Singular Value Analysis and Balanced Realizations for Nonlinear Systems

Kenji Fujimoto[1] and Jacquelien M.A. Scherpen[2]

[1] Department of Mechanical Science and Engineering, Nagoya University
fujimoto@nagoya-u.jp
[2] Faculty of Mathematics and Natural Sciences, University of Groningen
j.m.a.scherpen@rug.nl

1 Introduction

For linear control systems minimal realization theory and the related model reduction methods play a crucial role in understanding and handling the system. These methods are well established and have proved to be very successful, e.g., [Antoulas05, OA01, ZDG96]. In particular the method called balanced truncation gives a good reduced order model with respect to the input-output behavior, [Moore81, Glover84]. This method relies on the relation with the system Hankel operator, which plays a central role in minimal realization theory. Specifically, the Hankel operator supplies a set of similarity invariants, the so called Hankel singular values, which can be used to quantify the importance of each state in the corresponding input-output system [JS82]. The Hankel operator can also be factored into a composition of observability and controllability operators, from which Gramian matrices can be defined and the notion of balanced realization follows, first introduced in [Moore81] and further studied by many authors, e.g. [JS82, ZDG96]. This linear theory is rather complete and the relations between and interpretations in the state-space and input-output settings are fully understood.

A nonlinear extension of the state-space concept of balanced realizations has been introduced in [Scherpen93], mainly based on studying the past input nenergy and the future output energy. Since then, many results on state-space balancing, modifications, computational issues for model reduction and related minimality considerations for nonlinear systems have appeared in the literature, e.g., [GS01, HE02, LMG02, NK00, NK98, SG00, VG00]. In particular, singular value functions which are a nonlinear state-space extension of the Hankel singular values for linear systems play an important role for nonlinear balanced realizations. However, the original characterization in [Scherpen93] was incomplete in a sense that the defined singular value functions are not unique, the relation with the nonlinear Hankel operator was not clarified, and the resulting model reduction procedure gives different reduced order models depending on the choice of different set of singular value functions, e.g. [GS01].

Balanced realization and the related model order reduction technique rely on singular value analysis. This analysis investigates the singular values and the corresponding singular vectors for a given operator. The analysis is important since it extracts the gain structure of the operator, that is, it characterizes the largest input-output ratio and the corresponding input [Stewart93]. Since linear singular values are defined as eigenvalues of the composition of the given operator and its adjoint, it is natural to introduce a nonlinear version of adjoint operators to obtain a nonlinear counterpart of a singular value. There has been done quite some research on the nonlinear extension of adjoint operators, e.g. [Batt70, SG02, FSG02] and the references therein. Here we do not explicitly use these definitions of nonlinear adjoint operators. We rely on a characterization of singular values for nonlinear operators based on the gain structure as studied in [Fujimoto04]. The balanced realization based on this analysis yields a realization that is based on the singular values of the corresponding Hankel operator, and results in a method which can be viewed as a complete extension of the linear methods, both from an input-output and a state-space point of view, [FS05].

The related model order reduction technique, nonlinear balanced truncation, preserves several important properties of the original system and corresponding input-output operator, such as stability, controllability, observability and the gain structure [FS03].

This paper gives an overview of the series of research on balanced realization and the related model order reduction method based on nonlinear singular value analysis. Section 2 explains the taken point of view on singular value analysis for nonlinear operators. Section 3 briefly reviews the linear balancing method and balanced truncation in order to show the way of thinking for the nonlinear case. Section 4 treats the state-space balancing method stemming from [Scherpen93]. Then, in Section 5 we continue with balanced realizations based on the singular value analysis of the nonlinear Hankel operator. Furthermore, in Section 6 balanced truncation based on the method of Section 5 is presented. Finally, in Section 7 a numerical simulation illustrates how the proposed model order reduction method works for real-world systems.

2 Singular Value Analysis of Nonlinear Operators

Singular value analysis plays an important role in the characterizations of the principal behavior of linear operators. Here we formulate a nonlinear counterpart of singular value analysis. It is a basic ingredient for considering balanced realizations for nonlinear systems explained further on in this paper.

Let us consider a linear operator $A : U \to Y$ with Hilbert spaces U and Y. Then

$$A^* A\, v = \sigma^2 v \tag{1}$$

holds with $\sigma(\geq 0) \in \mathbb{R}$ and $v \in U$ where σ and v are called a *singular value* and a (right) *singular vector* of the operator A. Here A^* is the adjoint of A satisfying

$$\langle y, A\,u\rangle_Y = \langle A^* y, u\rangle_U \tag{2}$$

for all $u \in U$ and $y \in Y$ where $\langle\cdot,\cdot\rangle_X$ denotes the the inner product of the space X. For a finite dimensional signal space U, the operator A can be described by

$$A = \sum_{i=1}^{n} \sigma_i\, w_i\, v_i^*$$

with the singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, the corresponding right singular vectors v_i's. and the left singular vectors w_i's. Then we can obtain an approximation of A with rank $m < n$ by

$$A^a := \sum_{i=1}^{m} \sigma_i\, w_i\, v_i^*.$$

We can easily observe that this approximation preserves the gain of the original operator A

$$\|A^a\| = \sigma_1 = \|A\|.$$

Furthermore, the error bound is obtained by

$$\|A - A^a\| = \sigma_{m+1}.$$

For the generalization to nonlinear systems, we consider the following interpretation of singular values for linear operators. The largest singular value of the operator A characterizes the gain of the operator and the corresponding singular vector $v_{\max}$ represents the input maximizing the input-output ratio. Namely, the following equations hold.

$$\sigma_{\max} = \sup_{u\neq 0} \frac{\|A\,u\|}{\|u\|},\quad v_{\max} = \arg\sup_{u\neq 0} \frac{\|A\,u\|}{\|u\|} \tag{3}$$

Now, let us consider a smooth nonlinear operator $f : U \to Y$ with Hilbert spaces U and Y. How to define singular values of the nonlinear operator $f(u)$ is not immediately clear because there does not exist an operator $f^*(y)$ such that Equation (2) holds with $A = f$. Several papers define a nonlinear counterpart of an adjoint operator, e.g., [Batt70, SG02, FSG02]. For our nonlinear balancing purpose we generalize the linear way of thinking given by Equation (3). More precisely we consider the following definitions

$$\sigma^c_{\max} = \sup_{\|u\|=c} \frac{\|f(u)\|}{\|u\|},\quad v^c_{\max} = \arg\sup_{\|u\|=c} \frac{\|f(u)\|}{\|u\|} \tag{4}$$

where the gain of the operator f is characterized for each input magnitude c. The property that the gain of a nonlinear operator depends on the magnitude of its input is quite natural in the nonlinear setting and, for instance, this idea can be found in the input-to-state stability literature, e.g., [JTP94, SW96]. If $\sigma^c_{\max}$ is obtained, then we can calculate the largest singular value $\sigma_{\max}$ and the corresponding singular vector $v_{\max}$ of f by

