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Invariant Measures

Integral geometry, as it is used in this book for the treatment of random geometric structures with stationarity or isotropy properties, is based on the notion of an invariant measure. Here invariance refers to a group operation and thus to a homogeneous space. Invariant measures on topological groups and homogeneous spaces are known as Haar measures. The general theory of such measures can be found, for example, in Hewitt and Ross [342] and Nachbin [571]. However, we do not presuppose here any knowledge of the theory of Haar measure (with the exception of Section 13.3, which is rarely used in this book and could be dispensed with). For the topological groups and homogeneous spaces that are relevant for integral geometry in Euclidean spaces, the existence and uniqueness of invariant measures will be proved in Section 13.2 in a direct and elementary way, starting from Lebesgue measure and assuming only basic facts from measure theory.

13.1 Group Operations and Invariant Measures

A **topological group** is a group G together with a topology on G such that the map from $G \times G$ to G defined by $(x, y) \mapsto xy$ (the product of x and y) and the map from G to G defined by $x \mapsto x^{-1}$ are continuous. The topologies of all topological groups occurring in this book are assumed to be locally compact and second countable.

Let G be a group and E a nonempty set. An **operation** of G on E is a map $\varphi : G \times E \to E$ satisfying

$$\varphi(g, \varphi(g', x)) = \varphi(gg', x), \qquad \varphi(e, x) = x$$

for all $g, g' \in G$, the unit element e of G, and all $x \in E$. One also says that G **operates on** E, by means of φ. For $\varphi(g, x)$ one usually writes gx, provided that the operation is clear from the context. The group G **operates transitively** on E if for any $x, y \in E$ there exists $g \in G$ such that $y = gx$.

If G is a topological group, E is a topological space, and the operation φ is continuous, one says that G **operates continuously** on E.

The following situation often occurs: E is a nonempty set and G is a group of transformations (bijective mappings onto itself) of E, with the composition as group multiplication; the operation of G on E is given by $(g, x) \mapsto gx :=$ image of x under g. When transformation groups occur in the following, multiplication and operation are always understood in this sense.

A basic situation considered in integral geometry is the operation of a transformation group on some space together with the induced operation on a space of geometrically significant subsets. Generally, let the space E be as in Section 12.2, that is, locally compact and second countable, and let $\mathcal{F}$ be the space of closed subsets of E. Let G be a topological group operating continuously on E. For $g \in G$ and $F \subset E$, let

$$gF := \{gx : x \in F\}. \tag{13.1}$$

For each $g \in G$, the bijective map $x \mapsto gx$ is continuous, and so is its inverse, thus it is a homeomorphism. It follows that $gF \in \mathcal{F}$ for $F \in \mathcal{F}$. Hence, (13.1) defines an operation of G on the space $\mathcal{F}$. This operation is continuous.

Theorem 13.1.1. *If the topological group G operates continuously on E, then the map*

$$\begin{aligned} G \times \mathcal{F} &\to \mathcal{F} \\ (g, F) &\mapsto gF \end{aligned}$$

is continuous.

Proof. Suppose that $(g_i, F_i) \to (g, F)$ in $G \times \mathcal{F}$. We have to show that $g_i F_i \to gF$ in $\mathcal{F}$, and for this we use Theorem 12.2.2.

(α) Let $x \in gF$, thus $x = gy$ with $y \in F$. Since $F_i \to F$ in $\mathcal{F}$, there are $y_i \in F_i$ with $y_i \to y$. For $x_i := g_i y_i$ we have $x_i \in g_i F_i$ and $x_i \to x$.

(β) Let $(g_{i_k} F_{i_k})_{k \in \mathbb{N}}$ be a subsequence and let $x_{i_k} = g_{i_k} y_{i_k}$ with $y_{i_k} \in F_{i_k}$ be such that $x_{i_k} \to x$. Then $y_{i_k} \to y := g^{-1}x$, and $y \in F$, hence $x = gy \in gF$. $\square$

Let G be a topological group. A **homogeneous G-space** is a pair (E, φ) with the following properties: E is a topological space, $\varphi : G \times E \to E$ is a transitive continuous operation of G on E, and for (one and hence for all) $p \in E$, the mapping $\varphi(\cdot, p)$ is open. Up to isomorphism, all homogeneous G-spaces are obtained in the following way. Let H be a subgroup of G (with the trace topology) and let G/H be the factor space, that is, the space $\{aH : a \in G\}$ of left cosets of H in G, equipped with the quotient topology. The map $\pi : G \to G/H$ defined by $\pi(a) := aH$ for $a \in G$ is called the **natural projection**. The quotient topology on G/H is characterized by the properties that π is continuous and open. By

$$\zeta(g, aH) := gaH \qquad \text{for } g \in G,\ aH \in G/H,$$

one obtains a transitive continuous operation ζ of G on G/H; it is called the **natural operation** of G on G/H. The pair $(G/H, \zeta)$ is a homogeneous G-space. Conversely, let (E, φ) be a homogeneous G-space. For an arbitrarily chosen point $p \in E$ let S_p be the **stability group** of p, that is, the subgroup $S_p := \{g \in G : gp = p\}$ (with $gp := \varphi(g, p)$). Then the map

$$\begin{aligned} \beta : G/S_p &\to E \\ gS_p &\mapsto gp \end{aligned}$$

is a homeomorphism from G/S_p onto E with the property that $\beta(gaS_p) = g\beta(aS_p)$ for all $g \in G$ and all $aS_p \in G/S_p$. In this sense, the homogeneous G-spaces (E, φ) and $(G/S_p, \zeta)$ are isomorphic. Hence, if a homogeneous G-space is given, one can always assume that it is of the form G/H with a subgroup H and that the operation is the natural one. The subgroup H is closed if and only if G/H is a Hausdorff space.

We turn to invariant measures. Let the topological group G operate continuously on the topological space E. A Borel measure ρ on E is called G-**invariant** (or briefly **invariant**, if G is clear from the context) if

$$\rho(gA) = \rho(A) \qquad \text{for all } A \in \mathcal{B}(E) \text{ and all } g \in G.$$

This definition makes sense: for each $g \in G$, the mapping $x \mapsto gx$ is a homeomorphism, hence $A \in \mathcal{B}(E)$ implies $gA \in \mathcal{B}(E)$. An invariant regular Borel measure on a locally compact homogeneous space which is not identically zero is called a **Haar measure.**

For a measure on a group, several notions of invariance are natural. A topological group G operates on itself by means of the mapping $(g, x) \mapsto gx$ (multiplication in G) for $(g, x) \in G \times G$. The corresponding invariance of a measure on G is called left invariance. More generally, for $g \in G$ and $A \subset G$ we write

$$\begin{aligned} gA &:= \{ga : a \in A\}, \\ Ag &:= \{ag : a \in A\}, \\ A^{-1} &:= \{a^{-1} : a \in A\}. \end{aligned}$$

If $A \in \mathcal{B}(G)$, then also gA, Ag, A^{-1} are Borel sets, because multiplication from the left or the right and taking the inverse are homeomorphisms. Now let ρ be a measure on G. It is called **left invariant** if $\rho(gA) = \rho(A)$, and **right invariant** if $\rho(Ag) = \rho(A)$, for all $A \in \mathcal{B}(G)$ and all $g \in G$. The measure ρ is **inversion invariant** if $\rho(A^{-1}) = \rho(A)$ for all $A \in \mathcal{B}(G)$. If ρ has all three invariance properties, it is called **invariant**. A left invariant (right invariant, invariant) regular Borel measure which is not identically zero is called a **left Haar measure (right Haar measure, Haar measure)**.

Invariance properties of measures are equivalent to invariance properties of integrals. Let ρ be a regular Borel measure on the topological group G. If ρ is left invariant, then every measurable function $f \geq 0$ on G satisfies

$$\int_G f(ag)\,\rho(\mathrm{d}g) = \int_G f(g)\,\rho(\mathrm{d}g) \tag{13.2}$$

for all $a \in G$. This follows immediately from the definition of the integral. Conversely, if (13.2) holds for all continuous functions $f \geq 0$ with compact support, then the left invariance of ρ is obtained from (12.1), (12.2). Similarly, the right invariance of ρ is equivalent to

$$\int_G f(ga)\,\rho(\mathrm{d}g) = \int_G f(g)\,\rho(\mathrm{d}g)$$

for all $a \in G$, and the inversion invariance of ρ is equivalent to

$$\int_G f(g^{-1})\,\rho(\mathrm{d}g) = \int_G f(g)\,\rho(\mathrm{d}g),$$

in each case for all nonnegative functions $f \in \mathbf{C}_c(G)$.

