathbf{1}_{g_i A}(x) \, \lambda_E(\mathrm{d}x),$$

one shows as in the proof of Theorem 12.3.6 that the function $E \mapsto \lambda_E(A)$ is upper semicontinuous and thus measurable. By Theorem 3.1.2, also (4.29) is measurable. Now let $\mathcal{A}$ be the system of all Borel sets $A \subset mB^d$ for which (4.29) is measurable. We have shown that $\mathcal{A}$ contains all compact subsets of mB^d. Evidently, $\mathcal{A}$ is closed under disjoint countable unions and relative complements. Since $\mathcal{C}$ is $\cap$-stable, $\mathcal{A}$ contains the σ-algebra generated by the compact sets in mB^d, and, therefore, all Borel sets in mB^d. $\qquad\square$

The measurability being shown, we can define

$$\varphi_X(A) := \mathbb{E} \sum_{E \in X} \lambda_E(A), \qquad A \in \mathcal{B}(\mathbb{R}^d),$$

and thus obtain a locally finite measure φ_X. If X is stationary, φ_X is translation invariant and, therefore, of the form $\varphi_X = \alpha\lambda$ with a number $\alpha \in [0, \infty)$. The following theorem shows that this constant is precisely the intensity γ.

Theorem 4.4.3. *Let X be a stationary k-flat process in $\mathbb{R}^d$ with intensity γ. Then*

$$\mathbb{E} \sum_{E \in X} \lambda_E = \gamma\lambda.$$

Proof. Using the Campbell theorem and Theorem 4.4.2, we get

$$\mathbb{E} \sum_{E \in X} \lambda_E(A) = \int_{A(d,k)} \lambda_E(A)\,\Theta(\mathrm{d}E)$$

$$= \gamma \int_{G(d,k)} \int_{L^\perp} \lambda_{L+x}(A)\,\lambda_{L^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}L)$$

$$= \gamma \int_{G(d,k)} \lambda(A)\,\mathbb{Q}(\mathrm{d}L)$$

$$= \gamma\lambda(A),$$

as stated. $\qquad\square$

Further interpretations of the intensity will be obtained in Section 9.4. In particular, formula (9.33) provides $k + 1$ such interpretations.

Processes of k-flats satisfying Poisson assumptions again have particular properties. From Theorems 3.2.1 and 3.6.1, the following is immediately clear.

Theorem 4.4.4. *Let $\gamma \in (0, \infty)$ and let $\mathbb{Q}$ be a probability measure on $G(d,k)$. Then there is (up to equivalence) precisely one stationary Poisson k-flat process X in $\mathbb{R}^d$ with intensity γ and directional distribution $\mathbb{Q}$. The process X is isotropic if and only if $\mathbb{Q} = \nu_k$.*

In the next theorem, we collect some consequences of the independence properties of Poisson k-flat processes. We say that two linear subspaces L, L' of $\mathbb{R}^d$ are in **general position** if

$$\mathrm{lin}\,(L \cup L') = \mathbb{R}^d \quad \text{or} \quad \dim\,(L \cap L') = 0.$$

Two k-planes E, E' are said to be in general position if their direction spaces $\pi_0(E), \pi_0(E')$ are in general position.

Theorem 4.4.5. *Let X be a stationary Poisson k-flat process in $\mathbb{R}^d$.*

(a) *If $k < d/2$, then a.s. any two k-flats of X are disjoint.*

(b) *If the directional distribution $\mathbb{Q}$ of X has no atoms, then a.s. any two k-flats of the process X are not translates of each other.*

(c) *If the directional distribution of X is absolutely continuous with respect to the invariant measure ν_k, then a.s. any two k-planes of the process X are in general position.*

Proof. Let $A \in \mathcal{B}(A(d,k)^2)$. From Theorem 3.1.3, Corollary 3.2.4, Theorem 4.1.2 we get

$$\mathbb{E} \sum_{(E_1,E_2) \in X_{\neq}^2} \mathbf{1}_A(E_1, E_2) = \int_{A(d,k)^2} \mathbf{1}_A \, \mathrm{d}\Lambda^{(2)}$$

$$= \int_{A(d,k)} \int_{A(d,k)} \mathbf{1}_A(E_1, E_2)\, \Theta(\mathrm{d}E_1)\, \Theta(\mathrm{d}E_2)$$

$$= \gamma^2 \int_{G(d,k)} \int_{G(d,k)} \int_{L_2^\perp} \int_{L_1^\perp} \mathbf{1}_A(L_1 + x_1, L_2 + x_2)$$

$$\times \lambda_{L_1^\perp}(\mathrm{d}x_1)\, \lambda_{L_2^\perp}(\mathrm{d}x_2)\, \mathbb{Q}(\mathrm{d}L_1)\, \mathbb{Q}(\mathrm{d}L_2).$$

To prove (a), suppose that $k < d/2$ and choose

$$A := \{(E_1, E_2) \in A(d,k)^2 : E_1 \cap E_2 \neq \emptyset\}.$$

For fixed k-flats $L_1 \in G(d,k)$, $E_2 \in A(d,k)$, the integral

$$\int_{L_1^\perp} \mathbf{1}_A(L_1 + x_1, E_2)\, \lambda_{L_1^\perp}(\mathrm{d}x_1)$$

gives the $(d-k)$-dimensional Lebesgue measure of the image of E_2 under the orthogonal projection to $L_1^\perp$, which is zero. We deduce that

$$\mathbb{E} \sum_{(E_1,E_2) \in X_{\neq}^2} \mathbf{1}_A(E_1, E_2) = 0,$$

and from this the assertion (a) follows.

To prove (b), let $m \in \mathbb{N}$ and

$$A := \{(E_1, E_2) \in A(d,k)^2 : E_i \cap mB^d \neq \emptyset,\ i = 1, 2,\ E_1 \text{ is a translate of } E_2\}.$$

Then we get

$$\mathbb{E} \sum_{(E_1, E_2) \in X_{\neq}^2} \mathbf{1}_A(E_1, E_2)$$

$$\leq \left(\gamma m^{d-k} \kappa_{d-k}\right)^2 \int_{G(d,k)} \int_{G(d,k)} \mathbf{1}_A(L_1, L_2)\, \mathbb{Q}(\mathrm{d}L_1)\, \mathbb{Q}(\mathrm{d}L_2)$$

$$= \left(\gamma m^{d-k} \kappa_{d-k}\right)^2 \int_{G(d,k)} \mathbb{Q}(\{L_2\})\, \mathbb{Q}(\mathrm{d}L_2)$$

$$= 0,$$

since $\mathbb{Q}$ has no atoms. Assertion (b) follows, since $m \in \mathbb{N}$ was arbitrary.

To prove (c), suppose that $\mathbb{Q}$ has a density f with respect to ν_k. For $m \in \mathbb{N}$ we choose

$$A := \{(E_1, E_2) \in A(d,k)^2 : E_i \cap mB^d \neq \emptyset,\ i = 1, 2,$$

$$E_1, E_2 \text{ not in general position}\}.$$

As above, we obtain similarly

$$\mathbb{E} \sum_{(E_1, E_2) \in X_{\neq}^2} \mathbf{1}_A(E_1, E_2)$$

$$\leq \left(\gamma m^{d-k} \kappa_{d-k}\right)^2 \int_{G(d,k)} \int_{A(L_2)} f(L_1)\, \nu_k(\mathrm{d}L_1) f(L_2)\, \nu_k(\mathrm{d}L_2)$$

$$= 0,$$

since the set $A(L_2) := \{L_1 \in G(d,k) : (L_1, L_2) \in A\}$ satisfies $\nu_k(A(L_2)) = 0$, as can be deduced from Lemma 13.2.1. Since $m \in \mathbb{N}$ was arbitrary, assertion (c) follows. $\qquad \square$

When considering section processes in the following, we shall also meet j-flat processes for $j = 0$. These can be considered as ordinary point processes. Namely, we identify every one-pointed set $\{x\}$ with x, observing that the mapping $\{x\} \mapsto x$ maps the subspace $\{\{x\} : x \in \mathbb{R}^d\}$ of $\mathcal{F}'$ homeomorphically to $\mathbb{R}^d$.

Sections with a Fixed Plane

We turn to sections with fixed planes, a topic which, at least in small dimensions, is important for applications. Let X be a stationary k-flat process in $\mathbb{R}^d$ ($k \in \{1, \ldots, d-1\}$), and let S be a fixed $(d-k+j)$-flat with $0 \leq j \leq k-1$. We recall the definition of the section process,

$$(X \cap S)(\omega) := \sum_{E \in X,\, E \cap S \neq \emptyset} \delta_{E \cap S}.$$

As we shall see below, $X \cap S$ is a j-flat process. Its realizations lie in the section plane S and it is stationary with respect to S. Therefore, in the following $X \cap S$ is considered as a stationary j-flat process in S. The question arises how intensity and directional distribution of $X \cap S$ are related to the corresponding parameters of X. This will now be investigated.

Because of the stationarity of the process X it is no restriction to assume that $S \in G(d, d - k + j)$. The nonempty intersections $E \cap S$, $E \in X$, can be r-flats with $r \in \{j, \ldots, \min(k, d - k + j)\}$; we show, however, that almost surely they are j-flats. Let

$$A := \{E \in A(d, k) : \dim (E \cap S) > j\}$$

(with the usual convention that $\dim \emptyset := -1$). By (4.25),

$$\mathbb{E} X(A) = \Theta(A) = \gamma \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_A(L + x) \, \lambda_{L^\perp}(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}L).$$

If $\mathbf{1}_A(L + x) = 1$, then L and S span only a proper subspace U of $\mathbb{R}^d$, and we have $x \in U$ and $\dim (L^\perp \cap U) < \dim L^\perp$. This gives

$$\mathbb{E} X(A) \leq \gamma \int_{G(d,k)} \lambda_{L^\perp}(L^\perp \cap U) \, \mathbb{Q}(\mathrm{d}L) = 0.$$

Hence, almost surely we have $\dim (E \cap S) = j$ or $E \cap S = \emptyset$ for $E \in X$. Therefore, $X \cap S$ is a j-flat process in S (which may have intensity 0, though). Similarly we obtain that $X \cap S$ is a.s. simple. In fact, if a j-flat in S is generated as the intersection of two distinct k-flats E_1, E_2 with S, then $E_1 \cap E_2$ is an i-flat with $j \leq i \leq k - 1$. For given i, we consider all i-flats which are the intersection of two flats of X (counting every such i-flat only once, even if it is generated in two different ways). In this way, a process Y_i of i-flats is obtained (the measurability is not difficult to prove). The process Y_i is stationary. By the argument used above and because of $i \leq k - 1$, the flats of Y_i intersect the plane S a.s. in planes of dimension less than j. This shows that $X \cap S$ is a.s. simple.

First we consider now the case $\dim S = d - k$, where $X \cap S$ is an ordinary point process in S. In the next theorem, we determine the intensity of this point process. For the subspace determinant $[\cdot, \cdot]$ occurring in the following we refer to Section 14.1.

Theorem 4.4.6. *Let $k \in \{1, \ldots, d-1\}$, and let X be a stationary k-flat process in $\mathbb{R}^d$ with intensity γ and directional distribution $\mathbb{Q}$. Let $S \in G(d, d - k)$, and let $\gamma_{X \cap S}$ be the intensity of the point process $X \cap S$. Then*

$$\gamma_{X \cap S} = \gamma \int_{G(d,k)} [S, L] \, \mathbb{Q}(\mathrm{d}L).$$

Proof. Let B^{d-k} be the unit ball in S. By the definition of the intensity of the point process $X \cap S$,

$$
\begin{aligned}
\kappa_{d-k}\gamma_{X\cap S} &= \mathbb{E}(X \cap S)(\mathcal{F}_{B^{d-k}}) \\
&= \mathbb{E}X(\mathcal{F}_{B^{d-k}}) = \Theta(\mathcal{F}_{B^{d-k}}) \\
&= \gamma \int_{G(d,k)} \int_{L^{\perp}} \mathbf{1}_{\mathcal{F}_{B^{d-k}}}(L+x)\,\lambda_{L^{\perp}}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}L) \\
&= \gamma \int_{G(d,k)} \lambda_{L^{\perp}}(B^{d-k}|L^{\perp})\,\mathbb{Q}(\mathrm{d}L).
\end{aligned}
$$

Here $B^{d-k}|L^{\perp}$ is the image of B^{d-k} under the orthogonal projection to $L^{\perp}$. The $(d-k)$-volume of this image is given by $\lambda_S(B^{d-k})[S,L]$, from which the assertion follows.   □

In the cases $k = 1$ and $k = d-1$ it is convenient to replace the directional distribution $\mathbb{Q}$ by the **spherical directional distribution** φ. This is the measure on the unit sphere S^{d-1} which, for a set $A \in \mathcal{B}(S^{d-1})$ without antipodal points, is defined by

$$
\varphi(A) := \frac{1}{2}\mathbb{Q}(\{L(u) : u \in A\}) \qquad \text{if } k = 1,
$$

for $L(u) := \mathrm{lin}\{u\}$, respectively

$$
\varphi(A) := \frac{1}{2}\mathbb{Q}(\{u^{\perp} : u \in A\}) \qquad \text{if } k = d-1. \tag{4.30}
$$

(The factor $\frac{1}{2}$ appears here since $L(u) = L(-u)$ and $u^{\perp} = (-u)^{\perp}$.) By additivity, φ is then defined for all $A \in \mathcal{B}(S^{d-1})$. Thus, φ is an even probability measure on S^{d-1}. Writing $\gamma_X(u) := \gamma_{X\cap L(u)}$ if $k = 1$, respectively $\gamma_X(u) := \gamma_{X\cap u^{\perp}}$ if $k = d-1$, we then have

$$
\gamma_X(u) = \gamma \int_{S^{d-1}} |\langle u, v\rangle|\,\varphi(\mathrm{d}v). \tag{4.31}
$$

The right side of (4.31) defines the support function of a centrally symmetric convex body, which can be associated with the measure $\gamma\varphi$. This body belongs to the class of zonoids. Such associated zonoids will be studied and applied in Section 4.6.