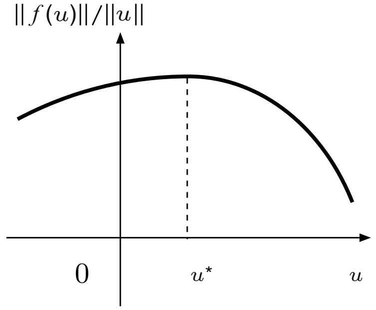

Fig. 1. Maximizing input $u = u^\star$ of $f(u)$

$$\sigma_{\max} = \sup_{c>0} \sigma^c_{\max}, \quad v_{\max} = v^c_{max}|_{c=\arg\sup_{c\neq 0} \sigma^c_{\max}}. \tag{5}$$

In the linear case, the largest singular value $\sigma_{\max}$ coincides with $\sigma^c_{\max}$ for all $c > 0$.

Now we are ready to define the singular value σ and the corresponding singular vector v for the operator f fulfilling the relationship (4). This is obtained by simply differentiating the condition in Equation (4). Figure 1 depicts the (locally) largest singular vector $u^\star$ when f is a mapping of $\mathbb{R} \to \mathbb{R}$. At the point $u = u^\star$ where the input-output takes its maximum value, the derivative of the input-output ratio has to be 0. Therefore the following equation has to hold for all u satisfying $\|u\| = c$.

$$\mathrm{d}\left(\frac{\|f(u)\|}{\|u\|}\right)(\mathrm{d}u) = 0 \tag{6}$$

Here the Fréchet derivative[1] is adopted to describe the problem. This equation is equivalent to

$$\langle(\mathrm{d}f(u))^* f(u) - \frac{\|f(u)\|^2}{\|u\|^2} u,\ \mathrm{d}u\rangle = 0. \tag{7}$$

On the other hand, the derivative of $\|u\| = c$ yields

$$\langle u,\ \mathrm{d}u\rangle = 0. \tag{8}$$

Combining Equations (7) and (8), we obtain the condition for the singular vector v.

Theorem 1. *[Fujimoto04] Consider a nonlinear operator $f : U \to Y$ with Hilbert spaces U and Y. Then the input-output ratio of $\|f(u)\|/\|u\|$ has a critical value for an arbitrary input magnitude $\|u\| = c$ if and only if*

$$(\mathrm{d}f(v))^* f(v) = \lambda\, v \tag{9}$$

with a scalar $\lambda \in \mathbb{R}$ and $v \in U$.

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[1] The Fréchet derivative of an operator $T : U \to Y$ is an operator $T : U \times U \to Y$ satisfying $f(u+v) = f(u) + \mathrm{d}f(u)(v) + o(\|v\|)$ such that $f(u)(v)$ depends linearly on v.

We now define v as a *singular vector* for a nonlinear operator f if it fulfills Equation (9). Immediate extension of the linear case by defining the singular value of f by $\sigma := \sqrt{\lambda}$ is not appropriate. This can be seen from the fact that, e.g., λ can be negative. A better extension is given by using the singular vector v, and defining the corresponding singular value by

$$\sigma = \frac{\|f(v)\|}{\|v\|}. \tag{10}$$

In the remainder of this paper, investigating the solutions of the pair of Equations (9) and (10) is called *singular value analysis* of the nonlinear operator f. Here σ is called a *singular value* of f, and v is called the corresponding *singular vector*. It can be readily observed that

$$\lambda = \sigma^2$$

holds in the linear case. However, this equation does not hold in the nonlinear case. Although the scalar λ is always real, it can be negative in the nonlinear case [Fujimoto04].

A more detailed discussion on nonlinear singular value analysis is given in [Fujimoto04].

3 Balanced Realization for Linear Systems

This section briefly reviews balanced realizations in the linear systems case in order to show the way of thinking in the nonlinear case. See standard textbooks for the detail, e.g., [OA01, ZDG96]. Consider the following controllable, observable, and asymptotically stable linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \; x(0) = 0 \\ y = Cx \end{cases} \tag{11}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. The controllability Gramian P and the observability Gramian Q of the system Σ in Equation (11) are obtained by the solutions to the Lyapunov equations

$$AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} = 0 \tag{12}$$
$$A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}C = 0. \tag{13}$$

It is known that the positive definiteness of the Gramians P and Q is equivalent to controllability and observability of the system Σ in Equation (11), respectively. Furthermore, the matrices P and Q themselves are quantitative indicators of the controllability and observability, that is, P and Q describe the behavior of input-to-state and that of state-to-output, respectively.

A *balanced realization* of Σ is a state-space realization which has the following Gramians

$$P = Q = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{14}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ are called *Hankel singular values*. Here the system is balanced because $P = Q$ implies that relation between input-to-state and state-to-output is balanced and diagonalized P and Q implies that the importance of each coordinate axis is balanced. There is another realization called an *input-normal form* which has the following Gramians

$$P = I, \ \ Q = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \tag{15}$$

where only the balancing between the coordinate axes is achieved.

If $\sigma_i > \sigma_j$ then the coordinate axis x_i is more important to the input-output behavior, i.e., better controllable and observable, than the axis x_j. Therefore if $\sigma_k \gg \sigma_{k+1}$ holds for a certain k ($1 \leq k < n$), then we can obtain a k-dimensional reduced order model by neglecting the dynamics of $x_{k+1}, \ldots, x_n$. This model reduction procedure is called *balanced truncation*. More precisely, balanced truncation is executed as follows. Suppose that the system is in a balanced realization and divide the coordinate as follows

$$\begin{aligned} x &= (x^a, x^b) \\ x^a &= (x_1, \ldots, x_k) \\ x^b &= (x_{k+1}, \ldots, x_n). \end{aligned}$$

Further divide the state-space system

$$\begin{pmatrix} \dot{x}^a \\ \dot{x}^b \end{pmatrix} = \begin{pmatrix} A^a & A^{ab} \\ A^{ba} & A^b \end{pmatrix} \begin{pmatrix} x^a \\ x^b \end{pmatrix} + \begin{pmatrix} B^a \\ B^b \end{pmatrix} u$$

$$y = \begin{pmatrix} C^a, & C^b \end{pmatrix} \begin{pmatrix} x^a \\ x^b \end{pmatrix}.$$