We prove some uniqueness results for invariant measures. They are only needed for the groups and homogeneous spaces of Euclidean geometry, but without additional effort we can prove them in a more general setting.

Theorem 13.1.2. *Every left Haar measure on a compact group G with a countable base is invariant.*

Proof. Let ν be a left Haar measure on a group G satisfying the assumptions. Since ν is finite on compact sets, we may assume $\nu(G) = 1$, without loss of generality. For a continuous function $f \geq 0$ on G and for $x \in G$ we have

$$\int f(y^{-1}x)\,\nu(\mathrm{d}y) = \int f((x^{-1}y)^{-1})\,\nu(\mathrm{d}y) = \int f(y^{-1})\,\nu(\mathrm{d}y). \tag{13.3}$$

Here the integrations extend over all of G; similar conventions will be adopted in the following. Fubini's theorem gives

$$\int f(y^{-1})\,\nu(\mathrm{d}y) = \int\int f(y^{-1}x)\,\nu(\mathrm{d}y)\,\nu(\mathrm{d}x)$$
$$= \int\int f(y^{-1}x)\,\nu(\mathrm{d}x)\,\nu(\mathrm{d}y) = \int f(x)\,\nu(\mathrm{d}x).$$

Hence, the measure ν is inversion invariant. Using this fact and (13.3), we obtain for $x \in G$ that

$$\int f(yx)\,\nu(\mathrm{d}y) = \int f(y^{-1}x)\,\nu(\mathrm{d}y)$$
$$= \int f(y^{-1})\,\nu(\mathrm{d}y) = \int f(y)\,\nu(\mathrm{d}y),$$

which shows that ν is also right invariant. $\square$

Clearly, in Theorem 13.1.2 the assumption 'left invariant' can be replaced by 'right invariant'.

The following uniqueness result for invariant measures makes special assumptions, but in this form it is sufficient for our purposes and is easy to prove.

Theorem 13.1.3. *Let G be a locally compact group with a countable base, let ν be a Haar measure and μ a left Haar measure on G. Then $\mu = c\nu$ with a constant c.*

Proof. For measurable functions $f, g \geq 0$ on G we have

$$\int f \, \mathrm{d}\nu \int g \, \mathrm{d}\mu = \int \int f(xy)g(y) \, \nu(\mathrm{d}x) \, \mu(\mathrm{d}y)$$

$$= \int \int f(xy)g(y) \, \mu(\mathrm{d}y) \, \nu(\mathrm{d}x) = \int \int f(y)g(x^{-1}y) \, \mu(\mathrm{d}y) \, \nu(\mathrm{d}x)$$

$$= \int f(y) \int g(x^{-1}y) \, \nu(\mathrm{d}x) \, \mu(\mathrm{d}y) = \int g \, \mathrm{d}\nu \int f \, \mathrm{d}\mu.$$

Here we have used, besides Fubini's theorem, the right and inversion invariance of ν and the left invariance of μ.

Since $\nu \neq 0$, there is a compact set $A_0 \subset G$ with $\nu(A_0) > 0$. For arbitrary $A \in \mathcal{B}(G)$ we put $f := \mathbf{1}_{A_0}$ and $g := \mathbf{1}_A$ and obtain $\nu(A_0)\mu(A) = \nu(A)\mu(A_0)$, hence $\mu = c\nu$ with $c := \mu(A_0)/\nu(A_0)$. □

Next, we prove a formula of integral geometric type, generalizing Theorem 5.2.1, which is useful for obtaining uniqueness results. It is slightly more general than needed.

Theorem 13.1.4. *Suppose that the compact group G operates continuously and transitively on the Hausdorff space E and that G and E have countable bases. Let ν be a Haar measure on G with $\nu(G) = 1$.*

Let $\rho \neq 0$ and α be locally finite Borel measures on E, let ρ be G-invariant. Then

$$\int_G \alpha(A \cap gB) \, \nu(\mathrm{d}g) = \alpha(A)\rho(B)/\rho(E)$$

for all $A, B \in \mathcal{B}(E)$.

Proof. If φ denotes the operation of G on E and if $x \in E$, the mapping $\varphi(\cdot, x) : G \to E$ is continuous and surjective, hence E is compact. Therefore, the measures α and ρ are finite. Let $A, B \in \mathcal{B}(E)$. The mapping $(g, x) \mapsto g^{-1}x$ from $G \times E$ to E is continuous and thus measurable, hence the mapping $(g, x) \mapsto \mathbf{1}_B(g^{-1}x)$ is measurable. From

$$\alpha(A \cap gB) = \int_E \mathbf{1}_{A \cap gB} \, \alpha(\mathrm{d}x) = \int_E \mathbf{1}_A(x)\mathbf{1}_B(g^{-1}x) \, \alpha(\mathrm{d}x)$$

it follows that $g \mapsto \alpha(A \cap gB)$ is a measurable mapping. Fubini's theorem yields

$$\int_G \alpha(A \cap gB)\,\nu(\mathrm{d}g) = \int_E \mathbf{1}_A(x) \int_G \mathbf{1}_B(g^{-1}x)\,\nu(\mathrm{d}g)\,\alpha(\mathrm{d}x). \tag{13.4}$$

The integral $\int_G \mathbf{1}_B(g^{-1}x)\,\nu(\mathrm{d}g)$ does not depend on x, since for $y \in E$ there exists $h \in G$ with $y = hx$ and therefore

$$\int_G \mathbf{1}_B(g^{-1}y)\,\nu(\mathrm{d}g) = \int_G \mathbf{1}_B((h^{-1}g)^{-1}x)\,\nu(\mathrm{d}g) = \int_G \mathbf{1}_B(g^{-1}x)\,\nu(\mathrm{d}g).$$

Hence, we obtain

$$\begin{aligned}\rho(E) \int_G \mathbf{1}_B(g^{-1}x)\,\nu(\mathrm{d}g) &= \int_E \int_G \mathbf{1}_B(g^{-1}x)\,\nu(\mathrm{d}g)\,\rho(\mathrm{d}x) \\ &= \int_G \int_E \mathbf{1}_B(g^{-1}x)\,\rho(\mathrm{d}x)\,\nu(\mathrm{d}g) = \int_G \rho(gB)\,\nu(\mathrm{d}g) = \rho(B).\end{aligned}$$

Inserting this in (13.4), we complete the proof. □

Theorem 13.1.5. *Suppose that the compact group G operates continuously and transitively on the Hausdorff space E and that G and E have countable bases. Let ν be a Haar measure on G with $\nu(G) = 1$.*

Then there exists a unique G-invariant Borel measure ρ on E with $\rho(E) = 1$. It can be defined by

$$\rho(B) = \nu(\{g \in G : gx_0 \in B)\}, \qquad B \in \mathcal{B}(E),$$

with arbitrary $x_0 \in E$.

Proof. Let ρ be a G-invariant Borel measure on E with $\rho(E) = 1$. We choose $x_0 \in E$ and let α be the Dirac measure on E concentrated at x_0. Then Theorem 13.1.4 with $A := \{x_0\}$ gives

$$\rho(B) = \nu(\{g \in G : g^{-1}x_0 \in B\})$$

for $B \in \mathcal{B}(E)$. Thus ρ is unique. Conversely, if ρ is defined in this way, it is clear that it is a G-invariant normalized measure. □

Notes for Section 13.1

1. For an extensive treatment of topological groups and homogeneous spaces, we refer to Hewitt and Ross [342], Nachbin [571], Gaal [242]. Information on invariant measures is also found in Bourbaki [119] and Cohn [177].

2. Results in the spirit of Theorem 13.1.4 (which extends Theorem 5.2.1) go back to Balanzat [55]. More general versions and further references are in Groemer [290] and Schneider [683].

13.2 Homogeneous Spaces of Euclidean Geometry

In this section we introduce the transformation groups and homogeneous spaces that occur in Euclidean integral geometry. Our main aim is to construct their Haar measures in an elementary way, presupposing only the knowledge of Lebesgue measure and its properties.

