11

Non-stationary Models

Although the main theme of this book is random geometric structures with invariance properties, such as stationarity or isotropy, we conclude with an outlook to some of the extensions that are possible without such assumptions. The invariance properties in previous chapters allowed us to employ integral geometric formulas for obtaining results on geometric mean values. Our set-up followed also the historical development of the field, where from the beginning stationarity and isotropy seemed to be natural and convenient conditions to get simple and applicable formulas. Their counterparts for non-isotropic random sets and particle processes are necessarily more complicated, as we have seen in some of the previous sections. However, once the step from isotropic to non-isotropic structures is made, the question arises whether a similar generalization from stationary to non-stationary structures is possible. Although random sets and point processes without any invariance properties have been studied by many authors under different aspects, one might get the impression that, for example, the mean value formulas for Boolean models, which are at the heart of stochastic geometry, rely on the invariance of the model. Surprisingly, this is not the case. As the dissertation of Fallert [222] showed (see also [223]), specific intrinsic volumes for Boolean models with convex or polyconvex grains can be introduced without any invariance requirements, and the formulas obtained in Section 9.1 transfer to this situation in a suitably generalized form. Even more astonishing is the fact that these local mean value formulas for non-stationary Boolean models (and Poisson particle processes) make heavy use of the iterated formulas of translative integral geometry, as we have discussed in Section 6.4. Thus, although we do not require that the distributions of our random structures are invariant with respect to the translation group, the corresponding integral geometric setting still plays an essential role.

Fallert's dissertation, which contained results on several non-stationary models (particle processes, Boolean models, processes of flats, random mosaics), initiated various further publications in which counterparts to formulas in the stationary case were established without the assumption of stationarity.

In this chapter, we present some of these generalizations, mostly concentrating on results which are in analogy to the ones discussed in previous sections.

11.1 Particle Processes and Boolean Models

We consider a particle process X on $\mathcal{C}'$ in $\mathbb{R}^d$. Although we do not assume any invariance of the distribution of X, we require some regularity of the intensity measure Θ (which is assumed to be locally finite, as always). Namely, we assume that a decomposition

$$\Theta(A) = \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_A(C+x)\eta(C,x)\,\lambda(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}C), \qquad A \in \mathcal{B}(\mathcal{C}'), \tag{11.1}$$

exists, with a probability measure $\mathbb{Q}$ on $\mathcal{C}_0$ and a measurable function $\eta \geq 0$ on $\mathcal{C}_0 \times \mathbb{R}^d$. How restrictive is this assumption? Due to the topological properties of $\mathcal{C}'$, respectively those of $\mathcal{C}_0 \times \mathbb{R}^d$, a locally finite measure Θ on $\mathcal{C}'$ always has a decomposition

$$\Theta(A) = \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_A(C+x)\,\rho(C,\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}C), \qquad A \in \mathcal{B}(\mathcal{C}'), \tag{11.2}$$

with a probability measure $\mathbb{Q}$ on $\mathcal{C}_0$ and a kernel $\rho : \mathcal{C}_0 \times \mathcal{B} \to \mathbb{R}^+$, that is, a function that is measurable in the first variable and is a locally finite measure in the second variable. This follows from the disintegration result for probability measures (see, e.g., Kallenberg [386, Th. 6.3]) by a simple extension argument (compare Kallenberg [387, Lemma 3.1]). Our additional assumption is that $\rho(C,\cdot)$ be absolutely continuous with respect to λ, for each C. In fact, if we assume this and denote the density by $\eta(C,\cdot)$, then the decomposition (11.2) transforms into (11.1).

We say that a locally finite measure Θ on $\mathcal{C}'$ admitting a decomposition (11.1) is **translation regular**. This name is chosen since Θ is translation regular if and only if it is absolutely continuous with respect to some translation invariant, locally finite measure $\widetilde{\Theta}$. In fact, for a given translation regular measure Θ with decomposition (11.1), one can choose $\widetilde{\Theta}$ as

$$\widetilde{\Theta}(A) = \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_A(C+x)\,\lambda(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}C), \qquad A \in \mathcal{B}(\mathcal{C}').$$

The other direction follows from Theorem 4.1.1. One should be aware of the fact that the decomposition (11.1) is not unique, in general. In fact, if $f > 0$ is a measurable function on $\mathcal{C}_0$ with $\int f \mathrm{d}\mathbb{Q} = 1$, then we can replace η by η/f and $\mathbb{Q}$ by $A \mapsto \int f \mathbf{1}_A \,\mathrm{d}\mathbb{Q}$, and (11.1) remains valid. We therefore say that the translation regular measure Θ is **represented by the pair** $(\eta, \mathbb{Q})$ if (11.1) holds.

It is sometimes convenient to modify this set-up slightly by imposing additional conditions. For example, we may require that η is continuous or that η depends only on the location, so that we have

$$\Theta(A) = \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_A(C + x)\eta(x)\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}C), \qquad A \in \mathcal{B}(\mathcal{C}'). \tag{11.3}$$

The former condition is sometimes helpful, since it implies that densities of geometric functionals exist at every point and not only almost everywhere. The latter condition has the advantage that it ensures that η and $\mathbb{Q}$ are uniquely determined. Namely, if we interpret X as a marked point process $\widehat{X}$ on $\mathbb{R}^d$ with mark space $\mathcal{C}_0$ (such that X is the image of $\widehat{X}$ under $(x, C) \mapsto C + x$), then $A \mapsto \int \eta \mathbf{1}_A \, \mathrm{d}\lambda$ is the intensity measure of the underlying unmarked point process in $\mathbb{R}^d$, and $\mathbb{Q}$ is the mark distribution. It is therefore natural to call the measure $\mathbb{Q}$ in (11.3) the **distribution of the typical grain** and η the **(spatial) intensity function** of X. If X is stationary, $\eta = \gamma$ is a constant.

If X is a Poisson process, (11.3) implies that $\widehat{X}$ is independently marked, whereas (11.1) allows dependencies between the marks (or between the marks and the points).

Up to here, we did not impose additional conditions on the shape of the particles and, in fact, some of the following results hold in this generality, for compact particles. This is particularly the case for the results on contact distributions of Boolean models, and we shall comment on these in the Notes. But since we now aim at defining specific intrinsic volumes, the restriction to convex particles seems natural. Some of the results can be generalized easily to particles in the convex ring, using the additivity of the functionals involved. This would require additional integrability conditions, therefore we leave such generalizations to the reader and assume convex grains, from now on. If the intensity measure Θ of a process X of convex particles has a representation (11.1), then its local finiteness is equivalent to

$$\int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}\{(K + x) \cap C \neq \emptyset\}\eta(K, x)\, \lambda(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}K) < \infty \quad \text{for } C \in \mathcal{C}. \tag{11.4}$$

If X is stationary, (11.4) is equivalent to (4.4).

General assumption. We assume throughout Sections 11.1 and 11.2 that the occurring particle processes satisfy (11.1) with locally finite Θ, and thus also (11.4).

In analogy to Sections 4.1 and 9.2, we now want to define densities of translation invariant and measurable functionals φ for the particle process X. Since these densities will depend on the location in space, they will be functions and not constants. Therefore, we need an appropriate local concept. As in Section 9.2, we start with a translation invariant, additive, and measurable functional $\varphi : \mathcal{R} \to \mathbb{R}$. In addition, we require that the restriction of φ to $\mathcal{K}$ is continuous and nonnegative. For simplicity, in this chapter, we call φ a **standard functional**. We say that φ has a **local extension** Φ if $\Phi : \mathcal{R} \times \mathcal{B} \to \mathbb{R}$ is a kernel, in the sense that $\Phi(\cdot, A)$ is a measurable function on $\mathcal{R}$ for each $A \in \mathcal{B}$ and $\Phi(K, \cdot)$ is a finite signed Borel measure on $\mathbb{R}^d$ for each $K \in \mathcal{R}$, and if Φ has the following properties:

- $\varphi(K) = \Phi(K, \mathbb{R}^d)$ for all $K \in \mathcal{R}$,
- $\Phi(K, \cdot) \geq 0$ for $K \in \mathcal{K}$,
- $\Phi(K, \cdot)$ is additive in K, for $K \in \mathcal{K}$,
- Φ is translation covariant, that is, satisfies $\Phi(K + x, A + x) = \Phi(K, A)$ for $K \in \mathcal{R}$, $A \in \mathcal{B}$, $x \in \mathbb{R}^d$,
- Φ is locally determined, that is, $\Phi(K, A) = \Phi(M, A)$ for $K, M \in \mathcal{R}$, $A \in \mathcal{B}$, if there is an open set $U \subset \mathbb{R}^d$ with $K \cap U = M \cap U$ and $A \subset U$,
- $K \mapsto \Phi(K, \cdot)$ is weakly continuous on $\mathcal{K}'$.

Typical examples of standard functionals having a local extension are, of course, the intrinsic volumes, but there are many others.

For a standard functional φ with local extension Φ, we define the **φ-density** $\overline{\varphi}(X, \cdot)$, as a function on $\mathbb{R}^d$, by

$$\overline{\varphi}(X, z) := \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \eta(K, z - x) \, \Phi(K, \mathrm{d}x) \, \mathbb{Q}(\mathrm{d}K).$$

If X is stationary, then

$$\overline{\varphi}(X, z) = \gamma \int_{\mathcal{K}_0} \varphi(K) \, \mathbb{Q}(\mathrm{d}K) = \overline{\varphi}(X)$$

is the φ-density defined in Section 9.2.

Theorem 11.1.1. *Let X be a process of convex particles in $\mathbb{R}^d$, and let φ be a standard functional with local extension Φ. Then*

$$\mathbb{E} \sum_{K \in X} \Phi(K, \cdot)$$

is a locally finite measure on $\mathbb{R}^d$ which is absolutely continuous with respect to λ, and $\overline{\varphi}(X, \cdot)$ is a corresponding density.

Moreover, we have

$$\overline{\varphi}(X, z) = \lim_{r \to 0} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{K \in X} \Phi(K, z + rW) \tag{11.5}$$

for λ-almost all $z \in \mathbb{R}^d$ and all $W \in \mathcal{K}$ with $V_d(W) > 0$.

Proof. In order to show the local finiteness, let $B \in \mathcal{B}$ be a bounded Borel set. Choose $r > 0$ with $B \subset \operatorname{int} rB^d$. Then, using Campbell's theorem, the facts that Φ is locally determined and that φ is continuous on $\mathcal{K}'$, we obtain

$$\mathbb{E} \sum_{K \in X} \Phi(K, B)$$

$$= \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \Phi(K + y, B) \eta(K, y) \, \lambda(\mathrm{d}y) \, \mathbb{Q}(\mathrm{d}K)$$

$$\le \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \varphi((K+y) \cap rB^d) \mathbf{1}\{(K+y) \cap rB^d \neq \emptyset\} \eta(K,y) \, \lambda(\mathrm{d}y) \, \mathbb{Q}(\mathrm{d}K)$$
$$\le c(rB^d) \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}\{(K+y) \cap rB^d \neq \emptyset\} \eta(K,y) \, \lambda(\mathrm{d}y) \, \mathbb{Q}(\mathrm{d}K)$$
$$< \infty,$$

by (11.4).

In a similar manner, we get

$$\begin{aligned}\mathbb{E} \sum_{K \in X} \Phi(K,B) &= \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \Phi(K+y,B) \eta(K,y) \, \lambda(\mathrm{d}y) \, \mathbb{Q}(\mathrm{d}K) \\ &= \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{B-y}(x) \eta(K,y) \, \Phi(K,\mathrm{d}x) \, \lambda(\mathrm{d}y) \, \mathbb{Q}(\mathrm{d}K) \\ &= \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{B}(z) \eta(K,z-x) \, \Phi(K,\mathrm{d}x) \, \lambda(\mathrm{d}z) \, \mathbb{Q}(\mathrm{d}K) \\ &= \int_B \left(\int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \eta(K,z-x) \, \Phi(K,\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}K) \right) \lambda(\mathrm{d}z)\end{aligned}$$

which proves the absolute continuity and the stated form of the density.

The limit relation follows from Lebesgue's differentiation theorem (see, e.g., Rudin [654, Th. 8.8] or Wheeden and Zygmund [811, Th. 7.2]). □

If $\eta(K,\cdot)$ is continuous, uniformly in K, then the function $\overline{\varphi}(X,\cdot)$ is continuous and, therefore, the limit relation (11.5) holds for all z.

As a first example of the application of Theorem 11.1.1, we choose $\varphi = V_j$, the jth intrinsic volume. The local extension of V_j is given by the curvature measure Φ_j. Thus, we obtain the following generalization of Corollary 9.4.2.

Corollary 11.1.1. *Let X be a process of convex particles in $\mathbb{R}^d$ and let $j \in \{0,\dots,d\}$. Then*

$$\mathbb{E} \sum_{K \in X} \Phi_j(K,\cdot)$$

is a locally finite measure on $\mathbb{R}^d$ which is absolutely continuous with respect to λ, and a density is given by

$$\begin{aligned}\overline{V}_j(X,z) &:= \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \eta(K,z-x) \, \Phi_j(K,\mathrm{d}x) \, \mathbb{Q}(\mathrm{d}K) \qquad (11.6) \\ &= \lim_{r \to 0} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{K \in X} \Phi_j(K, z+rW)\end{aligned}$$

for λ-almost all $z \in \mathbb{R}^d$ and all $W \in \mathcal{K}$ with $V_d(W) > 0$.