Then the reduced order model is obtained by

$$\Sigma^a : \begin{cases} \dot{x}^a &= A^a x^a + B^a u \\ y &= C^a x^a \end{cases}.$$

By balanced truncation it is readily obtained that several properties are preserved. This can be seen by studying the Lyapunov equations (12) and (13), and their truncated versions, e.g.,

Theorem 2. *[Moore81] The controllability Gramian P^a and the observability Gramian Q^a of the reduced order model Σ^a are given by*

$$P^a = Q^a = \text{diag}(\sigma_1, \ldots, \sigma_k).$$

The controllability operator $\mathcal{C}$ and the observability operator $\mathcal{O}$ of the system Σ as in (11) are given by

$$\mathcal{C} : u \mapsto x^0 := \int_0^\infty \mathrm{e}^{\tau A} B u(\tau) \mathrm{d}\tau$$
$$\mathcal{O} : x^0 \mapsto y := C\mathrm{e}^{tA} x^0.$$

Furthermore, their composition is defined as the *Hankel operator* $\mathcal{H}$ of the original system Σ.

$$\mathcal{H} = \mathcal{O}\,\mathcal{C} \tag{16}$$

These operators are closely related to the Gramians, i.e.,

$$P = \mathcal{C}\,\mathcal{C}^*$$
$$Q = \mathcal{O}^*\mathcal{O}$$

Now consider a linear system given by Equation (11), which is not necessarily observable and/or controllable. The relation between the Gramians and the observability and controllability operator allows one to prove the following theorem.

Theorem 3. *[ZDG96] The operator $\mathcal{H}^*\mathcal{H}$ and the matrix PQ have the same nonzero eigenvalues.*

Proof: The proof of this theorem is easily obtained and instructive for the nonlinear extension case. We first prove the '⇒' part. Due to (16), the eigenvalue problem of $\mathcal{H}^*\mathcal{H}$ reduces to

$$\mathcal{C}^*\mathcal{O}^*\mathcal{O}\,\mathcal{C}\,v = \lambda\,v, \quad v \in U, \ \lambda \in \mathbb{R}$$

with $\lambda = \sigma^2$. Defining $\xi := \mathcal{C}\,v \in \mathbb{R}^n$ and premultiplying $\mathcal{C}$ to the above equation, we obtain

$$\mathcal{C}\,\mathcal{C}^*\mathcal{O}^*\mathcal{O}\,\mathcal{C}\,v = \mathcal{C}\,\lambda\,v$$

which reduces to

$$PQ\,\xi = \lambda\,\xi \tag{17}$$

characterizing the eigenvalues of PQ. Furthermore, the '⇐' part can be proved in a similar way. Suppose that we have the above equation. Then premultiplying $\mathcal{C}^*\mathcal{O}\;\mathcal{O}^*$ and defining $\bar{v} := \mathcal{C}^*\mathcal{O}\;\mathcal{O}^*\xi$ we obtain

$$\mathcal{H}^*\mathcal{H}\,\bar{v} = \lambda\,\bar{v}$$

which coincides with the eigenvalue problem of $\mathcal{H}^*\mathcal{H}$. □

Thus the singular value problem of the operator $\mathcal{H}$ is closely related to the eigenvalue problem of the matrix PQ, and a singular vector v of $\mathcal{H}$ is characterized by an eigenvector ξ of PQ.

Due to this property, the constants σ_i's in Equation (14) are called Hankel singular values. Furthermore, the Hankel norm $\|\Sigma\|_H$ of the operator Σ is defined by the L_2 gain of the corresponding Hankel operator as

$$\|\Sigma\|_H := \sup_{\substack{u \in L_2[0,\infty) \\ u \neq 0}} \frac{\|\mathcal{H}(u)\|_{L_2}}{\|u\|_{L_2}} = \sigma_1. \tag{18}$$

Theorem 2 implies that the balanced truncation procedure preserves the Hankel norm of the original system, that is,

$$\|\Sigma^a\|_H = \|\Sigma\|_H. \tag{19}$$

It is also known that the error bound of this model order reduction procedure is given by

$$\|\Sigma - \Sigma^a\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i. \tag{20}$$

The relation between the Gramians and the Hankel, controllability and observability operators gives rise to both input-output operator interpretations as well as state-space interpretations of Hankel singular values and balanced truncation. These interpretations are crucial for the extension to nonlinear systems.

4 Basics of Nonlinear Balanced Realizations

This section gives a nonlinear extension of balanced realization introduced in the previous section. Let us consider the following asymptotically stable input-affine nonlinear system

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \ x(0) = x^0 \\ y = h(x) \end{cases} \tag{21}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. The controllability operator $\mathcal{C} : U \to X$ with $X = \mathbb{R}^n$ and $U = L_2^m[0,\infty)$, and the observability operator $\mathcal{O} : X \to Y$ with $Y = L_2^p[0,\infty)$ for this system are defined by

$$\mathcal{C} : u \mapsto x^0 : \begin{cases} \dot{x} \ = -f(x) - g(x)u \ x(\infty) = 0 \\ x^0 = x(0) \end{cases}$$

$$\mathcal{O} : x^0 \mapsto y : \begin{cases} \dot{x} = f(x) \ x(0) = x^0 \\ y = h(x) \end{cases} .$$

This definition implies that the observability operator $\mathcal{O}$ is a map from the initial condition $x(0) = x^0$ to the output L_2 signal when no input is applied. To interpret the meaning of $\mathcal{C}$, let us consider a time-reversal behavior of the $\mathcal{C}$ operator as

$$\mathcal{C} : u \mapsto x^0 : \begin{cases} \dot{x} \ = f(x) + g(x)u(-t) \ x(-\infty) = 0 \\ x^0 = x(0) \end{cases} . \tag{22}$$

Then the controllability operator $\mathcal{C}$ can be regarded as a mapping from the input L_2 signal to the terminal state $x(0) = x^0$ when the initial state is $x(-\infty) = 0$. Therefore, as in the linear case, $\mathcal{C}$ and $\mathcal{O}$ represent the input-to-state behavior and the state-to-output behavior, respectively. As in the linear case, the Hankel operator for the nonlinear operator Σ in (16) is given by the composition of $\mathcal{C}$ and $\mathcal{O}$