We consider three groups of bijective affine maps of $\mathbb{R}^d$ onto itself, the **translation group** T_d, the **rotation group** SO_d, and the **rigid motion group** G_d. The **translations** $t \in T_d$ are the maps of the form $t = t_x$ with $x \in \mathbb{R}^d$, where $t_x(y) = y + x$ for $y \in \mathbb{R}^d$. The mapping $\tau : x \mapsto t_x$ is an isomorphism of the additive group $\mathbb{R}^d$ onto T_d. Hence, we can identify T_d with $\mathbb{R}^d$, which we shall often do tacitly. In particular, T_d carries the topology inherited from $\mathbb{R}^d$ via τ. Since $t_x \circ t_y = t_{x+y}$ and $t_x^{-1} = t_{-x}$, composition and inversion are continuous, hence T_d is a topological group. In view of the topological properties of $\mathbb{R}^d$ we can thus state the following.

Theorem 13.2.1. *The translation group T_d is an abelian, locally compact topological group with countable base. The operation of T_d on $\mathbb{R}^d$ is continuous.*

The elements of the rotation group SO_d are the linear mappings $\vartheta : \mathbb{R}^d \to \mathbb{R}^d$ that preserve scalar product and orientation; they are called (**proper**) **rotations**. With respect to the standard (orthonormal) basis of $\mathbb{R}^d$, every rotation ϑ is represented by an orthogonal matrix $M(\vartheta)$ with determinant 1. The mapping $\mu : \vartheta \mapsto M(\vartheta)$ is an isomorphism of the group SO_d onto the group $\mathcal{SO}(d)$ of orthogonal (d,d)-matrices with determinant 1 under matrix multiplication. If we identify a (d,d)-matrix with the d^2-tuple of its entries, we can consider $\mathcal{SO}(d)$ as a subset of $\mathbb{R}^{d^2}$ (this identification defines the topology of $\mathcal{SO}(d)$). This set is bounded, since the rows of an orthogonal matrix are normalized, and it is closed in $\mathbb{R}^{d^2}$, hence compact. The mappings $(M,N) \mapsto MN$ from $\mathcal{SO}(d) \times \mathcal{SO}(d)$ to $\mathcal{SO}(d)$ and $M \mapsto M^{-1}$ from $\mathcal{SO}(d)$ to $\mathcal{SO}(d)$ are continuous, and so is the mapping $(M,x) \mapsto Mx$ (where x is considered as a $(d,1)$-matrix) from $\mathcal{SO}(d) \times \mathbb{R}^d$ into $\mathbb{R}^d$. Using the mapping μ^{-1} to transfer the topology from $\mathcal{SO}(d)$ to SO_d, we thus obtain the following.

Theorem 13.2.2. *The rotation group SO_d is a compact topological group with countable base. The operation of SO_d on $\mathbb{R}^d$ is continuous.*

The elements of the motion group G_d are the affine maps $g : \mathbb{R}^d \to \mathbb{R}^d$ that preserve the distances between points and the orientation; they are called **(rigid) motions**. Every rigid motion $g \in G_d$ can be represented uniquely as the composition of a rotation ϑ and a translation t_x, that is, $g = t_x \circ \vartheta$, or $gy = \vartheta y + x$ for $y \in \mathbb{R}^d$. The mapping

$$\begin{aligned} \gamma : \mathbb{R}^d \times SO_d \to &\quad G_d \\ (x, \vartheta) \quad &\mapsto t_x \circ \vartheta \end{aligned} \tag{13.5}$$

is bijective. We employ it to transfer the topology from $\mathbb{R}^d \times SO_d$ to G_d. Using Theorems 13.2.1 and 13.2.2, it is then easy to show the following.

Theorem 13.2.3. *G_d is a locally compact topological group with countable base. Its operation on $\mathbb{R}^d$ is continuous.*

After these topological groups, we consider the homogeneous spaces that play a role in Euclidean integral geometry.

The unit sphere S^{d-1} is obviously a homogeneous SO_d-space.

For $q \in \{0, \ldots, d\}$, let $G(d,q)$ be the set of all q-dimensional linear subspaces of $\mathbb{R}^d$, and let $A(d,q)$ be the set of all q-dimensional affine subspaces of $\mathbb{R}^d$. The natural operation of SO_d on $G(d,q)$ is given by $(\vartheta, L) \mapsto \vartheta L :=$ image of L under ϑ. Similarly, the natural operation of G_d on $A(d,q)$ is given by $(g, E) \mapsto gE :=$ image of E under g. We introduce suitable topologies on $G(d,q)$ and $A(d,q)$. For this, let $L_q \in G(d,q)$ be a fixed subspace and $L_q^\perp$ its orthogonal complement. The mappings

$$\begin{aligned} \beta_q : SO_d &\to G(d,q) \\ \vartheta \;&\mapsto\; \vartheta L_q \end{aligned} \tag{13.6}$$

and

$$\begin{aligned} \gamma_q : L_q^\perp \times SO_d &\to \quad A(d,q) \\ (x, \vartheta) \quad &\mapsto \vartheta(L_q + x) \end{aligned} \tag{13.7}$$

are surjective (but not injective). We endow $G(d,q)$ with the finest topology for which β_q is continuous, and $A(d,q)$ with the finest topology for which γ_q is continuous. Thus, a subset $A \subset A(d,q)$, for example, is open if and only if $\gamma_q^{-1}(A)$ is open. It is an elementary task to prove the following.

Theorem 13.2.4. *$G(d,q)$ is compact and has a countable base, the map β_q is open, and the operation of SO_d on $G(d,q)$ is continuous and transitive.*

Theorem 13.2.5. *$A(d,q)$ is locally compact and has a countable base, the map γ_q is open, and the operation of G_d on $A(d,q)$ is continuous and transitive.*

It should be remarked that the topologies on $G(d,q)$ and $A(d,q)$, as well as the invariant measures to be introduced soon, do not depend on the special choice of the subspace L_q. This follows easily from the fact that SO_d operates transitively on $G(d,q)$ and G_d operates transitively on $A(d,q)$.

The topological spaces $G(d,q)$ are called **Grassmann manifolds** or **Grassmannians**, and the spaces $A(d,q)$ are called **affine Grassmannians**.

It was convenient here to introduce the topologies on $G(d,q)$ and $A(d,q)$ as described. Generally in this book, we equip $\mathcal{F}(\mathbb{R}^d)$, the set of closed subsets of $\mathbb{R}^d$, with the topology of closed convergence, as summarized in Section 12.2. The trace of this topology on $G(d,q)$ or $A(d,q)$ coincides with the topology introduced above. To see this, for example for the case of $A(d,q)$, we first note that the mapping $g \mapsto gL_q$ from G_d into $\mathcal{F}$ is continuous, by Theorem 13.1.1. In order to show that the topology of closed convergence on $A(d,q)$ coincides with the one introduced above, it therefore suffices to show the following. If

$E_i, E \in A(d,q)$ and $E_i \to E$ in $\mathcal{F}$, then there exist motions $g_i, g \in G_d$ such that $E_i = g_i L_q$, $E = gL_q$ and $g_i \to g$ (in G_d). This is easy to see with the aid of Theorem 12.2.2.

By Theorem 13.1.1, the induced operation of the motion group G_d on the space $\mathcal{F} = \mathcal{F}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$ is also continuous. From this fact we draw two conclusions, which are used occasionally. First, applying Theorem 13.1.1 again, but now to the space $E = \mathcal{F}'$ $(= \mathcal{F}(\mathbb{R}^d) \setminus \{\emptyset\})$, we get:

Theorem 13.2.6. *The map*

$$(g, A) \mapsto gA := \{gF : F \in A\}, \qquad g \in G_d,\ A \in \mathcal{F}(\mathcal{F}'),$$

is continuous.

The operation of G_d on $\mathbb{R}^d$ induces also an operation on the space $\mathcal{C}'$ of nonempty compact subsets.

Theorem 13.2.7. *The map*

$$\begin{aligned} G_d \times \mathcal{C}' &\to \mathcal{C}' \\ (g, C) &\mapsto gC \end{aligned}$$

where $\mathcal{C}'$ is equipped with the Hausdorff metric, is continuous.