One could have expected that a locally defined intrinsic volume $\overline{V}_j(X,z)$ should satisfy

$$\overline{V}_j(X,z) = \lim_{r\to 0} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{K\in X} V_j(K \cap (z+rW)),$$

but, for $j \in \{0,\dots,d-1\}$, this does not even make sense for stationary and isotropic X, since the limit on the right side does not exist in general, as one can see from (9.32).

For the second example, we choose $\varphi(K) = \binom{d}{j} V(K[j], -M[d-j])$, $j \in \{1,\dots,d-1\}$, with fixed $M \in \mathcal{K}'$. According to (6.25), the local extension is given by the mixed measure $\Phi^{(0)}_{j,d-j}(K,M;\cdot \times \mathbb{R}^d)$.

Corollary 11.1.2. *Let X be a process of convex particles in $\mathbb{R}^d$, let $M \in \mathcal{K}'$ and $j \in \{0,\dots,d\}$. Then*

$$\mathbb{E} \sum_{K\in X} \Phi^{(0)}_{j,d-j}(K,M;\cdot \times \mathbb{R}^d)$$

is a locally finite measure on $\mathbb{R}^d$ which is absolutely continuous with respect to λ, and a density is given by

$$\begin{aligned}
&\binom{d}{j} \overline{V}(X[j], -M[d-j]; z) \\
&:= \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \eta(K, z-x)\, \Phi^{(0)}_{j,d-j}(K,M; \mathrm{d}x \times \mathbb{R}^d)\, \mathbb{Q}(\mathrm{d}K) \\
&= \lim_{r\to 0} \frac{1}{V_d(rW)} \, \mathbb{E} \sum_{K\in X} \Phi^{(0)}_{j,d-j}(K,M;(z+rW) \times \mathbb{R}^d)
\end{aligned}$$

for λ-almost all $z \in \mathbb{R}^d$ and all $W \in \mathcal{K}$ with $V_d(W) > 0$.

For $M = B^d$, Corollary 11.1.2 reduces to Corollary 11.1.1.

However, we may also let M vary and apply Theorem 11.1.1 a second time. Since this would involve independent copies of X, we state the corresponding result only for Poisson processes, to which Corollary 3.2.4 applies. Then we get a density for mixed volumes of the particle process $X^2_{\neq}$. For simplicity, we also omit the corresponding local limit relations in the following results.

Corollary 11.1.3. *Let X be a Poisson process of convex particles in $\mathbb{R}^d$ and let $j \in \{0,\dots,d\}$. Then*

$$\mathbb{E} \sum_{(K,M)\in X^2_{\neq}} \Phi^{(0)}_{j,d-j}(K,M;\cdot)$$

is a locally finite measure on $(\mathbb{R}^d)^2$ which is absolutely continuous with respect to λ^2, and a density is given by

$$\binom{d}{j}\overline{V}(X[j], -X[d-j]; z_1, z_2)$$
$$:= \int_{\mathcal{K}_0}\int_{\mathcal{K}_0}\int_{(\mathbb{R}^d)^2} \eta(K_1, z_1 - x_1)\eta(K_2, z_2 - x_2)\, \Phi^{(0)}_{j,d-j}(K_1, K_2; \mathrm{d}(x_1, x_2))$$
$$\times\, \mathbb{Q}(\mathrm{d}K_1)\, \mathbb{Q}(\mathrm{d}K_2)$$

for λ^2-almost all $(z_1, z_2) \in (\mathbb{R}^d)^2$.

As a further generalization of all three corollaries, we may consider the mixed functional $V^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k)$ introduced in Section 6.4. We can keep some of the K_i fixed and let the others vary in X. Repeated application of Theorem 11.1.1 to the Poisson process X, where the local extension at each step uses the mixed measures $\Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; \cdot)$ in a suitable way, yields the existence of the density $\overline{V}^{(j)}_{m_1,\dots,m_n,m_{n+1},\dots,m_k}(X, \dots, X, K_{n+1}, \dots, K_k; \cdot)$ as a function on $(\mathbb{R}^d)^n$. We formulate this result only for the case $n = k$.

Corollary 11.1.4. *Let X be a Poisson process of convex particles in $\mathbb{R}^d$, let $k \in \mathbb{N}$, $j \in \{0, \dots, d\}$ and $m_1, \dots, m_k \in \{j, \dots, d\}$ with*

$$\sum_{i=1}^{k} m_i = (k-1)d + j.$$

Then,

$$\mathbb{E} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}} \Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; \cdot)$$

is a locally finite measure on $(\mathbb{R}^d)^k$ which is absolutely continuous with respect to λ^k, and a density is given by

$$\overline{V}^{(j)}_{m_1,\dots,m_k}(X, \dots, X; z_1, \dots, z_k)$$
$$:= \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0}\int_{(\mathbb{R}^d)^k} \eta(K_1, z_1 - x_1) \cdots \eta(K_k, z_k - x_k)$$
$$\times\, \Phi^{(j)}_{m_1,\dots,m_k}(K_1, \dots, K_k; \mathrm{d}(x_1, \dots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k)$$

for λ^k-almost all $(z_1, \dots, z_k) \in (\mathbb{R}^d)^k$.

We remark that the densities $\overline{V}^{(j)}_{m_1,\dots,m_k}(X, \dots, X; \cdot, \dots, \cdot)$ inherit the important properties of the mixed functionals and mixed measures, namely they are symmetric with respect to a permutation of the indices $m_1, \dots, m_k$ (and the corresponding variables), and they obey a decomposition property: if $m_1 = d$, then

$$\overline{V}^{(j)}_{m_1,\dots,m_k}(X, \dots, X; z_1, \dots, z_k)$$
$$= \overline{V}_d(X, z_1)\overline{V}^{(j)}_{m_2,\dots,m_k}(X, \dots, X; z_2, \dots, z_k). \tag{11.7}$$

We now turn to Boolean models $Z = Z_X$, where X is a Poisson process on $\mathcal{K}'$ (we still assume that X satisfies (11.1) and (11.4)). Let Z be a Boolean model with convex grains in $\mathbb{R}^d$, let $K \in \mathcal{K}'$ and $\varphi : \mathcal{R} \to \mathbb{R}$ be a measurable, additive and conditionally bounded functional. Then we have

$$\mathbb{E}\,|\varphi(Z \cap K)| < \infty \tag{11.8}$$

and (recall that $K^x := K + x$)

$$\begin{aligned} &\mathbb{E}\,\varphi(Z \cap K) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}_0} \dots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^k} \varphi(K \cap K_1^{x_1} \cap \ldots \cap K_k^{x_k}) \\ &\qquad \times \eta(K_1, x_1) \cdots \eta(K_k, x_k)\, \lambda^k(\mathrm{d}(x_1, \ldots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k). \end{aligned} \tag{11.9}$$

This follows from (9.8), together with the special form of the intensity measure.

In addition, we now assume that φ is a standard functional (hence translation invariant) with local extension Φ. As in Section 6.4, we can infer that there are uniquely determined kernels $\Phi_{(k)} : \mathcal{K}^k \times \mathcal{B}((\mathbb{R}^d)^k) \to \mathbb{R}^+$, for $k = 1, 2, \ldots$, such that

$$\begin{aligned} &\int_{(\mathbb{R}^d)^{k-1}} \Phi(K_1 \cap K_2^{x_2} \cap \ldots \cap K_k^{x_k}, A_1 \cap A_2^{x_2} \cap \ldots \cap A_k^{x_k})\, \lambda^{k-1}(\mathrm{d}(x_2, \ldots, x_k)) \\ &= \Phi_{(k)}(K_1, \ldots, K_k; A_1 \times \ldots \times A_k) \end{aligned} \tag{11.10}$$

holds for all $k \in \mathbb{N}$, $K_1, \ldots, K_k \in \mathcal{K}$, $A_1, \ldots, A_k \in \mathcal{B}$. Namely, (11.10) for all Borel sets $A_1, \ldots, A_k$ is equivalent to

$$\begin{aligned} &\int_{(\mathbb{R}^d)^{k-1}} \int_{\mathbb{R}^d} g(x_1, x_1 - x_2, \ldots, x_1 - x_k)\, \Phi(K_1 \cap K_2^{x_2} \cap \ldots \cap K_k^{x_k}, \mathrm{d}x_1) \\ &\times \lambda^{k-1}(\mathrm{d}(x_2, \ldots, x_k)) \\ &= \int_{(\mathbb{R}^d)^k} g(x_1, \ldots, x_k)\, \Phi_{(k)}(K_1, \ldots, K_k; \mathrm{d}(x_1, \ldots, x_k)) \end{aligned} \tag{11.11}$$

for all continuous functions g on $(\mathbb{R}^d)^k$, provided that the measure on the right side exists. Due to the properties of Φ, the mapping

$$g \mapsto \int_{\mathbb{R}^d} g(x_1, x_1 - x_2, \ldots, x_1 - x_k)\, \Phi(K_1 \cap K_2^{x_2} \cap \ldots \cap K_k^{x_k}, \mathrm{d}x_1)$$

is continuous on $\mathbf{C}_c((\mathbb{R}^d)^k)$, for λ^{k-1}-almost all $(x_2, \ldots, x_k)$. Therefore, the left side of (11.11) defines a positive linear functional T on $\mathbf{C}_c((\mathbb{R}^d)^k)$ through

$$\begin{aligned} T(g) := &\int_{(\mathbb{R}^d)^{k-1}} \int_{\mathbb{R}^d} g(x_1, x_1 - x_2, \ldots, x_1 - x_k)\, \Phi(K_1 \cap K_2^{x_2} \cap \ldots \cap K_k^{x_k}, \mathrm{d}x_1) \\ &\times \lambda^{k-1}(\mathrm{d}(x_2, \ldots, x_k)). \end{aligned}$$

The existence and uniqueness of the measure $\Phi_{(k)}(K_1, \dots, K_k; \cdot)$ now follows from the Riesz representation theorem. Since

$$(K_1, \dots, K_k) \mapsto \Phi(K_1 \cap K_2^{x_2} \cap \dots \cap K_k^{x_k}, \cdot)$$

is continuous on $(\mathcal{K}')^k$, for λ^{k-1}-almost all $(x_2, \dots, x_k) \in (\mathbb{R}^d)^{k-1}$ (and by our assumptions on Φ), we obtain the continuity (and hence measurability) of

$$(K_1, \dots, K_k) \mapsto \Phi_{(k)}(K_1, \dots, K_k; \cdot).$$

Finally, $\Phi_{(k)}(K_1, \dots, K_k; A_1 \times \dots \times A_k)$ is invariant under simultaneous permutations of the bodies K_i and the sets A_i.

We call $\Phi_{(1)}, \Phi_{(2)}, \dots$ the **associated kernels** of Φ. We remark that, since Φ is locally determined, the same is true for the kernel $\Phi_{(k)}$. Therefore, we can replace the convex body K_i by an unbounded convex set, as long as the corresponding Borel set A_i is bounded. Also, the translation covariance of Φ implies that $\Phi_{(k)}$ is translation covariant in each variable K_i (with associated Borel set A_i).

Since φ and the local extension Φ are defined for sets $K \in \mathcal{R}$ and since Φ is locally determined, the value $\Phi(Z, A)$ exists for bounded Borel sets $A \in \mathcal{B}$ and yields a (random) signed Radon measure $\Phi(Z, \cdot)$. We now show that $\mathbb{E}\,\Phi(Z, \cdot)$ is absolutely continuous and prove a representation of the density.

Theorem 11.1.2. *Let Z be a Boolean model in $\mathbb{R}^d$ with convex grains and φ a standard functional with local extension Φ and associated kernels $\Phi_{(k)}$, $k \in \mathbb{N}$. Then*

$$\mathbb{E}\,\Phi(Z, \cdot)$$

is a signed Radon measure on $\mathbb{R}^d$ which is absolutely continuous with respect to λ. For λ-almost all $z \in \mathbb{R}^d$, its density $\overline{\varphi}(Z, \cdot)$ satisfies

$$\overline{\varphi}(Z, z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^k} \eta(K_1, z - x_1) \cdots \eta(K_k, z - x_k)$$
$$\times \Phi_{(k)}(K_1, \dots, K_k; \mathrm{d}(x_1, \dots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k).$$

Proof. For the local finiteness, let $B \in \mathcal{B}$ be a bounded Borel set with $B \subset \operatorname{int} rB^d$, for some $r > 0$. Applying (11.8) with $\varphi = \Phi(\cdot, B)$ and $K = rB^d$, we obtain

$$\mathbb{E}\,|\Phi(Z, B)| < \infty.$$

Moreover, from (11.9) and (11.10), it follows that

$$\mathbb{E}\,\Phi(Z, B)$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^k} \int_{\mathbb{R}^d} \mathbf{1}_B(x_0)\, \Phi(rB^d \cap K_1^{x_1} \cap \dots \cap K_k^{x_k}, \mathrm{d}x_0)$$
$$\times\, \eta(K_1, x_1) \cdots \eta(K_k, x_k)\, \lambda^k(\mathrm{d}(x_1, \dots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^{k+1}} \mathbf{1}_B(x_0)\eta(K_1, x_0 - x_1) \cdots \eta(K_k, x_0 - x_k)$$

$$\times \Phi_{(k+1)}(rB^d, K_1, \dots, K_k; \mathrm{d}(x_0, \dots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^{k+1}} \mathbf{1}_B(x_0)\eta(K_1, x_0 - x_1) \cdots \eta(K_k, x_0 - x_k)$$

$$\times \Phi_{(k+1)}(\mathbb{R}^d, K_1, \dots, K_k; \mathrm{d}(x_0, \dots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k).$$

Here we have used that $\Phi_{(k+1)}$ is locally determined.