A corresponding uniqueness theorem (Theorem 14.3.4) shows that the function γ_X in (4.31) uniquely determines the measure $\gamma\varphi$ (and therefore also γ and φ). In particular, for a stationary Poisson line or hyperplane process X, the distribution $\mathbb{P}_X$ is uniquely determined by the **section intensities** $\gamma_{X\cap S}$, $S \in G(d, d-1)$, respectively $S \in G(d, 1)$ (see also Section 4.6). For $1 < k < d-1$, however, a stationary Poisson k-flat process X is in general not uniquely determined by the section intensities $\gamma_{X\cap S}$, $S \in G(d, d-k)$; see Note 2 of this section.

Now we consider also the case of higher-dimensional section planes S, where we obtain in S an intersection process of j-flats with $j > 0$. Let X be a stationary k-flat process, and let $S \in G(d, d-k+j)$, with $j \in \{1, \ldots, k-1\}$, be a fixed plane. As shown above, $X \cap S$ is a.s. a j-flat process. Its intensity measure, $\Theta_{X \cap S}$, is concentrated on the space

$$G(S, j) := \{L \in G(d, j) : L \subset S\}.$$

Theorem 4.4.7. *Let $k \in \{2, \ldots, d-1\}$, and let X be a stationary k-flat process in $\mathbb{R}^d$ with intensity γ and directional distribution $\mathbb{Q}$. Let $j \in \{1, \ldots, k-1\}$ and $S \in G(d, d-k+j)$; let $\gamma_{X \cap S}$ be the intensity and $\mathbb{Q}_{X \cap S}$ the directional distribution of the j-flat process $X \cap S$. Then, for $A \in \mathcal{B}(G(d, j))$,*

$$\gamma_{X \cap S} \mathbb{Q}_{X \cap S}(A) = \gamma \int_{G(d,k)} \mathbf{1}_A(L \cap S)[L, S] \, \mathbb{Q}(\mathrm{d}L).$$

(If $\gamma_{X \cap S} = 0$, then $\mathbb{Q}_{X \cap S}$ is not defined, and the expression $\gamma_{X \cap S} \mathbb{Q}_{X \cap S}$ has to be read as the zero measure.)

Proof. Let $P \in \mathcal{B}(A(d, j))$. By Campbell's theorem and Theorem 4.4.2,

$$\Theta_{X \cap S}(P) = \mathbb{E}(X \cap S)(P) = \mathbb{E} \sum_{E \in X} \mathbf{1}_P(E \cap S)$$

$$= \int_{A(d,k)} \mathbf{1}_P(E \cap S) \, \Theta(\mathrm{d}E)$$

$$= \gamma \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_P((L + x) \cap S) \, \lambda_{L^\perp}(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}L).$$

The intensity measure $\Theta_{X \cap S}$ is concentrated on the j-flats in S and is invariant under the translations of S into itself. By (4.24) (applied in S) and the definition of intensity and directional distribution, for $A \in \mathcal{B}(G(d, j))$ and $B_S := B^d \cap S$ we get

$$\gamma_{X \cap S} \mathbb{Q}_{X \cap S}(A)$$

$$= \frac{1}{\kappa_{d-k}} \Theta_{X \cap S}(\mathcal{F}_{B_S} \cap \pi_0^{-1}(A))$$

$$= \frac{\gamma}{\kappa_{d-k}} \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_{\mathcal{F}_{B_S} \cap \pi_0^{-1}(A)}((L + x) \cap S) \, \lambda_{L^\perp}(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}L)$$

$$= \frac{\gamma}{\kappa_{d-k}} \int_{G(d,k)} \mathbf{1}_A(L \cap S) \lambda_{d-k}(B_S | L^\perp) \, \mathbb{Q}(\mathrm{d}L),$$

since $(L + x) \cap S \in \mathcal{F}_{B_S} \cap \pi_0^{-1}(A)$ obviously holds if and only if $L \cap S \in A$ and $x \in B_S | L^\perp$. If T denotes the orthogonal complement of $L \cap S$ in S, then

$$B_S | L^\perp = (B_S | T) | L^\perp = B_T | L^\perp.$$

The orthogonal projection from T to $L^\perp$ has the absolute determinant $[L, S]$, hence

$$\lambda_{d-k}(B_S|L^\perp) = \kappa_{d-k}[L, S].$$

This yields the assertion. □

Intersection Processes

By intersecting flats in a k-flat process among themselves, we obtain new lower-dimensional flat processes. We shall now study such intersection processes in the case of stationary Poisson flat processes. In particular, we are interested in how the intensity and the directional distribution of an intersection process depend on the data of the original process. We restrict ourselves to two cases: intersecting k-tuples of hyperplanes, or intersecting pairs of r-flats, where $r \geq d/2$. In some cases we shall be able to obtain sharp inequalities between the intensities of the intersection process and the original process; this will be explained in Section 4.6.

First we consider hyperplane processes. It is convenient to represent hyperplanes in the form

$$H(u, \tau) := \{x \in \mathbb{R}^d : \langle x, u \rangle = \tau\} \tag{4.32}$$

with a unit vector $u \in S^{d-1}$ and a number $\tau \in \mathbb{R}$. Every hyperplane $H \in A(d, d-1)$ has two such representations. Instead of $H(u, 0)$, we shall write $u^\perp$ again.

Let X be a stationary hyperplane process in $\mathbb{R}^d$ with intensity $\gamma \neq 0$ and directional distribution $\mathbb{Q}$. Using the spherical directional distribution φ introduced by (4.30), the decomposition of the intensity measure Θ given by Theorem 4.4.2 can be written in the form

$$\int_{A(d,d-1)} f \, d\Theta = \gamma \int_{S^{d-1}} \int_{-\infty}^{\infty} f(H(u, \tau)) \, d\tau \, \varphi(du). \tag{4.33}$$

Let $k \in \{2, \ldots, d\}$. For every realization of X, we consider the intersection of any k hyperplanes in the process which are in general position. We want to show that in this way we obtain a stationary $(d-k)$-flat process X_k; we shall call this the **intersection process of order k** of the process X. For $P \in \mathcal{B}(A(d, d-k))$, define the function $f_P : A(d, d-1)^k \to \mathbb{R}$ by

$$f_P(H_1, \ldots, H_k) := \begin{cases} 1, & \text{if } H_1 \cap \ldots \cap H_k \in P, \\ 0 & \text{else.} \end{cases} \tag{4.34}$$

The set of all $(H_1, \ldots, H_k) \in A(d, d-1)^k$ with $\dim(H_1 \cap \ldots \cap H_k) = d-k$ is open, and on this set the mapping $(H_1, \ldots, H_k) \mapsto H_1 \cap \ldots \cap H_k$ is continuous. Hence, f_P is measurable. By Theorem 3.1.3, the function

$$X_k(P) := \frac{1}{k!} \sum_{(H_1,\ldots,H_k) \in X_{\neq}^k} f_P(H_1,\ldots,H_k)$$

is measurable. If P is compact, there exists a ball that is hit by all $(d-k)$-flats in P and hence also by all hyperplanes $H_1,\ldots,H_k$ with $f_P(H_1,\ldots,H_k) = 1$. It follows that $X_k(P)$ is a.s. finite. Thus, X_k is a point process in $A(d,d-k)$. Obviously, it is stationary, but it may have intensity zero and need not be simple. If X is a stationary Poisson hyperplane process, then almost surely either the intersection $H_1 \cap \ldots \cap H_k$ is empty or $H_1,\ldots,H_k$ are in general position, as follows by the method used in the proof of Theorem 4.4.5. Therefore, X_k is a.s. simple. That X_k is not a Poisson process, in general, is already seen in the case $d = 2$, $k = 2$, since for a stationary Poisson point process in $\mathbb{R}^2$ a.s. no three points are collinear.

In the following, for vectors $u_1,\ldots,u_m \in \mathbb{R}^d$, $m \le d$, we denote by $\nabla_m(u_1,\ldots,u_m)$ the m-dimensional volume of the parallelepiped spanned by $u_1,\ldots,u_m$.

Theorem 4.4.8. *Let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity $\gamma \neq 0$ and spherical directional distribution φ. Let $k \in \{2,\ldots,d\}$, and let X_k be the intersection process of order k of X. Then the intensity γ_k and the directional distribution $\mathbb{Q}_k$ of X_k are given by*

$$\gamma_k \mathbb{Q}_k(A)$$
$$= \frac{\gamma^k}{k!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \mathbf{1}_A(u_1^\perp \cap \ldots \cap u_k^\perp) \nabla_k(u_1,\ldots,u_k)\, \varphi(\mathrm{d}u_1) \cdots \varphi(\mathrm{d}u_k)$$

for $A \in \mathcal{B}(G(d,d-k))$.

(For $k = d$, $\mathbb{Q}_k(A)$ and $\mathbf{1}_A(u_1^\perp \cap \ldots \cap u_k^\perp)$ have to be omitted from the formula. If $\gamma_k = 0$, the measure $\mathbb{Q}_k$ is not defined; then $\gamma_k \mathbb{Q}_k$ has to be read as the zero measure.)

Proof. Let Θ_k be the intensity measure of the intersection process X_k (the subsequent proof also yields that Θ_k is locally finite). For $P \in \mathcal{B}(A(d,d-k))$ let f_P be the function defined by (4.34). Then

$$\Theta_k(P) = \mathbb{E}X_k(P)$$
$$= \frac{1}{k!} \mathbb{E} \sum_{(H_1,\ldots,H_k) \in X_{\neq}^k} f_P(H_1,\ldots,H_k)$$
$$= \frac{1}{k!} \int_{A(d,d-1)^k} f_P \,\mathrm{d}\Lambda^{(k)},$$

by Theorem 3.1.3. Here $\Lambda^{(k)} = \Theta^k$ by Corollary 3.2.4. Together with (4.33) this gives

$$k! \Theta_k(P) = \int_{A(d,d-1)} \cdots \int_{A(d,d-1)} f_P(H_1, \ldots, H_k) \, \Theta(\mathrm{d}H_1) \cdots \Theta(\mathrm{d}H_k)$$

$$= \gamma^k \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_P(H(u_1, \tau_1), \ldots, H(u_k, \tau_k))$$

$$\times \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k \, \varphi(\mathrm{d}u_1) \cdots \varphi(\mathrm{d}u_k).$$

Let $A \in \mathcal{B}(G(d, d-k))$ and choose $P := \mathcal{F}_{B^d} \cap \pi_0^{-1}(A)$. By (4.26),

$$k! \gamma_k \mathbb{Q}_k(A) = \frac{k!}{\kappa_k} \Theta_k(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A))$$

$$= \frac{\gamma^k}{\kappa_k} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_P(H(u_1, \tau_1), \ldots, H(u_k, \tau_k))$$

$$\times \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k \, \varphi(\mathrm{d}u_1) \cdots \varphi(\mathrm{d}u_k),$$

where

$$f_P(H(u_1, \tau_1), \ldots, H(u_k, \tau_k))$$

$$= \mathbf{1}_A(u_1^\perp \cap \ldots \cap u_k^\perp) \mathbf{1}_{\mathcal{F}_{B^d}}(H(u_1, \tau_1) \cap \ldots \cap H(u_k, \tau_k)).$$

For the computation of the integral

$$I_k := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{1}_{\mathcal{F}_{B^d}}(H(u_1, \tau_1) \cap \ldots \cap H(u_k, \tau_k)) \, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k,$$

we first assume that $u_1, \ldots, u_k$ are linearly independent. Let $k = d$. For $\tau := (\tau_1, \ldots, \tau_d)$ let $T(\tau)$ be the intersection point of the hyperplanes $H(u_1, \tau_1), \ldots, H(u_d, \tau_d)$. Then I_d is the d-dimensional Lebesgue measure of the set $T^{-1}(B^d)$. The mapping T is injective, and its inverse is given by $T^{-1}(x) = (\langle x, u_1 \rangle, \ldots, \langle x, u_d \rangle)$; the Jacobian of T^{-1} is $\nabla_d(u_1, \ldots, u_d)$. Therefore,

$$I_d = \kappa_d \nabla_d(u_1, \ldots, u_d).$$

For $k < d$ we obtain

$$I_k = \kappa_k \nabla_k(u_1, \ldots, u_k),$$

by applying the obtained result in the space $\mathrm{lin}\,\{u_1, \ldots, u_k\}$. Thus we get

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_P(H(u_1, \tau_1), \ldots, H(u_k, \tau_k)) \, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k$$

$$= \mathbf{1}_A(u_1^\perp \cap \ldots \cap u_k^\perp) \kappa_k \nabla_k(u_1, \ldots, u_k).$$

This equation also holds if $u_1, \ldots, u_k$ are linearly dependent, since in that case both sides are zero. This completes the proof. $\qquad \square$

Now we consider a stationary process X of r-flats, where $d/2 \le r \le d-1$. In every realization of X, we take the intersection of any two flats in the process which are in general position. By similar arguments to those used for hyperplanes, we see that we obtain in this way a stationary process of $(2r-d)$-flats. We denote it by X_2 and call it the **intersection process of order** 2 of X.