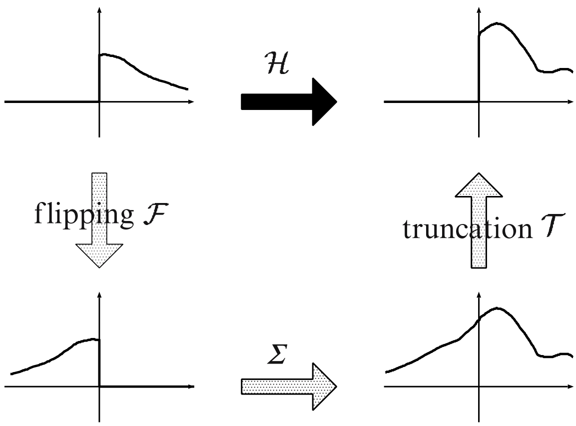

Fig. 2. Hankel operator $\mathcal{H}$ of Σ

$$\mathcal{H} := \mathcal{O} \circ \mathcal{C}. \tag{23}$$

The input-output mapping of a Hankel operator is explained in Figure 2. The lower side of the figure depicts the input-output behavior of the original operator Σ in Equation (21). The upper side depicts the input-output behavior of the Hankel operator of Σ, where the signal in the upper left side is the time-flipped signal of the lower left side signal. The flipping operator is defined by

$$\mathcal{F}(u(t)) := u(-t).$$

The upper right side signal is the truncated signal (to the space $L_2[0,\infty)$) of the lower left side signal. The corresponding truncation operator is given by

$$\mathcal{T}(y(t)) := \begin{cases} 0 & (t < 0) \\ y(t) & (t \geq 0) \end{cases}.$$

The definition of a Hankel operator implies that it describes the mapping from the input to the output generated by the state at $t = 0$. Hence we can analyze the relationship between the state and the input-output behavior of the original operator Σ by investigating its Hankel operator.

To this end, we need to define certain operators and functions related to Gramians in the linear case. First a norm-minimizing inverse $\mathcal{C}^\dagger : X \to U$ of $\mathcal{C}$ is introduced.

$$\mathcal{C}^\dagger : x^0 \mapsto u := \arg \min_{\mathcal{C}(u)=x^0} \|u\|$$

The operators $\mathcal{C}^\dagger$ and $\mathcal{O}$ yield the definitions of the controllability function $L_c(x)$ and the observability function $L_o(x)$ that are generalization of the controllability and observability Gramians, respectively.

$$L_c(x^0) := \frac{1}{2}\|\mathcal{C}^\dagger(x^0)\|^2 = \min_{\substack{u \in L_2(-\infty,0] \\ x(-\infty)=0, x(0)=x^0}} \frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt \tag{24}$$

$$L_o(x^0) := \frac{1}{2}\|\mathcal{O}(x^0)\|^2 = \frac{1}{2}\int_{0}^{\infty} \| y(t) \|^2 \, dt, \; {\scriptstyle x(0)=x^0,\; u(t)\equiv 0,\quad 0\le t<\infty.} \tag{25}$$

These definitions imply that the controllability function $L_c(x^0)$ is the minimum input energy (in the L_2 norm sense) required to move from the initial state $x(-\infty) = 0$ to the terminal state $x(0) = x^0$, and that the observability function $L_o(x^0)$ is the output energy generated by the initial state $x(0) = x^0$ with zero input, respectively. If the system Σ is linear as in (11), then those functions are described by

$$L_c(x) = \frac{1}{2}x^{\mathrm{T}}P^{-1}x, \;\; L_o(x) = \frac{1}{2}x^{\mathrm{T}}Qx \tag{26}$$

with the controllability Gramian P and the observability Gramian Q the solutions of the Lyapunov equations (12) and (13). Here the inverse of P appears in the above equation because $\mathcal{C}^\dagger$ appears in the definition (24), whereas $\mathcal{C}$ can be used in the linear case. In order to obtain those functions $L_c(x)$ and $L_o(x)$, we need to solve a Hamilton-Jacobi equation and a Lyapunov equation.

Theorem 4. *[Scherpen93] Consider the system (21). Suppose that* 0 *is an asymptotically stable equilibrium point and that a smooth observability function* $L_o(x)$ *exists. Then* $L_o(x)$ *is the unique smooth solution of*

$$\frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2}\, h(x)^{\mathrm{T}} h(x) = 0$$

with $L_o(0) = 0$*. Furthermore, assume that a smooth controllability function* $L_c(x)$ *exists. Then* $L_c(x)$ *is the unique smooth solution of*

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2}\,\frac{\partial L_c(x)}{\partial x} g(x)g(x)^{\mathrm{T}} \frac{\partial L_c(x)}{\partial x}^{\mathrm{T}} = 0$$

with $L_c(0) = 0$ *such that* 0 *is an asymptotically stable equilibrium point of* $\dot{x} = -f(x) - g(x)g(x)^{\mathrm{T}}(\partial L_c(x)/\partial x)^{\mathrm{T}}$.

Similar to the linear case, the positive definiteness of the controllability and observability functions implies strong reachability and zero-state observability of the system Σ in (21), respectively. Combining these two properties, we can obtain the following result on the minimality of the system.

Theorem 5. *[SG00] Consider the system (21). Suppose that*

$$0 < L_c(x) < \infty$$
$$0 < L_o(x) < \infty$$

hold for all $x \neq 0$*. Then the system is a minimal realization as defined in [Isidori95].*

Similar to the linear case, $L_c(x)$ and $L_o(x)$ can be used to "measure the minimality" of a nonlinear dynamical system. Furthermore, a basis for nonlinear balanced realization is obtained as a nonlinear generalization of the relationship (15) in the linear case. For that, a factorization of $L_o(x)$ into a semi-quadratic form needs to be done, i.e., in a convex neighborhood of the equilibrium point 0 we can write

$$L_o(x) = \frac{1}{2} x^{\mathrm{T}} M(x) x, \quad \text{with} \quad M(0) = \frac{\partial^2 L_o}{\partial x^2}(0). \tag{27}$$

Now, an input-normal/output-diagonal form can be obtained.

Theorem 6. *[Scherpen93] Consider the system (21) on a neighborhood W of 0. Suppose that 0 is an asymptotically stable equilibrium point, that it is zero-state observable, that smooth controllability and observability functions $L_c(x)$ and $L_o(x)$ exist on W, and that $(\partial^2 L_c/\partial x^2)(0) > 0$ and $(\partial^2 L_o/\partial x^2)(0) > 0$ hold. Furthermore, assume that the number of distinct eigenvalues of $M(x)$ is constant on W. Then there exists coordinates such that the controllability and observability functions $L_c(x)$ and $L_o(x)$ satisfy*

$$L_c(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \tag{28}$$

$$L_o(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \tau_i(x) \tag{29}$$

where $\tau_1(x) \geq \tau_2(x) \geq \ldots \geq \tau_n(x)$.