The translation covariance of $\Phi_{(k+1)}$ in the first variable shows that

$$\Phi_{(k+1)}(\mathbb{R}^d, K_1, \dots, K_k; \cdot) = \lambda \otimes \Phi_{(k)}(K_1, \dots, K_k; \cdot).$$

In fact, for bounded $A \in \mathcal{B}$,

$$\Phi_{(k+1)}(\mathbb{R}^d, K_1, \dots, K_k; A \times A_1 \times \ldots \times A_k)$$

$$= \int_{(\mathbb{R}^d)^k} \Phi(K_1^{x_1} \cap \ldots \cap K_k^{x_k}, A \cap A_1^{x_1} \cap \ldots \cap A_k^{x_k})\, \lambda^k(\mathrm{d}(x_1, \dots, x_k))$$

$$= \int_{(\mathbb{R}^d)^{k-1}} \int_{\mathbb{R}^d} \Phi(K_1 \cap K_2^{x_2} \cap \ldots \cap K_k^{x_k}, (A - x) \cap A_1 \cap A_2^{x_2} \ldots \cap A_k^{x_k})$$

$$\times \lambda(\mathrm{d}x)\, \lambda^{k-1}(\mathrm{d}(x_2, \dots, x_k))$$

$$= \lambda(A) \int_{(\mathbb{R}^d)^{k-1}} \Phi(K_1 \cap K_2^{x_2} \cap \ldots \cap K_k^{x_k}, A_1 \cap A_2^{x_2} \ldots \cap A_k^{x_k})$$

$$\times \lambda^{k-1}(\mathrm{d}(x_2, \dots, x_k)),$$

by Theorem 5.2.1.

Hence, we conclude from Fubini's theorem that

$$\mathbb{E}\, \Phi(Z, B)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^k}$$

$$\times \int_{\mathbb{R}^d} \mathbf{1}_B(x_0)\eta(K_1, x_0 - x_1) \cdots \eta(K_k, x_0 - x_k)\, \lambda(\mathrm{d}x_0)$$

$$\times \Phi_{(k)}(K_1, \dots, K_k; \mathrm{d}(x_1, \dots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k)$$

$$= \int_B \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^k} \eta(K_1, z - x_1) \cdots \eta(K_k, z - x_k) \right.$$

$$\left. \times \Phi_{(k)}(K_1, \dots, K_k; \mathrm{d}(x_1, \dots, x_k))\, \mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k) \right) \lambda(\mathrm{d}z).$$

This confirms the result. □

We apply Theorem 11.1.2 with $\varphi = V_j$. The local extension of $V_j(K)$ is the jth curvature measure $\Phi_j(K,\cdot)$. For the associated kernel $(\Phi_j)_{(k)}$, Theorem 6.4.1 yields

$$(\Phi_j)_{(k)}(K_1,\dots,K_k;\cdot) = \sum_{\substack{m_1,\dots,m_k=j\\ m_1+\dots+m_k=(k-1)d+j}}^{d} \Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\cdot).$$

Hence, $\mathbb{E}\,\Phi_j(Z,\cdot)$ is absolutely continuous with density a.e. given by

$$\begin{aligned}
&\overline{V}_j(Z,z)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{\substack{m_1,\dots,m_k=j\\ m_1+\dots+m_k=(k-1)d+j}}^{d} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^k} \eta(K_1, z-x_1)\cdots\\
&\qquad \times \eta(K_k, z-x_k)\, \Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k; \mathrm{d}(x_1,\dots,x_k))\, \mathbb{Q}(\mathrm{d}K_1)\cdots \mathbb{Q}(\mathrm{d}K_k).
\end{aligned}$$

From Corollary 11.1.4 we obtain

$$\begin{aligned}
&\overline{V}_j(Z,z)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{\substack{m_1,\dots,m_k=j\\ m_1+\dots+m_k=(k-1)d+j}}^{d} \overline{V}^{(j)}_{m_1,\dots,m_k}(X,\dots,X;z,\dots,z).
\end{aligned}$$

We use the decomposition property (11.7) and get, with arguments similar to those in the deduction of Theorem 9.1.3,

$$\begin{aligned}
&\overline{V}_j(Z,z)\\
&= \sum_{s=1}^{d-j} \sum_{r=0}^{\infty} \binom{r+s}{r} \frac{(-1)^{r+s-1}}{(r+s)!} \overline{V}_d(X,z)^r\\
&\qquad \times \sum_{\substack{m_1,\dots,m_s=j\\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X;z,\dots,z)\\
&= -\mathrm{e}^{-\overline{V}_d(X,z)} \sum_{s=1}^{d-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j\\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X;z,\dots,z)\\
&= \mathrm{e}^{-\overline{V}_d(X,z)} \Bigg(\overline{V}_j(X,z) - \sum_{s=2}^{d-j} \frac{(-1)^s}{s!}\\
&\qquad \times \sum_{\substack{m_1,\dots,m_s=j+1\\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X;z,\dots,z) \Bigg).
\end{aligned}$$

Hence, we arrive at the following result.

Theorem 11.1.3. *Let Z be a Boolean model in $\mathbb{R}^d$ with convex grains. Then, for λ-almost all z,*

$$\overline{V}_d(Z,z) = 1 - \mathrm{e}^{-\overline{V}_d(X,z)}, \tag{11.12}$$

$$\overline{V}_{d-1}(Z,z) = \mathrm{e}^{-\overline{V}_d(X,z)}\overline{V}_{d-1}(X,z),$$

and

$$\overline{V}_j(Z,z) = \mathrm{e}^{-\overline{V}_d(X,z)} \Bigg(\overline{V}_j(X,z) - \sum_{s=2}^{d-j} \frac{(-1)^s}{s!} \times \sum_{\substack{m_1,\dots,m_s=j+1 \\ m_1+\dots+m_s=(s-1)d+j}}^{d-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X;z,\dots,z) \Bigg),$$

for $j = 0,\dots,d-2$.

If Z is stationary, this reduces to Theorem 9.1.5, and if Z is also isotropic, we get Theorem 9.1.3.

Notes for Section 11.1

1. As we have already mentioned, specific intrinsic volumes for non-stationary (Poisson) particle processes and Boolean models were introduced by Fallert [222, 223]. There, one also finds Corollaries 11.1.1, 11.1.4 and Theorem 11.1.3. Corollaries 11.1.2, 11.1.3 are special cases of more general results in Weil [801]. Theorem 6 in [801] gives formulas for the density of mixed volumes,

$$\overline{V}(Z[j], M[d-j], z)$$

for a Boolean model Z with polyconvex grains and a fixed body $M \in \mathcal{K}'$, which are in analogy to Theorem 11.1.3. The proof of Theorem 6 contains some misprints ($z_1,\dots,z_k$ have to be replaced by $z,\dots,z$ and $\lambda(\mathrm{d}z_1)\cdots\lambda(\mathrm{d}z_k)$ by $\lambda(\mathrm{d}z)$). The paper [801] also presents more explicit formulas for the densities $\overline{V}_i(Z,\cdot)$, $i = 0,1,2$, for a planar Boolean model with circular grains.

Formulas for densities of some of the intrinsic volumes (volume density, surface area density) for non-stationary Boolean models of (deterministic or random) balls have also been obtained by Hahn, Micheletti, Pohlink and Stoyan [314], K. Mecke [505, 506], Micheletti and Stoyan [516], Quintanilla and Torquato [609, 610].

2. In Note 4 to Section 9.1 we have remarked that, for a stationary Boolean model Z with convex grains in dimensions 2 and 3, densities for mixed volumes of Z determine the intensity γ uniquely. For non-stationary Boolean models in $\mathbb{R}^2$, a corresponding result was obtained by Weil [799]. It was shown that the values $\overline{V}_0(Z,z)$, $\overline{V}(Z[1], M[1]; z)$, for all $M \in \mathcal{K}'$ and $\overline{V}_2(Z,z)$, determine the specific Euler characteristic $\overline{V}_0(X,z)$ at z uniquely. The corresponding three-dimensional case was settled in Goodey and Weil [280] under a symmetry condition. Without this, a uniqueness result for $\overline{V}_0(X,z)$ was shown, if instead of the local mean mixed volumes $\overline{V}(Z[1], M[2]; z)$ and $\overline{V}(Z[2], M[1]; z)$ the densities of support functions and surface area measures for Z at z are given (see also the following note).

3. We have applied formula (11.9) mainly to real functionals φ. It can also be applied to measure- or function-valued functionals. In particular, this yields the existence of the density $\overline{h}(Z, u; z)$ of the centered support function $h^*(K, u)$, $u \in \mathbb{R}^d$, and a formula expressing it in terms of densities of mixed centered support functions of the particles in X. In view of (6.28), the necessary local extension is given by the **support kernel** $\rho(K, u; \cdot)$, defined as

$$\rho(K, u; B) = \Phi^{(0)}_{1,d-1}(K, u^+; B \times A_{u^\perp}),$$

for $u \in S^{d-1}$ and $B \in \mathcal{B}$. This notion was first studied by Goodey and Weil [281]. Similarly, the existence of the density $\overline{S}(Z, B; z)$ of the surface area measure $S_{d-1}(K, B)$, $B \in \mathcal{B}(S^{d-1})$, follows. Its relation to the corresponding notion for X is given by

$$\overline{S}(Z, \cdot; z) = \mathrm{e}^{-\overline{V}_d(Z,z)}\, \overline{S}(X, \cdot; z).$$

The local extension is given here by a suitable support measure. These results were obtained in Goodey and Weil [280]. In contrast to the stationary case and due to the occurrence of the intensity function η, the function $\overline{h}(Z, \cdot; z)$ need no longer be centered (and for $d \geq 3$ also not convex), and the measure $\overline{S}(Z, \cdot; z)$ need no longer be a surface area measure. This indicates some of the difficulties arising in the non-stationary setting.

4. For a non-stationary particle process X and a functional φ, the densities $\overline{\varphi}(X, z)$ were introduced as functions depending on the location $z \in \mathbb{R}^d$, whereas, for stationary X, they do not depend on z. Conversely, one can ask whether invariance properties of X can be inferred from invariance properties of $\overline{\varphi}(X, z)$, for suitable functionals φ. Results of this type were obtained by Hoffmann [345, 347]. Assume that the intensity measure of the particle process X is of the form (11.3) with a continuous function η. Hoffmann defined the **generalized local mean normal measure** of X at $z \in \mathbb{R}^d$ by

$$\mu_z(A, B) := \int_{\mathcal{K}_0} \mathbf{1}_B(K) \int_{\mathbb{R}^d} \eta(z - x)\, \Xi_{d-1}(K, \mathrm{d}x \times A)\, \mathbb{Q}(\mathrm{d}K)$$

for $A \in \mathcal{B}(S^{d-1})$, $B \in \mathcal{B}(\mathcal{K}_0)$. An intuitive interpretation is obtained from

$$\mathbb{E} \sum_{K \in X} \mathbf{1}_B(K - c(K)) \mathcal{H}^{d-1}(C \cap \tau(K, A)) = \int_C \mu_z(A, B)\, \lambda(\mathrm{d}z)$$

for $C \in \mathcal{B}$, where $\tau(K, A)$ denotes the set of boundary points of K for which an outer normal vector belongs to A. Under the assumption that $\dim K \geq d - 1$ for $\mathbb{Q}$-almost all $K \in \mathcal{K}_0$ and that the support of $\mathbb{Q}$ contains some strictly convex body, Hoffmann proved that X is weakly stationary and weakly isotropic if and only if μ_z is rotation invariant, which means that $\mu_z(\vartheta A, \vartheta B) = \mu_z(A, B)$ for all $z \in \mathbb{R}^d$, $A \in \mathcal{B}(S^{d-1})$, $B \in \mathcal{B}(\mathcal{K}_0)$ and $\vartheta \in SO_d$. Hoffmann also showed a corresponding result for processes of convex cylinders. This comprises Theorem 1 of Schneider [707] (see Theorem 11.3.2 below), which was the motivation for Hoffmann's investigation.

5. Theorem 11.1.3 has been extended to Boolean models of cylinders by Hoffmann [345, 348]. Due to the local nature of the mixed measures, such an extension seems natural; the main effort went into finding the special form of the mixed measures for cylinders. Special cylinder processes were also studied by Spiess and Spodarev [732].

6. For a stationary Poisson process X on $\mathcal{K}'$ and the corresponding Boolean model Z the intersection density $\gamma_d(X)$ and the mean visible volume $\overline{V}_s(Z)$ were introduced and studied in Section 4.6, and sharp lower and upper estimates for the product $\gamma_d(X)\overline{V}_s(Z)$ were given in Theorem 4.6.3. Hoffmann [345] has studied intersection densities and mean visible volumes for non-stationary Poisson processes and Boolean models and has obtained some generalizations of Theorem 4.6.3. He has also considered intersection densities of a different kind, where the Hausdorff measure is replaced by a curvature measure.

11.2 Contact Distributions

We continue the investigation of general Boolean models Z with convex grains and consider generalized contact distributions. As an immediate generalization of the function introduced in Section 2.4, in the stationary case, we define the **contact distribution function** $H_B(x,\cdot)$ of a random closed set $Z \subset \mathbb{R}^d$ as the distribution function of the B-distance $d_B(x,Z)$ from a point $x \notin Z$ to Z, hence, for $r \geq 0$,

$$\begin{aligned} H_B(x,r) &:= \mathbb{P}((x+rB)\cap Z \neq \emptyset \,|\, x \notin Z) \\ &= \mathbb{P}(d_B(x,Z) \leq r \,|\, x \notin Z). \end{aligned}$$

Here the **gauge body** (or structuring element) B is a convex body containing 0, and we assume that the local volume fraction $\mathbb{P}(x \in Z) = \overline{V}_d(Z,x)$ is less than one (so that $\mathbb{P}(x \notin Z) > 0$).

