# Decision Making Based on Fuzzy Data Envelopment Analysis

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**Summary.** DEA (data envelopment analysis) is a non-parametric technique for measuring and evaluating the relative efficiencies of a set of entities with common crisp inputs and outputs. In fact, in a real evaluation problem input and output data of entities evaluated often fluctuate. These fluctuating data can be represented as linguistic variables characterized by fuzzy numbers for reflecting a kind of general feeling or experience of experts. Based on the fundamental CCR model, a fuzzy DEA model is proposed to deal with the efficiency evaluation problem with the given fuzzy input and output data. Furthermore, a fuzzy aggregation model for integrating multiple attribute fuzzy values of objects is proposed based on the fuzzy DEA model. Using the proposed fuzzy DEA models, the crisp efficiency in CCR model is generalized to be a fuzzy efficiency to reflect the inherent uncertainty in real evaluation problems. Using the proposed fuzzy aggregation models, the objects can be ranked objectively.

## 1 Introduction

Data envelopment analysis (DEA) initially proposed by Charnes et al. [3] is a non-parametric technique for measuring and evaluating the relative efficiencies of a set of entities, called decision making units (DMUs), with the common inputs and outputs. Examples include school, hospital, library and, more recently, whole economic and society systems, in which outputs and inputs are always multiple in character. Most of DEA papers make an assumption that input and output data are crisp ones without any variation. In fact, inputs and outputs of DMUs are ever-changeful. For example, for evaluating operation efficiencies of airlines, seat-kilometers available, cargo-kilometers available, fuel and labor are regarded as inputs and passenger-kilometers performed as an output [4]. It is common sense that these inputs and output are easy to change because of weather, season, operating state and so on. Because DEA

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is a 'boundary' method sensitive to outliers, it is very difficult to evaluate the efficiency of DMU with varying inputs and outputs by conventional DEA models. Some researchers have proposed several models to challenge how to deal with the variation of data in efficiency evaluation problems by stochastic frontier models [1,9,19]. On the other hand, in more general cases, the data for evaluation are often collected from investigation by polling where the natural language such as *good, medium* and *bad* are used to reflect a kind of general situation of the investigated entities rather than a specific case. In the above example, an expert can make a general conclusion that the airline A is about *200 passenger-kilometers* and fuel cost is *high* based on his rich experience. These fuzzy concepts are used to summarize the general situation of inputs and outputs and reflect the ambiguity of the experts' judgment. The center of a fuzzy number represents the most general case and the spread reflects some possibilities. Some DEA models under uncertainty have been research in papers [5, 6, 10–16, 18, 21, 24].

In this paper, a fuzzy DEA model is proposed which is an extension of CCR model for evaluating the fuzzy efficiency of DMU with the given fuzzy input and output data. The crisp efficiency in CCR model is generalized to be a fuzzy number to reflect the inherent uncertainty in real evaluation problems.

Aggregation operators play an important role in information integration and decision analysis, which offer the synthesized one-dimensional information from the high-dimensional space to facilitate an overall judgment in the decision-making procedure. Several kinds of aggregation operators have been researched in papers [2,7,8,17,20,22,23]. In essence, these methods are sorts of weighted aggregation operators. That is, aggregation is represented as a kind of generalized weighted sum where weight factors of attributes are predetermined by decision-makers to represent their preference or a sort of threshold. It is obvious that different weight factors lead to different aggregation results. Generally speaking, it is very difficult to choose suitable weight factors because of the existence of inherent uncertainty and subjectivity for determining them. In particular, sometimes we need some objective rather than subjective assessment by aggregation operators. In other words, there is no such authority (decision-maker) with the right to determine the weight factors of attributes in advance. Let us give a scenario for explaining this viewpoint. A motorcycle company has designed five kinds of new products and wants to know which is the most popular so that they can make a decision for mass-production. In so doing, a demonstration can be held where the questionnaires on attributes related with sales, such as, price, beauty, comfort and fuel cost etc. are collected from visitors. In this case, it is unimaginable that this company can predetermine the weight factors of attributes because buying or not is completely decided by customers not this company. However, it is certain that the company can give some suggestion on attributes, for example, "the price is the most important attribute for a good sale". Meanwhile, customers also can't determine the weight factors of attributes because producing which kind of motorcycle is completely decided by the company rather than the individual preference of some customer. However, customers can express their comments on the attributes of motorcycles. In a word, there is no authority to determine some specified weight factors of attributes in this example. The weight factors of attributes should objectively reflect the inherent characteristic of the information from customers and the company. This kind of evaluation system is called agent-clients evaluation (ACE) system. In ACE systems the agent (company) can collect some information on the evaluated objects from clients (customers) and decide which action should be taken to meet clients' preference. The ACE systems greatly differ from multi-criteria decision-making systems in the sense that there is an agent rather than an authority that has right to specify weight factors of attributes in advance. An aggregation model for ACE system, called Self-organizing fuzzy aggregation model, is proposed in the paper [11].

In this paper, an aggregation model for integrating multiple attribute fuzzy values of objects is proposed based on the fuzzy DEA model, in which the fuzzy multi-input values of all DMUs become the crisp value 1.

This paper is organized as follows: Section 2 is devoted to a brief introduction of DEA. In Sect. 3, fuzzy DEA models are proposed. In Sect. 4, the methods for evaluating the objects with multiple fuzzy attribute values are proposed. For illustration of our methods, numerical examples are given in Sects. 3 and 4. Section 5 makes some concluding remarks for this paper.