A state-space realization satisfying the conditions (28) and (29) is called an *input-normal form*, and the functions $\tau_i(x)$, $i = 1, 2, \ldots, n$ are called singular value functions. We refer to [Scherpen93] for the construction of the coordinate transformation that brings the system in the form of Theorem 6. If a singular value function $\tau_i(x)$ is larger than $\tau_j(x)$, then the coordinate axis x_i plays more important role than the coordinate axis x_j does. Thus this realization is similar to the linear input-normal/output-diagonal realization (15), and it directly yields a tool for model order reduction of a nonlinear systems. However, a drawback of the above realization is that the the singular value functions $\tau_i(x)$'s and consequently, the corresponding realization are not unique, e.g. [GS01]. For example, if the observability function is given by

$$L_o(x) = \frac{1}{2}(x_1^2 \tau_1(x) + x_2^2 \tau_2(x)) = \frac{1}{2}(2x_1^2 + x_2^2 + x_1^2 x_2^2),$$

with the state-space $x = (x_1, x_2)$, then the corresponding singular value functions are

$$\tau_1(x) = 2 + k x_2^2$$
$$\tau_2(x) = 1 + (1 - k) x_1^2$$

with an arbitrary scalar constant k. This example reveals that the singular value function are not uniquely determined by this characterization. To overcome these problems, balanced realization based on nonlinear singular value analysis introduced in Section 2 is investigated in the following section.

5 Balanced Realizations Based on Singular Value Analysis of Hankel Operators

In this section, application of singular value analysis to nonlinear Hankel operators determines a balanced realization with a direct input-output interpretation whereas the balanced realization of Theorem 6 is completely determined based on state-space considerations only. To this end, we consider the Hankel operator $\mathcal{H} : U \to Y$ as defined in (23) with $U = L_2^m[0, \infty)$ and $Y = L_2^p[0, \infty)$. Then Equation (9) yields

$$(\mathrm{d}\mathcal{H}(v))^* \, \mathcal{H}(v) = \lambda \, v, \quad \lambda \in \mathbb{R}, \;\; v \in U. \tag{30}$$

Since we consider a singular value analysis problem on L_2 spaces, we need to find state trajectories of certain Hamiltonian dynamics, see e.g., [FS05]. In the linear case, Theorem 3 shows that we only need to solve an eigenvalue problem (17) on a finite dimensional space $X = \mathbb{R}^n$ to obtain the singular values and singular vectors of the Hankel operator $\mathcal{H}$. Here we provide its nonlinear counterpart as follows.

Theorem 7. *[FS05] Consider the Hankel operator defined by Equation (23). Suppose that the operators $\mathcal{C}^\dagger$ and $\mathcal{O}$ exist and are smooth. Suppose moreover that $\lambda \in \mathbb{R}$ and $\xi \in X$ satisfy the following equation*

$$\frac{\partial L_o(\xi)}{\partial \xi} = \lambda \, \frac{\partial L_c(\xi)}{\partial \xi}, \quad \lambda \in \mathbb{R}, \;\; \xi \in X. \tag{31}$$

Then λ and

$$v := \mathcal{C}^\dagger(\xi) \tag{32}$$

satisfy Equation (9). That is, v defined above is a singular vector of $\mathcal{H}$.

Though the original singular value analysis problem (9) is a nonlinear problem on an infinite dimensional signal space $U = L_2^m[0, \infty)$, the problem to be solved in the above theorem is a nonlinear algebraic equation on a finite dimensional space $X = \mathbb{R}^n$ which is also related to a nonlinear eigenvalue problem on X, see [Fujimoto04].

In the linear case, where $L_c(x)$ and $L_o(x)$ are given by (26), Equation (31) reduces to

$$\xi^{\mathrm{T}} Q = \lambda \, \xi^{\mathrm{T}} P^{-1}$$

where P and Q are the controllability and observability Gramians. This equation is equivalent to (17), i.e., λ and ξ are an eigenvalue and an eigenvector of PQ. Furthermore, Equation (32) characterizes the relationship between a singular vector v of $\mathcal{H}$ and an eigenvector ξ of PQ as in the linear case result. Thus Theorem 7 can be regarded as a nonlinear counterpart of Theorem 3.

In the linear case, there always exist n independent pairs of eigenvalues and eigenvectors of PQ. What happens in the nonlinear case? The answer is provided in the following theorem.

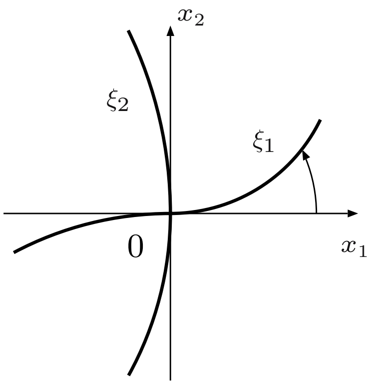

Fig. 3. Configuration of $\xi_1(s)$ and $\xi_2(s)$ in the case $n = 2$

Theorem 8. *[FS05] Consider the system Σ in (21) and the Hankel operator $\mathcal{H}$ in Equation (23) with $X = \mathbb{R}^n$. Suppose that the Jacobian linearization of the system has n distinct Hankel singular values. Then Equation (31) has n independent solution curves $\xi = \xi_i(s)$, $s \in \mathbb{R}$, $i = 1, 2, \ldots, n$ intersecting to each other at the origin and satisfying the condition*

$$\|\xi_i(s)\| = |s|.$$

In the linear case, the solutions of Equation (31) are the lines (orthogonally) intersecting to each other at the origin. The above theorem shows that instead of these lines, in the nonlinear case n independent curves $x = \xi_i(s)$, $i = 1, 2, \ldots, n$ exist. For instance, if the dimension of the state is $n = 2$, the solution of Equation (31) is illustrated in Figure 3.