For the Boolean model $Z = Z_X$, we use the same notations as in the previous section. Since

$$H_B(x,r) = \frac{\overline{V}_d(Z-rB,x) - \overline{V}_d(Z,x)}{1-\overline{V}_d(Z,x)},$$

we obtain from (11.12) and (11.6)

$$\begin{aligned} H_B(x,r) &= \frac{\mathrm{e}^{-\overline{V}_d(X,x)} - \mathrm{e}^{-\overline{V}_d(X-rB,x)}}{\mathrm{e}^{-\overline{V}_d(X,x)}} \\ &= 1 - \exp\left(-\int_{\mathcal{K}_0}\int_{(K-rB)\setminus K} \eta(K,x-y)\,\lambda(\mathrm{d}y)\,\mathbb{Q}(\mathrm{d}K)\right). \quad (11.13) \end{aligned}$$

To the inner integral, we apply formula (14.26) involving the relative support measures $\Xi_j(K;B;\cdot)$ and get

$$\begin{aligned} &\int_{(K-rB)\setminus K} \eta(K,x-y)\,\lambda(\mathrm{d}y) \quad (11.14) \\ &= \sum_{j=0}^{d-1}(d-j)\kappa_{d-j}\int_0^r\int_{(\mathbb{R}^d)^2} t^{d-1-j}\eta(K,x-y-tb)\,\Xi_j(K;B;\mathrm{d}(y,b))\,\mathrm{d}t. \end{aligned}$$

The definition of the relative support measures requires that K and B have independent support sets (see Section 14.3). This is satisfied, for example, if one of the bodies K, B is strictly convex.

Inserting (11.14) into (11.13), we obtain the following theorem.

Theorem 11.2.1. *Let Z be a Boolean model in $\mathbb{R}^d$ with convex grains and let B be a gauge body. Assume that K and B have independent support sets, for $\mathbb{Q}$-almost all K. Then*

$$H_B(x,r) = 1 - \exp\left(-\int_0^r h_B(x,t)\,\mathrm{d}t\right)$$

for $r \geq 0$, with

$$h_B(x,t) := \sum_{j=0}^{d-1}(j+1)\kappa_{j+1}t^j \int_{\mathcal{K}_0}\int_{(\mathbb{R}^d)^2} \eta(K, x-y-tb) \times \Xi_{d-1-j}(K;B;\mathrm{d}(y,b))\,\mathbb{Q}(\mathrm{d}K).$$

If $B = B^d$, the measure $\Xi_j(K;B;\cdot)$ is the (ordinary) support measure $\Xi_j(K,\cdot)$ of K. Hence, we obtain a formula for the **spherical contact distribution function** $H(x,\cdot)$ of Z.

Corollary 11.2.1. *For a Boolean model Z with convex grains, we have*

$$H(x,r) = 1 - \exp\left(-\int_0^r h(x,t)\,\mathrm{d}t\right)$$

for $r \geq 0$, with

$$h(x,t) := \sum_{j=0}^{d-1}(j+1)\kappa_{j+1}t^j \int_{\mathcal{K}_0}\int_{(\mathbb{R}^d)^2} \eta(K,x-y-tb)\,\Xi_{d-1-j}(K,\mathrm{d}(y,b))\,\mathbb{Q}(\mathrm{d}K).$$

Theorem 11.2.1 shows that $H_B(x,\cdot)$ is differentiable. In particular, if for $\mathbb{Q}$-almost all K the function $\eta(K,\cdot)$ is continuous, we get

$$\begin{aligned}\frac{\partial}{\partial r}H(x,r)\Big|_{r=0} &= h(x,0)\\ &= 2\int_{\mathcal{K}_0}\int_{\mathbb{R}^d} \eta(K,x-y)\,\Phi_{d-1}(K,\mathrm{d}y)\,\mathbb{Q}(\mathrm{d}K)\\ &= 2\overline{V}_{d-1}(X,x)\end{aligned}$$

and thus

$$2\overline{V}_{d-1}(Z,x) = (1-\overline{V}_d(Z,x))\frac{\partial}{\partial r}H(x,r)\Big|_{r=0}.$$

Now we consider generalized contact distribution functions, involving directions and local geometric information in the contact points. As we have shown in Lemma 9.5.1, the distance $d_B(x, Z)$ is almost surely attained at a single particle Z_i of the underlying Poisson process X, thus $d_B(x, Z) = d_B(x, Z_i)$. This implies $x \in \mathrm{bd}\,(Z_i - rB)$ with $r := d_B(x, Z)$. If Z_i and B have almost surely independent support sets (as we shall assume below), the decomposition $x = z + rb$, $z \in \mathrm{bd}\, Z_i$, $b \in \mathrm{bd}\,(-B)$, is unique. With the notation introduced before Theorem 14.3.2, we have $z =: p_B(Z, x)$ and $b =: u_B(Z, x)$, thus

$$p_B(Z, x) = x - d_B(x, Z)u_B(Z, x).$$

We call $p_B(Z, x)$ the B**-contact point** in Z and $-u_B(Z, x)$ the B**-direction** from x to Z. For simplicity, we just speak of the contact point and the direction.

It is possible to exploit additional local information at the contact point. For this, we assume that a mapping $\rho : \mathcal{S} \times \mathbb{R}^d \to \mathbb{R}$ (where $\mathcal{S}$ is the extended convex ring) is given which is measurable and translation covariant, that is, satisfies $\rho(F + y, x + y) = \rho(F, x)$ for $F \in \mathcal{S}$ and $x, y \in \mathbb{R}^d$. Moreover, we assume that $\rho(F, x) = 0$ if $x \notin \mathrm{bd}\, F$ and that ρ is 'local' in the sense that, for any $x \in \mathbb{R}^d$ and any neighborhood U of x, we have $\rho(F, x) = \rho(F \cap U, x)$. For example, $\rho(F, x)$ could be the value of a curvature function of $\mathrm{bd}\, F$ at x, and 0 if this is not defined. In the following, we write

$$l_B(Z, x) := \rho(Z, p_B(Z, x)).$$

As a generalization of Theorem 11.2.1 (and also of Theorem 9.5.2) we show the following result.

Theorem 11.2.2. *Let Z be a Boolean model in $\mathbb{R}^d$ with convex grains and let B be a gauge body. Assume that K and B have independent support sets, for $\mathbb{Q}$-almost all K. Let $g \geq 0$ be a measurable function on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$. Then, for $x \in \mathbb{R}^d$ with $\mathbb{P}(x \notin Z) > 0$, we have*

$$\begin{aligned} &\mathbb{E}\left(\mathbf{1}\{d_B(x, Z) < \infty\} g(d_B(x, Z), u_B(Z, x), l_B(Z, x)) \,|\, x \notin Z\right) \\ &= \sum_{j=0}^{d-1} (j+1)\kappa_{j+1} \int_0^\infty t^j (1 - H_B(x, t)) \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^2} g(t, b, \rho(K, y)) \\ &\quad \times \eta(K, x - y - tb)\, \Xi_{d-1-j}(K; B; \mathrm{d}(y, b))\, \mathbb{Q}(\mathrm{d}K)\, \mathrm{d}t. \end{aligned}$$

Proof. We fix x with $\mathbb{P}(x \notin Z) > 0$. The following arguments are quite similar to those employed in the proof of Theorem 9.5.2; we even use some of the notation introduced there. Namely, for an enumeration $X = \{Z_1, Z_2, \ldots\}$ we define the events

$$A_n := \{0 < d_B(x, Z_n) < \infty\},$$

$$B_n := \{d_B(x, U(X \setminus \{Z_n\})) > d_B(x, Z_n)\}$$

and

$$C_n := \{(B, Z_n) \in \mathcal{K}^2_{ind}\},$$

where $\mathcal{K}^2_{ind}$ denotes the set of pairs $(K, M) \in (\mathcal{K}')^2$ of convex bodies with independent support sets. Then

$$(d_B(x, Z), u_B(Z, x), l_B(Z, x)) = (d_B(x, Z_n), u_B(Z_n, x), l_B(Z_n, x))$$

on $A_n \cap B_n \cap C_n$ and almost surely

$$\{0 < d_B(x, Z) < \infty\} = \bigcup_{n=1}^{\infty} (A_n \cap B_n \cap C_n).$$

We abbreviate

$$\widetilde{g}(K) := g(d_B(x, K), u_B(K, x), l_B(K, x))$$

for $K \in \mathcal{K}'$. Using Theorem 3.2.5 and formula (14.27), we obtain

$$\begin{aligned}
&\mathbb{E}\left(\mathbf{1}\{0 < d_B(x, Z) < \infty\} g(d_B(x, Z), u_B(Z, x), l_B(Z, x))\right) \\
&= \mathbb{E} \sum_{n=1}^{\infty} \mathbf{1}_{A_n \cap B_n \cap C_n} \widetilde{g}(Z_n) \\
&= \mathbb{E}\Bigg(\sum_{K \in X} \mathbf{1}\{0 < d_B(x, K) < \infty\} \mathbf{1}\{(B, K) \in \mathcal{K}^2_{ind}\} \widetilde{g}(K) \\
&\qquad \times \mathbf{1}\{d_B(x, U(X \setminus \{K\})) > d_B(x, K)\} \Bigg) \\
&= \int_{\mathcal{K}'} \mathbf{1}\{0 < d_B(x, K) < \infty\} \mathbf{1}\{(B, K) \in \mathcal{K}^2_{ind}\} \widetilde{g}(K) \\
&\qquad \times \mathbb{P}(d_B(x, U(X)) > d_B(x, K))\, \Theta(\mathrm{d}K) \\
&= \mathbb{P}(x \notin Z) \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbf{1}\{0 < d_B(x, z + K) < \infty\} \widetilde{g}(z + K) \\
&\qquad \times (1 - H_B(x, d_B(x, z + K))) \eta(K, z)\, \lambda(\mathrm{d}z)\, \mathbb{Q}(\mathrm{d}K) \\
&= \mathbb{P}(x \notin Z) \sum_{j=0}^{d-1} (j+1) \kappa_{j+1} \int_0^{\infty} \int_{\mathcal{K}_0} \int_{(\mathbb{R}^d)^2} g(t, b, \rho(K, y)) \eta(K, x - y - tb) \\
&\qquad \times (1 - H_B(x, t)) t^j\, \Xi_{d-1-j}(K; B; \mathrm{d}(y, b))\, \mathbb{Q}(\mathrm{d}K)\, \mathrm{d}t.
\end{aligned}$$

Here we have used that $(y, b) \in \operatorname{supp} \Xi_{d-1-j}(K; B; \cdot)$ implies $\widetilde{g}(x - y - tb + K) = g(t, b, \rho(K, y))$. Division by $\mathbb{P}(x \notin Z)$ yields the assertion. □

We mention some special cases of this result. First, if g depends only on the distance $d_B(x, Z)$, Theorem 11.2.2 reduces to Theorem 11.2.1. This follows from the exponential formula of Lebesgue–Stieltjes calculus (see the corresponding more detailed argument given in Section 9.5). Next, for $B = B^d$, we obtain a result for the spherical contact distribution function, as a generalization of Corollary 11.2.1. Note that, for the spherical contact distribution, the condition $d(x, Z) < \infty$ is satisfied almost surely.

Corollary 11.2.2. *For a Boolean model Z with convex grains, a point $x \in \mathbb{R}^d$ with $\mathbb{P}(x \notin Z) > 0$, and a measurable function $g \geq 0$ on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$, we have*

$$\begin{aligned}&\mathbb{E}\left(g(d(x,Z), u(Z,x), l(Z,x)) \,|\, x \notin Z\right)\\&= \sum_{j=0}^{d-1}(j+1)\kappa_{j+1}\int_0^\infty t^j(1-H(x,t))\int_{\mathcal{K}_0}\int_{\mathbb{R}^d\times\mathbb{R}^d} g(t,u,\rho(K,y))\\&\quad\times \eta(K, x-y-tu)\,\Xi_{d-1-j}(K, \mathrm{d}(y,u))\,\mathbb{Q}(\mathrm{d}K)\,\mathrm{d}t.\end{aligned}$$

If Z is stationary, the formulas in Theorem 11.2.2 and Corollary 11.2.2 simplify only slightly, in that x can be replaced by 0 and the function η by the constant γ. Theorem 11.2.1 and Corollary 11.2.1 then reduce to the corresponding results in Theorem 9.1.1. A further simplification of Corollary 11.2.2 is possible if, for stationary Z (and $x = 0$), the function g depends only on $d(0, Z)$ and $u(Z, 0)$. Then, the support measure $\Xi_{d-1-j}(K, \cdot)$ can be replaced by its image under $(y, u) \mapsto u$, the area measure $\Psi_{d-1-j}(K, \cdot)$.

Corollary 11.2.3. *For a stationary Boolean model Z with convex grains and a measurable function $g \geq 0$ on $\mathbb{R}^+ \times S^{d-1}$, we have*

$$\begin{aligned}&\mathbb{E}\left(g(d(0,Z), u(Z,0)) \,|\, 0 \notin Z\right)\\&= \gamma\sum_{j=0}^{d-1}(j+1)\kappa_{j+1}\int_0^\infty t^j(1-H(t))\int_{\mathcal{K}_0}\int_{S^{d-1}} g(t,u)\\&\quad\times \Psi_{d-1-j}(K, \mathrm{d}u)\,\mathbb{Q}(\mathrm{d}K)\,\mathrm{d}t.\end{aligned}$$

Notes for Section 11.2

1. The results on contact distributions for non-stationary Boolean models and their generalized versions have been obtained in Hug [356], Hug and Last [357], and Hug, Last and Weil [358]; a survey with additional results is Hug, Last and Weil [359] (see also shorter presentations in Weil [802, 803]). In Hug, Last and Weil [358], an even more general version of Theorem 11.2.1 and the subsequent results is obtained, where the point x is replaced by a test body $L \in \mathcal{K}'$. The considered function g can then also depend on the contact point $p_B(L, Z)$ in which the B-distance of L from Z is realized (provided this point is unique). The formula is proved in exactly the same way, but uses a more general Steiner-type result which involves mixed relative support measures depending on three convex bodies.