## 2 Data Envelopment Analysis

DEA (data envelopment analysis) is a non-parametric technique for measuring and evaluating the relative efficiencies of a set of entities with common crisp inputs and outputs. CCR model, a basic DEA model, is a linear programming (LP) based method proposed by Charnes et al. [3]. In CCR model the efficiency of the entity evaluated is obtained as a ratio of its weighted output to its weighted input subject to the condition that the ratio for each entity is not greater than 1. Mathematically, it is described as follows:

$$\begin{aligned} \max_{\mu,\nu} \quad & \frac{\mu^t \mathbf{y}_o}{\nu^t \mathbf{x}_o} \\ \text{s. t.} \quad & \frac{\mu^t \mathbf{y}_j}{\nu^t \mathbf{x}_j} \le 1 \ (j = 1, \ldots, n), \\ & \mu \ge \mathbf{0}, \\ & \nu \ge \mathbf{0}. \end{aligned} \tag{1}$$

Here the evaluated entities (DMUs) form a reference set and $n$ is the number of DMUs. $\mathbf{y}_j = [y_{j1}, \ldots, y_{jm}]^t$ and $\mathbf{x}_j = [x_{j1}, \ldots, x_{js}]^t$ in (1) are the given positive output and input vectors of the $j$th DMU, respectively, and $m$ and $s$ are the numbers of outputs and inputs of DMU, respectively. $\mu$ and $\nu$ in (1) are the coefficient vectors of $\mathbf{y}_j$ and $\mathbf{x}_j$, respectively and the index $o$ indicates the evaluated DMU. $\mu \geq \mathbf{0}$ represents the vector whose elements are not smaller than zero but at least one element is positive value whereas $\mu > \mathbf{0}$ represents the vector with positive elements.

The model (1) is equivalent to the following LP problem.

$$\begin{aligned} \max_{\mu,\nu} \quad & \mu^t \mathbf{y}_o \\ \text{s. t.} \quad & \nu^t \mathbf{x}_o = 1, \\ & \mu^t \mathbf{y}_j \leq \nu^t \mathbf{x}_j \ (j = 1, \ldots, n), \\ & \mu \geq \mathbf{0}, \\ & \nu \geq \mathbf{0}. \end{aligned} \tag{2}$$

It can be seen from (2) that the essence of CCR model is that the DMU evaluated tries to find out its own weight vector to maximize its weighted output with the constraints that its weighted input is fixed as unity and the weighted output is not greater than the weighted input for all DMUs. In other words, each DMU seeks its favorite weight vector to its own advantage.

## 3 Fuzzy DEA Models

If the input and output data are fuzzy numbers for representing the judgment of persons, let us consider how to evaluate the efficiencies of DMUs. Firstly, the basic concepts of fuzzy sets are introduced in the following section.

### 3.1 Preliminaries of Fuzzy Sets

**Definition 1.** *A fuzzy number $A$ is called L–L fuzzy number and denoted as $(a, c, d)_L$ if its membership function is defined by*

$$\Pi_A(x) = \begin{cases} L((a-x)/c), & x \leq a \\ 1, & x = a\,, \\ L((x-a)/d), & x \geq a \end{cases} \tag{3}$$

*where $c > 0,\ d > 0$ and reference functions $L : [0, +\infty) \to [0, 1]$ is a strictly decreasing functions with $L(0) = 1$. An L–L fuzzy number $(a, c, d)_L$ with $L(x) = \max(0, 1 - |x|)$ is called triangular fuzzy number, denoted as (a,c,d). A symmetrical L–L fuzzy number is denoted as $(a, c)_L$ for the case of $c = d$.*

An $n$-dimensional vector $\mathbf{x} = [x_1, \ldots, x_n]^t$ can be fuzzified as a symmetrical L–L fuzzy vector $\mathbf{A}$ whose membership function is defined as

$$\Pi_{\mathbf{A}}(\mathbf{x}) = \Pi_{A_1}(x_1) \wedge \ldots \wedge \Pi_{A_n}(x_n), \tag{4}$$

where $\Pi_{A_i}(x_i)$ is the membership function of a symmetrical L–L fuzzy number, denoted as $(a_i, c_i)_L$. An $n$-dimensional L–L fuzzy vector is denoted as $\mathbf{A} = (\mathbf{a}, \mathbf{c})_L$ with $\mathbf{a} = [a_1, \ldots, a_n]^t$ and $\mathbf{c} = [c_1, \ldots, c_n]^t$.

Consider a fuzzy linear system

$$Y = A_1 x_1 + \cdots + A_n x_n = \mathbf{A}^t \mathbf{x}, \tag{5}$$

where $x_i$ is a real number ($i = 1, \ldots, n$) and $\mathbf{A}$ is an $n$-dimensional symmetrical L–L fuzzy vector whose element is $(a_i, c_i)_L$. From the extension principle, it is known that $Y$ is a symmetrical L–L fuzzy number as follows.

$$Y = \left( \sum_{i=1,\ldots,n} x_i a_i, \sum_{i=1,\ldots,n} |x_i| c_i \right)_L = (\mathbf{a}^t \mathbf{x}, \mathbf{c}^t\, |\mathbf{x}|)_L. \tag{6}$$

Its $h$-level set, denoted as $[Y]_h$, is as follows.

$$[Y]_h = [\mathbf{a}^t\mathbf{x} - L^{-1}(h)\mathbf{c}^t|\mathbf{x}|, \mathbf{a}^t\mathbf{x} + L^{-1}(h)\mathbf{c}^t|\mathbf{x}|], \tag{7}$$

where $|\mathbf{x}| = [|x_1|, \ldots, |x_n|]^t$ and $0 < h \leq 1$.