We can relate the solutions $\xi_i(s)$ to the singular values of the Hankel operator $\mathcal{H}$. Let $v_i(s)$ and $\sigma_i(s)$ denote the singular vector and the singular value parameterized by s corresponding to $\xi_i(s)$. Then we have

$$\begin{aligned} v_i(s) &:= \mathcal{C}^\dagger(\xi_i(s)) \\ \sigma_i(s) &:= \frac{\|\mathcal{H}(v_i(s))\|_{L_2}}{\|v_i(s)\|_{L_2}} = \frac{\|\mathcal{O}(\xi_i(s))\|_{L_2}}{\|\mathcal{C}^\dagger(\xi_i(s))\|_{L_2}} \\ &= \sqrt{\frac{L_o(\xi_i(s))}{L_c(\xi_i(s))}}. \end{aligned}$$

By this equation, we can obtain an explicit expression of the singular values $\sigma_i(s)$'s of the Hankel operator $\mathcal{H}$. These functions $\sigma_i(s)$'s are called *Hankel singular values*. Without loss of generality we assume that the following equation holds for $i = 1, 2, \ldots, n$ in a neighborhood of the origin

$$\min\{\sigma_i(s), \sigma_i(-s)\} > \max\{\sigma_{i+1}(s), \sigma_{i+1}(-s)\}. \tag{33}$$

As in the linear case, the solution curves $\xi_i(s)$'s play the roles of the coordinate axes of balanced realization. By applying an isometric coordinate transformation which maps the solution curves $\xi_i(s)$'s into the coordinate axes, we obtain a realization whose (new) coordinate axes x_i are the solution of Equation (31), i.e.,

$$\left.\frac{\partial L_o(x)}{\partial x}\right|_{x=(0,\ldots,0,x_i,0,\ldots,0)} = \lambda \left.\frac{\partial L_c(x)}{\partial x}\right|_{x=(0,\ldots,0,x_i,0,\ldots,0)} \tag{34}$$

$$\sigma_i(x_i) = \sqrt{\frac{L_o(0,\ldots,0,x_i,0,\ldots,0)}{L_c(0,\ldots,0,x_i,0,\ldots,0)}}. \tag{35}$$

Equation (35) implies that the new coordinate axes x_i, $i = 1,\ldots,n$ are the solutions of Equation (31) for Hankel singular value analysis. Therefore the Hankel norm defined in (18) can be obtained by

$$\begin{aligned}\|\Sigma\|_H &= \sup_{u\neq 0} \frac{\|\mathcal{H}(u)\|_{L_2}}{\|u\|_{L_2}} \\ &= \sup_{s\in\mathbb{R}} \max_i \sigma_i(s) \\ &= \sup_{x_1\in\mathbb{R}} \sqrt{\frac{L_o(x_1,0,\ldots,0)}{L_c(x_1,0,\ldots,0)}}\end{aligned}$$

provided the ordering condition (33) holds for all $s \in \mathbb{R}$. Furthermore, apply this coordinate transformation recursively to all lower dimensional subspaces such as $(x_1, x_2, \ldots, x_k, 0, \ldots, 0)$, then we can obtain a state-space realization satisfying Equation (35) and

$$x_i = 0 \iff \frac{\partial L_o(x)}{\partial x_i} = 0 \iff \frac{\partial L_c(x)}{\partial x_i} = 0. \tag{36}$$

This property is crucial for balanced realization and model order reduction. Using tools from differential topology, e.g. [Milnor65], we can prove that this realization is diffeomorphic to the following *precise* input-normal/output-diagonal realization.

Theorem 9. *[FS03] Consider the system Σ in (21). Suppose that the assumptions in Theorem 8 hold. Then the there exists a coordinates in a neighborhood of the origin such that the system is in input-normal/output-diagonal form satisfying*

$$\begin{aligned}L_c(x) &= \frac{1}{2}\sum_{i=1}^{n} x_i^2 \\ L_o(x) &= \frac{1}{2}\sum_{i=1}^{n} x_i^2\, \sigma_i(x_i)^2\end{aligned}$$

This realization is much more precise than that in Theorem 6 in the following senses: (a) The solutions of Equation (31) coincide with the coordinate axes, that is, Equation (34) holds. (b) The ratio of the observability function L_o to the controllability function L_c equals the singular values $\sigma_i(x_i)$'s on the coordinate axes, that is Equation (35) holds. (c) Furthermore, an exact balanced realization can be obtained by a coordinate transformation

$$z_i = \phi_i(x_i) := x_i \sqrt{\sigma_i(x_i)} \tag{37}$$

which is well-defined in a neighborhood of the origin.

Corollary 1. *[FS03] The coordinate change (37) transforms the input-normal realization in Theorem 9 into the following form*

$$L_c(z) = \frac{1}{2} \sum_{i=1}^{n} \frac{z_i^2}{\sigma_i(z_i)}$$
$$L_o(z) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \sigma_i(z_i).$$

Since we only use the coordinate transformation (37) preserving the coordinate axes, the realization obtained here also satisfies the properties (a) and (b) explained above. The controllability and observability functions can be written as

$$L_c(z) = \frac{1}{2} z^{\mathrm{T}} \underbrace{\mathrm{diag}(\sigma_1(z_1), \ldots, \sigma_n(z_n))^{-1}}_{P(z)} z$$
$$L_o(z) = \frac{1}{2} z^{\mathrm{T}} \underbrace{\mathrm{diag}(\sigma_1(z_1), \ldots, \sigma_n(z_n))}_{Q(z)} z$$

Here $P(z)$ and $Q(z)$ can be regarded as nonlinear counterparts of the controllability and observability Gramians as observed in Equation (14) with the relation (26) since

$$P(z) = Q(z) = \mathrm{diag}(\sigma_1(z_1), \sigma_2(z_2), \ldots, \sigma_n(z_n)). \tag{38}$$

The axes of this realization are uniquely determined. We call this state-space realization a *balanced realization* of the nonlinear system Σ in Equation (21). As in the linear case, both the relationship between the input-to-state and state-to-output behavior and that among the coordinate axes are balanced.

6 Model Order Reduction

An important application of balanced realizations is that it is a tool for model order reduction called *balanced truncation*. Here, a model order reduction method preserving the Hankel norm of the original system is proposed. Suppose that the system (21) is balanced in the sense that it satisfies Equations (35) and (36). Note that the realizations in Theorem 9 and Corollary 1 satisfy these conditions. Suppose moreover that

$$\min\{\sigma_k(s), \sigma_k(-s)\} \gg \max\{\sigma_{k+1}(s), \sigma_{k+1}(-s)\}$$

holds with a certain k ($1 \leq k < n$). Divide the state into two vectors $x = (x^a, x^b)$

$$\begin{aligned} x^a &:= (x_1, \dots, x_k) \in \mathbb{R}^k \\ x^b &:= (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}, \end{aligned}$$

and the vector field into two vector fields accordingly

$$\begin{aligned} f(x) &= \begin{pmatrix} f^a(x) \\ f^b(x) \end{pmatrix} \\ g(x) &= \begin{pmatrix} g^a(x) \\ g^b(x) \end{pmatrix}, \end{aligned}$$

and truncate the state by substituting $x^b = 0$. Then we obtain a k-dimensional state-space model Σ^a with the state x^a (with a $(n-k)$-dimensional residual model Σ^b with the state x^b).

$$\Sigma^a : \begin{cases} \dot{x}^a = f^a(x^a, 0) + g^a(x^a, 0)u^a \\ y^a = h(x^a, 0) \end{cases} \tag{39}$$

$$\Sigma^b : \begin{cases} \dot{x}^b = f^b(0, x^b) + g^b(0, x^b)u^b \\ y^b = h(0, x^b) \end{cases} \tag{40}$$

This procedure is called *balanced truncation*. The obtained reduced order models have preserved the following properties.