Proof. Suppose that $(g_i, C_i) \to (g, C)$ in $G_d \times \mathcal{C}'$. Then $C_i \to C$ in $(\mathcal{C}', \delta)$, hence $C_i \to C$ in $\mathcal{F}$ by Theorem 12.3.2. From Theorem 13.1.1 it follows that $g_i C_i \to gC$ in $\mathcal{F}$. Since the sequence $(C_i)_{i \in \mathbb{N}}$ is uniformly bounded and $g_i \to g$, also the sequence $(g_i C_i)_{i \in \mathbb{N}}$ is uniformly bounded. By Theorem 12.3.3, $g_i C_i \to gC$ in $(\mathcal{C}', \delta)$. ☐

Now we construct invariant measures on the introduced groups and homogeneous spaces. (Since these are locally compact, second countable spaces, all Borel measures on them are regular; see, for example, Cohn [177, Proposition 7.2.3].) We start from Lebesgue measure on $\mathbb{R}^d$ and construct further measures by means of continuous (and hence measurable) mappings. The local finiteness of the image measures has to be checked in every case. We shall, however, not mention this fact explicitly, when it is easy to see.

The measures ρ to be considered below will depend on the dimension d of the space $\mathbb{R}^d$. If different dimensions occur, corresponding measures and other objects will be distinguished by lower indices. Symbols without lower index always refer to the dimension d.

We suppose that the reader is familiar with the construction and the properties of the Lebesgue measure λ on $\mathbb{R}^d$, including the following uniqueness theorem. Recall that $C^d = [0,1]^d$ is the unit cube in $\mathbb{R}^d$.

Theorem 13.2.8. *The Lebesgue measure λ is the only translation invariant measure on $\mathbb{R}^d$ with $\lambda(C^d) = 1$.*

Since the Lebesgue measure λ is rigid motion invariant (as well as invariant under reflections), it is the Haar measure on the homogeneous G_d-space $\mathbb{R}^d$, normalized in a special way. We note that

$$\lambda(B^d) =: \kappa_d = \frac{\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)}.$$

The Haar measure on the homogeneous SO_d-space S^{d-1}, the unit sphere, is easily derived from the Lebesgue measure. For $A \in \mathcal{B}(S^{d-1})$ we define

$$\widehat{A} := \{\alpha x \in \mathbb{R}^d : x \in A,\ \ 0 \le \alpha \le 1\}.$$

A standard argument shows that $\widehat{A} \in \mathcal{B}(\mathbb{R}^d)$, hence we can define $\sigma(A) := d\lambda(\widehat{A})$. This yields a finite measure σ on S^{d-1} for which

$$\sigma(S^{d-1}) =: \omega_d = d\kappa_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

The rotation invariance of λ implies the rotation invariance of σ. As the Lebesgue measure, σ is invariant under the full group $O(d)$ of orthogonal transformations (proper and improper rotations). We call σ the **spherical Lebesgue measure**. Up to a constant factor, σ is the only rotation invariant Borel measure on S^{d-1}. This follows from Theorem 13.1.5.

The spherical Lebesgue measure can be used to show the existence of the Haar measure for the rotation group.

Theorem 13.2.9. *On the rotation group SO_d, there is a unique Haar measure ν with $\nu(SO_d) = 1$.*

Proof. The uniqueness follows from Theorem 13.1.3. To prove the existence, we denote by LI_d the set of linearly independent d-tuples of vectors from S^{d-1}. We define a map $\psi : LI_d \to SO_d$ in the following way. Let $(x_1, \dots, x_d) \in LI_d$. By Gram–Schmidt orthonormalization we transform $(x_1, \dots, x_d)$ into the d-tuple $(z_1, \dots, z_d)$; then we denote by $(\bar{z}_1, \dots, \bar{z}_d)$ the positively oriented d-tuple for which $\bar{z}_i = z_i$ for $i = 1, \dots, d-1$ and $\bar{z}_d = z_d$ or $-z_d$. If $(e_1, \dots, e_d)$ denotes the standard orthonormal basis of $\mathbb{R}^d$, there is a unique rotation $\vartheta \in SO_d$ satisfying $\vartheta e_i = \bar{z}_i$ for $i = 1, \dots, d$. We define $\psi(x_1, \dots, x_d) := \vartheta$.

Explicitly, we have $z_i = y_i/\|y_i\|$ with $y_1 = x_1$ and

$$y_k = x_k - \sum_{j=1}^{k-1} \langle x_k, y_j \rangle \frac{y_j}{\|y_j\|^2}, \qquad k = 2, \dots, d.$$

From this representation, the following is evident. If $\rho \in SO_d$ is a rotation and if the d-tuple $(x_1, \dots, x_d) \in LI_d$ is transformed into $(z_1, \dots, z_d)$ and then into $(\bar{z}_1, \dots, \bar{z}_d)$, then the d-tuple $(\rho x_1, \dots, \rho x_d)$ is transformed into $(\rho z_1, \dots, \rho z_d)$ and subsequently into $(\rho \bar{z}_1, \dots, \rho \bar{z}_d)$. Thus we have

$$\psi(\rho x_1, \ldots, \rho x_d) = \rho\psi(x_1, \ldots, x_d).$$

For $(x_1, \ldots, x_d) \in (S^{d-1})^d \setminus LI_d$ we define $\psi(x_1, \ldots, x_d) := \mathrm{id}$. For the product measure

$$\sigma^d := \underbrace{\sigma \otimes \ldots \otimes \sigma}_{d},$$

the set $(S^{d-1})^d \setminus LI_d$ has measure zero; hence for any $\rho \in SO_d$ the equality $\psi(\rho x_1, \ldots, \rho x_d) = \rho\psi(x_1, \ldots, x_d)$ holds σ^d-almost everywhere. The mapping $\psi : (S^{d-1})^d \to SO_d$ is measurable, since LI_d is open and ψ is continuous on LI_d and constant on $(S^{d-1})^d \setminus LI_d$.

Now we define $\bar{\nu}$ as the image measure of σ^d under ψ, thus $\bar{\nu} = \psi(\sigma^d)$. Then $\bar{\nu}$ is a finite measure on SO_d, and for $\rho \in SO_d$ and measurable $f \geq 0$ we obtain

$$\begin{aligned}
&\int_{SO_d} f(\rho\vartheta)\,\bar{\nu}(\mathrm{d}\vartheta) \\
&= \int_{(S^{d-1})^d} f(\rho\psi(x_1, \ldots, x_d))\,\sigma^d(\mathrm{d}(x_1, \ldots, x_d)) \\
&= \int_{S^{d-1}} \cdots \int_{S^{d-1}} f(\psi(x_1, \ldots, x_d))\,\sigma(\mathrm{d}x_1) \cdots \sigma(\mathrm{d}x_d) \\
&= \int_{SO_d} f(\vartheta)\,\bar{\nu}(\mathrm{d}\vartheta).
\end{aligned}$$

Here we have used the rotation invariance of the spherical Lebesgue measure. We have proved that the measure $\bar{\nu}$ is left invariant and thus invariant, by Theorem 13.1.2. The measure $\nu := \bar{\nu}/\bar{\nu}(SO_d)$ is invariant and normalized. □

From now on, ν will always denote the normalized invariant measure on SO_d.

The following special result on ν null sets is needed at several instances. Two linear subspaces L, L' of $\mathbb{R}^d$ are in **general position** if

$$\dim(L \cap L') = \max\{0, \dim L + \dim L' - d\},$$

otherwise they are said to be in **special position**. The latter is equivalent to

$$\mathrm{lin}(L \cup L') \neq \mathbb{R}^d \qquad \text{and} \qquad \dim(L \cap L') > 0.$$

Lemma 13.2.1. *Let L, L' be linear subspaces of $\mathbb{R}^d$, and let $A \subset SO_d$ be the set of all rotations ϑ for which L and $\vartheta L'$ are in special position. Then $\nu(A) = 0$.*

Proof. We may assume that $\dim L + \dim L' < d$, since otherwise we can pass to orthogonal complements. Let $v_1, \ldots, v_m$ be an orthonormal basis of L' and put $L_i := \mathrm{lin}\{v_1, \ldots, v_i\}$ for $i = 1, \ldots, m$ and $L_0 := \{0\}$. Then

$$
\begin{aligned}
&\nu(\{\vartheta \in SO_d : \dim(L \cap \vartheta L') > 0\}) \\
&= \nu\left(\bigcup_{i=1}^{m} \{\vartheta \in SO_d : \dim(L \cap \vartheta L_{i-1}) = 0,\ \dim(L \cap \vartheta L_i) > 0\}\right) \\
&= \sum_{i=1}^{m} \nu(\{\vartheta \in SO_d : \vartheta v_i \in \operatorname{lin}(L \cup \vartheta L_{i-1})\}).
\end{aligned}
$$