### 3.2 Fuzzy DEA Based on CCR Model

Considering fuzzy input and output data, CCR model (2) can be naturally generalized to be the following fuzzy DEA model.

$$\begin{aligned} \max_{\mu,\gamma} \quad & \mu^t \mathbf{Y}_o \\ \text{s. t.} \quad & \nu^t \mathbf{X_o} \approx \tilde{1}, \\ & \mu^t \mathbf{Y}_j \lesssim \nu^t \mathbf{X}_j \ (j = 1, \ldots, n), \\ & \mu \geq \mathbf{0}, \\ & \nu \geq \mathbf{0}, \end{aligned} \tag{8}$$

where $\mathbf{X}_j = (\mathbf{x}_j, \mathbf{c}_j)_L$ and $\mathbf{Y}_j = (\mathbf{y}_j, \mathbf{d}_j)_L$ are an $s$-dimensional L–L fuzzy input vector and an $m$-dimensional fuzzy output vector of the $j$th DMU, respectively, which generalize crisp input and output vectors in (2). Meanwhile, "equal", "smaller than" and "maximizing crisp output" in (2) are extended to be "almost equal", "almost smaller than" and "maximizing a fuzzy number", respectively. Moreover, 1 in (2) becomes a fuzzy number $\tilde{1} = (1, e)_L$ where $e \leq 1$ is the predefined spread of $\tilde{1}$. In what follows, we interpret the concepts of "$\mu^t \mathbf{Y}_j \lesssim \nu^t \mathbf{X}_j$", "max $\mu^t \mathbf{Y}_o$" and "$\nu^t \mathbf{X_o} \approx \tilde{1}$" in sequence.

**Definition 2.** *Given two L–L fuzzy numbers $Z_1 = (z_1, w_1)_L$ and $Z_2 = (z_2, w_2)_L$ , the relation $Z_1 \tilde{<}_h Z_2$ $(0 < h \leq 1)$ holds if and only if the following inequalities are true for any possibility level $k \in [h, 1]$.*

$$z_1 - L^{-1}(k) w_1 \leq z_2 - L^{-1}(k) w_2, \tag{9}$$

$$z_1 + L^{-1}(k) w_1 \leq z_2 + L^{-1}(k) w_2, \tag{10}$$

*where $L^{-1}(\cdot)$ is the inverse function of $L(\cdot)$.*

**Theorem 1.** *The necessary and sufficient conditions that* (9) *and* (10) *hold for any* $k \in [h, 1]$ *are as follows:*

$$z_1 - L^{-1}(h)w_1 \leq z_2 - L^{-1}(h)w_2, \tag{11}$$

$$z_1 + L^{-1}(h)w_1 \leq z_2 + L^{-1}(h)w_2, \tag{12}$$

*Proof.* It is trivial to prove the necessity. Let us now prove the sufficiency. If $h = 1$, the (11) and (12) are equivalent to (9) and (10), respectively. The sufficiency obviously holds for $h = 1$. Thus, we only consider the case of $h < 1$ in what follows. Taking the sum of (11) and (12) leads to

$$z_2 \geq z_1. \tag{13}$$

(11) is equivalent to

$$z_2 - z_1 \geq L^{-1}(h)(w_2 - w_1). \tag{14}$$

It is straightforward that the relation $0 \leq L^{-1}(k)/L^{-1}(h) \leq 1$ holds for $0 < h \leq k$. Thus,

$$z_2 - z_1 \geq L^{-1}(h)(w_2 - w_1) \geq L^{-1}(k)(w_2 - w_1). \tag{15}$$

(15) is equivalent to

$$z_1 - L^{-1}(k)w_1 \leq z_2 - L^{-1}(k)w_2. \tag{16}$$

Likewise, we can prove that

$$z_1 + L^{-1}(k)w_1 \leq z_2 + L^{-1}(k)w_2. \tag{17}$$

It proves this theorem.

Now, let us consider maximizing a fuzzy number. Referring to Definition 2, "Maximizing an L–L fuzzy number $Z = (z, w)_L$" can be explained as simultaneously maximizing $z - L^{-1}(h)w$ and $z + L^{-1}(h)w$. Here, the following weighted function

$$\lambda_1(z - L^{-1}(h)w) + \lambda_2(z + L^{-1}w), \tag{18}$$

is introduced to obtain some compromise solution where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ are the weights of left and right endpoints of the $h$-level set of Z, respectively, with $\lambda_1 + \lambda_2 = 1$. Taking $\lambda_1 = 1$ is regarded as a pessimistic opinion of maximizing Z because the worst situation is considered, whereas taking $\lambda_2 = 1$ is regarded as an optimistic opinion because the best situation is concerned with.

Next, let us consider the relation $\mathbf{v}^t\mathbf{X}_o \approx \tilde{1}$ in (8) which plays the same role as $\mathbf{v}^t\mathbf{x_o} = 1$ in (2). The crisp input vector $\mathbf{x}_0$ in CCR model becomes a fuzzy vector $\mathbf{X}_0$ so that $\mathbf{v}^t\mathbf{x}_0 = 1$ is generalized to be $\mathbf{v}^t\mathbf{X}_0 \approx \tilde{1}$ where $\tilde{1} = (1, e)_L$ is a fuzzy unity given by decision-makers. Different from the crisp case, that is, $\mathbf{v}^t\mathbf{x_o} = 1$, where the vector $\mathbf{v}$ can be found out to satisfy this equality, the