Theorem 10. *[FS01, FS06] Suppose that the system Σ satisfies Equations (35) and (36) and apply the balanced truncation procedure explained above. Then the controllability and observability functions of the reduced order models Σ^a and Σ^b denoted by L_c^a, L_c^b, L_o^a and L_o^b, respectively, satisfy the following equations*

$$\begin{aligned} L_c^a(x^a) &= L_c(x^a, 0), \quad L_o^a(x^a) = L_o(x^a, 0) \\ L_c^b(x^b) &= L_c(0, x^b), \quad L_o^b(x^b) = L_o(0, x^b) \end{aligned}$$

which implies

$$\begin{aligned} \sigma_i^a(x_i^a) &= \sigma_i(x_i^a), \quad i = 1, 2, \dots, k \\ \sigma_i^b(x_i^b) &= \sigma_{i+k}(x_i^b), \quad i = 1, 2, \dots, n-k \end{aligned}$$

with the singular values σ^a's of the system Σ^a and the singular values σ^b of the system Σ^b. In particular, if σ_1 is defined globally, then

$$\|\Sigma^a\|_H = \|\Sigma\|_H. \tag{41}$$

Theorem 10 states that the important characteristics of the original system such as represented by the controllability and observability functions and Hankel singular values are preserved. Moreover, by Theorem 5, this implies that the controllability, observability, minimality and the gain property is preserved under the model reduction. These preservation properties hold for truncation of any realization satisfying the conditions (35) and (36), such as the realizations in Theorem 9 and Corollary 1 [FS01]. Furthermore, concerning the stability, (global) Lyapunov stability and local asymptotic stability are preserved with this procedure as well. Note that this theorem is a natural nonlinear counterpart of Theorem 2 and Equation (19). However, a nonlinear counterpart of the error bound of the reduced order model as in (20) has not been found yet.

7 Numerical Example

In this section, we apply the proposed model order reduction procedure to a double pendulum (an underactuated two degrees of freedom robot manipulator) as depicted in Figure 4.

Here m_i denotes the mass located at the end of the i-th link, l_i denotes the length of the i-th link, μ_i denotes the friction coefficient of the i-th link, and x_i denotes the angle of the i-th link. We select the physical parameters as $l_1 = l_2 = 1$, $m_1 = m_2 = 1$, $\mu_1 = \mu_2 = 1$, $g_0 = 9.8$ with g_0 the gravity coefficient. The dynamics of this system can be described by an input-affine nonlinear system model (21) with 4 dimensional state-space

$$x = (x_1, x_2, x_3, x_4) := (x_1, x_2, \dot{x}_1, \dot{x}_2). \tag{42}$$

The input u denotes the torque applied to the first link at the first joint and the output y denotes the horizontal and the vertical coordinates of the position of the mass m_2. The potential energy $V(x)$ and the kinetic energy $T(x)$ for this system are described by

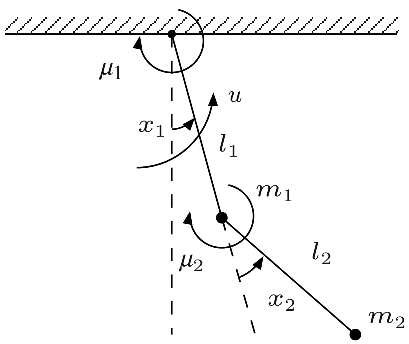

Fig. 4. The double pendulum

$$V(x) = -m_1 g_0 l_1 \cos x_1 - m_2 g_0 l_1 \cos x_1 - m_2 g_0 l_2 \cos(x_1 + x_2)$$
$$T(x) = \frac{1}{2}(\dot{x}_1, \dot{x}_2) M(x) \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$
$$M(x) =$$
$$\begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2 m_2 l_1 l_2 \cos x_2 & m_2 l_2^2 + m_2 l_1 l_2 \cos x_2 \\ m_2 l_2^2 + m_2 l_1 l_2 \cos x_2 & m_2 l_2^2 \end{pmatrix} \tag{43}$$

where $M(x)$ denotes the inertia matrix. Then the dynamics of this system is obtained by the Lagrange's method as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L(x)}{\partial(\dot{x}_1, \dot{x}_2)}^{\mathrm{T}} - \frac{\partial L(x)}{\partial(x_1, x_2)}^{\mathrm{T}} = \begin{pmatrix} u - \mu_1 \dot{x}_1 \\ -\mu_2 \dot{x}_2 \end{pmatrix} \tag{44}$$

with the Lagrangian $L(x) := T(x) - V(x)$. This equation reduces to the system (21) with

$$f(x) = \begin{pmatrix} x_3 \\ x_4 \\ M^{-1} \left(\frac{\partial(T-V)}{\partial(x_1,x_2)}^{\mathrm{T}} - \dot{M} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} - \begin{pmatrix} \mu_1 \dot{x}_1 \\ \mu_2 \dot{x}_2 \end{pmatrix} \right) \end{pmatrix}$$
$$g(x) = \begin{pmatrix} 0 \\ 0 \\ M^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$
$$h(x) = \begin{pmatrix} l_1 \sin x_1 + l_2 \sin(x_1 + x_2) \\ l_1(1 - \cos x_1) + l_2(1 - \cos(x_1 + x_2)) \end{pmatrix}.$$

See [FT06] for the details of the model.

For computing L_o and L_c, we use the method based on Taylor series expansion proposed in [Lukes69]. Then we need to solve the nonlinear algebraic equation (31). Although it is much easier to be solved compared with the original singular value analysis problem in (30), it is still difficult to obtain a closed form solution. Again using Taylor series expansion we can prove that the computation of Equation (31) reduces to solving linear algebraic equations recursively. Applying this procedure and calculating the balancing coordinate transformation up to the 4-th order terms of the Taylor series expansion, results in the following Hankel singular value functions.

$$\sigma_1(x_1)^2 = 1.98 \times 10^{-1} + 4.14 \times 10^{-4} x_1^2 + o(|x_1|^3)$$
$$\sigma_2(x_2)^2 = 1.72 \times 10^{-1} + 3.28 \times 10^{-4} x_2^2 + o(|x_2|^3)$$
$$\sigma_3(x_3)^2 = 5.83 \times 10^{-5} + 1.51 \times 10^{-4} x_3^2 + o(|x_3|^3)$$
$$\sigma_4(x_4)^2 = 9.37 \times 10^{-6} + 9.22 \times 10^{-6} x_4^2 + o(|x_4|^3)$$

These functions are depicted in Figure 5 where the solid line denotes σ_1, the dotted line denotes σ_2, the dashed line denotes σ_3 and the dashed and dotted line denotes σ_4.