2. As was shown in Hug and Last [357] and Hug, Last and Weil [358], [359], the results on generalized contact distributions, which we proved here for Boolean models, hold true for random closed sets Z which are the union of a point process X on $\mathcal{K}'$, where the intensity measure Θ of X satisfies (11.4) and the second factorial moment measure $\Lambda^{(2)}$ of X has a certain smoothness property. The resulting formulas for Z are then formulated and proved with the Palm distribution of X. In this general framework, (generalized) contact distributions of Gibbs processes, Cox processes, Poisson cluster processes and more general cluster models (grain models where the underlying ordinary point process is a Poisson cluster process) can be subsumed; the results are surveyed in Hug, Last and Weil [359].

3. In Section 9.5, we have already stated and proved a special case of Theorem 11.2.1, namely Theorem 9.5.2. The latter result concerned a stationary Boolean model Z, where the grains are balls with random radius. In the discussion, we remarked that the intensity γ and the radius distribution $\mathbb{G}$ are determined by the generalized contact distributions of Z. One may expect that this result holds for more general Boolean models Z. The question, which information on the intensity function η and the distribution of the typical grain $\mathbb{Q}$ can be inferred from the generalized contact distributions of Z, is discussed in detail in Hug, Last and Weil [358, §4], and several uniqueness results are given.

4. Boolean models with compact grains. For the spherical contact distribution function and its variants, a far-reaching generalization was obtained by Hug, Last and Weil [361]. They proved a Steiner formula for arbitrary closed sets $F \subset \mathbb{R}^d$, by which support measures $\Xi_j(F,\cdot)$, $j = 0,\dots,d-1$, of F are defined. The latter are signed Radon-type measures on the normal bundle $\operatorname{Nor} F$ of F; they are defined on Borel sets $A \subset \operatorname{Nor} F$, for which the reach function $\delta(F,\cdot)$ is bounded away from 0 and ∞. Here, the **reach** $\delta(F,x,u)$, $(x,u) \in \operatorname{Nor} F$, is the largest $r \geq 0$ such that $x + ru$ has a unique projection point x in F. With the help of this Steiner formula, Boolean models with arbitrary compact grains can be considered (satisfying a condition analogous to (11.1)). The following is one of the results obtained in Hug, Last and Weil [361] (d_{dj} are given constants).

Let Z be a stationary Boolean model with compact grains. Then

$$H(r) = 1 - \exp\left(-\int_0^r h(t)\,\mathrm{d}t\right), \qquad r \geq 0,$$

with

$$h(t) := \sum_{j=0}^{d-1} d_{dj} t^j \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d \times S^{d-1}} \mathbf{1}\{t < \delta(C,x,u)\}\, \Xi_{d-1-j}(C, \mathrm{d}(x,u))\, \mathbb{Q}(\mathrm{d}C).$$

5. The general Steiner formula also yields results for (generalized) contact distributions of arbitrary stationary random closed sets $Z \subset \mathbb{R}^d$. Namely, let

$$H(t,A) := \mathbb{P}(d(0,Z) \leq t, u(0,Z) \in A \,|\, 0 \notin Z), \qquad t \geq 0,\ A \in \mathcal{B}(S^{d-1}).$$

Then,

$$(1-p)H(t,A) = \sum_{j=0}^{d-1} c_{dj} \int_0^t s^j \Gamma_{d-1-j}(A \times (s,\infty])\,\mathrm{d}s$$

with constants c_{di} and

$$\Gamma_i(\cdot) := \mathbb{E} \int_{C^d \times S^{d-1}} \mathbf{1}\{(u, \delta(Z, x, u)) \in \cdot\}\, \Xi_i(Z, \mathrm{d}(x, u)).$$

Thus, $H(\cdot, A)$ is absolutely continuous and we have an explicit formula for the density.

Moreover, $H(\cdot, A)$ is differentiable with the exception of at most countably many points, but it need not be differentiable at 0. If

$$\mathbb{E}|\Xi_i|(Z, B \times S^{d-1}) < \infty \qquad \text{for some } B \in \mathcal{B}(\mathbb{R}^d),\ \lambda(B) > 0,\ i = 0, \dots, d-1,$$

(which excludes fractal behavior, for example), then

$$\lim_{t \to 0+} t^{-1}(1 - p) H(t, A) = \overline{S}_{d-1}(Z, A),$$

for $A \in \mathcal{B}(S^{d-1})$, where

$$\overline{S}_{d-1}(Z, A) := 2\mathbb{E}\, \Xi_{d-1}(Z; C^d \times A) \ < \infty.$$

In particular, for such random closed sets we have

$$(1 - p) H'(0) = 2\overline{V}_{d-1}(Z) := \overline{S}_{d-1}(Z, S^{d-1}).$$

Hence, for a stationary random set Z fulfilling the expectation condition above, the surface area density is defined and, even more, a mean surface area measure $\overline{S}_{d-1}(Z, \cdot)$ exists. The normalized measure

$$R(Z, \cdot) = \frac{\overline{S}_{d-1}(Z, \cdot)}{\overline{S}_{d-1}(Z, S^{d-1})}$$

is called the **rose of directions** of Z. It is the distribution of the (outer) normal in a typical point of $\mathrm{bd}\, Z$.

6. Characterization of convex grains. For a stationary Boolean model Z with convex grains and a gauge body B, Theorem 9.1.1 shows that

$$-\ln(1 - H_B(r))$$

is a polynomial in $r \geq 0$ (of degree d). As we have mentioned earlier, this can be used, for $B = B^d$ or $B = [0, u]$, to obtain simple estimators for the intensity γ, the specific intrinsic volumes $\overline{V}_j(X)$ and other mean functionals, such as $\overline{S}_{d-1}(X, \cdot)$. For non-convex grains, the occurrence of the reach function in the formula, explained in Note 4 above, shows that we can no longer expect a polynomial behavior of contact distributions. This was made more precise by Heveling, Hug and Last [338] and Hug, Last and Weil [362] and yields a possibility to check the convexity of the grains.

Namely, let us consider a stationary Boolean model Z with compact and regular grains (the latter means that $C = \mathrm{cl\,int}\, C$ holds for $\mathbb{Q}$-almost all C). We assume that

$$\int_{\mathcal{C}_0} V_d(\mathrm{conv}\, K + B^d)\, \mathbb{Q}(\mathrm{d}K) < \infty \tag{11.15}$$

and define the **ALLC-function** (average logarithmic linear contact distribution function) L of Z by

$$L(r) := -\int_{S^{d-1}} \ln(1 - H_{[0,u]}(r))\, \sigma_{d-1}(\mathrm{d}u), \qquad r \geq 0.$$

The following result was presented in Hug, Last and Weil [362].

Let Z be a stationary Boolean model in $\mathbb{R}^d$ with regular compact grains satisfying (11.15)*. Then the ALLC-function L of Z is linear if and only if the grains are almost surely convex.*

We sketch the proof of the non-obvious direction. Thus, we assume that L is linear. Then

$$f := \overline{V}_d(X) + L$$

is a polynomial and

$$f(r) = \gamma \int_{S^{d-1}} \int_{\mathcal{C}_0} \lambda(K + r[0,u])\, \mathbb{Q}(\mathrm{d}K)\, \sigma_{d-1}(\mathrm{d}u).$$

We have

$$\begin{aligned} f(r) \leq \widetilde{f}(r) &:= \gamma \int_{S^{d-1}} \int_{\mathcal{C}_0} \lambda(K_u + r[0,u])\, \mathbb{Q}(\mathrm{d}K)\, \sigma_{d-1}(\mathrm{d}u) \\ &= \gamma \int_{S^{d-1}} \int_{\mathcal{C}_0} \Big(\lambda(K_u) + \lambda_{d-1}(K \,|\, u^{\perp}) r\Big)\, \mathbb{Q}(\mathrm{d}K)\, \sigma_{d-1}(\mathrm{d}u) \\ &= \widetilde{a} + \widetilde{b} r, \end{aligned}$$

where K_u is the convexification of K in direction u (for each line l in direction u, we replace $K \cap l$ by its convex hull). Hence,

$$f(r) = a + br \qquad \text{with } a \leq \widetilde{a},\ b \leq \widetilde{b}.$$

For sufficiently large r, we have $K + r[0,u] = K_u + r[0,u]$, uniformly in u. This implies $f = \widetilde{f}$, for large r and therefore for all $r \geq 0$. But then $K = K_u$, for all u, which implies convexity. (Notice that in some of these arguments the regularity of the grains is used.)

There is a corresponding result in Hug, Last and Weil [362] for the two-dimensional unit disk B^2, which concerns the **ALDC-function** (average logarithmic disk contact distribution function) D of Z,

$$D(r) := -\int_{SO_d} \ln(1 - H_{\vartheta B^2}(r))\, \nu(\mathrm{d}\vartheta), \qquad r \geq 0.$$

Instead of (11.15), we need the stronger assumption of uniformly bounded grains.

Let Z be a stationary Boolean model in $\mathbb{R}^d$ with regular, uniformly bounded compact grains. Then the ALDC-function D of Z is a polynomial if and only if the grains are almost surely convex.

The proof is more complicated and is based on the following steps.

First, by Fubini's theorem, it is sufficient to show the result for $d = 2$ and a fixed (regular) grain $K \in \mathcal{C}'$. More precisely, it is sufficient to show that $K \subset \mathbb{R}^2$ is convex if $\lambda_2(K + rB^2)$ is a polynomial in $r \geq 0$.

As in the linear case, we compare $\lambda_2(K(r)) = \sum_{i=0}^m a_i r^i$, $K(r) := K + rB^2$, with the volume of the convex hull $\bar{K}(r) := \bar{K} + rB^2$, $\bar{K} := \text{conv}K$,

$$\lambda_2(\bar{K}(r)) = V_2(\bar{K}) + 2rV(\bar{K}, B^2) + r^2 V_2(B^2).$$

Since $\lambda_2(rB^2) \leq \lambda_2(K(r)) \leq \lambda_2(\bar{K}(r))$, we obtain

$$\lambda_2(K(r)) = a_0 + a_1 r + a_2 r^2, \qquad r \geq 0,$$

with $a_2 = V_2(B^2)$ and $a_0 + a_1 r \leq V_2(\bar{K}) + 2rV(\bar{K}, B^2)$.

For λ_1-almost all $r \geq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}r} V_2(K(r)) = \int_{\mathrm{bd}\, K(r)} h(B^2, u_{K(r)}(x))\, \mathcal{H}^1(\mathrm{d}x),$$

where $u_{K(r)}$ is the ($\mathcal{H}^1$-almost everywhere existing) outer unit normal vector in x and $h(M, \cdot)$ is the support function of the convex body M. Moreover, $K(r)$ is star-shaped, for sufficiently large r. Hence,

$$\int_{\mathrm{bd}\, K(r)} h(B^2, u_{K(r)}(x))\, \mathcal{H}^1(\mathrm{d}x) \leq \int_{\mathrm{bd}\, \bar{K}(r)} h(B^2, u_{\bar{K}(r)}(x))\, \mathcal{H}^1(\mathrm{d}x),$$

which, under the spherical image map, transforms to

$$\int_{S^1} h(B^2, u)\, S_1(K(r), \mathrm{d}u) \leq \textstyle\int_{S^1} h(B^2, u)\, S_1(\bar{K}(r), \mathrm{d}u). \tag{11.16}$$

Here, the image measure $S_1(\bar{K}(r), \cdot)$ is the surface area measure of $\bar{K}(r)$ and $S_1(K(r), \cdot)$ is, by Minkowski's theorem, also the surface area measure of some convex body $\widetilde{K}(r)$, the **convexification** of $K(r)$.

As one can show, $K(r) \subset \widetilde{K}(r)$, after a suitable translation, and therefore $\bar{K}(r) \subset \widetilde{K}(r)$. This implies $h(\bar{K}(r), \cdot) \leq h(\widetilde{K}(r), \cdot)$. On the other hand, (11.16) and the symmetry of planar mixed volumes yield

$$\begin{aligned}
\int_{S^1} h(\bar{K}(r), u)\, \sigma_1(\mathrm{d}u) &= \int_{S^1} h(B^2, u)\, S_1(\bar{K}(r), \mathrm{d}u) \\
&\geq \int_{S^1} h(B^2, u)\, S_1(K(r), \mathrm{d}u) \\
&= \int_{S^1} h(B^2, u)\, S_1(\widetilde{K}(r), \mathrm{d}u) \\
&= \int_{S^1} h(\widetilde{K}(r), u)\, \sigma_1(\mathrm{d}u).
\end{aligned}$$

Therefore $h(\bar{K}(r), \cdot) = h(\widetilde{K}(r), \cdot)$, hence $\bar{K}(r) = \widetilde{K}(r)$, which implies that $K(r)$ and the convex hull $\bar{K}(r)$ have the same boundary length, $\mathcal{H}^1(K(r)) = \mathcal{H}^1(\bar{K}(r))$. For a planar star-shaped set this implies $K(r) = \bar{K}(r)$. Consequently, $K = \bar{K} = \text{conv}\, K$.