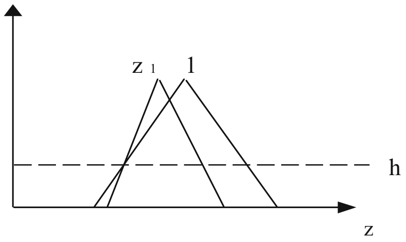


**Fig. 1.** Explanation of $Z \approx \tilde{1}$

vector $\nu$ can not always be found out to make the equality $\nu^t\mathbf{X}_o = \tilde{1}$ hold in the sense that $\nu^t\mathbf{X}_o$ and $\tilde{1}$ have the same membership function. As a result, finding out a vector $\nu$ to make $\nu^t\mathbf{X}_o = \tilde{1}$ is translated into finding out $\nu$ to make the fuzzy number $\nu^t\mathbf{X}_o$ approach $\tilde{1}$ as much as possible, simply denoted by $\nu^t\mathbf{X}_o \approx \tilde{1}$. Considering Definition 2, the fuzzy number $\nu^t\mathbf{X}_o$ that satisfies $\nu^t\mathbf{X}_o \approx \tilde{1}$ can be regarded as an upper bound subject to $\nu^t\mathbf{X}_o \underset{\sim}{<} \tilde{1}$. It means that the left endpoints of the $h$-level sets of $\nu^t\mathbf{X}_o$ and $\tilde{1}$ overlap while the right endpoint of $\nu^t\mathbf{X}_o$ expands rightwards as much as possible but is not larger than that of $\tilde{1}$ shown in Fig. 1. Thus, with considering the formulations (5) and (7), the problem for finding out $\nu$ such that $\nu^t\mathbf{X}_o \approx \tilde{1}$, i.e., $Z = (\nu^t\mathbf{x}_o, \nu^t\mathbf{c}_o)_L \approx \tilde{1}$, can be converted into the following optimization problem.

$$\begin{aligned} \max_{\nu} \quad & \nu^t\mathbf{c}_o \\ \text{s. t.} \quad & \nu^t\mathbf{x}_o - L^{-1}(h)\nu^t\mathbf{c}_o = 1 - L^{-1}(h)e, \\ & \nu^t\mathbf{x}_o + L^{-1}(h)\nu^t\mathbf{c}_o \leq 1 + L^{-1}(h)e, \\ & \nu \geq \mathbf{0}. \end{aligned} \tag{19}$$

*Remarks.* The optimization problem (19) is used to find out the maximum $Z = \nu^t\mathbf{X}_o$ constrained by $\nu^t\mathbf{X}_o \leq \tilde{1}$ with the same left endpoint as the one of fuzzy number $\tilde{1}$ in $h$-level sets. This procedure can be regarded as a generalization of the procedure that seeking a value $x$ such that $x = 1$ is equivalent to finding out the biggest $x$ subject to $x \leq 1$.

Using (9), (10), (18) and (19) and considering (5) and (7), the fuzzy optimization problem (8) can be transformed into the following LP problem with a primary objective function and a secondary objective function.

$$\begin{aligned} \max_{\mu,\nu} \quad & \lambda_1(\mu^t\mathbf{y}_o - L^{-1}(h)\mu^t\mathbf{d}_o) + \lambda_2(\mu^t\mathbf{y}_o + L^{-1}(h)\mu^t\mathbf{d}_o) \\ \text{s. t.} \quad & \max_{\nu} \quad \nu^t\mathbf{c}_o \\ & \text{s. t.} \quad \nu^t\mathbf{x}_o - L^{-1}(h)\nu^t\mathbf{c}_o = 1 - L^{-1}(h)e, \\ & \qquad\;\; \nu^t\mathbf{x}_o + L^{-1}(h)\nu^t\mathbf{c}_o \leq 1 + L^{-1}(h)e, \\ & \qquad\;\; \nu \geq \mathbf{0}, \end{aligned} \tag{20}$$

$$\boldsymbol{\mu}^t \mathbf{y}_j - L^{-1}(h)\boldsymbol{\mu}^t \mathbf{d}_j \le \boldsymbol{\nu}^t \mathbf{x}_j - L^{-1}(h)\boldsymbol{\nu}^t \mathbf{c}_j \ (j = 1, \ldots, n),$$
$$\boldsymbol{\mu}^t \mathbf{y}_j + L^{-1}(h)\boldsymbol{\mu}^t \mathbf{d}_j \le \boldsymbol{\nu}^t \mathbf{x}_j + L^{-1}(h)\boldsymbol{\nu}^t \mathbf{c}_j \ (j = 1, \ldots, n),$$
$$\boldsymbol{\mu} \ge \mathbf{0}.$$

It should be noted that the optimization problem (19) is embedded into (20) to obtain $\boldsymbol{\nu}$ such that $\boldsymbol{\nu}^t \mathbf{X}_o \approx \tilde{1}$. The obtained optimal vectors from (20) are denoted as $\boldsymbol{\nu}^*$ and $\boldsymbol{\mu}^*$.

*Remarks.* It can be seen that when $\mathbf{c}_i = \mathbf{0}$, $\mathbf{d}_i = \mathbf{0}$ and $e = 0$, the fuzzy DEA (8) just becomes CCR model. It means that the model (8) can evaluates the efficiencies of DMUs in more general way, by which the crisp, fuzzy and hybrid inputs and outputs can be handled homogeneously.

Assuming that the optimal value of the objective function of (19) is $g_o$, the optimization problem (20) can be rewritten as the following LP problem.

$$\begin{aligned} \max_{\boldsymbol{\mu},\boldsymbol{\nu}} \quad & \lambda_1(\boldsymbol{\mu}^t \mathbf{y}_o - L^{-1}(h)\boldsymbol{\mu}^t \mathbf{d}_o) + \lambda_2(\boldsymbol{\mu}^t \mathbf{y}_o + L^{-1}(h)\boldsymbol{\mu}^t \mathbf{d}_o) \\ \text{s. t.} \quad & \boldsymbol{\nu}^t \mathbf{x}_o \ge g_o \\ & \boldsymbol{\mu}^t \mathbf{y}_j - L^{-1}(h)\boldsymbol{\mu}^t \mathbf{d}_j \le \boldsymbol{\nu}^t \mathbf{x}_j - L^{-1}(h)\boldsymbol{\nu}^t \mathbf{c}_j \ (j = 1, \ldots, n), \\ & \boldsymbol{\mu}^t \mathbf{y}_j + L^{-1}(h)\boldsymbol{\mu}^t \mathbf{d}_j \le \boldsymbol{\nu}^t \mathbf{x}_j + L^{-1}(h)\boldsymbol{\nu}^t \mathbf{c}_j \ (j = 1, \ldots, n), \\ & \boldsymbol{\mu} \ge \mathbf{0}, \\ & \boldsymbol{\nu} \ge \mathbf{0}. \end{aligned} \tag{21}$$