We shall show that here each summand is zero. Let $i \in \{1, \dots, m\}$ be fixed and write $\operatorname{lin}(L \cup \vartheta L_{i-1}) =: M(\vartheta)$. Put $H := L_{i-1} \cap S^{d-1}$ and $H' := L_{i-1}^{\perp} \cap S^{d-1}$. For $x \in S^{d-1} \setminus (H \cup H')$ there is a unique decomposition $x = tv + \sqrt{1-t^2}\, x'$ with $v \in H$, $x' \in H'$, $t \in (0,1)$. Since $\vartheta v \in M(\vartheta)$ for $v \in H$, we have $\vartheta x \in M(\vartheta)$ if and only if $\vartheta x' \in M(\vartheta)$. Moreover, for $x' \in H'$ there exists a rotation $\rho_x \in SO_d$ with $\rho_x v_i = x'$ and $\rho_x L_{i-1} = L_{i-1}$. We obtain

$$
\begin{aligned}
\nu(\{\vartheta \in SO_d : \vartheta x \in M(\vartheta)\}) &= \nu(\{\vartheta \in SO_d : \vartheta x' \in M(\vartheta)\}) \\
&= \nu(\{\vartheta \in SO_d : \vartheta \rho_x v_i \in M(\vartheta \rho_x)\}) \\
&= \nu(\{\vartheta \in SO_d : \vartheta v_i \in M(\vartheta)\})
\end{aligned}
$$

by the invariance of ν. Integration with the spherical Lebesgue measure and Fubini's theorem yield

$$
\begin{aligned}
&\sigma(S^{d-1})\nu(\{\vartheta \in SO_d : \vartheta v_i \in M(\vartheta)\}) \\
&= \int_{S^{d-1} \setminus (H \cup H')} \nu(\{\vartheta \in SO_d : \vartheta x \in M(\vartheta)\})\, \sigma(\mathrm{d}x) \\
&= \int_{SO_d} \int_{S^{d-1} \setminus (H \cup H')} \mathbf{1}_{M(\vartheta)}(\vartheta x)\, \sigma(\mathrm{d}x)\, \nu(\mathrm{d}\vartheta) = 0,
\end{aligned}
$$

since $\dim M(\vartheta) \le \dim L + i - 1 \le d - 1$. □

With the aid of the invariant measures λ and σ, we can construct the Haar measure on the rigid motion group G_d. Since G_d it is not compact, an invariant measure μ on G_d cannot be finite, as is easy to see. In order to normalize μ, we specify the compact set $A_0 := \gamma(C^d \times SO_d)$ and require that $\mu(A_0) = 1$.

Theorem 13.2.10. *On the motion group G_d there is a Haar measure μ with $\mu(A_0) = 1$. Up to a constant factor, it is the only left Haar measure on G_d.*

Proof. The uniqueness assertion is a special case of Theorem 13.1.3. We define μ as the image measure of the product measure $\lambda \otimes \nu$ under the homeomorphism $\gamma : \mathbb{R}^d \times SO_d \to G_d$ defined by (13.5). Then μ is a Borel measure on G_d with $\mu(\gamma(C^d \times SO_d)) = \lambda(C^d)\nu(SO_d) = 1$.

To show the left invariance of μ, let $f \ge 0$ be a continuous function on G_d and let $g' \in G_d$. With $g' = \gamma(t', \vartheta')$ we have

$$\begin{aligned}\int_{G_d} f(g'g)\,\mu(\mathrm{d}g) &= \int_{SO_d}\int_{\mathbb{R}^d} f(\gamma(t',\vartheta')\gamma(t,\vartheta))\,\lambda(\mathrm{d}t)\,\nu(\mathrm{d}\vartheta)\\ &= \int_{SO_d}\int_{\mathbb{R}^d} f(\gamma(t'+\vartheta't,\vartheta'\vartheta))\,\lambda(\mathrm{d}t)\,\nu(\mathrm{d}\vartheta)\\ &= \int_{SO_d}\int_{\mathbb{R}^d} f(\gamma(t,\vartheta))\,\lambda(\mathrm{d}t)\,\nu(\mathrm{d}\vartheta)\\ &= \int_{G_d} f(g)\,\mu(\mathrm{d}g),\end{aligned}$$

where we have used the motion invariance of λ and the left invariance of ν. Hence, μ is left invariant. Analogously, the right invariance of ν implies via

$$\int_{G_n} f(gg')\,\mu(\mathrm{d}g) = \int_{SO_d}\int_{\mathbb{R}^d} f(\gamma(t+\vartheta t',\vartheta\vartheta'))\,\lambda(\mathrm{d}t)\,\nu(\mathrm{d}\vartheta)$$
$$= \int_{SO_d}\int_{\mathbb{R}^d} f(\gamma(t,\vartheta))\,\lambda(\mathrm{d}t)\,\nu(\mathrm{d}\vartheta) = \int_{G_d} f(g)\,\mu(\mathrm{d}g)$$

the right invariance of μ, and from

$$\int_{G_d} f(g^{-1})\,\mu(\mathrm{d}g) = \int_{SO_d}\int_{\mathbb{R}^d} f(\gamma(-\vartheta^{-1}t,\vartheta^{-1}))\,\lambda(\mathrm{d}t)\,\nu(\mathrm{d}\vartheta)$$
$$= \int_{SO_d}\int_{\mathbb{R}^d} f(\gamma(t,\vartheta))\,\lambda(\mathrm{d}t)\,\nu(\mathrm{d}\vartheta) = \int_{G_d} f(g)\,\mu(\mathrm{d}g),$$

where the inversion invariance of ν was used, we obtain the inversion invariance of μ. $\square$

The notation μ for the Haar measure on G_d, normalized as above, will be maintained in the following. The decomposition of the measure μ inherent in its construction is often used in the form

$$\begin{aligned}\int_{G_d} f\,\mathrm{d}\mu &= \int_{SO_d}\int_{\mathbb{R}^d} f(t_x\circ\vartheta)\,\lambda(\mathrm{d}x)\,\nu(\mathrm{d}\vartheta)\\ &= \int_{SO_d}\int_{\mathbb{R}^d} f(\vartheta\circ t_x)\,\lambda(\mathrm{d}x)\,\nu(\mathrm{d}\vartheta)\end{aligned} \tag{13.8}$$

for μ-integrable functions f on G_d; here t_x is the translation by the vector x. The last equality follows from $\vartheta\circ t_x = t_{\vartheta x}\circ\vartheta$ and the invariance of the Lebesgue measure λ under the rotation ϑ.

Now we turn to invariant measures on the Grassmannian $G(d,q)$ of q-dimensional linear subspaces and on the affine Grassmannian $A(d,q)$ of q-dimensional affine subspaces of $\mathbb{R}^d$. As above, we suppose that $q\in\{0,\dots,d\}$ and $L_q\in G(d,q)$ is a fixed q-dimensional linear subspace, and we use the maps β_q and γ_q defined by (13.6) and (13.7), respectively.

On $G(d,q)$ and $A(d,q)$, some of the transformation groups introduced above operate continuously, for example on $A(d,q)$ the groups T_d, SO_d and G_d. Only the operations of G_d on $A(d,q)$ and of SO_d on $G(d,q)$ are transitive. Therefore, by an **invariant measure** on $A(d,q)$ we understand a rigid motion invariant (G_d-invariant) measure on $A(d,q)$, and an **invariant measure** on $G(d,q)$ is rotation invariant (SO_d-invariant).

Theorem 13.2.11. *On $G(d,q)$ there is a unique Haar measure ν_q, normalized by $\nu_q(G(d,q)) = 1$.*

This is just a special case of Theorem 13.1.5. We also notice that ν_q is the image measure of ν under the mapping β_q.

A corresponding assertion for $A(d,q)$ requires a normalization on a suitable compact subset A_0^q. We choose

$$A_0^q := \{E \in A(d,q) : E \cap B^d \neq \emptyset\}.$$

Theorem 13.2.12. *On $A(d,q)$ there is a unique Haar measure μ_q, normalized by $\mu_q(A_0^q) = \kappa_{d-q}$.*

It satisfies

$$\int_{A(d,q)} f \, \mathrm{d}\mu_q = \int_{G(d,q)} \int_{L^\perp} f(L+y) \, \lambda_{d-q}(\mathrm{d}y) \, \nu_q(\mathrm{d}L) \tag{13.9}$$

for every measurable function $f \geq 0$ on $A(d,q)$.