The result holds in a more general version, with the unit disk B^2 replaced by a smooth planar body B, and the proof is essentially the same.

However, as was shown in Heveling, Hug and Last [338], a corresponding result for three-dimensional gauge bodies B is wrong. In particular, for $d = 3$ and $B = B^3$, $\ln(1 - H)$ can be a polynomial without Z having convex grains. An example is given by a Boolean model Z, the primary grain of which is the union of two touching balls of equal radius.

11.3 Processes of Flats

Our aim in this section is to see how some of the results on flat processes obtained in Sections 4.4 and 4.6 carry over to the non-stationary case. For simplicity, we assume that all k-flat processes occurring in the following are simple. If we omit the stationarity assumption, some regularity property of the intensity measure will be necessary, similarly to Section 11.1, to get smooth results. We say that a measure on the space $A(d,k)$ of k-flats in $\mathbb{R}^d$ is **translation regular** if it is absolutely continuous with respect to some translation invariant, locally finite measure on $A(d,k)$.

Let $k \in \{1,\dots,d-1\}$, and let X be a k-flat process in $\mathbb{R}^d$ with a translation regular intensity measure $\Theta \neq 0$ (assumed to be locally finite, as always). By assumption, there exist a locally finite, translation invariant measure $\widetilde{\Theta}$ on $A(d,k)$ and a nonnegative, locally $\widetilde{\Theta}$-integrable function η on $A(d,k)$ such that

$$\Theta(A) = \int_A \eta \, \mathrm{d}\widetilde{\Theta}$$

for $A \in \mathcal{B}(A(d,k))$. The density η is only determined $\widetilde{\Theta}$-almost everywhere. If $\widetilde{\Theta}$ and η can be chosen such that η is continuous on $A(d,k)$, then we say that Θ is translation regular **with continuous density**.

By Theorem 4.4.1, the measure $\widetilde{\Theta}$ has a decomposition

$$\widetilde{\Theta}(A) = \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_A(L+x)\, \lambda_{L^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}L)$$

for $A \in \mathcal{B}(A(d,k))$, with a finite measure $\mathbb{Q}$ on $G(d,k)$, without loss of generality a probability measure (since $\mathbb{Q} \neq 0$, and $\widetilde{\Theta}$ and η can be changed by constant factors). For the intensity measure of X, this yields the representation

$$\Theta(A) = \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_A(L+x)\eta(L+x)\, \lambda_{L^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}L). \tag{11.17}$$

As in Section 11.1 we say that Θ is **represented by the pair** $(\eta, \mathbb{Q})$ (which, *nota bene*, is not uniquely determined).

For a stationary k-flat process X, the intensity γ, given by

$$\mathbb{E} \sum_{E \in X} \lambda_E = \gamma\lambda$$

(Theorem 4.4.3), and the directional distribution determine the intensity measure, by (4.25). For a k-flat process with a translation regular intensity measure there are corresponding quantities, but depending on the location. They are obtained from the following result. For $E \in A(d,k)$ we denote here by $E_0 \in G(d,k)$ the translate of E through 0.

Theorem 11.3.1. *Let X be a k-flat process in $\mathbb{R}^d$ with a translation regular intensity measure represented by $(\eta, \mathbb{Q})$. Let $A \in \mathcal{B}(G(d,k))$. Then*

$$\mathbb{E} \sum_{E \in X} \mathbf{1}_A(E_0)\lambda_E = \int_{(\cdot)} \int_{G(d,k)} \mathbf{1}_A(L)\eta(L+z)\,\mathbb{Q}(\mathrm{d}L)\,\lambda(\mathrm{d}z).$$

In particular, the measure

$$\mathbb{E} \sum_{E \in X} \lambda_E$$

has a density with respect to Lebesgue measure, given by

$$\gamma(z) := \int_{G(d,k)} \eta(L+z)\,\mathbb{Q}(\mathrm{d}L) \tag{11.18}$$

for almost all $z \in \mathbb{R}^d$.

Proof. Let $B \in \mathcal{B}(\mathbb{R}^d)$. Using Campbell's theorem and (11.17), we obtain

$$\begin{aligned}
&\mathbb{E} \sum_{E \in X} \mathbf{1}_A(E_0)\lambda_E(B) \\
&= \int_{A(d,k)} \mathbf{1}_A(E_0)\lambda_E(B)\,\Theta(\mathrm{d}E) \\
&= \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_A(L)\lambda_{L+x}(B)\eta(L+x)\,\lambda_{L^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}L) \\
&= \int_{G(d,k)} \int_{L^\perp} \int_L \mathbf{1}_B(y+x)\mathbf{1}_A(L)\eta(L+x)\,\lambda_L(\mathrm{d}y)\,\lambda_{L^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}L) \\
&= \int_{G(d,k)} \int_B \mathbf{1}_A(L)\eta(L+z)\,\lambda(\mathrm{d}z)\,\mathbb{Q}(\mathrm{d}L) \\
&= \int_B \int_{G(d,k)} \mathbf{1}_A(L)\eta(L+z)\,\mathbb{Q}(\mathrm{d}L)\,\lambda(\mathrm{d}z),
\end{aligned}$$

which completes the proof. □

The function γ is called the **intensity function** of the k-flat process X. It replaces the constant intensity appearing in the stationary case.

If the density η is continuous on $A(d,k)$, then it follows from the compactness of $G(d,k)$ and the finiteness of $\mathbb{Q}$ that there is a uniquely determined continuous version of the intensity function on $\mathbb{R}^d$. In this case an intuitive interpretation is easily obtained as follows. If K is a convex body with interior points, then, for $z \in \mathbb{R}^d$ and $r > 0$,

$$\mathbb{E} \sum_{E \in X} \lambda_E(rK+z) = \int_{A(d,k)} \lambda_E(rK+z)\,\Theta(\mathrm{d}E)$$

$$= \int_{G(d,k)} \int_{L^\perp} \lambda_{L+x}(rK+z)\eta(L+x)\, \lambda_{L^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}L)$$

$$= \int_{G(d,k)} \int_{L^\perp} \int_L \mathbf{1}_{rK+z}(x+y)\eta(L+x)\, \lambda_L(\mathrm{d}y)\, \lambda_{L^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}L)$$

$$= \int_{G(d,k)} \int_{\mathbb{R}^d} \mathbf{1}_{rK+z}(t)\eta(L+t)\, \lambda(\mathrm{d}t)\, \mathbb{Q}(\mathrm{d}L).$$

Since η is continuous, the inner integral is equal to $\eta(L+z_{r,L})V_d(rK)$ for some point $z_{r,L} \in rK+z$. The continuity of η now gives

$$\gamma(z) = \lim_{r\to 0} \frac{1}{V_d(rK)} \mathbb{E} \sum_{E\in X} \lambda_E(rK+z).$$

It is also easy to see that

$$\gamma(z) = \lim_{r\to 0} \frac{1}{\kappa_{d-k} r^{d-k}} \mathbb{E} X(\mathcal{F}_{rB^d+z})$$

holds for all $z \in \mathbb{R}^d$. This extends (4.27).

As a counterpart to the directional distribution in the stationary case, we define a measure $\varphi(z,\cdot)$ on $G(d,k)$ by

$$\varphi(z,\cdot) := \int_{(\cdot)} \eta(L+z)\, \mathbb{Q}(\mathrm{d}L)$$

for $z \in \mathbb{R}^d$, which is finite almost everywhere. From Theorem 11.3.1 it follows that

$$\mathbb{E} \sum_{E\in X} \mathbf{1}_A(E_0)\lambda_E(B) = \int_B \varphi(z,A)\, \lambda(\mathrm{d}z)$$

for $A \in \mathcal{B}(G(d,k))$ and $B \in \mathcal{B}(\mathbb{R}^d)$. This relation, together with the fact that the σ-algebra $\mathcal{B}(G(d,k))$ is countably generated, shows that for λ-almost all z the measure $\varphi(z,\cdot)$ is uniquely determined and hence depends only on the process X and not on the choice of $\widetilde{\Theta}$ and η. The measure $\varphi(z,\cdot)$ is called the **directional measure** of X at z. At the points z with $0 < \gamma(z) < \infty$, the **directional distribution** $\varphi(z,\cdot)/\gamma(z)$ can be defined.

If X is stationary, then the directional measure $\varphi(z,\cdot)$ does not depend on z. If X is also isotropic, then $\varphi(z,\cdot)$ is rotation invariant. We prove a certain converse statement. Here we use the terminology introduced in the remark at the end of Section 9.2.

Theorem 11.3.2. *Let X be a k-flat process in $\mathbb{R}^d$ whose intensity measure is translation regular with a continuous density. If the directional measure $\varphi(z,\cdot)$ is rotation invariant for all $z \in \mathbb{R}^d$, then X is weakly stationary and weakly isotropic.*

Proof. Under the assumptions, the intensity function γ is continuous, hence the set

$$M := \{z \in \mathbb{R}^d : \gamma(z) > 0\}$$

is open (and not empty). Let $z \in M$. The finite, rotation invariant measure $\varphi(z, \cdot)$ is a multiple of ν_k. Since $\varphi(z, G(d,k)) = \gamma(z)$, it follows that

$$\varphi(z, \cdot) = \gamma(z)\nu_k,$$

hence from (11.18) we get

$$\nu_k(A) = \int_A \frac{\eta(L+z)}{\gamma(z)} \, \mathbb{Q}(\mathrm{d}L) \tag{11.19}$$

for $A \in \mathcal{B}(G(d,k))$. Let also $y \in M$, then

$$\int_A \frac{\eta(L+z)}{\gamma(z)} \, \mathbb{Q}(\mathrm{d}L) = \int_A \frac{\eta(L+y)}{\gamma(y)} \, \mathbb{Q}(\mathrm{d}L)$$

for $A \in \mathcal{B}(G(d,k))$, hence

$$\frac{\eta(L+z)}{\gamma(z)} = \frac{\eta(L+y)}{\gamma(y)} \tag{11.20}$$

for all $L \in \operatorname{supp} \mathbb{Q}$, by the continuity of η. By (11.19), $\operatorname{supp} \mathbb{Q} = G(d,k)$, since $\nu_k(A) > 0$ for every nonempty open set $A \subset G(d,k)$.

The set

$$N_z := \{L \in G(d,k) : \eta(L+z) = 0\}$$

satisfies $\nu_k(N_z) = 0$, by (11.19). Let $U \subset M$ be a neighborhood of y. The directions E_0 of the k-flats E through z and a point of U fill a nonempty open set in $G(d,k)$. Therefore, $y_1 \in U$ can be chosen such that there exists a subspace $L \in G(d,k)$ with $L + z = L + y_1$ and $\eta(L+z) > 0$. Relation (11.20) implies that $\gamma(y_1) = \gamma(z)$, and since y_1 can be chosen arbitrarily close to y, we deduce that $\gamma(y) = \gamma(z)$, by the continuity of γ.

We have proved that the continuous intensity function γ is constant on the set M where it is positive. Hence, γ is constant on $\mathbb{R}^d$. From (11.17), (11.19) and (11.20) we conclude that

$$\Theta(A) = \gamma \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_A(L+x) \, \lambda_{L^\perp}(\mathrm{d}x) \, \nu_k(\mathrm{d}L)$$

for $A \in \mathcal{B}(G(d,k))$. This shows that Θ is invariant under translations and rotations and thus completes the proof. □

The rest of this section is devoted to Poisson hyperplane processes. We want to extend Theorem 4.6.5 on intersection densities to the non-stationary, translation regular case. This requires the introduction of associated zonoids depending on the location.

For hyperplanes, we use the representation (4.32), but we consider only hyperplanes not passing through 0. Every such hyperplane has a unique representation

$$H(u,\tau) := \{x \in \mathbb{R}^d : \langle x,u\rangle = \tau\}$$

with $u \in S^{d-1}$ and $\tau > 0$.

Let X be a hyperplane process in $\mathbb{R}^d$ with a translation regular intensity measure represented by the pair $(\eta, \mathbb{Q})$. It is convenient to use the function $g : S^{d-1} \times (0,\infty) \to [0,\infty)$ defined by

$$g(u,\tau) := \eta(H(u,\tau))$$

for $u \in S^{d-1}$ and $\tau > 0$ and by $g(u,\tau) := 0$ for $\tau \le 0$ or if $\eta(H(u,\tau))$ is not defined. Instead of $\mathbb{Q}$, we use the measure ϕ on the sphere S^{d-1} with the property

$$\phi(A) = \frac{1}{2}\mathbb{Q}(\{H(u,0) : u \in A\}$$

for $A \in \mathcal{B}(S^{d-1})$ without antipodal points. The intensity measure of X is then given by

$$\Theta(A) = 2\int_{S^{d-1}}\int_0^\infty \mathbf{1}_A(H(u,\tau))g(u,\tau)\,\mathrm{d}\tau\,\phi(\mathrm{d}u) \tag{11.21}$$

for $A \in \mathcal{B}(A(d,d-1))$.