**Definition 3.** *The fuzzy efficiency of an evaluated DMU with the L–L fuzzy input vector* $\mathbf{X}_o = (\mathbf{x}_o, \mathbf{c}_o)_L$ *and output vector* $\mathbf{Y}_0 = (\mathbf{y}_o, \mathbf{d}_o)_L$ *is defined as an L–L fuzzy number* $E = (w_l, \eta, w_r)_L$ *as follows:*

$$\eta = \frac{\boldsymbol{\mu}^{*t}\mathbf{y}_o}{\boldsymbol{\nu}^{*t}\mathbf{x}_o},$$
$$w_l = \eta - \frac{\boldsymbol{\mu}^{*t}(\mathbf{y}_o - \mathbf{d}_o L^{-1}(h))}{\boldsymbol{\nu}^{*t}(\mathbf{x}_o + \mathbf{c}_o L^{-1}(h))},$$
$$w_r = \frac{\boldsymbol{\mu}^{*t}(\mathbf{y}_o + \mathbf{d}_o L^{-1}(h))}{\boldsymbol{\nu}^{*t}(\mathbf{x}_o - \mathbf{c}_o L^{-1}(h))} - \eta.$$

*It is obvious that the uncertainty from the inputs and outputs of DMUs characterized by fuzzy numbers is transferred to the uncertainty of the evaluated efficiency, which is very close to human thinking.*

**Definition 4.** *The DMU with* $\eta + w_r \ge 1$ *for a given possibility level* $h$ *is called an h-possibilistic D efficient DMU (PD DMU). On the contrary, the DMU with* $\eta + w_r < 1$ *for a given possibility level* $h$ *is called an h-possibilistic D inefficient DMU (PDI DMU). The set of all PD DMUs is called the h-possibilistic nondominated set, denoted by* $S_h$.

*It is obvious that the h-possibilistic D efficient DMUs (PD DMUs) and the h-possibilistic D inefficient DMUs (PDI DMUs) in the case of* $h = 1$ *become the conventional D efficient DMUs and D inefficient DMUs in CCR model.*

**Theorem 2.** *The center of the fuzzy efficiency of any DMU obtained from (20) is not greater than 1.*

*Proof.* Suppose that $\mu^{\circ}$ and $\nu^{\circ}$ are obtained from (20) for an evaluated DMU. Thus the following inequalities hold.

$$\mu^{ot}\mathbf{y}_j - L^{-1}(h)\mu^{ot}\mathbf{d}_j \leq \nu^{ot}\mathbf{x}_j - L^{-1}(h)\nu^{ot}\mathbf{c}_j \ (j = 1, \ldots, n), \tag{22}$$

$$\mu^{ot}\mathbf{y}_j + L^{-1}(h)\mu^{ot}\mathbf{d}_j \leq \nu^{ot}\mathbf{x}_j + L^{-1}(h)\nu^{ot}\mathbf{c}_j \ (j = 1, \ldots, n). \tag{23}$$

Taking the sum of (22) and (23), the following inequalities hold.

$$\mu^{\circ t}\mathbf{y}_j \leq \nu^{\circ t}\mathbf{x}_j \ (j = 1, \ldots, n). \tag{24}$$

Then,

$$\eta = \frac{\mu^{\circ t}\mathbf{y}_o}{\nu^{\circ t}\mathbf{x}_o} \leq 1, \tag{25}$$

which proves Theorem 2.

The formulation (25) means that evaluating fuzzy efficiencies of DMUs by the model (20) is similar to evaluating crisp efficiencies of DMUs by CCR model. Both of them seek the nondominated one by other DMUs.

Now, we discuss the given possibility level $h$. If we take a large value for $h$, it means that we consider a relatively narrow range of input and output data where all of the data considered have high possibilistic grades. Conversely, if we take a small value for $h$, it means that we investigate the input and output data in relatively wide range.

Let us consider a special case of Definition 1, that is, the symmetrical triangular fuzzy number, denoted as $(a, c)$ where its membership function is defined as follows:

$$\pi_A(x) = \begin{cases} 1-|x-a|/c, a-c \leq x \leq a+c, c > 0 \\ 0 \qquad\qquad\qquad , \qquad otherwise \end{cases} . \tag{26}$$

Assume the given fuzzy inputs and outputs of the $i$th DMU are symmetrical triangular fuzzy vectors, denoted as $(\mathbf{x}_i,\ \mathbf{c}_i)$ and $(\mathbf{y}_i,\ \mathbf{d}_i)$, respectively, the optimization problem (20) can be rewritten as follows [10]:

$$\begin{aligned}
\max_{\mu,\nu}\quad & \lambda_1(\mu^t\mathbf{y}_o - (1-h)\mu^t\mathbf{d}_o) + \lambda_2(\mu^t\mathbf{y}_o + (1-h)\mu^t\mathbf{d}_o) \qquad (27)\\
\text{s. t.}\quad & \max_{\nu}\quad \nu^t\mathbf{c}_o \\
& \text{s. t.}\quad \nu^t\mathbf{x}_o - (1-h)\nu^t\mathbf{c}_o = 1-(1-h)e, \\
& \qquad\quad \nu^t\mathbf{x}_o + (1-h)\nu^t\mathbf{c}_o \leq 1+(1-h)e, \\
& \qquad\quad \nu \geq \mathbf{0}, \\
& \mu^t\mathbf{y}_j - (1-h)\mu^t\mathbf{d}_j \leq \nu^t\mathbf{x}_j - (1-h)\nu^t\mathbf{c}_j \ (j = 1, \ldots, n), \\
& \mu^t\mathbf{y}_j + (1-h)\mu^t\mathbf{d}_j \leq \nu^t\mathbf{x}_j + (1-h)\nu^t\mathbf{c}_j \ (j = 1, \ldots, n), \\
& \mu \geq \mathbf{0}.
\end{aligned}$$

The value of $e$ in (27) is take as

$$e = \max_{j=1,\ldots,n} (\max_{k=1,\ldots,s} c_{jk}/x_{jk}). \tag{28}$$

### 3.3 Numerical Examples

First, a simple numerical example is considered where input and output are symmetrical triangular fuzzy numbers. The data are listed in Table 1.