Theorem 4.4.9. *Let $d/2 \le r \le d-1$, let X be a stationary Poisson process of r-flats in $\mathbb{R}^d$ with intensity $\gamma \ne 0$ and directional distribution $\mathbb{Q}$. Let X_2 be the intersection process of order 2 of X. Then the intensity γ_2 and the directional distribution $\mathbb{Q}_2$ of X_2 are given by*

$$\gamma_2 \mathbb{Q}_2(A) = \frac{\gamma^2}{2} \int_{G(d,r)} \int_{G(d,r)} \mathbf{1}_A(E \cap F)[E,F]\, \mathbb{Q}(\mathrm{d}E)\, \mathbb{Q}(\mathrm{d}F)$$

for $A \in \mathcal{B}(G(d, 2r-d))$.

(If $\gamma_2 = 0$, the measure $\mathbb{Q}_2$ is not defined; then $\gamma_2 \mathbb{Q}_2$ has to be read as the zero measure.)

Proof. Let Θ_2 be the intensity measure of X_2. For $A \in \mathcal{B}(G(d, 2r-d))$ we obtain, similarly to the proof of Theorem 4.4.8,

$$\gamma_2 \mathbb{Q}_2(A)$$
$$= \frac{1}{\kappa_{2(d-r)}} \Theta_2(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A))$$
$$= \frac{\gamma^2}{2\kappa_{2(d-r)}} \int_{G(d,r)} \int_{G(d,r)} \int_{E^\perp} \int_{F^\perp} \mathbf{1}_A(E \cap F)\mathbf{1}_{\mathcal{F}_{B^d}}((E+x) \cap (F+y))$$
$$\times \lambda_{F^\perp}(\mathrm{d}y)\, \lambda_{E^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}E)\, \mathbb{Q}(\mathrm{d}F).$$

In the integral

$$I(x) := \int_{F^\perp} \mathbf{1}_{\mathcal{F}_{B^d}}((E+x) \cap (F+y))\, \lambda_{F^\perp}(\mathrm{d}y),$$

the integrand is equal to 1 if and only if $y \in (B^d \cap (E+x))|F^\perp$. As in the proof of Theorem 4.4.7 (observing that $B^d \cap (E+x)$ is now a ball of radius $\sqrt{1 - \|x\|^2}$), we obtain

$$I(x) = \kappa_{d-r}[E,F](1 - \|x\|^2)^{(d-r)/2}.$$

This gives

$$\int_{E^\perp} \int_{F^\perp} \mathbf{1}_{\mathcal{F}_{B^d}}((E+x) \cap (F+y))\, \lambda_{F^\perp}(\mathrm{d}y)\, \lambda_{E^\perp}(\mathrm{d}x)$$
$$= \kappa_{d-r}[E,F] \int_{B^d \cap E^\perp} (1 - \|x\|^2)^{(d-r)/2}\, \lambda_{E^\perp}(\mathrm{d}x)$$
$$= \kappa_{2(d-r)}[E,F].$$

This yields the assertion. □

The Proximity of Non-intersecting Poisson Flats

The considered intersection densities of stationary Poisson hyperplane processes are examples of real parameters that describe the geometric behavior of such processes and are not determined by the intensity alone. We now suggest a similar parameter for r-flat processes, where $r < d/2$. For these, we cannot work with intersections. The proposed parameter is a means to measure how close flats in general position of the process approach each other, in the mean. (If the directional distribution of a Poisson r-flat process is absolutely continuous, then by Theorem 4.4.5 almost surely any two flats in the process are in general position.)

Let $1 \le r < d/2$ and $E_1, E_2 \in A(d, r)$. If E_1, E_2 are in general position, there are uniquely determined points $x_1 \in E_1$ and $x_2 \in E_2$ such that

$$d(E_1, E_2) := \|x_1 - x_2\| = \inf\{\|y_1 - y_2\| : y_1 \in E_1,\ y_2 \in E_2\}.$$

We call the point

$$m(E_1, E_2) := \frac{1}{2}(x_1 + x_2)$$

the **midpoint** of E_1 and E_2.

Let X be a stationary r-flat process in $\mathbb{R}^d$, where $1 \le r < d/2$. For every realization of X we take the midpoint $m(E_1, E_2)$ of any two flats E_1, E_2 of the realization which are in general position and satisfy $d(E_1, E_2) \le 1$. (The bound 1 for the distance is only chosen for convenience; for a Poisson process, a different bound would result in an additional factor in (4.35).) In this way, we obtain a stationary point process in $\mathbb{R}^d$, the **midpoint process** of X. Its intensity is denoted by $\pi(X)$ and called the **proximity** of the flat process X. (Here $\pi(X) = 0$ is possible, for example, if the directional distribution of X is degenerate.)

Theorem 4.4.10. *Let $1 \le r < d/2$, and let X be a stationary Poisson r-flat process in $\mathbb{R}^d$ with intensity $\gamma > 0$ and directional distribution $\mathbb{Q}$. Then the proximity of X is given by*

$$\pi(X) = \frac{1}{2}\kappa_{d-2r}\gamma^2 \int_{G(d,r)} \int_{G(d,r)} [E, F]\, \mathbb{Q}(\mathrm{d}E)\, \mathbb{Q}(\mathrm{d}F). \qquad (4.35)$$

Proof. For $E_1, E_2 \in A(d, r)$, define

$$g(E_1, E_2) := \begin{cases} 1, & \text{if } E_1, E_2 \text{ are in general position,} \\ & \quad d(E_1, E_2) \le 1 \text{ and } m(E_1, E_2) \in B^d, \\ 0 & \text{otherwise.} \end{cases}$$

Since the proximity $\pi(X)$ is the intensity of the midpoint process of X, it is given by the expectation

$$\pi(X) = \frac{1}{2\kappa_d} \mathbb{E} \sum_{(E_1,E_2)\in X^2_{\neq}} g(E_1, E_2).$$

By Theorems 3.1.3, 4.4.2 and Corollary 3.2.4, we obtain

$$\pi(X) = \frac{1}{2\kappa_d} \int_{A(d,r)} \int_{A(d,r)} g(E_1, E_2)\,\Theta(\mathrm{d}E_1)\,\Theta(\mathrm{d}E_2)$$

$$= \frac{\gamma^2}{2\kappa_d} \int_{G(d,r)} \int_{G(d,r)} \int_{E^\perp} \int_{F^\perp} g(E + x, F + y)$$

$$\times \lambda_{F^\perp}(\mathrm{d}y)\,\lambda_{E^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}E)\,\mathbb{Q}(\mathrm{d}F).$$

We compute the inner double integral

$$I(E, F) := \int_{E^\perp} \int_{F^\perp} g(E + x, F + y)\,\lambda_{F^\perp}(\mathrm{d}y)\,\lambda_{E^\perp}(\mathrm{d}x)$$

for two fixed subspaces $E, F \in G(d,r)$ in general position.

Let $E + F =: V$ and $U := V^\perp$. Vectors $x \in E^\perp$ and $y \in F^\perp$ have unique decompositions

$$x = x_1 + x_2, \quad x_1 \in E^\perp \cap V,\ x_2 \in U,$$

$$y = y_1 + y_2, \quad y_1 \in F^\perp \cap V,\ y_2 \in U,$$

which gives

$$I(E, F) = \int_U \int_U J(E, F, x_2, y_2)\,\lambda_U(\mathrm{d}x_2)\,\lambda_U(\mathrm{d}y_2)$$

with

$$J(E, F, x_2, y_2)$$
$$= \int_{E^\perp \cap V} \int_{F^\perp \cap V} g(E + x_1 + x_2, F + y_1 + y_2)\,\lambda_{F^\perp \cap V}(\mathrm{d}y_1)\,\lambda_{E^\perp \cap V}(\mathrm{d}x_1).$$

To compute this double integral, let $z \in V$ be the intersection point of $E + x_1$ and $F + y_1$. The distance of $E + x_1 + x_2$ and $F + y_1 + y_2$ is realized by the points $z + x_2$ and $z + y_2$, hence $d(E + x_1 + x_2, F + y_1 + y_2) = \|x_2 - y_2\|$ and $m(E + x_1 + x_2, F + y_1 + y_2) = z + (x_2 + y_2)/2$. Thus, $J(E, F, x_2, y_2) = 0$ if $\|x_2 - y_2\| > 1$. Assume that $\|x_2 - y_2\| \leq 1$. Then $g(E + x_1 + x_2, F + y_1 + y_2) = 1$ if and only if $z + (x_2 + y_2)/2 \in B^d$. The set $V \cap (B^d - (x_2 + y_2)/2)$ is a $2r$-dimensional ball with radius $(1 - \|(x_2 + y_2)/2\|^2)^{1/2}$. It follows that

$$J(E, F, x_2, y_2) = \kappa_{2r}(1 - \|(x_2 + y_2)/2\|^2)^r [E, F],$$

if $\|x_2 - y_2\| \leq 1$ and $\|(x_2 + y_2)/2\| \leq 1$, and 0 otherwise. This yields

$$\pi(X) = \frac{\kappa_{2r}}{2\kappa_d}\gamma^2 \int_{G(d,r)} \int_{G(d,r)} [E, F] \, \mathbb{Q}(\mathrm{d}E) \, \mathbb{Q}(\mathrm{d}F) \cdot K$$

with

$$K := \int_{U^2} \mathbf{1}\{\|x_2 - y_2\| \leq 1\}\mathbf{1}\{\|(x_2 + y_2)/2\| \leq 1\}$$
$$\times (1 - \|(x_2 + y_2)/2\|^2)^r \, \lambda_U^2(\mathrm{d}(x_2, y_2)).$$

The substitution $x_2 - y_2 = u$, $(x_2 + y_2)/2 = v$ allows us to compute this integral, which completes the proof. $\qquad\square$

Remark. Let X be as in Theorem 4.4.10. Let $\mathbb{Q}^\perp$ be the image measure of $\mathbb{Q}$ under the mapping $L \mapsto L^\perp$ from $G(d,r)$ to $G(d, d-r)$. There is a stationary Poisson $(d-r)$-flat process $X^\perp$ with directional distribution $\mathbb{Q}^\perp$ and intensity γ. A comparison of Theorems 4.4.9 and 4.4.10 shows that the second intersection density $\gamma_2(X^\perp)$ of the process $X^\perp$ and the proximity $\pi(X)$ of the process X are related by

$$\frac{1}{2}\kappa_{d-2r}\gamma_2(X^\perp) = \pi(X).$$

Therefore, inequalities for the second intersection density of a stationary Poisson flat process, as they are treated in Section 4.6, can be transferred to the proximity.

Notes for Section 4.4

1. Flat processes, in particular under Poisson assumptions, were first studied intensively by Miles [521, 523] and Matheron [460, 461, 462]. In the book by Matheron [462] one finds most of the results of Section 4.4, though partially with different proofs. For example, Theorem 4.4.1 appears there (p. 66) with a proof involving an extension of conditional probabilities, whereas we have preferred to give a direct and more elementary proof.

2. In the discussion following Theorem 4.4.6, we have mentioned the result (first pointed out by Matheron) that the distribution $\mathbb{P}_X$ of a stationary Poisson k-flat process X is uniquely determined by the section intensities $\gamma_{X \cap S}$, $S \in G(d, d-k)$, if either $k = 1$ or $k = d - 1$. That there is no corresponding uniqueness result for $1 < k < d - 1$, was shown by Goodey and Howard [271]. Sections with planes S of dimension $d-k+j$, $j \in \{1, \ldots, k-1\}$, raise at least two questions: whether the section intensities $\gamma_{X \cap S}$, or whether the intensity measures of $X \cap S$, $S \in G(d, d-k+j)$, are sufficient to determine the distribution of the Poisson k-flat process X. These questions were answered partially by Goodey and Howard [271, 272] and completely by Goodey, Howard and Reeder [273].

The distribution $\mathbb{P}_X$ of a stationary Poisson hyperplane process X is, more generally, determined by the section intensities $\gamma_{X \cap S}$, $S \in G(d, r)$, for fixed

$r \in \{1, \ldots, d-1\}$. Corresponding inversion formulas (for the intensity measure of X) are discussed in Spodarev [733, 734] (in a purely analytic setting, more general inversion formulas are treated by Rubin [653]).

3. Theorem 4.4.10, together with Theorem 4.6.6, is found in Schneider [699], though with a factor $1/2$ missing.

4. In analogy to the idea of proximity, Spodarev [733, 735] introduced the **rose of neighborhood** γ_{kr} of a stationary k-flat process X, as a function on $G(d, r)$ where $k+r < d$. For $S \in G(d, r)$, $\gamma_{kr}(S)$ is the intensity of the (stationary) process of points in S arising as projections of midpoints $m(E, S), E \in X$, with distance $d(E, S) \leq 1$ (say). Relating X to a 'dual' process X' of $(d-k)$-flats, $\gamma_{kr}(S)$ transforms into the section intensity $\gamma_{X' \cap S^\perp}$ of X'. Therefore, the uniqueness, respectively non-uniqueness, results for section intensities (see Note 2 above) carry over to the roses of neighborhood.

For a similar situation, a process X of k-flats and a fixed r-plane S with $r+s < d$, Hug, Last and Weil [360] discussed the question whether distance measurements from S to (the union set of) the flats in X suffice to determine the directional distribution of X. Their results also hold for non-stationary processes X (see the Notes to Section 11.3).