Proof. We define

$$\mu_q := \gamma_q(\lambda_{d-q} \otimes \nu).$$

If $A \subset A(d,q)$ is compact, the sets

$$\gamma_q(\{x \in L_q^\perp : \|x\| < k\} \times SO_d), \qquad k \in \mathbb{N},$$

constitute an open covering of A, hence A is included in one of these sets. It follows that $\mu_q(A) < \infty$.

Let $g = \gamma(x, \vartheta) \in G_d$ and let $f \geq 0$ be a measurable function on $A(d,q)$. Denoting by Π the orthogonal projection to $L_q^\perp$, we obtain

$$\begin{aligned}
&\int_{A(d,q)} f(gE) \, \mu_q(\mathrm{d}E) \\
&= \int_{SO_d} \int_{L_q^\perp} f(g\rho(L_q + y)) \, \lambda_{d-q}(\mathrm{d}y) \, \nu(\mathrm{d}\rho) \\
&= \int_{SO_d} \int_{L_q^\perp} f(\vartheta\rho(L_q + y + \Pi(\rho^{-1}\vartheta^{-1}x))) \, \lambda_{d-q}(\mathrm{d}y) \, \nu(\mathrm{d}\rho) \\
&= \int_{SO_d} \int_{L_q^\perp} f(\vartheta\rho(L_q + y)) \, \lambda_{d-q}(\mathrm{d}y) \, \nu(\mathrm{d}\rho)
\end{aligned}$$

$$= \int_{SO_d} \int_{L_q^\perp} f(\rho(L_q + y))\, \lambda_{d-q}(\mathrm{d}y)\, \nu(\mathrm{d}\rho)$$
$$= \int_{A(d,q)} f(E)\, \mu_q(\mathrm{d}E),$$

where we have used the invariance properties of λ_{d-q} and ν. This shows the invariance of μ_q.

We observe that we may also write

$$\int_{A(d,q)} f\, \mathrm{d}\mu_q = \int_{SO_d} \int_{L_q^\perp} f(\rho(L_q + x))\, \lambda_{d-q}(\mathrm{d}x)\, \nu(\mathrm{d}\rho)$$
$$= \int_{SO_d} \int_{(\rho L_q)^\perp} f(\rho L_q + y)\, \lambda_{d-q}(\mathrm{d}y)\, \nu(\mathrm{d}\rho).$$

Since ν_q is the image measure of ν under β_q, this can be written in the form (13.9).

From the representation (13.9) we infer that μ_q does not depend on the choice of the subspace L_q. We also deduce that $\mu(A_0^q) = \kappa_{d-q}$.

To prove the uniqueness, we assume that τ is another Haar measure on $A(d,q)$. Let $\widetilde{G}(d,q)$ (respectively $\widetilde{A}(d,q)$) be the open set of all $L \in G(d,q)$ (respectively $E \in A(d,q)$) that intersect $L_q^\perp$ at precisely one point. The mapping

$$\begin{aligned} \delta_q : L_q^\perp \times \widetilde{G}(d,q) &\to \widetilde{A}(d,q) \\ (x, L) &\mapsto L + x \end{aligned}$$

is a homeomorphism. For fixed $B \in \mathcal{B}(\widetilde{G}(d,q))$ and arbitrary $A \in \mathcal{B}(L_q^\perp)$ we define $\eta(A) := \tau(\delta_q(A \times B))$. Then η is a Borel measure on $L_q^\perp$, which is invariant under the translations of $L_q^\perp$ into itself. Theorem 13.2.8 implies that $\eta(A) = \lambda_{d-q}(A)\alpha(B)$ with a constant $\alpha(B) \geq 0$. Hence we have

$$\tau(\delta_q(A \times B)) = \lambda_{d-q}(A)\alpha(B)$$

for arbitrary $A \in \mathcal{B}(L_q^\perp)$ and $B \in \mathcal{B}(\widetilde{G}(d,q))$. Obviously this equality defines a finite measure α on $\mathcal{B}(\widetilde{G}(d,q))$, and $\delta_q^{-1}(\tau) = \lambda_{d-q} \otimes \alpha$. For a measurable function $f \geq 0$ on $\widetilde{A}(d,q)$ we obtain

$$\int_{\widetilde{A}(d,q)} f\, \mathrm{d}\tau = \int_{\widetilde{G}(d,q)} \int_{L_q^\perp} f(L + x)\, \lambda_{d-q}(\mathrm{d}x)\, \alpha(\mathrm{d}L)$$
$$= \int_{\widetilde{G}(d,q)} \int_{L^\perp} f(L + y)\, \lambda_{d-q}(\mathrm{d}y)\, \varphi(\mathrm{d}L) \tag{13.10}$$

with a new measure φ on $\widetilde{G}(d,q)$, defined by $\mathrm{d}\varphi(L) = D(L_q^\perp, L^\perp)^{-1}\mathrm{d}\alpha(L)$, where $D(L_q^\perp, L^\perp)$ is the absolute determinant of the orthogonal projection from $L_q^\perp$ onto $L^\perp$.

Now let $B \in \mathcal{B}(G(d,q))$ and

$$B' := \{L + y : L \in B,\ \ y \in L^\perp \cap B^d\}.$$

By $\beta(B) := \tau(B')$ we define a rotation invariant finite measure β on $G(d,q)$. According to Theorem 13.2.11 it is a multiple of ν_q. On the other hand, (13.10) gives $\tau(B') = \kappa_{d-q}\varphi(B)$ for $B \subset \widetilde{G}(d,q)$. Hence there is a constant c with $\varphi(B) = c\nu_q(B)$ for all Borel sets $B \subset \widetilde{G}(d,q)$. From (13.10) and (13.9) we deduce that $\tau(A) = c\mu_q(A)$ for all Borel sets $A \subset \widetilde{A}(d,q)$. Since μ_q does not depend on the choice of the subspace $L_q \in G(d,q)$, it is easy to see that $\tau = c\mu_q$. □

From the introduced homogeneous spaces of flats and their invariant measures, we derive other ones, which are used occasionally in integral geometry.

For $p, q \in \{0, \dots, d\}$ and a fixed linear subspace $L \in G(d,p)$, let

$$G(L,q) := \begin{cases} \{L' \in G(d,q) : L' \subset L\} & \text{if } \ q \le p, \\ \{L' \in G(d,q) : L' \supset L\} & \text{if } \ q > p. \end{cases}$$

Similarly, for $E \in A(d,p)$ let

$$A(E,q) := \begin{cases} \{E' \in A(d,q) : E' \subset E\} & \text{if } \ q \le p, \\ \{E' \in A(d,q) : E' \supset E\} & \text{if } \ q > p. \end{cases}$$

In the case $q \le p$ the spaces $G(L,q)$ and $A(E,q)$ are obviously homeomorphic to $G(p,q)$ and $A(p,q)$, respectively. For $q > p$, the situation is slightly different. Here $G(L,q)$ is homeomorphic to $G(d-p, q-p)$, because each $L' \in G(L,q)$ is of the form $L' = L + L''$ with a unique subspace $L'' \in G(L^\perp, q-p)$, so that $G(L,q)$ is homeomorphic, in a natural way, to $G(L^\perp, q-p)$; the latter space is homeomorphic to $G(d-p, q-p)$, by the preceding remark. The space $A(E,q)$ with $q > p$ is evidently homeomorphic to $G(L,q)$, where L is the translate of E through the origin. Thus $A(E,q)$, too, is homeomorphic to $G(d-p, q-p)$.

On these spaces, we introduce invariant measures in the natural way. For a linear subspace $L \in G(d,q)$ we first put

$$SO(L) := \{\rho \in SO_d : \rho L = L,\ \rho x = x \text{ for } x \in L^\perp\},$$

which is the subgroup of all proper rotations of $\mathbb{R}^d$ mapping L into itself and fixing each point of $L^\perp$. Since $SO(L)$ is isomorphic to SO_p, it carries a unique normalized invariant measure, which we denote by ν_L. As usual, we consider ν_L as a measure defined on the whole group SO_d. We have

$$\nu_{\vartheta L}(\vartheta A \vartheta^{-1}) = \nu_L(A) \tag{13.11}$$

for $A \in \mathcal{B}(SO_d)$ and arbitrary rotations $\vartheta \in SO_d$; this can be deduced, for example, from Theorem 13.1.3.