The fuzzy efficiencies of DMUs (A, B, C, D, E) were obtained by the model (27) with $\lambda_1 = 1, \lambda_2 = 0$ for the different $h$ values and illustrated in Table 2, where $e = 0.25$. Table 2 shows that as the value of $h$ increases, the center of fuzzy efficiency becomes larger and the width of fuzzy efficiency becomes smaller. For the case of $h = 1$, the fuzzy efficiencies of DMUs become crisp values which are the same as the ones obtained from CCR model. From Table 2, we have $S_1 = S_{0.75} = S_{0.5} = \{B\}$ and $S_0 = \{B, D\}$. It means that decreasing the value of $h$ offers more opportunities for $PD$ DMUs in this example. It can be seen from the simulation results that the inherent fuzziness from input and output data has been reflected by fuzzy efficiencies evaluated.

Next, an example with two symmetrical triangular fuzzy inputs and two symmetrical triangular fuzzy outputs illustrated in Table 3 is considered. Fuzzy efficiencies obtained from the model (27) with $\lambda_1 = 1, \lambda_2 = 0$ for different $h$ values are listed in Table 4. The results in Table 4 show that with $h$ being higher the center of fuzzy efficiency almost increases except DMU

**Table 1.** DMUs with single fuzzy input and single fuzzy output

| Branches | A | B | C | D | E |
|---|---|---|---|---|---|
| inputs | (2.0,0.5) | (3.0,0.5) | (3.0,0.6) | (5.0,1.0) | (5.0,0.5) |
| outputs | (1.0,0.3) | (3.0,0.7) | (2.0,0.4) | (4.0,1.0) | (2.0,0.2) |

**Table 2.** Fuzzy efficiencies of DMUs with different $h$ values

| h | A | B | C | D | E |
|---|---|---|---|---|---|
| 0 | (0.21,0.47,0.35) | (0.32,0.95,0.45) | (0.21,0.63,0.32) | (0.28,0.76,0.43) | (0.07,0.38,0.08) |
| 0.5 | (0.12,0.49,0.15) | (0.18,0.97,0.21) | (0.12,0.65,0.14) | (0.16,0.78,0.19) | (0.04,0.39,0.04) |
| 0.75 | (0.06,0.49,0.07) | (0.09,0.98,0.10) | (0.06,0.66,0.07) | (0.08,0.79,0.09) | (0.02,0.39,0.02) |
| 1 | (0.0,0.5,0.0) | (0.0,1.0,0.0) | (0.0,0.67,0.0) | (0.0,0.8,0.0) | (0.0,0.4,0.0) |

**Table 3.** DMUs with two fuzzy inputs and two fuzzy outputs

| Branches | A | B | C | D | E |
|---|---|---|---|---|---|
| x1 | (4.0,0.5) | (2.9,0.0) | (4.9,0.5) | (4.1,0.7) | (6.5,0.6) |
| x2 | (2.1,0.2) | (1.5,0.1) | (2.6,0.4) | (2.3,0.1) | (4.1,0.5) |
| y1 | (2.6,0.2) | (2.2,0.0) | (3.2,0.5) | (2.9,0.4) | (5.1,0.7) |
| y2 | (4.1,0.3) | (3.5,0.2) | (5.1,0.8) | (5.7,0.2) | (7.4,0.9) |

**Table 4.** The fuzzy efficiencies of DMUs with different $h$ values

| h | A | B | C | D | E |
|---|---|---|---|---|---|
| 0 | (0.15,0.81,0.18) | (0.10,0.98,0.11) | (0.22,0.82,0.3) | (0.22,0.93,0.32) | (0.18,0.79,0.23) |
| 0.5 | (0.08,0.83,0.09) | (0.03,0.97,0.03) | (0.12,0.83,0.14) | (0.12,0.97,0.15) | (0.10,0.82,0.11) |
| 0.75 | (0.04,0.84,0.04) | (0.03,0.99,0.03) | (0.06,0.83,0.07) | (0.06,0.98,0.07) | (0.05,0.83,0.06) |
| 1 | (0.0,0.85,0.0) | (0.0,1.0,0.0) | (0.0,0.86,0.0) | (0.0,1.0,0.0) | (0.0,1.0,0.0) |

$B$ in the case of $h = 0.5$ and the width becomes smaller as in the first example. In this example, the nondominated sets with different $h$ values are $S_0 = \{B, C, D, E\}$, $S_{0.5} = \{B, D\}$, $S_{0.75} = \{B, D\}$ and $S_1 = \{B, D, E\}$. It can be seen that $h = 0.0$ gives the most opportunities for $PD$ DMUs and the increasing of the value of $h$ can not always lead to the increasing of the number of $PD$ DMUs. These phenomena indicate that efficiency evaluation via fuzzy DEA models is more complex than the normal DEA because of the inherent fuzziness contained in inputs and outputs.