5. The complementary theorem of Miles (see Note 5 of Section 3.2), in its versions for Poisson flat processes due to Miles [523] and to Møller and Zuyev [555], was considerably extended by Baumstark and Last [86]. They considered stationary Poisson processes of k-flats ($k \in \{0, \ldots, d-1\}$) in $\mathbb{R}^d$ and obtained that the integral geometric contents of several closed sets constructed on such processes have conditional Gamma distributions.

4.5 Surface Processes

After studying processes of k-dimensional flats, it is a natural next step to consider processes of k-dimensional surfaces. Since unbounded surfaces can be represented as unions of countably many bounded surfaces, we may restrict ourselves to the latter. In particular, we shall consider particle processes where the particles are compact surfaces. For example, a **surface process** in $\mathbb{R}^3$ is obtained if the particles are almost surely two-dimensional surfaces, and a particle process consisting of curves is a **curve process** or **fiber process**, etc. The technical requirements for a theory of particle processes of k-dimensional surfaces depend very much on the generality of the notion of k-surface that is employed. A suitable general concept is that of a $\mathcal{H}^k$-rectifiable closed set. We refer to Section 14.5 for the definition of $\mathcal{H}^k$-rectifiable sets, and to Zähle [823] for a proof of the fact that the system $\mathcal{X}^{(k)}$ of $\mathcal{H}^k$-rectifiable closed sets in $\mathbb{R}^d$ is a measurable subset of $\mathcal{F}$. Therefore, a k-surface process can be defined as a point process in $\mathcal{F}$ the intensity measure of which is concentrated on $\mathcal{X}^{(k)}$. The treatment of such processes, however, requires methods from geometric measure theory, which are outside the scope of this book. For that reason, in the following we treat, with complete proofs, only an elementary version of

surface processes, namely special processes in the convex ring $\mathcal{R}$. For this, we consider k-dimensional surfaces ($k = 1, \ldots, d-1$) that can be represented as finite unions of k-dimensional compact convex sets, for example, polyhedral surfaces of dimension k.

For $k \in \{1, \ldots, d-1\}$, we denote by $\mathcal{K}^{(k)}$ the set of all convex sets $K \in \mathcal{K}$ of dimension k and by $\mathcal{R}^{(k)} \subset \mathcal{R}$ the set of all finite unions of elements from $\mathcal{K}^{(k)}$. Elements of $\mathcal{R}^{(k)}$ are briefly called k-**surfaces** in the following. Obvious modifications of the proof of Theorem 2.4.2 show that $\mathcal{K}^{(k)}$ and $\mathcal{R}^{(k)}$ are Borel subsets of $\mathcal{F}$. By a k-**surface process** in $\mathbb{R}^d$ we understand a particle process with intensity measure concentrated on $\mathcal{R}^{(k)}$. This elementary case is sufficient for demonstrating the typical questions and results about surface processes. The extension to more general models then requires no principally new ideas, but is technically more involved. The methods and results from [823] allow us to obtain the results below with the system $\mathcal{R}^{(k)}$ of elementary k-surfaces replaced by the system $\mathcal{X}^{(k)}$ of $\mathcal{H}^k$-rectifiable closed sets, but this is not carried out here.

Let X be a stationary k-surface process with intensity measure $\Theta \neq 0$. According to Theorem 4.1.1, this process has an intensity γ and a grain distribution $\mathbb{Q}$. The intensity γ has to be distinguished from the k-volume density or specific k-volume. The latter is the intensity of the induced random k-volume measure (and is, therefore, by some authors called the 'intensity' of X). It can be introduced as follows. First we note that for $C \in \mathcal{R}^{(k)}$ we have

$$V_k(C) = \mathcal{H}^k(C), \tag{4.36}$$

where V_k is the additive extension of the kth intrinsic volume to the convex ring $\mathcal{R}$ (see Sections 14.2 and 14.4) and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. This follows by additivity, since (4.36) is true for $C \in \mathcal{K}^{(k)}$. The function $C \mapsto V_k(C)$, $C \in \mathcal{R}^{(k)}$, which we call the k-volume, is measurable by Theorem 14.4.4 (the additive extension of V_k is measurable on $\mathcal{R}$). By (4.36) it is nonnegative.

According to (4.6), the k-volume density of X is defined by

$$\overline{V}_k(X) := \gamma \int_{\mathcal{C}_0} V_k \, d\mathbb{Q}. \tag{4.37}$$

We call $\overline{V}_k(X)$ the **specific** k-**volume** of X (the possibility of $\overline{V}_k(X) = \infty$ is not excluded).

We define a random measure η by

$$\eta := \sum_{C \in X} \mathcal{H}^k \llcorner C.$$

Almost surely η is locally finite, since X is a particle process, and each particle $C \in \mathcal{R}^{(k)}$ satisfies $\mathcal{H}^k(C) < \infty$. The following theorem shows that the specific k-volume, if finite, can be interpreted as the intensity of the stationary random measure η.

Theorem 4.5.1. *Let X be a stationary k-surface process in $\mathbb{R}^d$ with $\overline{V}_k(X) < \infty$. Then*

$$\sum_{C \in X} \mathcal{H}^k \llcorner C$$

is a stationary random measure, and $\overline{V}_k(X)$ is its intensity, that is,

$$\overline{V}_k(X) = \frac{1}{\lambda(A)} \, \mathbb{E} \sum_{C \in X} \mathcal{H}^k(C \cap A) \tag{4.38}$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda(A) < \infty$.

Proof. Let $A \in \mathcal{B}(\mathbb{R}^d)$ be given. We define

$$f(C) := (\mathcal{H}^k \llcorner C)(A) = \mathcal{H}^k(A \cap C) \qquad \text{for } C \in \mathcal{R}^{(k)}.$$

Assume, first, that A is compact and that $K_i, K \in \mathcal{K}^{(k)}$ satisfy $K_i \to K$ in the Hausdorff metric. Let $E \in A(d, k)$ be the plane with $K \subset E$. There exist rigid motions g_i, converging to the identity, such that $g_i K_i \subset E$ and $g_i K_i \to K$. For every $x \in \mathbb{R}^d$,

$$\limsup \mathbf{1}_{g_i(A \cap K_i)}(x) \leq \mathbf{1}_{A \cap K}(x).$$

As in the proof of Theorem 12.3.6, we get

$$\mathcal{H}^k(A \cap K) = \int_E \mathbf{1}_{A \cap K}(x) \, \mathcal{H}^k(\mathrm{d}x) \geq \int_E \limsup \mathbf{1}_{g_i(A \cap K_i)}(x) \, \mathcal{H}^k(\mathrm{d}x)$$

$$\geq \limsup \int_E \mathbf{1}_{g_i(A \cap K_i)}(x) \, \mathcal{H}^k(\mathrm{d}x) = \limsup \mathcal{H}^k(A \cap K_i).$$

Thus, on $\mathcal{K}^{(k)}$ the function f is upper semicontinuous and, therefore, measurable. Modifying the proof of Theorem 14.4.4, we see that f is measurable on $\mathcal{R}^{(k)}$. Since this holds for all compact sets A, it holds for all Borel sets A. Now the Campbell theorem shows that $\sum_{C \in X} (\mathcal{H}^k \llcorner C)(A)$ is measurable. It follows that $\sum_{C \in X} \mathcal{H}^k \llcorner C$ is a random measure; clearly it is stationary. Campbell's theorem further shows that

$$\mathbb{E} \sum_{C \in X} (\mathcal{H}^k \llcorner C)(A) = \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathcal{H}^k(A \cap (C + x)) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \gamma \int_{\mathcal{C}_0} V_k(C) \lambda(A) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \overline{V}_k(X) \lambda(A),$$

where Theorem 5.2.1 (with $\alpha := \mathcal{H}^k \llcorner C$) and (4.37) were used. This proves (4.38). Now the assumption $\overline{V}_k(X) < \infty$ implies that $\sum_{C \in X} \mathcal{H}^k \llcorner C$ has locally finite intensity measure and intensity $\overline{V}_k(X)$. $\qquad \square$

A k-surface has, at $\mathcal{H}^k$-almost every point, a k-dimensional tangent plane. For a k-surface process, this leads to the notion of its directional distribution. Let $C \in \mathcal{R}^{(k)}$, and let $C = \bigcup_{i=1}^m C_i$ be a representation with $C_i \in \mathcal{K}^{(k)}$ for $i = 1, \ldots, m$. The set of all $y \in C$ lying in some C_i and some C_j, where C_i and C_j have different affine hulls, is of $\mathcal{H}^k$-measure zero. For the remaining $y \in C$, we can choose i with $y \in C_i$ and then define the **tangent plane** T_yC of C at y as the linear subspace of $\mathbb{R}^d$ which is parallel to the affine hull of C_i. Thus, at $\mathcal{H}^k$-almost all $y \in C$, the tangent plane is uniquely determined.

Theorem 4.5.2. *Let X be a stationary k-surface process in $\mathbb{R}^d$ with specific k-volume satisfying $0 < \overline{V}_k(X) < \infty$. Then there is a unique probability measure $\mathbb{T}$ on the Grassmannian $G(d, k)$ satisfying*

$$\mathbb{E} \sum_{C \in X} \int_{B \cap C} \mathbf{1}_A(T_yC) \, \mathcal{H}^k(\mathrm{d}y) = \overline{V}_k(X)\lambda(B)\mathbb{T}(A)$$

for all $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \lambda(B) < \infty$ and all $A \in \mathcal{B}(G(d,k))$.

Proof. From the Campbell theorem and from Theorem 5.2.1 we obtain

$$\mathbb{E} \sum_{C \in X} \int_{B \cap C} \mathbf{1}_A(T_yC) \, \mathcal{H}^k(\mathrm{d}y)$$

$$= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \int_{B \cap (C+x)} \mathbf{1}_A(T_y(C + x)) \, \mathcal{H}^k(\mathrm{d}y) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \int_{(B-x) \cap C} \mathbf{1}_A(T_yC) \, \mathcal{H}^k(\mathrm{d}y) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}C)$$

$$= \gamma\lambda(B) \int_{\mathcal{C}_0} \int_C \mathbf{1}_A(T_yC) \, \mathcal{H}^k(\mathrm{d}y) \, \mathbb{Q}(\mathrm{d}C).$$

The mapping

$$A \mapsto \gamma \int_{\mathcal{C}_0} \int_C \mathbf{1}_A(T_yC) \, \mathcal{H}^k(\mathrm{d}y) \, \mathbb{Q}(\mathrm{d}C), \qquad A \in \mathcal{B}(G(d,k)),$$

is a measure η with $\eta(G(d,k)) = \overline{V}_k(X)$. Defining $\mathbb{T} := \eta/\overline{V}_k(X)$, we obtain the assertion. The uniqueness is clear. $\qquad\qquad\square$

For later use, we note that

$$\overline{V}_k(X)\mathbb{T}(A) = \gamma \int_{\mathcal{C}_0} \int_C \mathbf{1}_A(T_yC) \, \mathcal{H}^k(\mathrm{d}y) \, \mathbb{Q}(\mathrm{d}C). \qquad (4.39)$$

We call the probability measure $\mathbb{T}$ the **directional distribution** of the k-surface process X (another common name is **rose of directions**, in particular for fiber processes). The directional distribution can be interpreted as the distribution of the tangent plane in a typical point of the surface process.

Now we consider section processes derived from k-surface processes. Let X be a stationary k-surface process with positive, finite specific k-volume, and let $S \in G(d, d-k+j)$ be a $(d-k+j)$-plane, where $0 \leq j \leq k-1$. In Section 3.6 we have defined the section process $X \cap S$. It is a particle process in S. Because of the elementary notion of k-surface that we employ, it is not difficult to show that $X \cap S$ is almost surely a j-surface process in S. We do not carry out the proof here, since very similar arguments were already employed when we treated processes of flats (before Theorem 4.4.6). It is clear that the j-surface process $X \cap S$ is stationary in S.