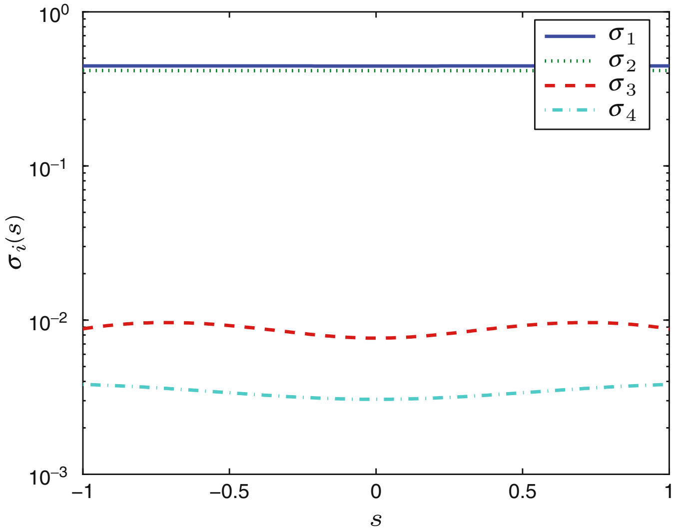

Fig. 5. Hankel singular value functions $\sigma_i(s)$, $i = 1, \ldots, 4$

From this figure, we conclude that in a neighborhood of 0 $\sigma_2(x_2) \gg \sigma_3(x_3)$, and thus that an appropriate dimension of the reduced order model is 2. We can now apply the balanced truncation procedure as presented in the previous section.

We have executed some simulations of the original and reduced models to evaluate the effectiveness of the proposed model order reduction method. Here the time responses for impulsive inputs are depicted in the figures, i.e., Figure 6 describes the response of the horizontal movement and Figure 7 describes the response of the vertical movement. In the figures, the solid line denotes the response of the original system, the dashed line denotes the response of the linearized reduced order model, and the dashed/dotted line denotes the response of the nonlinear reduced order model.

In Figure 6, all trajectories are identical which indicates that both linear and nonlinear reduced order models can approximate the behavior of the original model well. However, in Figure 7, one can observe that the trajectory of the linear reduced order model is quite different from the original whereas the trajectory of the nonlinear reduced order model is almost identical with that of the original system. This is due to the fact that the linearization of the vertical displacement of the mass m_2 is 0 since it consists of a cosine function of the state. These simulations demonstrate the effectiveness of our nonlinear balanced truncation method. It is noted that the proposed computation algorithm is currently only applicable to systems whose size is relatively small. A big progress on computation of nonlinear balanced realization is required to make it be applicable to real-world large scale systems.

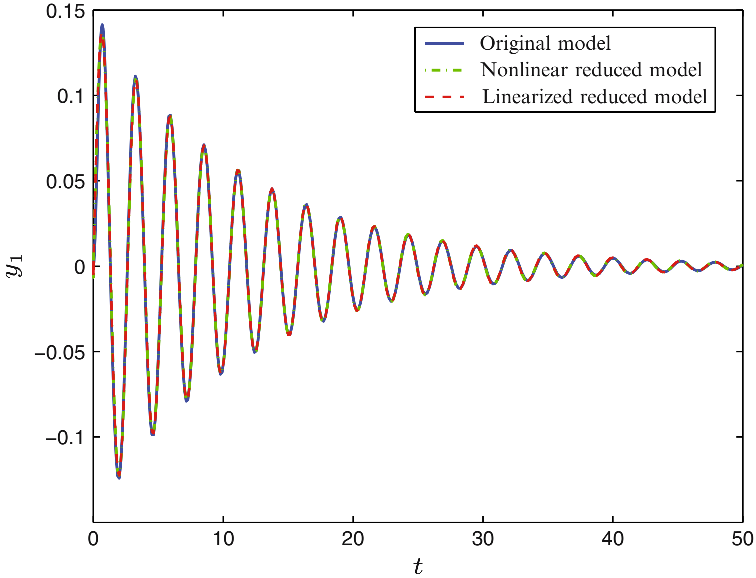

Fig. 6. The horizontal displacement

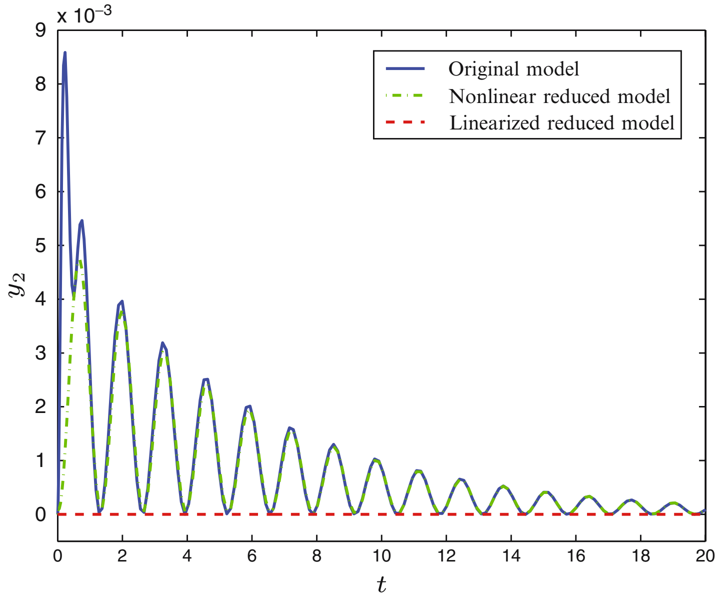

Fig. 7. The vertical displacement

8 Conclusion

In this paper, we have presented an overview of singular value analysis of nonlinear operators and its application to balanced realizations and model order reduction methods for nonlinear systems. Recent development in this area of research provides a precise and complete basis for model oder reduction of nonlinear dynamical systems. A reduced order model derived by this technique preserves many important properties of the original system such as controllability, observability, stability and the Hankel norm. Compared with the theoretical results, however, computational developments are still in their infancy, meaning that large scale nonlinear systems are still difficult to handle. Future research should thus include a strong focus on the computational algorithms for making nonlinear balanced truncation a useful tool in large scale applications.

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