## 4 Evaluation of Objects with Multiple Fuzzy Attribute Values

### 4.1 Fuzzy Aggregation Models Based on Fuzzy DEA

Let us now consider an evaluation system $D = (O, A, Y)$, where $O = \{o_1, \ldots, o_n\}$ is a set of the objects evaluated, $A = \{A_1, \ldots, A_m\}$ is a set of the attributes of $o_i$ $(i = 1, \ldots, n)$ and $Y$ is a mapping defined as:

$$Y : O \times A \to V, \tag{29}$$

where $V$ is a set of all fuzzy numbers defined on the space $R^1$. $\mathbf{Y}_j$ is an $m$-dimensional fuzzy vector whose element is a realization of the mapping $Y$ to represent an attribute value of $o_j$. For the sake of simplicity, the L–L fuzzy vector is used to represent $\mathbf{Y}_j$, denoted as $\mathbf{Y}_j = (\mathbf{y}_j, \mathbf{d}_j)_L$. It should be noted that $\mathbf{Y}_j$ is the evaluation vector rather than the original attribute vector. For example, there are three motorcycles A, B and C, their prices are 5,000$, 3,000$ and 1,000$, respectively. The evaluations of them from an evaluator may be "high", "middle" and "low" instead of "5000$", "3000$" and "1000$".

The problem for evaluating objects with multiple attributes can be regarded as a special case of the FDEA model (8) with unity input shown as follows [11].

$$\begin{aligned}
\max_{\mathbf{u}_o} \quad & \boldsymbol{\mu}_o^t \mathbf{Y}_o \\
\text{s. t.} \quad & \boldsymbol{\mu}_o^t \mathbf{Y}_j \underset{\sim}{<} 1 \ (j = 1, \ldots, n), \\
& \mu_{oi} - \mu_{oj} \geq d(i,j) \geq 0 (i \neq j, \ (i,j) \in B \subset \{1, \ldots, m\}^2) \\
& \mu_{oi} \geq \varepsilon \ (i = 1, \ldots, m),
\end{aligned} \tag{30}$$

where $\varepsilon$ is a positive constant. The constraint $\mu_{oi} - \mu_{oj} \geq d(i,j) \geq 0$ represents some suggestion from an evaluator, namely, the minimum difference of importance degrees between the attributes $A_i$ and $A_j$. For example, that motorcycle company can make such a suggestion that price is more important than beauty for sale. If no such suggestion, these constraints will disappear. The constraints $\mu_{oi} \geq \varepsilon$ $(i = 1, \ldots, m)$ mean that the weight factors of the attributes are at least larger than $\varepsilon$ which plays a crucial role to prevent the dominance effect of some large-valued attribute, which will be explained later. Denote the optimal solution of (30) as $\boldsymbol{\mu}_o^*$. The value of objective function $\boldsymbol{\mu}_o^{*t}\mathbf{Y}_o$ is the aggregated evaluation of the object $o$. The essential feature of (30) is that each evaluated object tries to find out the weight factors of attributes to its own advantage under the same constraint conditions. Thus the weight factors can be regarded as the results of fair competition rather than the one predetermined by an evaluator.

If an evaluator can suggest a linearly ordered attribute set $A_{order}$ whose $i$th element is the $i$th most important attribute in $A$, we can detail (30) as follows.

$$\begin{aligned} \max_{\mathbf{u}_o} \quad & \boldsymbol{\mu}_o^t \mathbf{Y}_o \\ \text{s. t.} \quad & \boldsymbol{\mu}_o^t \mathbf{Y}_j \precsim 1, \ (j = 1, \ldots, n), \\ & \mu_{oi} - \mu_{o(i+1)} \geq \varepsilon_i \geq 0 \ (i = 1, \ldots, m-1), \\ & \mu_{om} \geq \varepsilon_m > 0, \end{aligned} \tag{31}$$

where $\mathbf{Y}_j$ is reordered to correspond to $A_{order}$ and $\varepsilon_i$ $(i = 1, \ldots, m-1)$ are positive constants reflecting the differences of important degrees between two consecutive attributes in $A_{order}$ and $\varepsilon_m$ represents the lowest limit of weight factors.

If "$\precsim$" is explained by Definition 2, the model (31) can be transformed into the following optimization problem with considering (5), (7), (9), (10) and (18).

$$\begin{aligned} \max_{\mathbf{u}_o} \quad & \lambda_1(\boldsymbol{\mu}_o^t \mathbf{y}_o - L^{-1}(h)\boldsymbol{\mu}_o^t \mathbf{d}_o) + \lambda_2(\boldsymbol{\mu}_o^t \mathbf{y}_o - L^{-1}(h)\boldsymbol{\mu}_o^t \mathbf{d}_o) \\ \text{s. t.} \quad & \boldsymbol{\mu}_o^t \mathbf{y}_j + L^{-1}(h)\boldsymbol{\mu}_o^t \mathbf{d}_j \leq 1 \ (j = 1, \ldots, n), \\ & \mu_{oi} - \mu_{o(i+1)} \geq \varepsilon_i \geq 0 \ (i = 1, \ldots, m-1), \\ & \mu_{om} \geq \varepsilon_m > 0. \end{aligned} \tag{32}$$

In order to clarify the role of the constraints $\mu_{oi} \geq \varepsilon$ $(i = 1, \ldots, m)$ in (31), let us consider the following LP problem.

$$\begin{aligned} \max_{\mathbf{u}_o} \quad & \boldsymbol{\mu}_o^t \mathbf{y}_o \\ \text{s. t.} \quad & \boldsymbol{\mu}_o^t \mathbf{y}_j \leq 1 \ (j = 1, \ldots, n), \\ & \boldsymbol{\mu}_o \geq \mathbf{0}. \end{aligned} \tag{33}$$

It is a special case of (31) for $h = 1$. The constraints $\mu_{oi} - \mu_{o(i+1)} \geq \varepsilon_i \geq 0$ $(i = 1, \ldots, m-1)$ and $\mu_{om} \geq \varepsilon_m > 0$ in (31) are simply replaced by $\boldsymbol{\mu}_o \geq \mathbf{0}$. As a result, some large-valued attribute will dominate the rank so that the result is unacceptable to commonsense. For example, the evaluation of three objects with three attributes are $\{(0.4,\ 0,\ 0),(0.3,\ 0.9,\ 0.9),(0.3,\ 0.5,\ 0.7)\}$. Using (33) the object 1 with (0.4, 0, 0) is in the first rank because the value of attribute 1 of the object 1 dominates the values of the same attribute of other two objects even if other two attribute values of object 1 are very poor. If the weight factor $\mu_i$ $(i = 1,\ 2,\ 3)$ are limited to be more than 0.2, then the rank becomes 2, 3, 1 which is harmony with the common feeling.