Theorem 4.5.3. *Let X be a stationary k-surface process in $\mathbb{R}^d$ with positive, finite specific k-volume $\overline{V}_k(X)$ and with directional distribution $\mathbb{T}$. Let $S \in G(d, d-k+j)$, $0 \leq j \leq k-1$, and let $\overline{V}_j(X \cap S)$ be the specific j-volume of the section process $X \cap S$. Then*

$$\overline{V}_j(X \cap S) = \overline{V}_k(X) \int_{G(d,k)} [S, L]\, \mathbb{T}(\mathrm{d}L).$$

Proof. Let $A \subset S$ be a compact set with $\lambda_S(A) = 1$. By Theorem 4.5.1 and Campbell's theorem,

$$\overline{V}_j(X \cap S) = \mathbb{E} \sum_{C \in X} \mathcal{H}^j(C \cap A)$$

$$= \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathcal{H}^j((C + x) \cap A)\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}C).$$

Let $C = \bigcup_{i=1}^m C_i$ with $C_i \in \mathcal{K}^{(k)}$. By the inclusion–exclusion principle (with the notation used in (14.48)) we have

$$\int_{\mathbb{R}^d} \mathcal{H}^j((C + x) \cap A)\, \lambda(\mathrm{d}x) = \sum_{v \in S(m)} (-1)^{|v|-1} \int_{\mathbb{R}^d} \mathcal{H}^j((C_v + x) \cap A)\, \lambda(\mathrm{d}x)$$

$$= \sum_{v \in S(m)} (-1)^{|v|-1} [S, \mathrm{aff}\, C_v] V_k(C_v)$$

$$= \sum_{v \in S(m)} (-1)^{|v|-1} \int_{C_v} [S, T_y C_v]\, \mathcal{H}^k(\mathrm{d}y)$$

$$= \int_C [S, T_y C]\, \mathcal{H}^k(\mathrm{d}y)$$

(observe that $V_k(C_v) = 0$ if $\dim C_v < k$). This yields

$$\overline{V}_j(X \cap S) = \gamma \int_{\mathcal{C}_0} \int_C [S, T_y C]\, \mathcal{H}^k(\mathrm{d}y)\, \mathbb{Q}(\mathrm{d}C)$$

$$= \overline{V}_k(X) \int_{G(d,k)} [S, L]\, \mathbb{T}(\mathrm{d}L),$$

where we have used (4.39), extended from indicator functions to nonnegative measurable functions on $G(d, k)$. □

In the cases $k = 1$ (fiber processes) and $k = d-1$ (hypersurface processes) it is again convenient (as after Theorem 4.4.6) to interpret the directional distribution as an even measure on the sphere S^{d-1}. For a unit vector $u \in S^{d-1}$, $L(u)$ denotes the one-dimensional linear subspace spanned by u, and $u^{\perp}$ is the $(d-1)$-dimensional linear subspace orthogonal to u. Corresponding to a directional distribution $\mathbb{T}$ we define a spherical directional distribution φ by setting, for a set $A \in \mathcal{B}(S^{d-1})$ without pairs of antipodal points,

$$\varphi(A) := \frac{1}{2} \mathbb{T}(\{L(u) : u \in A\}) \qquad \text{if } k = 1$$

and

$$\varphi(A) := \frac{1}{2} \mathbb{T}(\{u^{\perp} : u \in A\}) \qquad \text{if } k = d - 1.$$

For the specific 0-volumes (intersection point densities) of the section processes found in Theorem 4.5.3, we now obtain, for $v \in S^{d-1}$,

$$\overline{V}_0(X \cap v^{\perp}) = \overline{V}_1(X) \int_{S^{d-1}} |\langle u, v \rangle| \, \varphi(\mathrm{d}u) \qquad \text{if } k = 1 \qquad (4.40)$$

and

$$\overline{V}_0(X \cap L(v)) = \overline{V}_{d-1}(X) \int_{S^{d-1}} |\langle u, v \rangle| \, \varphi(\mathrm{d}u) \qquad \text{if } k = d - 1. \qquad (4.41)$$

Note for Section 4.5

The investigation of the directional distribution (also called 'rose of directions') of fiber and surface processes was initiated in papers by Mecke and Stoyan, beginning with [501], which was generalized by Mecke and Nagel [495]. Pohlmann, Mecke and Stoyan [606] treated stereological formulas for stationary surface processes. For a very general investigation of fiber and surface processes (using Hausdorff rectifiable sets), we refer to Zähle [822].

4.6 Associated Convex Bodies

For a stationary particle process X in $\mathbb{R}^d$ and a suitable translation invariant function φ on $\mathcal{C}_0$, the φ-density $\overline{\varphi}$ was defined in Section 4.1 by

$$\overline{\varphi}(X) := \gamma \int_{\mathcal{C}_0} \varphi \, \mathrm{d}\mathbb{Q}.$$

This procedure is not restricted to real-valued functions φ. In particular, on the space of convex bodies, there are some geometrically meaningful translation invariant mappings into spaces of functions or measures which can be

employed. In this way one can associate with a particle process, besides intensities of real-valued functionals, also measures or convex bodies as describing parameters. Similar procedures are possible for other geometric processes, such as processes of flats, fibers, or surfaces, or even for certain random closed sets. One motivation for this comes from the fact that associated measures or convex bodies contain more information than real-valued parameters, and may yet be accessible to estimation procedures. Another reason for introducing auxiliary convex bodies lies in the observation that sometimes the application of results from convex geometry to associated auxiliary bodies leads to results, for example to solutions of extremal problems, which otherwise would be out of reach. Such results from convex geometry are applied in this section; they are collected in Section 14.3, with references to sources where proofs can be found.

Processes of Convex Particles

First we consider a stationary process X of convex particles in $\mathbb{R}^d$ with intensity $\gamma > 0$ and grain distribution $\mathbb{Q}$.

Since a convex body K is determined by its support function $h(K, \cdot)$, defined by

$$h(K, u) := \max\{\langle x, u \rangle : x \in K\}, \qquad u \in \mathbb{R}^d,$$

it appears natural to consider the density of the functional $h(\cdot, u)$ for $u \in \mathbb{R}^d$. However, the support function is not translation invariant. This is remedied by introducing the **centered support function**, by

$$h^*(K, u) := h(K, u) - \langle s(K), u \rangle = h(K - s(K), u), \qquad (4.42)$$

where $s(K)$ is the Steiner point of K (see (14.28)). We have

$$h^*(K + x, \cdot) = h^*(K, \cdot) \qquad \text{for } x \in \mathbb{R}^d$$

and $h^*(K, \cdot) \geq 0$. From (14.7) and (14.28) we obtain an estimate of the form $h^*(K, u) \leq c(d)V_1(K)\|u\|$ with a constant $c(d)$. Since V_1 is $\mathbb{Q}$-integrable by Theorem 4.1.2, $h^*(\cdot, u)$ is $\mathbb{Q}$-integrable. Hence, we can define

$$\overline{h}(X, u) := \gamma \int_{\mathcal{K}_0} h^*(K, u)\, \mathbb{Q}(\mathrm{d}K) \qquad \text{for } u \in \mathbb{R}^d.$$

Obviously, the function $\overline{h}(X, \cdot)$ is again convex and positively homogeneous, hence it is the support function of a uniquely determined convex body. We denote this body by $M(X)$ and call it the **mean body** of the particle process X.

In a similar way, the surface area measure $S_{d-1}(K, \cdot)$ (see (14.22)) can be employed. For $A \in \mathcal{B}(S^{d-1})$, the function $S_{d-1}(\cdot, A)$ is measurable and translation invariant. Further, $0 \leq S_{d-1}(K, \cdot) \leq S_{d-1}(K, S^{d-1}) = 2V_{d-1}(K)$,

where V_{d-1} is one of the intrinsic volumes (see Section 14.2). Since V_{d-1} is $\mathbb{Q}$-integrable by Theorem 4.1.2, we can define

$$\overline{S}_{d-1}(X, A) := \gamma \int_{\mathcal{K}_0} S_{d-1}(K, A)\, \mathbb{Q}(\mathrm{d}K) \tag{4.43}$$

for $A \in \mathcal{B}(S^{d-1})$. By monotone convergence, $\overline{S}_{d-1}(X, \cdot)$ is a measure.

We indicate how this measure-valued parameter can be interpreted in the case where the particles of the process X are a.s. of dimension d. From the process X we then also obtain a hypersurface process, by replacing each particle by its boundary. For such a hypersurface process, a directional distribution can be defined, similarly to Section 4.1. In contrast to the case $k = d-1$ of Theorem 4.5.2, we now consider an oriented directional distribution, taking into account that for the boundary of a d-dimensional convex body one can distinguish between an outer and an inner normal direction. For the boundary hypersurface $\mathrm{bd}\, K$ of a convex body it is convenient to describe the direction of a tangent hyperplane by its outer normal vector. For $\mathcal{H}^{d-1}$-almost all $y \in \mathrm{bd}\, K$, the outer unit normal vector $n_K(y)$ of K at y is uniquely determined. For a Borel set $A \subset S^{d-1}$, we have $S_{d-1}(K, A) = \mathcal{H}^{d-1}(n_K^{-1}(A))$. For $A \in \mathcal{B}(S^{d-1})$ and $B \in \mathcal{B}(\mathbb{R}^d)$ with $\lambda(B) < \infty$, the mapping $K \mapsto \mathcal{H}^{d-1}(B \cap n_K^{-1}(A))$ is measurable (as follows from Schneider [695, Theorem 4.2.1]), and the Campbell theorem together with (4.3) gives

$$\mathbb{E} \sum_{K \in X} \mathcal{H}^{d-1}\left(B \cap n_K^{-1}(A)\right)$$

$$= \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathcal{H}^{d-1}\left(B \cap n_{K+x}^{-1}(A)\right) \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K)$$

$$= \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathcal{H}^{d-1}\left((B - x) \cap n_K^{-1}(A)\right) \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K)$$

$$= \gamma \int_{\mathcal{K}_0} \lambda(B) \mathcal{H}^{d-1}\left(n_K^{-1}(A)\right) \mathbb{Q}(\mathrm{d}K)$$

$$= \gamma \lambda(B) \int_{\mathcal{K}_0} S_{d-1}(K, A)\, \mathbb{Q}(\mathrm{d}K),$$

where Theorem 5.2.1 was used. Thus, for $B \in \mathcal{B}(\mathbb{R}^d)$ with $\lambda(B) = 1$ we have

$$\overline{S}_{d-1}(X, A) = \mathbb{E} \sum_{K \in X} \mathcal{H}^{d-1}\left(B \cap n_K^{-1}(A)\right).$$

For this reason, the normalized measure $\overline{S}_{d-1}(X, \cdot)/2\overline{V}_{d-1}(X)$ can be interpreted as the **distribution of the normal vector in a typical boundary point** of the particle process X.

The measure $\overline{S}_{d-1}(X, \cdot)$ is called the **mean normal measure** of X (also in the case where the particles are not necessarily d-dimensional).

Starting from the measure-valued parameter $\overline{S}_{d-1}(X, \cdot)$, we now associate two convex bodies with the particle process X. This requires a preliminary consideration.

For a convex body K and for $u \in \mathbb{R}^d \setminus \{0\}$, we denote by $V_{d-1}(K|u^\perp)$ the $(d-1)$-dimensional volume of the orthogonal projection of K to $u^\perp$. The density of the function $K \mapsto V_{d-1}(K|u^\perp)$ for the particle process X is denoted by $\overline{V}_{d-1}(X|u^\perp)$, thus

$$\overline{V}_{d-1}(X|u^\perp) = \gamma \int_{\mathcal{K}_0} V_{d-1}(K|u^\perp) \, \mathbb{Q}(\mathrm{d}K).$$

With Fubini's theorem for kernels and with (14.41), for unit vectors $u \in S^{d-1}$ we get

$$\overline{V}_{d-1}(X|u^\perp) = \frac{\gamma}{2} \int_{\mathcal{K}_0} \int_{S^{d-1}} |\langle u, v \rangle| \, S_{d-1}(K, \mathrm{d}v) \, \mathbb{Q}(\mathrm{d}K)$$

$$= \frac{1}{2} \int_{S^{d-1}} |\langle u, v \rangle| \, \overline{S}_{d-1}(X, \mathrm{d}v). \qquad (4.44)$$

From the Campbell theorem and Theorem 4.1.2, we get for $r > 0$

$$\mathbb{E} \sum_{K \in X, \, c(K) \in rB^d} V_{d-1}(K|u^\perp)$$

$$= \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}_{rB^d}(c(K + x)) V_{d-1}((K + x)|u^\perp) \, \lambda(\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}K)$$

$$= \kappa_d r^d \, \overline{V}_{d-1}(X|u^\perp).$$

Thus, $\overline{V}_{d-1}(X|u^\perp) = 0$ holds if and only if

$$\sum_{K \in X} V_{d-1}(K|u^\perp) = 0$$

almost surely. If there exists a vector $u \in S^{d-1}$ with this property, we say that the particle process X is **degenerate**.

We assume now that X is not degenerate. Then (4.44) shows that the measure $\overline{S}_{d-1}(X, \cdot)$ is not concentrated on a great subsphere. Since

$$\int_{S^{d-1}} u \, S_{d-1}(K, \mathrm{d}u) = 0$$

always holds, we also have

$$\int_{S^{d-1}} u \, \overline{S}_{d-1}(X, \mathrm{d}u) = 0.$$

By the Theorem of Minkowski (Theorem 14.3.1), there exists a uniquely determined convex body $B(X) \in \mathcal{K}_0$ with

$$S_{d-1}(B(X),\cdot) = \overline{S}_{d-1}(X,\cdot). \tag{4.45}$$

We call $B(X)$ the **Blaschke body** of the particle process X. (The name reflects the fact that the addition of surface area measures induces the so-called Blaschke addition of the corresponding convex bodies.)

For a convex body K, we denote by Π_K its projection body (see Section 14.3, in particular (14.40)). The projection body of the Blaschke body, that is,

$$\Pi_X := \Pi_{B(X)}, \tag{4.46}$$

is called the **associated zonoid** of the particle process X. (The name refers to the fact that projection bodies belong to the special class of convex bodies known as zonoids; these are precisely the bodies which can be approximated by vector sums of line segments.) Using (14.40), (14.41), (4.44), (4.45), we get

$$\begin{aligned}
h(\Pi_X, u) &= \frac{1}{2} \int_{S^{d-1}} |\langle u, v \rangle| \, \overline{S}_{d-1}(X, \mathrm{d}v) \\
&= \overline{V}_{d-1}(X | u^{\perp}) \\
&= \gamma \int_{\mathcal{K}_0} h(\Pi_K, u) \, \mathbb{Q}(\mathrm{d}K).
\end{aligned} \tag{4.47}$$

Thus, the support function of the associated zonoid has a simple geometric meaning: on unit vectors, it represents the density of the projection volume in the direction of the vector. Moreover, Π_X can be interpreted as the mean projection body of the particle process X.