We assume now in addition that X is a Poisson process. For $k \in \{1,\dots,d\}$, let X_k be the intersection process of order k of the process X. Modifying the method of proof for Theorem 4.4.5, one can show that X_k is a.s. simple. Let Θ_k be its intensity measure. As in the proof of Theorem 4.4.8 (where the stationarity assumption is not needed in the beginning), one obtains

$$\Theta_k(A) = \frac{1}{k!}\int_{A(d,d-1)^{k*}} \mathbf{1}_A(H_1 \cap \ldots \cap H_k)\,\Theta^k(\mathrm{d}(H_1,\dots,H_k)) \tag{11.22}$$

for $A \in \mathcal{B}(A(d,d-k))$, where $A(d,d-1)^{k*}$ denotes the set of k-tuples of hyerplanes with linearly independent normal vectors. Thus, $k!\Theta_k$ is the image measure of $\Theta^k \llcorner A(d,d-1)^{k*}$ under the intersection mapping $(H_1,\dots,H_k) \mapsto H_1 \cap \ldots \cap H_k$. It follows that Θ_k is locally finite. Since Θ is absolutely continuous with respect to a translation invariant measure $\widetilde{\Theta}$ on $A(d,d-1)$, the measure Θ_k is absolutely continuous with respect to the image measure of $\widetilde{\Theta}^k \llcorner A(d,d-1)^{k*}$ under the same intersection mapping. Hence, the intersection process X_k has a translation regular intensity measure, too. We compute its intensity function. Let $B \in \mathcal{B}(\mathbb{R}^d)$ and $\lambda(B) < \infty$. Then, using (11.22) and (11.21),

$$k!\mathbb{E}\sum_{E \in X_k} \lambda_E(B)$$

$$= k!\int_{A(d,d-k)} \lambda_{d-k}(B \cap E)\,\Theta_k(\mathrm{d}E)$$

$$= \int_{A(d,d-1)^{k*}} \lambda_{d-k}(B \cap H_1 \cap \ldots \cap H_k)\, \Theta^k(\mathrm{d}(H_1, \ldots, H_k))$$

$$= 2^k \int_{(S^{d-1})^{k*}} \int_{(0,\infty)^k} \lambda_{d-k}(B \cap H(u_1, \tau_1) \cap \ldots \cap H(u_k, \tau_k))$$

$$\times g(u_1, \tau_1) \cdots g(u_k, \tau_k)\, \mathrm{d}(\tau_1, \ldots, \tau_k)\, \phi^k(\mathrm{d}(u_1, \ldots, u_k)),$$

where $(S^{d-1})^{k*}$ denotes the set of k-tuples of linearly independent unit vectors. We use the same transformation as at the end of the proof of Theorem 4.4.8. If $u_1, \ldots, u_k \in S^{d-1}$ are linearly independent and $H(u_1, \tau_1) \cap \ldots \cap H(u_k, \tau_k) =: L$, then

$$\int_{(0,\infty)^k} \lambda_{d-k}(B \cap H(u_1, \tau_1) \cap \ldots \cap H(u_k, \tau_k)) g(u_1, \tau_1) \cdots g(u_k, \tau_k)$$

$$\times \mathrm{d}(\tau_1, \ldots, \tau_k)$$

$$= \int_{L^\perp} \lambda_{d-k}(B \cap (L + x)) g(u_1, \langle u_1, x\rangle) \cdots g(u_k, \langle u_k, x\rangle)\, \lambda_{L^\perp}(\mathrm{d}x)$$

$$\times \nabla_k(u_1, \ldots, u_k)$$

$$= \int_B g(u_1, \langle u_1, z\rangle) \cdots g(u_k, \langle u_k, z\rangle)\, \lambda(\mathrm{d}z) \cdot \nabla_k(u_1, \ldots, u_k).$$

Since $\nabla_k(u_1, \ldots, u_k) = 0$ if $u_1, \ldots, u_k$ are linearly dependent, we conclude that

$$\mathbb{E} \sum_{E \in X_k} \lambda_E(B) = \frac{2^k}{k!} \int_{(S^{d-1})^k} \int_B g(u_1, \langle u_1, z\rangle) \cdots g(u_k, \langle u_k, z\rangle)\, \nabla_k(u_1, \ldots, u_k)$$

$$\times \lambda(\mathrm{d}z)\, \phi^k(\mathrm{d}(u_1, \ldots, u_k))$$

$$= \int_B \gamma_k(z)\, \lambda(\mathrm{d}z)$$

with

$$\gamma_k(z) = \frac{2^k}{k!} \int_{(S^{d-1})^k} \nabla_k(u_1, \ldots, u_k)$$

$$\times g(u_1, \langle u_1, z\rangle) \cdots g(u_k, \langle u_k, z\rangle)\, \phi^k(\mathrm{d}(u_1, \ldots, u_k)). \qquad (11.23)$$

This is the intensity function of the intersection process X_k. We rewrite it, using the measure $\widetilde{\varphi}_z$ on S^{d-1} defined by

$$\widetilde{\varphi}_z(A) := 2 \int_A g(u, \langle u, z\rangle)\, \phi(\mathrm{d}u)$$

for $A \in \mathcal{B}(S^{d-1})$. Then

$$\gamma_k(z) = \frac{1}{k!} \int_{(S^{d-1})^k} \nabla_k(u_1, \ldots, u_k)\, \widetilde{\varphi}_z^k(\mathrm{d}(u_1, \ldots, u_k)). \tag{11.24}$$

Now we define the **local associated zonoid** $\Pi_X(z)$ of X at z as the convex body with support function given by

$$h(\Pi_X(z), u) := \frac{1}{2} \int_{S^{d-1}} |\langle u, v\rangle|\, \widetilde{\varphi}_z(\mathrm{d}v)$$

for $u \in \mathbb{R}^d$.

From (11.24) and (14.35) we obtain the generalization of formula (4.63) to Poisson hyperplane processes with translation regular intensity measure, namely

$$\gamma_k(z) = V_k(\Pi_X(z)) \tag{11.25}$$

for $z \in \mathbb{R}^d$. We can state the following result.

Theorem 11.3.3. *Let X be a Poisson hyperplane process in $\mathbb{R}^d$ with a translation regular intensity measure. Let $k \in \{2, \ldots, d\}$, and let X_k be the intersection process of X of order k. Let γ be the intensity function of X and γ_k the intensity function of X_k. Then*

$$\gamma_k(z) \leq \frac{\binom{d}{k}\kappa_{d-1}^k}{d^k \kappa_{d-k}\kappa_d^{k-1}} \gamma(z)^k \tag{11.26}$$

for almost all $z \in \mathbb{R}^d$.

If the intensity measure of X is translation regular with a continuous density and if equality holds in (11.26) *for all z, then the process X is stationary and isotropic.*

Proof. The inequality (11.26) follows from (11.25) in the same way as in the stationary case (see Theorem 4.6.5). Assume that X is translation regular with a continuous density and that equality holds in (11.26) for all $z \in \mathbb{R}^d$. Then, for each z, the local associated zonoid $\Pi_X(z)$ is a ball (possibly one-pointed). Hence, the even part of the measure $\widetilde{\varphi}_z$ is proportional to the spherical Lebesgue measure. Since

$$\varphi(z, A) = \widetilde{\varphi}_z(\{u \in S^{d-1} : H(u, 0) \in A\} \qquad \text{for } A \in \mathcal{B}(G(d, d-1)),$$

it follows that $\varphi(z, \cdot)$ is rotation invariant. Now Theorem 11.3.2 shows that X is weakly stationary and weakly isotropic. Since the intensity measure of a Poisson process determines its distribution, X is stationary and isotropic. □

A remarkable aspect of Theorem 11.3.3 can be seen in the fact that together with isotropy also the stationarity, and thus invariance under all rigid motions, is characterized by an extremal property.

Notes for Section 11.3

1. The results of this section are taken from Schneider [707]. Fallert [222] has studied k-flat processes which instead of (11.17) satisfy

$$\Theta(A) = \int_{G(d,k)} \int_{L^\perp} \mathbf{1}_A(L+x) f(x)\, \lambda_{L^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}L)$$

with a locally integrable nonnegative function f. For $0 < k < d-1$, this assumption is more restrictive than the assumption (11.17) with a locally integrable nonnegative function η.

2. Intersections with fixed flats. Let X be a translation regular k-flat process, let $j \in \{0, \dots, k-1\}$, and let $S \in A(d-k+j)$. The intersection process $X \cap S$, defined by

$$X \cap S := \sum_{E_i \cap S \neq \emptyset} \delta_{E_i \cap S} \quad \text{if} \quad X = \sum \delta_{E_i},$$

is a translation regular j-flat process in S. Its intensity function at $z \in S$ is given by

$$\gamma_{X \cap S}(z) = \int_{G(d,k)} [L, S]\, \varphi(z, \mathrm{d}L),$$

where $\varphi(z, \cdot)$ is the direction measure of X at z. See Schneider [707], also for some results on the determination of translation regular flat processes from information on section processes.

3. Hoffmann [346] has given a common generalization of results of Schneider [707] and of Wieacker [817], by investigating intersection densities and local associated zonoids for non-stationary Poisson processes of hypersurfaces, which are cylinders with $(\mathcal{H}^k, k)$-rectifiable bases.

4. Hug, Last and Weil [360] study generalized contact distribution functions, in the sense of Section 11.2, for **Poisson networks** Z. The latter are union sets of Poisson k-flat processes X with translation regular intensity measures. They prove an analog of Theorem 11.2.2. As a consequence, it is shown that, for processes X with continuous density η and $z \in \mathbb{R}^d$, the distribution of $(d(z,Z), u(Z,z))$ determines the Radon transform $R_{k,d-1}\varphi(z,\cdot)$. Hence for line or hyperplane networks, the directional measure $\varphi(z,\cdot)$ is determined by measuring the distance and direction from the point z to Z. Various generalizations are also treated in [360]. For example, the point z is replaced by a flat $F \in A(d,j)$, with $k+j<d$, and the intensity measure Θ^F of the process of midpoints $m(E,F)$, $E \in X$, is assumed to be given. As another generalization, z is replaced by a flag of (linear) subspaces with increasing dimension and a uniqueness result for stationary Poisson networks is proved.

11.4 Tessellations

The purpose of this section is to extend from the stationary to the non-stationary case a basic result on random hyperplane mosaics, namely the relations of Theorem 10.3.1 between the specific intrinsic volumes of the face processes. However, for reasons explained later, we do this only under Poisson assumptions.

Let $\widehat{X}$ be a Poisson hyperplane process in $\mathbb{R}^d$ with a translation regular intensity measure. It generates a random tessellation X of $\mathbb{R}^d$, and our first aim will be to formulate a condition which ensures that the cells of the tessellation are a.s. bounded. A given point x is a.s. contained in a unique cell, denoted by Z_x, of the mosaic; this follows from the translation regularity.

Definition 11.4.1. *The hyperplane process* $\widehat{X}$ *is* **nondegenerate** *if the following holds.*

(a) *With positive probability, the zero cell* Z_0 *is bounded.*

(b) *If* $U \subset S^{d-1}$ *is a measurable set and if* $\widehat{X}$ *contains with positive probability a hyperplane with normal vector in* U*, then* $\widehat{X}$ *contains with positive probability infinitely many such hyperplanes.*

This is an appropriate geometric condition for obtaining a random mosaic, as the following theorem shows.

Theorem 11.4.1. *Let* $\widehat{X}$ *be a nondegenerate Poisson hyperplane process in* $\mathbb{R}^d$ *with a translation regular intensity measure. The system* X *of the induced cells is a random mosaic in general position. The process* $X^{(k)}$ *of the* k*-faces of* X *has a translation regular intensity measure, for* $k = 1, \dots, d$.

Proof. The intensity measure $\widehat{\Theta}(A)$ of $\widehat{X}$ has the representation (11.21),

$$\widehat{\Theta}(A) = 2 \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(H(u,\tau)) g(u,\tau)\,\mathrm{d}\tau\,\phi(\mathrm{d}u)$$

for $A \in \mathcal{B}(A(d,d-1))$, with a locally integrable, nonnegative function g and a finite measure ϕ. We put

$$P := \left\{ u \in S^{d-1} : \int_0^\infty g(u,\tau)\,\mathrm{d}\tau > 0 \right\}$$

and assume without loss of generality that the measure ϕ is **reduced**, in the sense that

$$\phi(S^{d-1} \setminus P) = 0.$$

Let $U := \operatorname{supp} \phi$ and suppose that $0 \notin \operatorname{int} \operatorname{conv} U$. Then $\operatorname{conv} U$ and 0 can be separated weakly by a hyperplane, hence there is a vector $v \in S^{d-1}$ with $\langle u, v \rangle \le 0$ for $u \in U$. We denote by M the set of hyperplanes $H(u,\tau)$, $\tau > 0$, meeting the ray $R := \{\lambda v : \lambda > 0\}$. If $H(u,\tau) \cap R \neq \emptyset$ and (w.l.o.g.) $R \not\subset H(u,\tau)$, then $\langle u, v \rangle > 0$, hence

$$\widehat{\Theta}(M) = 2 \int_{S^{d-1}} \int_0^\infty \mathbf{1}_M(H(u,\tau)) g(u,\tau)\,\mathrm{d}\tau\,\phi(\mathrm{d}u) = 0.$$

But then $R \subset Z_0$ a.s., which contradicts the assumption that Z_0 is bounded with positive probability. It follows that $0 \in \operatorname{int} \operatorname{conv} U$. As in the proof of

Theorem 10.3.2, this implies the existence of vectors $u_1, \dots, u_{2d} \in U$ and neighborhoods U_i of u_i in S^{d-1} for $i = 1, \dots, 2d$ such that

$$0 \in \operatorname{int} \operatorname{conv}\{v_1, \dots, v_{2d}\} \qquad \text{for all } (v_1, \dots, v_{2d}) \in U_1 \times \dots \times U_{2d}.$$

Let $x \in \mathbb{R}^d$, and let $A_i(x)$ be the set of all hyperplanes $H(u, \tau)$ with $u \in U_i$ and $\tau > \max\{\langle x, u_i \rangle, 0\}$ for $i = 1, \dots, 2d$. Then

$$\widehat{\Theta}(A_i(x)) = 2 \int_{U_i} \int_{\langle x, u_i \rangle}^{\infty} g(u, \tau) \, \mathrm{d}\tau \, \phi(\mathrm{d}u).$$

We have

$$\widehat{\Theta}(A_i(0)) = 2 \int_{U_i} \int_0^{\infty} g(u, \tau) \, \mathrm{d}\tau \, \phi(\mathrm{d}u) > 0,$$

since $\phi(U_i) > 0$, which follows from $u_i \in \operatorname{supp} \phi$ and the assumption that ϕ is reduced. Since $\widehat{X}$ is nondegenerate, this implies $\widehat{\Theta}(A(0)) = \infty$, thus

$$\widehat{\Theta}(A_i(x)) + 2 \int_{U_i} \int_0^{\langle x, u_i \rangle} g(u, \tau) \, \mathrm{d}\tau \, \mathrm{d}\phi(u) = \infty.$$

Here the second summand is finite since $\widehat{\Theta}$ is finite on compact sets. We conclude that $\widehat{\Theta}(A_i(x)) = \infty$.