**Theorem 3.** **[11]**. *There exits an optimal solution in* (32) *if and only if the constants* $\varepsilon_i$ $(i = 1, \ldots, m)$ *satisfy the following inequalities*

$$\mathbf{r}^t(\mathbf{y}_j + L^{-1}(h)\mathbf{d}_j) \leq 1\ (j = 1, \ldots, n) \tag{34}$$

*where*

$$r_i = \sum_{j=i,\ldots,m} \varepsilon_j, \tag{35}$$

*Proof.* Necessary condition: Suppose there is a feasible solution in (32) and $\mu_{om}$ satisfies the following relation

$$\mu_{om} = x \geq \varepsilon_m. \tag{36}$$

Thus, the following relations hold.

$$\begin{aligned} &\mu_{o(m-1)} \geq x + \varepsilon_{m-1}, \\ &\mu_{o(m-2)} \geq x + \varepsilon_{m-1} + \varepsilon_{m-2}, \\ &\quad\cdots\cdots \\ &\mu_{o1} \geq x + \sum_{i=1,\ldots,m-1} \varepsilon_i. \end{aligned} \tag{37}$$

Then

$$\boldsymbol{\mu}_o^t\mathbf{y}_j + L^{-1}(h)\boldsymbol{\mu}_o^t\mathbf{d}_j \geq \mathbf{r}^t(\mathbf{y}_j + L^{-1}(h)\mathbf{d}_j), \tag{38}$$

where $\mathbf{r}$ is defined by (35). Considering the constraint $\mathbf{u}_o^t\mathbf{y}_j + L^{-1}(h)\mathbf{u}_o^t\mathbf{d}_j \leq 1$ in (32), the following inequality should hold.

$$\mathbf{r}^t(\mathbf{y}_j + L^{-1}(h)\mathbf{d}_j) \leq 1\ (j = 1, \ldots, n). \tag{39}$$

It proves the necessity condition.

Sufficient condition: Suppose (34) holds, it is easy to check that there is a feasible solution in the constraint conditions of (32). That is

$$\mu_i = \sum_{j=i,\ldots,m} \varepsilon_j\ (i = 1, \ldots, m). \tag{40}$$

**Table 5.** Evaluation from an evaluator

| Objects | Attribute 1 | Attribute 2 | Attribute 3 | Attribute 4 |
|---|---|---|---|---|
| A | (0.3,0.1) | (0.5,0.2) | (0.7,0.2) | (0.9,0.1) |
| B | (0.2,0.1) | (0.9,0.1) | (0.7,0.2) | (0.4,0.2) |
| C | (0.5,0.3) | (0.5,0.2) | (0.9,0.1) | (0.6,0.3) |
| D | (0.7,0.3) | (0.8,0.1) | (0.8,0.1) | (0.9,0.1) |
| E | (0.4,0.1) | (0.6,0.2) | (0.3,0.2) | (0.5,0.2) |

Moreover, the constraint condition of (32) is a bounded closed set (compact set). Thus, there exists an optimal solution in (32). It proves the sufficiency condition.

**Corollary [11].** *If* $\varepsilon_i = a \ (i = 1, \ldots, m)$, *then* $a$ *satisfies*

$$a \leq 1/(\mathbf{m}^t(\mathbf{y}_j + L^{-1}(h)\mathbf{d}_j)) \ (j = 1, \ldots, n), \tag{41}$$

*where* $\mathbf{m} = [m, m-1, \ldots, 1]^t$.

### 4.2 Numerical Example

In Table 5, the evaluations of five objects from an evaluator are given. Symmetrical triangular fuzzy numbers are used for represent the evaluations.

Suppose that from attributes 1 to 4 their important degrees decrease and $\varepsilon_i$ $(i = 1, \ldots, 3)$ take 0.001 which offer some difference of weight factors between two consecutive attributes. $\varepsilon_4$ takes 0.001 to give the lowest limit of weight factors. The aggregated evaluations of objects A, B, C, D and E obtained by (32) for $h = 0.6$, $\lambda_1 = 1$ and $\lambda_2 = 0$ are (0.70, 0.17), (0.72, 0.16), (0.76, 0.24), (0.93, 0.17) and (0.60, 0.18), respectively. Let us simply analyze the evaluation results obtained. If only considering the center value, the rank of objects is {D, C, B, A, E}. It is obvious that D is the best one and C is the second one among all objects from Table 5. B is better than A because though B is a litter bit worse than A for the most import attribute 1 and worse than A for the unimportant attribute 4, it is remarkably better than A for the second important attribute and has the same value as A for the third important attribute. Compared with A, E has the almost same values for the first and second important attributes but remarkably small values for the third and fourth important values so that E is inferior to A. It can be concluded that the obtained result is very close to human intuition.

## 5 Conclusions

In this paper, fuzzy DEA models are proposed for evaluating the efficiencies of DMUs with fuzzy input and output data. The obtained efficiencies are fuzzy numbers to reflect the inherent fuzziness in evaluation problems. It can be concluded that the proposed fuzzy DEA models extend CCR model to more general forms where crisp, fuzzy and hybrid data can be handled easily. Moreover, based on the fuzzy DEA model, an aggregation model for integrating multiple attributes fuzzy value of objects is proposed. Using the proposed fuzzy aggregation models, the objects can be ranked objectively. Because uncertainty always exists in human thinking and judgment, fuzzy DEA models can play an important role in perceptual evaluation problems comprehensively existing in the real world.

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