Rewriting (4.47) in the form

$$h(\Pi_X, u) = \frac{\gamma}{2} \int_{\mathcal{K}_0} \int_{S^{d-1}} |\langle u, v \rangle| \, S_{d-1}(K, \mathrm{d}v) \, \mathbb{Q}(\mathrm{d}K),$$

integrating over S^{d-1} with respect to the spherical Lebesgue measure, and observing $S_{d-1}(K, S^{d-1}) = 2V_{d-1}(K)$ as well as (14.7), we obtain the identity

$$V_1(\Pi_X) = 2\overline{V}_{d-1}(X), \tag{4.48}$$

The intrinsic volume V_1 appearing here is essentially the mean width; hence, the identity says that the mean width of the associated zonoid is, up to a constant factor, the surface area density of the particle process X.

Next we show how further geometric quantities of the particle process X are related to the associated zonoid Π_X. First we determine

$$f(u) := \mathbb{E} \sum_{K \in X} \mathrm{card}\left([0, u] \cap \mathrm{bd}\, K \right),$$

the expected number of points in which the segment with endpoints 0 and $u \in \mathbb{R}^d \setminus \{0\}$ meets the boundaries of the bodies of the particle process.

Thus, for a unit vector u, the value $f(u)$ gives the intensity $\gamma_{L(u)}$ of the point process that is generated by intersecting the hypersurface process induced by the boundaries of the particles with the line $L(u)$ (as considered similarly in Theorem 4.5.3 for a different class of surface processes). With the Campbell theorem and the decomposition (4.2) we obtain

$$
\begin{aligned}
f(u) &= \gamma \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathrm{card}\left([0,u] \cap \mathrm{bd}\,(K+x)\right) \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K) \\
&= 2\gamma \int_{\mathcal{K}_0} \|u\| V_{d-1}(K|u^\perp)\, \mathbb{Q}(\mathrm{d}K) \\
&= 2\|u\| \overline{V}_{d-1}(X|u^\perp) \\
&= 2h(\Pi_X, u).
\end{aligned}
$$

Thus, we have

$$
h(\Pi_X, u) = \frac{1}{2}\mathbb{E} \sum_{K \in X} \mathrm{card}\left([0,u] \cap \mathrm{bd}\,K\right) = \frac{1}{2}\|u\|\gamma_{L(u)}, \qquad (4.49)
$$

which provides a further interpretation of the support function of the associated zonoid. In particular, for the intersection intensity we obtain from (4.47) the formula

$$
\gamma_{L(u)} = \int_{S^{d-1}} |\langle u, v \rangle|\, \overline{S}_{d-1}(X, \mathrm{d}v). \qquad (4.50)
$$

Since $\overline{S}_{d-1}(X, \cdot)/2\overline{V}_{d-1}(X)$ is a probability measure, this equation is analogous to (4.41). However, it must be observed that X in (4.41) is a hypersurface process, whereas in (4.50) it is a process of convex particles (for a convex body K, $2V_{d-1}(K)$ is the surface area).

Applications to Boolean Models

The associated zonoid is particularly useful when dealing with stationary Boolean models with convex grains. Therefore, we assume now in addition that X is a Poisson process. We still assume that X is nondegenerate, that is, it satisfies $\overline{V}_{d-1}(X|u^\perp) \neq 0$ for all $u \in S^{d-1}$. In this case, also the Boolean model $Z = Z_X$ is called nondegenerate (this property depends only on Z, since Z determines the particle process X up to equivalence). Hence, the Boolean model Z is degenerate if and only if there is a direction u such that the orthogonal projection of Z to $u^\perp$ a.s. has Lebesgue measure zero.

For $F \in \mathcal{F}$ and $x \in \mathbb{R}^d$, we write

$$
S_x(F) := \{y \in \mathbb{R}^d : [x,y] \cap F = \emptyset\}
$$

for the region visible from x; here F is regarded as opaque. The set $S_x(F)$ is open and star-shaped with respect to x; it is empty if $x \in F$. For the stationary Boolean model $Z = Z_X$, the conditional expectation

$$\overline{V}_s(Z) := \mathbb{E}(\lambda(S_0(Z)) \mid 0 \notin Z)$$

is called the **mean visible volume** outside Z (note that we always have $\mathbb{P}(0 \notin Z) > 0$, by (9.5)). The measurability of the function $\lambda(S_0(Z))$ and of the function $(u, \omega) \mapsto s_u(Z(\omega))$ used below follows from the measurability of the set

$$\left\{ (\omega, u, \alpha) \in \Omega \times S^{d-1} \times \mathbb{R}_0^+ : [0, \alpha u] \cap Z(\omega) = \emptyset \right\}.$$

The quantity $\overline{V}_s(Z)$ is a further simple parameter which, besides volume and surface area density, can be used for the description of a Boolean model. (Here, volume and surface area density refer to the underlying particle process X; a connection with corresponding parameters of the union set Z_X will be established later in Section 9.1.)

First we observe that also the visible domain $S_0(Z)$ can itself be averaged in a natural way, namely by averaging its radial function $\rho(S_0(Z), \cdot)$. For $u \in S^{d-1}$,

$$s_u(Z) := \rho(S_0(Z), u) = \sup\{\alpha \geq 0 : [0, \alpha u] \cap Z = \emptyset\}$$

defines the **visibility range** from 0 in direction u. For $r \geq 0$, we have

$$\mathbb{P}(s_u(Z) \leq r \mid 0 \notin Z) = H_l^{(u)}(r) = 1 - \mathrm{e}^{-r\overline{V}_{d-1}(X|u^\perp)},$$

as will be proved in Theorem 9.1.1. Thus, the visibility range $s_u(Z)$ has (under the condition $0 \notin Z$) an exponential distribution with parameter $\overline{V}_{d-1}(X|u^\perp)$ (which is positive, since X was assumed to be nondegenerate). Therefore, the kth moment of the visibility range $s_u(Z)$ is equal to $k!\overline{V}_{d-1}(X|u^\perp)^{-k}$; in particular, the expectation is $\overline{V}_{d-1}(X|u^\perp)^{-1}$. We define the **mean visible region** $\overline{K}_s$ outside Z as the star-shaped set with radial function

$$\rho(\overline{K}_s, \cdot) = \mathbb{E}(\rho(S_0(Z), \cdot) \mid 0 \notin Z),$$

thus

$$\overline{K}_s = \{\alpha u : u \in S^{d-1}, 0 \leq \alpha \leq \mathbb{E}(s_u(Z) \mid 0 \notin Z)\}.$$

Because of

$$\rho(\overline{K}_s, u) = \overline{V}_{d-1}(X|u^\perp)^{-1} = h(\Pi_X, u)^{-1}$$

for $u \in S^{d-1}$, the set $\overline{K}_s$ is the polar body of the associated zonoid, which in the following will be denoted by Π_X^o. In particular, it follows that the mean visible region is convex.

The volume of the visible region $S_0(Z)$ is given by

$$V_d(S_0(Z)) = \frac{1}{d} \int_{S^{d-1}} s_u(Z)^d \, \sigma(\mathrm{d}u),$$

where σ denotes spherical Lebesgue measure. Therefore, for the mean visible volume outside Z we obtain

$$\overline{V}_s(Z) = \mathbb{E}(\lambda(S_0(Z)) \mid 0 \notin Z)$$

$$= \frac{1}{d} \int_{S^{d-1}} \mathbb{E}(s_u(Z)^d \mid 0 \notin Z) \, \sigma(\mathrm{d}u)$$

$$= (d-1)! \int_{S^{d-1}} \overline{V}_{d-1}(X|u^\perp)^{-d} \, \sigma(\mathrm{d}u)$$

$$= d! V_d(\Pi_X^o).$$

We resume this as a theorem.

Theorem 4.6.1. *Let $Z = Z_X$ be a nondegenerate stationary Boolean model with convex grains in $\mathbb{R}^d$. The mean visible region outside Z is the polar body Π_X^o of the associated zonoid of X; the mean visible volume outside Z is given by*

$$\overline{V}_s(Z) = (d-1)! \int_{S^{d-1}} \overline{V}_{d-1}(X|u^\perp)^{-d} \, \sigma(\mathrm{d}u) = d! V_d(\Pi_X^o). \qquad (4.51)$$

We are now in a position to establish a few sharp inequalities between different parameters of the Boolean model Z_X, respectively of the corresponding particle process X. From (4.48) and (14.43) we obtain the inequality

$$\overline{V}_s(Z) \geq d! \kappa_d \left(\frac{\kappa_{d-1}}{d\kappa_d} 2\overline{V}_{d-1}(X) \right)^{-d}. \qquad (4.52)$$

Here, equality holds if and only if the associated zonoid Π_X is a ball. This occurs, for instance, if the density $\overline{S}_{d-1}(X, \cdot)$ of the surface area measure is rotation invariant (and, hence, the Blaschke body is a ball). Therefore, we can formulate the following result.

Theorem 4.6.2. *Let $Z = Z_X$ be a nondegenerate stationary Boolean model with convex grains in $\mathbb{R}^d$, generated by a Poisson particle process X with given surface area density. The mean visible volume outside Z_X is minimal if the process is isotropic.*

This raises the question whether there also exists an upper estimate of the mean visible volume $\overline{V}_s(Z)$ in terms of a functional density of X. In terms of the surface area density this is not possible, as can be shown by examples. A suitable functional for such an estimate is given by $V_d^{1-1/d}$ (which is of the same degree of homogeneity as the surface area). For this, we employ the Blaschke body $B(X)$. Applying successively (14.23), (4.45), (14.23), (14.30) and making use of mixed volumes (see Section 14.2) we obtain

$$V_d(B(X)) = \frac{1}{d} \int_{S^{d-1}} h(B(X), u) \, S_{d-1}(B(X), \mathrm{d}u)$$

$$= \frac{\gamma}{d} \int_{\mathcal{K}_0} \int_{S^{d-1}} h(B(X), u) \, S_{d-1}(K, \mathrm{d}u) \, \mathbb{Q}(\mathrm{d}K)$$

$$= \gamma \int_{\mathcal{K}_0} V(B(X), K, \ldots, K) \, \mathbb{Q}(\mathrm{d}K)$$

$$\geq V_d(B(X))^{1/d} \gamma \int_{\mathcal{K}_0} V_d(K)^{1-1/d} \, \mathbb{Q}(\mathrm{d}K),$$

hence

$$V_d(B(X))^{1-1/d} \geq \overline{V_d^{1-1/d}}(X).$$

From (4.51), (4.46), (14.44) we now obtain, as a counterpart to (4.52), the inequality

$$\overline{V}_s(Z) \leq d! \left(\frac{\kappa_{d-1}}{\kappa_d} \overline{V_d^{1-1/d}}(X) \right)^{-d}. \tag{4.53}$$

Here, equality holds if and only if the grain distribution $\mathbb{Q}$ is concentrated on a set of homothetic ellipsoids. This follows from the available information about the equality cases in the inequalities (14.30) and (14.44).

As a further parameter for a geometric description of a Poisson particle process X we introduce the intersection density of the boundaries. For a bounded Borel set $B \subset \mathbb{R}^d$, let $s(X, B)$ be the number of points in B arising as intersection points of the boundaries of any d distinct bodies of the process. The **intersection density** of X is the number $\gamma_d(X)$ satisfying

$$\mathbb{E}\, s(X, B) = \gamma_d(X)\lambda(B)$$

for all bounded Borel sets B. In order to show its existence and to compute it, we use Theorem 3.1.3, Corollary 3.2.4, and Theorem 4.1.1 and obtain

$$\mathbb{E}\, s(X, B) = \frac{1}{d!} \mathbb{E} \sum_{(K_1, \ldots, K_d) \in X_{\neq}^d} \operatorname{card}\left(B \cap \operatorname{bd} K_1 \cap \ldots \cap \operatorname{bd} K_d\right)$$

$$= \frac{1}{d!} \int_{\mathcal{K}^d} \operatorname{card}\left(B \cap \operatorname{bd} K_1 \cap \ldots \cap \operatorname{bd} K_d\right) \Lambda^{(d)}(\mathrm{d}(K_1, \ldots, K_d))$$

$$= \frac{\gamma^d}{d!} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} I(K_1, \ldots, K_d) \, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_d)$$

with

$$I(K_1, \ldots, K_d) :=$$

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \operatorname{card}\left(B \cap \operatorname{bd}\left(K_1 + x_1\right) \cap \ldots \cap \operatorname{bd}\left(K_d + x_d\right)\right) \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_d).$$

We abbreviate $B \cap \operatorname{bd}\left(K_1 + x\right) =: F_x$ and $\operatorname{bd} K_i =: F_i$ for $i = 2, \ldots, d$. Using the body Π_{F_x} given by (5.31) and applying Theorem 5.4.4, (14.21) and Theorem 5.2.1, we get

$$I(K_1, \ldots, K_d)$$

$$= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \operatorname{card} \left(F_x \cap (F_2 + x_2) \cap \ldots \cap (F_d + x_d) \right) \lambda(\mathrm{d}x_2) \cdots \lambda(\mathrm{d}x_d) \, \lambda(\mathrm{d}x)$$

$$= d! \int_{\mathbb{R}^d} V(\Pi_{F_x}, \Pi_{K_2}, \ldots, \Pi_{K_d}) \, \lambda(\mathrm{d}x)$$

$$= (d-1)! \int_{\mathbb{R}^d} \int_{S^{d-1}} h(\Pi_{F_x}, u) \, S(\Pi_{K_2}, \ldots, \Pi_{K_d}, \mathrm{d}u) \, \lambda(\mathrm{d}x)$$

$$= (d-1)! \int_{S^{d-1}} \int_{\mathbb{R}^d} h(\Pi_{F_x}, u) \, \lambda(\mathrm{d}x) \, S(\Pi_{K_2}, \ldots, \Pi_{K_d}, \mathrm{d}u)$$

$$= (d-1)! \int_{S^{d-1}} h(\Pi_{K_1}, u) \, S(\Pi_{K_2}, \ldots, \Pi_{K_d}, \mathrm{d}u) \, \lambda(B)$$

$$= d! V(\Pi_{K_1}, \Pi_{K_2}, \ldots, \Pi_{K_d}) \lambda(B).$$

Thus, we obtain

$$\mathbb{E}\, s(X, B) = \gamma^d \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} V(\Pi_{K_1}, \ldots, \Pi_{K_d}) \, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_d) \, \lambda(B).$$