Now we can continue as in the proof of Theorem 10.3.2 and deduce that the cell Z_x is almost surely bounded. The rest of that proof also carries over, showing that the system of induced cells is a random mosaic in general position.

Let $X^{(k)}$ be the system of k-faces of X, $k \in \{0, \dots, d\}$. As in Section 10.1 one sees that $X^{(k)}$ is a particle process. Since the intersection processes of $\widehat{X}$ have locally finite intensity measures, the proof of Theorem 10.3.1 shows that $X^{(k)}$ has locally finite intensity measure. It remains to show that this measure is translation regular.

Let $\widehat{X}_s$ be the stationary Poisson hyperplane process with spherical directional distribution ϕ and with intensity 1. It exists by Theorem 4.4.4 and has intensity measure

$$\widehat{\Theta}_s(A) = 2 \int_{S^{d-1}} \int_0^{\infty} \mathbf{1}_A(H(u, \tau)) \, \mathrm{d}\tau \, \phi(\mathrm{d}u),$$

for $A \in \mathcal{B}(A(d, d-1))$. The random hyperplane mosaic generated by $\widehat{X}_s$ is denoted by X_s, and the particle process of its k-faces by $X_s^{(k)}$. Let $\Theta_s^{(k)}$ be the intensity measure of $X_s^{(k)}$. We will show that $\Theta^{(k)}$ is absolutely continuous with respect to $\Theta_s^{(k)}$.

Let $A \in \mathcal{B}(\mathcal{K}')$ be a set with

$$\Theta_s^{(k)}(A) = 0. \tag{11.27}$$

In order to show that

$$\Theta^{(k)}(A) = 0, \tag{11.28}$$

it is sufficient to show that $\Theta^{(k)}(A_r) = 0$ for each $r \in \mathbb{N}$, where $A_r := \{K \in A : K \subset rB^d\}$. Let $\mathcal{H}_r$ be the set of hyperplanes meeting rB^d. To prove (11.28), it is sufficient to prove for $r, m \in \mathbb{N}$ that

$$\mathbb{E}(X^{(k)}(A_r) \mid \widehat{X}(\mathcal{H}_r) = m) = 0.$$

We choose r so large that $\widehat{\Theta}(\mathcal{H}_r) \neq 0$, which is possible since $\widehat{\Theta} \neq 0$. The process $\widehat{X}$ restricted to $\mathcal{H}_r$ and under the condition $\widehat{X}(\mathcal{H}_r) = m$ is stochastically equivalent to the process defined by m independent, identically distributed hyperplanes with distribution $\widehat{\Theta} \llcorner \mathcal{H}_r / \widehat{\Theta}(\mathcal{H}_r)$ (Theorem 3.2.2). We denote by $f(u_1, \dots, u_m, \tau_1, \dots, \tau_m)$ the number of k dimensional polytopes in the set A_r that are faces of the tessellation of $\mathbb{R}^d$ generated by the hyperplanes $H(u_1, \tau_1), \dots, H(u_m, \tau_m)$. Then

$$\begin{aligned}
&\mathbb{E}(X^{(k)}(A_r) \mid \widehat{X}(\mathcal{H}_r) = m) \\
&= \widehat{\Theta}(\mathcal{H}_r)^{-m} 2^m \int_{(S^{d-1})^m} \int_{(0,\infty)^m} f(u_1, \dots, u_m, \tau_1, \dots, \tau_m) \\
&\quad \times g(u_1, \tau_1) \cdots g(u_m, \tau_m) \, \mathrm{d}(\tau_1, \dots, \tau_m) \, \phi^m(\mathrm{d}(u_1, \dots, u_m)).
\end{aligned}$$

Similarly, for the stationary Poisson hyperplane process $\widehat{X}_s$ we get

$$\begin{aligned}
&\mathbb{E}(X_s^{(k)}(A_r) \mid \widehat{X}_s(\mathcal{H}_r) = m) \\
&= \widehat{\Theta}_s(\mathcal{H}_r)^{-m} 2^m \int_{(S^{d-1})^m} \int_{(0,\infty)^m} f(u_1, \dots, u_m, \tau_1, \dots, \tau_m) \\
&\quad \times \mathrm{d}(\tau_1, \dots, \tau_m) \, \phi^m(\mathrm{d}(u_1, \dots, u_m)).
\end{aligned}$$

Let M be the set of all m-tuples $(u_1, \dots, u_m) \in (S^{d-1})^m$ for which

$$\int_{(0,\infty)^m} f(u_1, \dots, u_m, \tau_1, \dots, \tau_m) g(u_1, \tau_1) \cdots g(u_m, \tau_m) \, \mathrm{d}(\tau_1, \dots, \tau_m) > 0.$$

For $(u_1, \dots, u_m) \in M$ we also have

$$\int_{(0,\infty)^m} f(u_1, \dots, u_m, \tau_1, \dots, \tau_m) \, \mathrm{d}(\tau_1, \dots, \tau_m) > 0.$$

Since $\mathbb{E}(X_s^{(k)}(A_r) \mid \widehat{X}_s(\mathcal{H}_r) = m) = 0$ by (11.27), it follows that $\phi^m(M) = 0$ and, therefore, that $\mathbb{E}(X^{(k)}(A_r) \mid \widehat{X}(\mathcal{H}_r) = m) = 0$. This proves (11.28). Thus $\Theta^{(k)}$ is absolutely continuous with respect to the translation invariant measure $\Theta_s^{(k)}$. This shows that the face process $X^{(k)}$ has a translation regular intensity measure. $\square$

Let $\widehat{X}$ and $\widehat{\Theta}$ be as in the previous theorem. Since the induced hyperplane mosaic $X = X^{(d)}$ and its processes $X^{(k)}$ of k-faces ($k = 0, \dots, d-1$) have (locally finite) translation regular intensity measures, they admit specific intrinsic volumes $\overline{V}_j(X^{(k)}, z) =: d_j^{(k)}(z)$, $j = 0, \dots, k$, satisfying

$$d_j^{(k)}(z) = \overline{V}_j(X^{(k)}, z) = \lim_{r \to 0} \frac{1}{V_d(rW)} \mathbb{E} \sum_{K \in X^{(k)}} \Phi_j(K, rW + z) \tag{11.29}$$

for λ-almost all z, where $W \in \mathcal{K}$ with $V_d(W) > 0$; see Corollary 11.1.1. We write

$$\gamma^{(k)}(z) := d_j^{(0)}(z)$$

and call $\gamma^{(k)}$ the **intensity function** of the k-face process $X^{(k)}$.

The following result extends Theorem 10.3.1, for Poisson processes. This restriction was made since it allows us to deduce the translation regularity of the intensity measures of the face processes, which otherwise would have to be an additional assumption.

Theorem 11.4.2. *Let $\widehat{X}$ be a nondegenerate Poisson hyperplane process in $\mathbb{R}^d$ with a translation regular intensity measure, let X be the induced hyperplane mosaic, and let $X^{(k)}$ be its k-face process, for $k = 0, \dots, d$. For $0 \le j \le k \le d$, the relation*

$$d_j^{(k)} = \binom{d-j}{d-k} d_j^{(j)}$$

holds λ-almost everywhere, in particular

$$\gamma^{(k)} = \binom{d}{k} \gamma^{(0)}.$$

Proof. Let $j \in \{0, \dots, d-1\}$, $z \in \mathbb{R}^d$, and $r > 0$. Given a realization of $\widehat{X}$ inducing a mosaic X in general position (without loss of generality), we can choose finitely many cells $S_1, \dots, S_p$ of X such that $P := \bigcup_{i=1}^p S_i$ is a convex polytope with $rB^d + z \subset \operatorname{int} P$. Then $\Phi_j(P, rB^d + z) = 0$ since $j < d$. Since the curvature measure Φ_j is additive on the convex ring $\mathcal{R}$, the inclusion–exclusion principle gives

$$\begin{aligned} 0 &= \Phi_j(P, rB^d + z) \\ &= \Phi_j\left(\bigcup_{i=1}^p S_i, rB^d + z\right) \\ &= \sum_{m=1}^p (-1)^{m-1} \sum_{i_1 < \dots < i_m} \Phi_j(S_{i_1} \cap \dots \cap S_{i_m}, rB^d + z). \end{aligned}$$

Each intersection $S_{i_1} \cap \dots \cap S_{i_m}$ is either empty or an i-face of the mosaic X and thus an element of $X^{(i)}$ for some $i \in \{1, \dots, d\}$. Conversely, each element of

$X^{(i)}$ meeting the ball $rB^d + z$ is obtained in this way. For a face F, let $\nu(F, m)$ denote the number of m-tuples $(S_{i_1}, \ldots, S_{i_m})$ with $S_{i_1} \cap \ldots \cap S_{i_m} = F$. Taking into account the fact that $\Phi_j(F, \cdot) = 0$ if $\dim F < j$, we deduce that

$$
\begin{aligned}
0 &= \sum_{i=j}^{d} \sum_{F \in X^{(i)}} \Phi_j(F, rB^d + z) \sum_{m=1}^{p} (-1)^{m-1} \nu(F, m) \\
&= \sum_{i=j}^{d} (-1)^{d-i} \sum_{F \in X^{(i)}} \Phi_j(F, rB^d + z) \qquad (11.30)
\end{aligned}
$$

(compare the proof of Theorem 10.1.4). Taking the expectation, dividing by $V_d(rB^d)$ and letting r tend to 0, we obtain from (11.29) the relation

$$\sum_{i=j}^{d} (-1)^{d-i} d_j^{(i)}(z) = 0$$

for almost all $z \in \mathbb{R}^d$ and for $j = 0, \ldots, d-1$.

Let $k \in \{1, \ldots, d-1\}$, $j \in \{0, \ldots, k-1\}$, and $r > 0$. Let E be a k-plane of the intersection process $\widehat{X}_{d-k}$. We apply (11.30) to the mosaic induced in the k-plane E. This gives

$$0 = \sum_{i=j}^{k} (-1)^{k-i} \sum_{F \in X^{(i)},\, F \subset E} \Phi_j(F, rB^d + z).$$

We sum over all k-planes $E \in \widehat{X}_{n-k}$ and observe that X is almost surely in general position. Hence, each i-face of X is contained in precisely $\binom{d-i}{d-k}$ k-planes of $\widehat{X}_{d-k}$. This gives

$$0 = \sum_{i=j}^{k} (-1)^{k-i} \binom{d-i}{d-k} \sum_{F \in X^{(i)}} \Phi_j(F, rB^d + z).$$

As above, (11.29) yields

$$\sum_{i=j}^{k} (-1)^{i} \binom{d-i}{d-k} d_j^{(i)}(z) = 0$$

for almost all $z \in \mathbb{R}^d$. The remaining part of the proof is identical to that of Theorem 10.3.1. □

Finally, we observe that the specific j-volume $d_j^{(j)}$ of the j-faces can be expressed in a different way. Let $\widehat{X}_{d-j}$ be the intersection process of order $d-j$ of the hyperplane process $\widehat{X}$, and let $\widehat{\gamma}_{d-j}$ be the intensity function of

$\widehat{X}_{d-j}$. It is the Radon–Nikodym derivative of the measure $\mathbb{E}\sum_{E\in\widehat{X}_{d-j}}\lambda_E$ and hence can be obtained by differentiation, in particular

$$\lim_{r\to 0}\frac{1}{V_d(rB^d}\,\mathbb{E}\sum_{E\in\widehat{X}_{d-j}}\lambda_E(rB^d+z)=\widehat{\gamma}_{d-j}(z)$$

for almost all z. Since

$$\sum_{E\in\widehat{X}_{d-j}}\lambda_E(rB^d+z)=\sum_{K\in X^{(j)}}\Phi_j(K,rB^d+z),$$

we deduce that $d^{(j)}=\widehat{\gamma}_{d-j}$ almost everywhere. Together with (11.26), this yields the inequality

$$d_j^{(k)}(z)\le\binom{d-j}{d-k}\binom{d}{j}\frac{\kappa_{d-1}^{d-j}}{d^{d-j}\kappa_j\kappa_d^{d-j-1}}\widehat{\gamma}(z)^{d-j}.$$

Equality holds if the hyperplane process $\widehat{X}$ is stationary and isotropic.

Notes for Section 11.4

1. The results of this section are taken from Schneider [707]. The proof of Theorem 11.4.2 uses ideas from Weiss [807] and Weiss and Zähle [809].

2. Fallert [222, chap. 6] has investigated Voronoi and Delaunay mosaics induced by Poisson processes in $\mathbb{R}^d$ with translation regular intensity measures.