Let $p, q \in \{0, \dots, d\}$ and $L \in G(d,q)$. We fix a subspace $L_q \in G(L,q)$. By means of the map (13.6), that is, $\beta_q : SO_d \to G(d,q)$, $\vartheta \mapsto \vartheta L_q$, we define

$$\nu_q^L := \beta_q(\nu_L)$$

for $q < p$ and

$$\nu_q^L := \beta_q(\nu_{L^\perp})$$

for $q \geq p$. Then we have

$$\nu_q^L(A) = \nu_L(\{\rho \in SO(L) : \rho L_q \in A\})$$

if $q < p$ and

$$\nu_q^L(A) = \nu_{L^\perp}(\{\rho \in SO(L^\perp) : \rho L_q \in A\})$$

if $q \geq p$, in each case for all $A \in \mathcal{B}(G(d,q))$. Thus ν_q^L is a normalized measure concentrated on $G(L,q)$; it does not depend on the choice of L_q and is invariant under $SO(L)$ and $SO(L^\perp)$. Moreover,

$$\nu_q^{\vartheta L}(\vartheta A) = \nu_q^L(A) \tag{13.12}$$

for $A \in \mathcal{B}(G(d,q))$ and all rotations $\vartheta \in SO_d$, as follows from (13.11).

For a fixed flat $E \in G(d,p)$ we choose $t \in \mathbb{R}^d$ with $E - t =: L \in G(d,p)$ and then a subspace $L_q \in G(L,q)$. If $q < p$, let λ_{p-q} be the Lebesgue measure on $L_q^\perp \cap L$. We define

$$\begin{aligned} \gamma_{q,t} : (L_q^\perp \cap L) \times SO(L) &\to \quad A(d,q) \\ (x, \vartheta) \qquad &\mapsto \vartheta(L_q + x) + t \end{aligned}$$

and

$$\mu_q^E := \gamma_{q,t}(\lambda_{p-q} \otimes \nu_L).$$

If $q \geq p$, we define

$$\begin{aligned} \gamma_{q,t} : SO(L^\perp) &\to A(d,q) \\ \vartheta \quad &\mapsto \vartheta L_q + t \end{aligned}$$

and

$$\mu_q^E := \gamma_{q,t}(\nu_{L^\perp}).$$

The measure μ_q^E is independent of the choice of t and L_q; it is concentrated on $A(E,q)$ and is invariant under the rigid motions of $\mathbb{R}^d$ that map E into itself. Moreover,

$$\mu_q^{gE}(gA) = \mu_q^E(A)$$

for $A \in \mathcal{B}(A(d,q))$ and all rigid motions $g \in G_d$.

Let $L \in G(d,q)$. In analogy to (13.9) we see that for given $t \in \mathbb{R}^d$ and for measurable functions $f \geq 0$ on $A(d,q)$ we have for $q < p$ the representation

$$\int_{A(L+t,q)} f \, \mathrm{d}\mu_q^{L+t} = \int_{G(L,q)} \int_{M^\perp \cap L} f(M + x + t) \, \lambda_{p-q}(\mathrm{d}x) \, \nu_q^L(\mathrm{d}M). \tag{13.13}$$

The corresponding equality for $q \geq p$ is

$$\int_{A(L+t,q)} f \, \mathrm{d}\mu_q^{L+t} = \int_{G(L,q)} f(M+t) \, \nu_q^L(\mathrm{d}M). \tag{13.14}$$

For the following measurability assertion we recall that the measure μ_p^F is concentrated on $A(F,p)$, but defined on all of $A(d,p)$. A similar remark concerns the measure ν_p^L. We write

$$A(d,p,q) := \{(E,F) \in A(d,p) \times A(d,q) : E \subset F\},$$
$$G(d,p,q) := \{(L,M) \in G(d,p) \times G(d,q) : L \subset M\}.$$

Lemma 13.2.2. *Let $0 \leq p < q \leq d$, and let $f : A(d,p,q) \to \mathbb{R}$ be a nonnegative measurable function. Then the maps*

$$F \mapsto \int_{A(F,p)} f(E,F) \, \mu_p^F(\mathrm{d}E), \qquad F \in A(d,q), \tag{13.15}$$

and

$$E \mapsto \int_{A(E,q)} f(E,F) \, \mu_q^E(\mathrm{d}F), \qquad E \in A(d,p), \tag{13.16}$$

are measurable.

Analogous statements hold for the measures ν_p^M, ν_q^L and nonnegative measurable functions on $G(d,p,q)$.

Proof. First let $f : A(d,p) \to \mathbb{R}$ be a continuous function with compact support. Let $(F_i)_{i\in\mathbb{N}}$ be a sequence in $A(d,q)$ converging to F. Then there is a sequence $(g_i)_{i\in\mathbb{N}}$ in the motion group G_d, converging to the identity id and such that $g_i^{-1}F = F_i$. We have

$$\begin{aligned} \int_{A(d,p)} f(E) \, \mu_p^{F_i}(\mathrm{d}E) &= \int_{A(d,p)} f(E) \, \mu_p^{g_i^{-1}F}(\mathrm{d}E) \\ &= \int_{A(d,p)} f(E) \, \mu_p^F(\mathrm{d}g_i E) \\ &= \int_{A(d,p)} f(g_i^{-1}E) \, \mu_p^F(\mathrm{d}E). \end{aligned}$$

The functions $f_i : E \mapsto f(g_i^{-1}E)$ converge to f for $i \to \infty$. If A denotes the compact support of f and $C \subset G_d$ is a compact set with $g_i \in C$ for all i, then CA is compact and $|f(g_i^{-1}E)| \leq \mathbf{1}_{CA}(E) \max |f|$. The dominated convergence theorem yields

$$\int_{A(d,p)} f(E) \, \mu_p^{F_i}(\mathrm{d}E) \to \int_{A(d,p)} f(E) \, \mu_p^F(\mathrm{d}E)$$

for $i \to \infty$. Thus the map

$$F \mapsto \int_{A(d,p)} f(E)\, \mu_p^F(\mathrm{d}E), \qquad F \in A(d,q),$$

is continuous and hence measurable.

We remark that Lemma 12.1.1 now shows that the function $F \mapsto \mu_p^F(B)$ is measurable for each $B \in \mathcal{B}(A(d,p))$, hence the mapping $(F,B) \mapsto \mu_p^F(B)$, $F \in A(d,q)$, $B \in \mathcal{B}(A(d,p))$, is a kernel.

Lemma 12.1.2 with $E = A(d,p)$ and $T = A(d,q)$ gives the measurability of the map (13.15) for nonnegative measurable functions f on $A(d,p,q)$.

The measurability of the map (13.16) and the proofs of the remaining assertions follow in a completely analogous manner. □

Notes for Section 13.2

1. The proof of Lemma 13.2.1 was communicated to us by Jürgen Kampf. A different proof appears in Goodey and Schneider [274]. The point in both cases was to give an elementary proof, avoiding an explicit representation of the invariant measure ν and using little more than its invariance.

2. Arguments similar to those used in the uniqueness proof of Theorem 13.2.12 appear, for example, also in the book by Ambartzumian [35].

13.3 A General Uniqueness Theorem

It was our aim in the preceding two sections to treat the special Haar measures that are needed for the purposes of Euclidean integral geometry, in a direct and elementary way. We present now a proof of a general uniqueness theorem for relatively invariant measures. This proof, in which the existence of general Haar measures is taken for granted, is adapted from Nachbin [571, pp. 138 ff].

In the following, we assume that G is a locally compact topological group and H is a closed subgroup; then the factor space G/H is locally compact. The following result is fundamental. *On every locally compact group there is a left Haar measure, and it is unique up to a positive factor.* The uniqueness admits a fairly quick proof (see, for example, Cohn [177], also for a lucid existence proof). Not every locally compact homogeneous space carries a Haar measure. This is shown, for example, by the standard operation of the affine group G_{aff} on $\mathbb{R}^d$. Any affine-invariant Borel measure on $\mathbb{R}^d$ is, in particular, translation invariant and thus, if it is locally finite, a multiple of the Lebesgue measure λ, but this is not affine-invariant. The Lebesgue measure satisfies

$$\lambda(gA) = |\det g|\lambda(A)$$

for $g \in G_{\mathrm{aff}}$ and $A \in \mathcal{B}(\mathbb{R}^d)$, and here the factor is independent of A. This motivates the following definition.