Here we have, by (14.21) and (4.47),

$$\gamma \int_{\mathcal{K}_0} V(\Pi_{K_1}, \ldots, \Pi_{K_d}) \, \mathbb{Q}(\mathrm{d}K_1)$$

$$= \frac{\gamma}{d} \int_{\mathcal{K}_0} \int_{S^{d-1}} h(\Pi_{K_1}, u) \, S(\Pi_{K_2}, \ldots, \Pi_{K_d}, \mathrm{d}u) \, \mathbb{Q}(\mathrm{d}K_1)$$

$$= \frac{1}{d} \int_{S^{d-1}} h(\Pi_X, u) \, S(\Pi_{K_2}, \ldots, \Pi_{K_d}, \mathrm{d}u)$$

$$= V(\Pi_X, \Pi_{K_2}, \ldots, \Pi_{K_d}).$$

Repeating this procedure, we finally get

$$\mathbb{E}\, s(X, B) = V_d(\Pi_X) \lambda(B),$$

and thus the existence of the intersection density, together with the representation

$$\gamma_d(X) = V_d(\Pi_X). \tag{4.54}$$

From (4.54), (4.48) and (14.31) we obtain *a sharp inequality between the intersection density and the surface area density*, namely

$$\gamma_d(X) \leq \kappa_d \left(\frac{2\kappa_{d-1}}{d\kappa_d} \overline{V}_{d-1}(X) \right)^d. \tag{4.55}$$

Here equality holds if and only if the associated zonoid Π_X is a ball, which occurs, for example, if the Poisson particle process X is isotropic.

It is intuitively plausible that a large intersection density indicates that much overlapping of particles occurs and that, therefore, the mean visible volume must be small. This intuition is indeed precise, in so far as the product $\gamma_d(X)\overline{V}_s(X)$ does not depend on the intensity of the process. For this quantity, we are able to establish sharp inequalities.

Theorem 4.6.3. *Let $Z = Z_X$ be a nondegenerate stationary Boolean model with convex grains in $\mathbb{R}^d$. The intersection density and the mean visible volume satisfy the inequalities*

$$4^d \leq \gamma_d(X)\overline{V}_s(Z) \leq d!\kappa_d^2. \tag{4.56}$$

On the right side, equality holds if the process X is isotropic. On the left side, equality holds if and only if the particles of X are almost surely parallelepipeds with edges of d fixed directions.

Proof. The inequalities follow from (4.51), (4.54), and (14.45). On the right side, equality holds if and only if the associated zonoid Π_X is an ellipsoid, thus in particular if the process is isotropic. On the left side, equality holds if and only if Π_X is a parallelepiped. This is equivalent to the existence of d linearly independent vectors $v_1, \ldots, v_d \in S^{d-1}$ such that the measure $\overline{S}_{d-1}(X, \cdot)$ is concentrated on $\{\pm v_i : i = 1, \ldots, d\}$. By (4.43), this holds if and only if for $\mathbb{Q}$-almost all $K \in \mathcal{K}_0$ the measure $S_{d-1}(K, \cdot)$ is concentrated on $\{\pm v_i : i = 1, \ldots, d\}$, hence if K is a parallelepiped with facet normal vectors $\pm v_i$. We conclude that equality in the left inequality of (4.56) holds if and only if the particles of X are almost surely parallelepipeds whose facet normals are parallel to d fixed directions. The facet normals determine also the directions of the edges. $\square$

Processes of Flats

We turn now to processes of flats and want to show how associated zonoids can be utilized for them. We describe a general construction, which is not only applicable to flat processes, but also, for instance, to fiber and surface processes. We begin with a finite Borel measure τ on the space $G(d, k)$ of k-dimensional linear subspaces of $\mathbb{R}^d$, $k \in \{1, \ldots, d-1\}$. There exists a convex body $\Pi^k(\tau)$ with support function

$$h(\Pi^k(\tau), \cdot) = \frac{1}{2} \int_{G(d,k)} h(L^\perp \cap B^d, \cdot) \, \tau(\mathrm{d}L). \tag{4.57}$$

That this is indeed a support function, is clear, since the integrand is a support function. Since $L^\perp \cap B^d$ is a ball (of dimension $d-k$) and thus a zonoid, $\Pi^k(\tau)$ is a zonoid, too. We have $h(L^\perp \cap B^d, u) = [L^\perp, u^\perp] = [L, L(u)]$ for $u \in S^{d-1}$, hence also

$$h(\Pi^k(\tau), u) = \frac{1}{2} \int_{G(d,k)} [L, L(u)] \, \tau(\mathrm{d}L), \qquad u \in S^{d-1}. \qquad (4.58)$$

Now we consider, first, a stationary hyperplane process X in $\mathbb{R}^d$. Let $\gamma > 0$ be its intensity and $\mathbb{Q}$ its directional distribution. We put

$$\Pi_X := \Pi^{d-1}(\gamma \mathbb{Q})$$

and call Π_X the **associated zonoid** of the hyperplane process X. By (4.58), we have

$$h(\Pi_X, u) = \frac{\gamma}{2} \int_{S^{d-1}} |\langle u, v \rangle| \, \varphi(\mathrm{d}v), \qquad u \in \mathbb{R}^d, \qquad (4.59)$$

where φ is the spherical directional distribution of X. According to Theorem 14.3.4, Π_X determines the measure $\gamma \varphi$ uniquely, hence also the intensity γ and the spherical directional distribution φ of X are uniquely determined by Π_X. In particular, a Poisson process X is isotropic if and only if Π_X is a ball. From Theorem 4.4.1 we get the following.

Theorem 4.6.4. *For every centered zonoid $Z \subset \mathbb{R}^d$ there is up to equivalence precisely one stationary Poisson hyperplane process X with associated zonoid Z.*

The support function of the associated zonoid is again connected with intersection densities. As in Section 4.4, let $\gamma_{X \cap L(u)}$ denote the intensity of the point process $X \cap L(u)$. By (4.59) and (4.34) we have

$$2h(\Pi_X, u) = \|u\| \gamma_{X \cap L(u)} = \mathbb{E}X(\mathcal{F}_{[0,u]}) \qquad \text{for } u \in \mathbb{R}^d. \qquad (4.60)$$

From (4.60) we see immediately how to obtain the associated zonoid of a section process. For an r-dimensional linear subspace $S \in G(d,r)$ with $r \in \{1, \ldots, d-1\}$, let $X \cap S$ be the section process (see Section 4.4). Its associated zonoid $\Pi_{X \cap S}$ is defined as a convex body in S. For $u \in S$ we have, by (4.60),

$$2h(\Pi_{X \cap S}, u) = \mathbb{E}(X \cap S)(\mathcal{F}_{[0,u]}) = \mathbb{E}X(\mathcal{F}_{[0,u]}) = 2h(\Pi_X, u).$$

For the orthogonal projection $\Pi_X | S$, we have $h(\Pi_X | S, u) = h(\Pi_X, u)$ for $u \in S$, hence

$$\Pi_{X \cap S} = \Pi_X | S. \qquad (4.61)$$

This means *that the associated zonoid of the section process $X \cap S$ is the orthogonal projection of the associated zonoid of X to the linear subspace S.*

Now we assume, in particular, that X is a stationary Poisson hyperplane process with intensity $\gamma > 0$. With the aid of the associated zonoid, we can obtain information on the intersection processes of X of higher order. In Section 4.4 we have defined, for $k \in \{2, \ldots, d\}$, the intersection process of order k of X, as the $(d-k)$-flat process X_k that is obtained if one takes the intersection

of any k hyperplanes of the process X which are in general position. Similar intersection processes can be formed more generally for stationary surface processes. As a different special case, we have previously considered the intersection point density γ_d of the boundary hypersurfaces of a stationary process of convex particles. Now, for the stationary Poisson hyperplane process X, let γ_k be the intensity and $\mathbb{Q}_k$ the directional distribution of the intersection process X_k. By Theorem 4.4.8 we then have, for $A \in \mathcal{B}(G(d, d - k))$,

$$\gamma_k \mathbb{Q}_k(A)$$
$$= \frac{\gamma^k}{k!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \mathbf{1}_A \left(u_1^\perp \cap \ldots \cap u_k^\perp \right) \nabla_k(u_1, \ldots, u_k) \, \varphi(\mathrm{d}u_1) \cdots \varphi(\mathrm{d}u_k),$$

where φ is the spherical directional distribution of X. The associated zonoid Π_X of X is given by

$$h(\Pi_X, u) = \int_{S^{d-1}} |\langle u, v \rangle| \, \rho(\mathrm{d}v) \qquad \text{for } u \in \mathbb{R}^d,$$

where $\rho := \gamma \varphi / 2$. Hence, if $\rho_{(k)}$ is the kth projection generating measure of Π_X, as defined by (14.36), then, for $A \in \mathcal{B}(G(d, d - k))$,

$$\gamma_k \mathbb{Q}_k(A) = \kappa_k \int_{G(d,k)} \mathbf{1}_A(L^\perp) \, \rho_{(k)}(\mathrm{d}L) = \kappa_k \rho_{(k)}^\perp(A),$$

where $\rho_{(k)}^\perp$ is the image measure of $\rho_{(k)}$ under the mapping $L \mapsto L^\perp$ from $G(d, k)$ to $G(d, d - k)$. Therefore, we have

$$\gamma_k \mathbb{Q}_k = \kappa_k \rho_{(k)}^\perp, \tag{4.62}$$

saying that *the intensity measure of the intersection process of order k of X is determined by the kth projection generating measure of the associated zonoid Π_X.*

The intensity γ_k of the intersection process of order k of X is called the kth **intersection density** of X. Here, $\gamma_1 = \gamma$. By the definition of the intersection processes and by Theorem 4.4.3, the kth intersection density is given by

$$\gamma_k = \frac{1}{d! \kappa_d} \mathbb{E} \sum_{(H_1, \ldots, H_k) \in X^k} \lambda_{d-k}^*(H_1 \cap \ldots \cap H_k \cap B^d).$$

Here $\lambda_{d-k}^*(A)$ is the $(d - k)$-dimensional volume of A if $\dim A = d - k$, and is zero otherwise. For the intersection densities, we can obtain inequalities. They are based on the fact that $\gamma_k = \kappa_k \rho_{(k)}(G(d, k))$ by (4.62) and hence, by (14.37),

$$\gamma_k = V_k(\Pi_X); \tag{4.63}$$

thus, *the kth intersection density is the kth intrinsic volume of the associated zonoid.* In particular, the dth intersection density, which is the density of the

intersection points generated by X, is nothing but the volume of the associated zonoid. An analog of this fact is (4.54); both equations are special cases of a corresponding result for general stationary hypersurface processes (with the k-volume density instead of γ_k). Also (4.55) and (4.56) can be generalized in this sense.

Now, from (14.31) we obtain the inequality

$$\left(\frac{\kappa_{d-j}}{\binom{d}{j}}\gamma_j\right)^k \geq \kappa_d^{k-j}\left(\frac{\kappa_{d-k}}{\binom{d}{k}}\gamma_k\right)^j \tag{4.64}$$

for $1 \leq j < k \leq d$. If $\gamma_j > 0$, then equality in (4.64) holds if and only if Π_X is a ball. By the uniqueness theorem 14.3.4, this holds if and only if the spherical directional distribution φ of X is the normalized spherical Lebesgue measure, hence, if and only if the Poisson hyperplane process X is isotropic. The case $\gamma_j = 0$ occurs, by (4.63), if and only if $\dim \Pi_X < j$, hence if and only if the spherical directional distribution φ is concentrated on $S^{d-1} \cap L$ for some subspace $L \in G(d, j-1)$. An equivalent condition is that the hyperplanes of the process almost surely contain a translate of the $(d+1-j)$-dimensional plane $L^\perp$.

We formulate the special case $j = 1$ as a theorem.

Theorem 4.6.5. *The kth intersection density, $k \in \{2, \ldots, d\}$, of a stationary Poisson hyperplane process of intensity $\gamma > 0$ in $\mathbb{R}^d$ satisfies the inequality*

$$\gamma_k \leq \frac{\binom{d}{k}\kappa_{d-1}^k}{d^k \kappa_{d-k}\kappa_d^{k-1}}\gamma^k.$$

Equality holds if and only if the process is isotropic.

Thus, the isotropic processes are characterized here by an extremal property of isoperimetric type: for given intensity, they have maximal intersection densities.

If X_1 and X_2 are independent stationary Poisson hyperplane processes, then their superposition $X_1 + X_2$ is also a stationary Poisson process, with intensity measure $\Theta_1 + \Theta_2$, if Θ_i is the intensity measure of X_i. It follows that the associated zonoids also add:

$$\Pi_{X_1+X_2} = \Pi_{X_1} + \Pi_{X_2}.$$

If $\gamma_k(X)$ denotes the kth intersection density of X, then (4.63) and (14.32) yield the inequality

$$\gamma_k(X_1 + X_2)^{1/k} \geq \gamma_k(X_1)^{1/k} + \gamma_k(X_2)^{1/k}, \tag{4.65}$$

for $k = 2, \ldots, d$. Equality in (4.65) holds at least if the hyperplane processes X_1 and X_2 have the same directional distribution, since then their associated zonoids are homothetic.

We can also derive a sharp estimate for the proximity (defined before Theorem 4.4.10) of a Poisson line process.

Theorem 4.6.6. *Let X be a stationary Poisson line process of given intensity $\gamma > 0$. The proximity $\pi(X)$ attains its maximum if and only if X is isotropic.*

Proof. This follows from the remark after Theorem 4.4.10 and the case $k = 2$ of Theorem 4.6.5.   □

For a k-flat process X and a fixed $(d - k)$-plane S, we have considered in Section 4.4 the section process $X \cap S$. There it was mentioned that a stationary Poisson k-flat process is uniquely determined, up to stochastic equivalence, by its intersection densities $\gamma_{X \cap S}$, $S \in G(d, d - k)$, if either $k = 1$ or $k = d - 1$, but not in the cases $1 < k < d - 1$. However, for kth-order intersection processes of stationary Poisson hyperplane processes, there exists a corresponding uniqueness result. An even stronger assertion is expressed by the subsequent theorem. Here it has to be observed that

$$\gamma_{X_k \cap S} = \gamma_k(X \cap S).$$

A stationary hyperplane process is **nondegenerate** if the hyperplanes of the process are not almost surely parallel to a fixed line.

Theorem 4.6.7. *Let X be a nondegenerate stationary Poisson hyperplane process of intensity γ in $\mathbb{R}^d$, let $r \in \{1, \ldots, d-1\}$ and $k \in \{1, \ldots, r\}$. Then X is uniquely determined (up to stochastic equivalence) by the kth intersection densities $\gamma_k(X \cap S)$ of the section processes $X \cap S$, $S \in G(d, r)$.*

Proof. By (4.63) and (4.61) we have

$$\gamma_k(X \cap S) = V_k(\Pi_{X \cap S}) = V_k(\Pi_X | S)$$

for $S \in G(d, r)$. Since X is nondegenerate, $\dim \Pi_X \geq d$, as was remarked earlier. By a theorem from convex geometry (Aleksandrov's Projection Theorem; see Gardner [244, Theorem 3.3.6]), the convex body Π_X, which is centrally symmetric with respect to 0, is uniquely determined by the intrinsic volumes $V_k(\Pi_X | S)$, $S \in G(d, r)$. Now the assertion follows from Theorem 4.6.4.   □

Flat Processes Hitting Convex Bodies

Now we consider more general flat processes. Let X be a stationary k-flat process of intensity $\gamma > 0$ in $\mathbb{R}^d$. We suppose that a convex 'test body' $K \in \mathcal{K}'$ is hit by the flats of the process, and we want to measure in different ways how intensively it is hit. We could, for example, be interested in deciding which shape a convex body of given volume must have so that in the mean it is hit by as few flats as possible. This depends on how the intensity of hitting is measured. For instance, if we use the k-dimensional volume of the intersections as a measure, then Theorem 4.4.3 gives the answer

$$\mathbb{E} \sum_{E \in X} V_k(K \cap E) = \gamma V_d(K),$$

saying that the left side is independent of the shape of K. Instead of the k-dimensional volume of the intersections, we could ask for the number of nonempty intersections or, more generally, using the jth intrinsic volume V_j, $j \in \{0, \dots, k\}$, ask for

$$\mathbb{E} \sum_{E \in X} V_j(K \cap E). \tag{4.66}$$

The number of nonempty intersections is included here, for $j = 0$. If X is in addition isotropic, then Theorem 9.4.8, to be proved later, gives the result

$$\mathbb{E} \sum_{E \in X} V_j(K \cap E) = \gamma c_{j,d}^{k,d-k+j} V_{d+j-k}(K),$$

with certain constants $c_{j,d}^{k,d-k+j}$. For $j < k$ and given positive volume, the functional $V_{d+j-k}(K)$ attains its minimum if and only if K is a ball (cf. (14.31)). Here the assumption of isotropy cannot be deleted; without it, the quantity (4.66) will not only depend on the intrinsic volume $V_{d+j-k}(K)$, but the shape of K will play an essential role. To see this, at least in some special cases, we first compute the expectation (4.66). Let Θ be the intensity measure and $\mathbb{Q}$ the directional distribution of X. From the Campbell theorem and Theorem 4.4.2 we get

$$\mathbb{E} \sum_{E \in X} V_j(K \cap E) = \int_{A(d,k)} V_j(K \cap E)\,\Theta(\mathrm{d}E)$$

$$= \gamma \int_{G(d,k)} \int_{L^\perp} V_j(K \cap (L+x))\,\lambda_{L^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}L).$$

Using the integral geometric formula (6.39), we obtain

$$\mathbb{E} \sum_{E \in X} V_j(K \cap E)$$

$$= \frac{\binom{d}{k-j}}{\kappa_{k-j}} \gamma \int_{G(d,k)} V(K[d+j-k], (L \cap B^d)[k-j])\,\mathbb{Q}(\mathrm{d}L), \tag{4.67}$$

where the integrand is a mixed volume. For $j = 0$, the integral geometric formula is not needed, and one obtains directly

$$\mathbb{E} \sum_{E \in X} V_0(K \cap E) = \gamma \int_{G(d,k)} V_{d-k}(K|L^\perp)\,\mathbb{Q}(\mathrm{d}L). \tag{4.68}$$

A further treatment of the integral (4.67) has only been successful in special cases. First we consider the case $j = k - 1$, that is, the surface area of the k-dimensional sections $K \cap E$. If $S_{d-1}(K, \cdot)$ denotes the surface area measure of K, then formula (14.23) for mixed volumes and (4.57) give

$$\int_{G(d,k)} V(K, \ldots, K, L \cap B^d) \, \mathbb{Q}(\mathrm{d}L)$$

$$= \frac{1}{d} \int_{G(d,k)} \int_{S^{d-1}} h(L \cap B^d, u) \, S_{d-1}(K, \mathrm{d}u) \, \mathbb{Q}(\mathrm{d}L)$$

$$= \frac{1}{d} \int_{S^{d-1}} \int_{G(d,k)} h(L \cap B^d, u) \, \mathbb{Q}(\mathrm{d}L) \, S_{d-1}(K, \mathrm{d}u).$$

We define a zonoid $\Pi_k(\mathbb{Q})$ by

$$h(\Pi_k(\mathbb{Q}), u) = \frac{1}{2} \int_{G(d,k)} h(L \cap B^d, u) \, \mathbb{Q}(\mathrm{d}L).$$

Thus,

$$\Pi_k(\mathbb{Q}) = \Pi^{d-k}(\mathbb{Q}^\perp),$$

where $\mathbb{Q}^\perp$ is the image measure of $\mathbb{Q}$ under the mapping $L \mapsto L^\perp$ from $G(d, k)$ to $G(d, d-k)$. Then we put

$$\Pi^X := \gamma \Pi_k(\mathbb{Q}).$$

We obtain

$$\int_{G(d,k)} V(K, \ldots, K, L \cap B^d) \, \mathbb{Q}(\mathrm{d}L)$$

$$= \int_{G(d,k)} \frac{1}{d} \int_{S^{d-1}} h(L \cap B^d, u) \, S_{d-1}(K, \mathrm{d}u) \, \mathbb{Q}(\mathrm{d}L)$$

$$= \frac{2}{d} \int_{S^{d-1}} h(\Pi_k(\mathbb{Q}), u) \, S_{d-1}(K, \mathrm{d}u)$$

$$= 2V(\Pi_k(\mathbb{Q}), K, \ldots, K),$$

hence

$$\mathbb{E} \sum_{E \in X} V_{k-1}(K \cap E) = dV(\Pi^X, K, \ldots, K).$$

From Minkowski's inequality (14.30), we now deduce the following extremal property.

Theorem 4.6.8. *Let X be a stationary k-flat process of intensity $\gamma > 0$ in $\mathbb{R}^d$, and let K be a convex body of given positive volume. Then the expected value*

$$\mathbb{E} \sum_{E \in X} V_{k-1}(K \cap E)$$

is minimal if and only if K is homothetic to the zonoid Π^X.

In the case of a line process ($k = 1$), the quantity $\mathbb{E} \sum_{E \in X} V_{k-1}(K \cap E)$ is just the expected number of lines hitting the body K.

Further information on the expected number of hitting k-planes is available in the case $k = d - 1$. Let X be a stationary hyperplane process of intensity $\gamma > 0$ and with spherical directional distribution φ. We assume that X is nondegenerate, so that the hyperplanes of the process are not almost surely parallel to a fixed line. Under this assumption, the measure φ is not concentrated on a great subsphere, and it follows that $V_d(\Pi_X) > 0$. Since φ is an even measure, it follows from Theorem 14.3.1 that there exists a uniquely determined convex body $B(X)$, centrally symmetric with respect to 0, for which

$$S_{d-1}(B(X), \cdot) = \gamma \varphi.$$

We call $B(X)$ the **Blaschke body** of the hyperplane process X. Thus, by (4.59) we have

$$\Pi_X = \Pi_{B(X)},$$

in analogy to (4.46).

Now from (4.68) and (14.23), we obtain

$$\mathbb{E} \sum_{E \in X} V_0(K \cap E) = \gamma \int_{G(d,d-1)} V_1(K|L^{\perp}) \, \mathbb{Q}(\mathrm{d}L)$$

$$= \gamma \int_{S^{d-1}} [h(K, u) + h(K, -u)] \, \varphi(\mathrm{d}u)$$

$$= 2 \int_{S^{d-1}} h(K, u) \, S_{d-1}(B(X), \mathrm{d}u)$$

$$= 2dV(K, B(X), \ldots, B(X)). \tag{4.69}$$

Again, we can apply Minkowski's inequality (14.30), and deduce the following result.

Theorem 4.6.9. *Let X be a nondegenerate stationary hyperplane process of intensity $\gamma > 0$ in $\mathbb{R}^d$, and let $K \in \mathcal{K}$ be a convex body with given volume $V_d(K) > 0$. The expected number of hyperplanes of the process X hitting the convex body K is minimal if and only if K is homothetic to the Blaschke body $B(X)$ of X.*

Notes for Section 4.6

1. The associated zonoid of a stationary process X of convex particles was introduced here as the projection body of the Blaschke body of X; this is equivalent to the original definition. The construction of the Blaschke body requires Minkowski's existence theorem (Theorem 14.3.1). This existence theorem was first used in Schneider [686] for associating a convex body with a directional distribution (of finitely many

random hyperplanes in that case) and then applying results from convex geometry. Extensive use of this association in the case of random hypersurfaces was made by Wieacker [817, 818]. The Blaschke body $B(X)$ of a particle process X was introduced in Weil [796]; that paper also provides information on the mean body $M(X)$. In Weil [797], Blaschke bodies were also suggested and investigated for random closed sets with values in the extended convex ring. For Boolean models, for example, this paper established a connection between the Blaschke body and the contact distribution function.

2. The associated zonoid of a stationary Poisson hyperplane process was introduced by Matheron [461, 462], under the name of 'Steiner compact set'. The book by Matheron [462] has already the formulas (4.61) and (4.63), and in principle also (4.62). Associated zonoids for random hyperplanes were also used in Schneider [685, 686]. Generalization and systematic application of associated zonoids then followed in the work of Wieacker [816, 817, 818]; see also Sections 6 and 7 in the survey article of Weil and Wieacker [806] and Section 6 of Schneider and Wieacker [720]. Wieacker has introduced different types of associated zonoids, and he has applied them to random surfaces, surface processes, k-flat processes, particle processes, and random mosaics. In Wieacker [817] one finds, for example, the assertions (4.48), (4.49), Theorem 4.6.1 (essentially), (4.52), (4.54), (4.55), (4.56) (the right-hand inequality), and a generalization of Theorem 4.6.5. For extensions and supplements (such as (4.56) (left side) and (4.53), see Schneider [689]; that paper treats Poisson processes of convex cylinders, which includes as special cases flat processes and processes of convex particles. Compare also inequality (10.52) and Note 4 for Section 10.4.

3. Inequality (4.64) and with it Theorem 4.6.5 are due to Thomas [756]. Similar arguments, but with different interpretations, appear in Schneider [686, 697]. Alternative proofs for special cases were found by Mecke [480, 484].

Theorem 4.6.5 raises the question which stationary Poisson k-flat processes of given (positive) intensity, where $d/2 \leq k < d-1$, have maximal second intersection density. According to Theorem 4.4.9, this amounts to finding the maximum of the integral

$$\int_{G(d,k)} \int_{G(d,k)} [E, F] \, \mathbb{Q}(\mathrm{d}E) \, \mathbb{Q}(\mathrm{d}F)$$

over all probability measures $\mathbb{Q}$ on $G(d,k)$. For $k = d-1$, the maximum is attained precisely by the rotation invariant probability measures, by Theorem 4.6.5. For $k < d-1$, however, the maximum is not attained by invariant measures, as was discovered by Mecke and Thomas [504] (see also Mecke [489]). Mecke [486, 487] was able to determine explicitly the extremal measures for $d = 2k$. Keutel [401] has completely settled the case where $k < d-2$ and $d-k$ divides d. The general case is still open.

Theorem 4.6.6 goes back, in principle, to Janson and Kallenberg [378], though with a different approach.

Theorems 4.6.8 and 4.6.9 are special cases of considerably more general assertions in Wieacker [818]. Theorem 4.6.7 and related results were first published in Schneider and Weil [717].