For the rest of this section, a measure on G or G/H is always a locally finite Borel measure. The measure ρ on the homogeneous space G/H is called **relatively invariant** if it is regular and not identically zero and if there exists a function $\chi : G \to \mathbb{R}$ such that

$$\rho(gA) = \chi(g)\rho(A) \qquad \text{for } g \in G \text{ and } A \in \mathcal{B}(G/H).$$

In that case, the function χ is called the **multiplier** of ρ. Obviously, χ is a homomorphism from G into the multiplicative group of positive real numbers. It can be shown that χ is continuous (cf. Hewitt and Ross [342, p. 204] or Gaal [242, p. 265]).

The study of relatively invariant measures is equivalent to the study of relatively invariant integrals. Here an **integral** on the locally compact space E is a positive linear functional on the space $\mathbf{C}_c(E)$ that is not identically zero. For $f \in \mathbf{C}_c(G)$ and $a \in G$ one writes $(a.f)(x) := f(a^{-1}x)$ for $x \in G$; then $a.f \in \mathbf{C}_c(G)$. The integral I on G is **left invariant** if $I(a.f) = I(f)$ for all $f \in \mathbf{C}_c(G)$ and all $a \in G$. For $f \in \mathbf{C}_c(G/H)$ and $a \in G$ one defines $(a.f)(xH) := f(a^{-1}xH)$, then $a.f \in \mathbf{C}_c(G/H)$. The integral I on G/H is called **relatively invariant with multiplier** χ if $I(a.f) = \chi(a)I(f)$ for all $f \in \mathbf{C}_c(G/H)$ and all $a \in G$. Every measure $\rho \neq 0$ on G/H induces an integral I via $I(f) = \int_{G/H} f \, \mathrm{d}\rho$ for $f \in \mathbf{C}_c(G/H)$. Conversely, the Riesz representation theorem implies that each integral I on G/H is generated in this way, by a uniquely determined regular Borel measure ρ. An integral is relatively invariant with multiplier χ if and only if the same holds for the corresponding measure.

We want to show that on a locally compact homogeneous space G/H there is, up to a constant factor, at most one relatively invariant measure with a given multiplier. For this, we first establish a relation between the spaces $\mathbf{C}_c(G)$ and $\mathbf{C}_c(G/H)$. For a function $f \in \mathbf{C}_c(G)$ we define

$$f'(x) := \int_H f(xy)\, \eta(\mathrm{d}y) \qquad \text{for } x \in G,$$

where η is a left Haar measure on H (so here we make use of its existence). The function f' is constant on the left cosets of H, since for $x \in zH$, which means $x = zh$ with $h \in H$, we have $f'(x) = \int f(zhy)\, \eta(\mathrm{d}y) = \int f(zy)\, \eta(\mathrm{d}y) = f'(z)$. Hence, there is a unique function $f^+ : G/H \to \mathbb{R}$ satisfying $f'(x) = f^+(xH)$ and thus

$$f^+(\pi(x)) = \int_H f(xy)\, \eta(\mathrm{d}y).$$

In this way, a linear map $f \mapsto f^+$ from $\mathbf{C}_c(G)$ into the vector space of real functions on G/H has been defined.

Lemma 13.3.1. *The correspondence $f \mapsto f^+$ maps $\mathbf{C}_c(G)$ onto $\mathbf{C}_c(G/H)$.*

Proof. Let $f \in \mathbf{C}_c(G)$. The function f' is continuous, because f is uniformly continuous. Since π is an open map, f^+ is continuous. If $f^+(xH) \neq 0$, there is an element $y \in H$ with $f(xy) \neq 0$ and thus $xy \in \operatorname{supp} f$ (where supp denotes the support), hence $xH \in \pi(\operatorname{supp} f)$. We conclude that $\operatorname{supp} f^+ \subset \pi(\operatorname{supp} f)$ and therefore $f^+ \in \mathbf{C}_c(G/H)$.

To prove that the mapping $f \mapsto f^+$ is surjective, let $h \in \mathbf{C}_c(G/H)$ and $K := \operatorname{supp} h$. Let V be a compact neighborhood of the unit element of G. For $g \in G$ the set $\pi(Vg)$ is a neighborhood of $\pi(g)$. Since K is compact, there are finitely many elements $g_1, \ldots, g_k \in G$ with $K \subset \bigcup_{j=1}^k \pi(Vg_j)$. Then the set $A := (Vg_1 \cup \ldots \cup Vg_k) \cap \pi^{-1}(K)$ is a compact subset of G with the property that $\pi(A) = K$. We can choose a function $u \in \mathbf{C}_c(G)$ with $u(A) = \{1\}$ and $0 \leq u \leq 1$. For $z \in G/H$ we define

$$\psi(z) := \begin{cases} \dfrac{h(z)}{u^+(z)} & \text{if} \quad u^+(z) \neq 0, \\ 0 & \text{if} \quad u^+(z) = 0. \end{cases}$$

Then $\psi u^+ = h$. Since ψ vanishes outside K and K is contained in the open set $\{z \in G/H : u^+(z) = 0\}$, the function ψ is continuous. Let $f := (\psi \circ \pi)u$; then $f \in \mathbf{C}_c(G)$ and

$$\begin{aligned} f^+(xH) &= \int_H f(xy)\, \eta(\mathrm{d}y) = \int_H \psi(xyH)u(xy)\, \eta(\mathrm{d}y) \\ &= \psi(xH) \int_H u(xy)\, \eta(\mathrm{d}y) \\ &= h(xH). \end{aligned}$$

Thus $f^+ = h$, which completes the proof. □

Theorem 13.3.1. *On a locally compact homogeneous space G/H there is, up to a constant factor, at most one relatively invariant measure with a given multiplier.*

Proof. Let ρ be a relatively invariant measure on G/H with multiplier χ. Let $a \in G$. The relative invariance of ρ implies that

$$\int_{G/H} a.h \, \mathrm{d}\rho = \chi(a) \int_{G/H} h \, \mathrm{d}\rho$$

for $h \in \mathbf{C}_c(G/H)$. For $f \in \mathbf{C}_c(G)$ we have $(a.f)^+ = a.f^+$, as follows immediately from the definitions. Since χ is a homeomorphism, we have $\chi = \chi(a)a.\chi$ and hence

$$\left(\frac{a.f}{\chi}\right)^+ = \frac{1}{\chi(a)} \left(\frac{a.f}{a.\chi}\right)^+ = \chi(a^{-1})a.\left(\frac{f}{\chi}\right)^+.$$

Now we define

$$I(f) := \int_{G/H} \left(\frac{f}{\chi}\right)^+ \mathrm{d}\rho \qquad \text{for } f \in \mathbf{C}_c(G).$$

Then I is a positive linear functional on $\mathbf{C}_c(G)$. For $a \in G$ we get

$$I(a.f) = \int_{G/H} \left(\frac{a.f}{\chi}\right)^+ \mathrm{d}\rho = \chi(a^{-1}) \int_{G/H} a.\left(\frac{f}{\chi}\right)^+ \mathrm{d}\rho$$
$$= \chi(a^{-1})\chi(a) \int_{G/H} \left(\frac{f}{\chi}\right)^+ \mathrm{d}\rho = I(f).$$

Thus I is a left invariant integral on $\mathbf{C}_c(G)$ and is, therefore, unique up to a constant factor; as mentioned, this uniqueness is equivalent to that of the left Haar measure. If now $\bar{\rho}$ is another relatively invariant measure on G/H with multiplier χ, then

$$\int_{G/H} \left(\frac{f}{\chi}\right)^+ \mathrm{d}\rho = c \int_{G/H} \left(\frac{f}{\chi}\right)^+ \mathrm{d}\bar{\rho}$$

for all $f \in \mathbf{C}_c(G)$ with some constant c. Since by Lemma 13.3.1 the function $(f/\chi)^+$ can be any element of $\mathbf{C}_c(G/H)$, we conclude that $\rho = c\bar{\rho}$. This completes the proof of the theorem. □