On Partially Wellfounded Generic Ultrapowers

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Dedicated to Boaz Trakhtenbrot on the occasion of his 85-th birthday.

Abstract. We construct a model without precipitous ideals but so that for each $\tau < \aleph_3$ there is a normal ideal over $\aleph_1$ with generic ultrapower wellfounded up to the image of τ.

1 Introduction

Let κ be a regular uncountable cardinal. For $f, g \in {}^{\kappa}On$ set

$$f <^* g \text{ iff } \{\alpha < \kappa \mid f(\alpha) < g(\alpha)\} \text{ contains a closed unbounded subset.}$$

The Galvin-Hajnal rank $\|g\|$ of a function $g \in {}^{\kappa}On$ is defined as follows

$$\|g\| = \sup\{\|f\| + 1 \mid f <^* g\}.$$

By induction on α, the αth canonical function h_α is defined (if it exists) as the $<^*$-least function greater than each $h_\beta, \beta < \alpha$. If h_α exists then it is unique modulo the nonstationary ideal over κ. First κ^+ canonical functions always exist. Hajnal (see [4], 27.11) showed that already in L the ω_2nd canonical function for $\kappa = \omega_1$ does not exist. By Jech and Shelah [6], the existence of ω_2nd canonical function is not a large cardinal property. Note that the existence of $f \in {}^{\kappa}\kappa$ with $\|f\| = \kappa^+$ does not necessary imply the existence of κ^+ canonical function over κ. Just, for example, in L there are many functions of the rank ω_2 without the least such function. On the other hand non existence of such f implies large cardinals. Thus, Donder and Koepke [1] showed that then $\kappa \geq \aleph_2$ implies $0^\dagger$ exists and $\kappa = \aleph_1$ implies $\aleph_2$ is almost $< \aleph_1$-Erdős cardinal in the core model $\mathcal{K}$.

An ideal I over κ is called precipitous if every its generic ultrapower is well founded. It is not hard to see that if every generic ultrapower of I is well founded up to the image of $(2^\kappa)^+$ then I is precipitous.
Suppose now that for each $\tau < (2^\kappa)^+$ there is an ideal over κ with generic ultrapowers well founded up to the image τ. Does this imply the existence of a precipitous ideal?

Our aim is to provide a negative answer. We will show the following:

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Theorem 1. *Suppose that*

1. $2^{\aleph_1} = \aleph_2$
2. *there is an* $\aleph_1$*-Erdős cardinal*
3. *there is a function* $f : \omega_1 \to \omega_1$ *with* $\|f\| \geq \omega_2$.

Then *for every* $\tau < \omega_3$ *there exists a normal ideal over* $\aleph_1$ *with a generic ultrapower wellfounded up to the image of* τ.

Remark 2. 1. Note that in general it is impossible to allow $\tau = \omega_3$. Thus, the cardinality of the forcing is only ω_2. Hence, if a generic ultrapower is wellfounded up to the image of $\tau = (\omega_3)^V$, then it is fully wellfounded (just taking a big enough elementary submodel (in V) of cardinality ω_2 arbitrary functions to those with the ranges being subsets of ω_3). But this implies an inner model in which ω_1 is a measurable cardinal, see [4]. The original V does not need to have even an inner model with a Ramsey cardinal.

2. The assumption 3 is not very restrictive. Thus by [1], if there is no such a function, then $\aleph_2$ is almost $< \aleph_1$-Erdős cardinal in the core model $\mathcal{K}$. In the last case we can assume that $V = \mathcal{K}$ or just collapse first a non $< \aleph_1$-Erdős cardinal in $\mathcal{K}$ to be new $\aleph_2$.
3. Note that up to $(\aleph_2)^V$ (not its image!) a generic ultrapower by the nonstationary ideal is always wellfounded , just due to the existence of canonical functions. It is possible (consistently) to get to the image of $\aleph_1$ using the canonical functions, if the nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated or consistently using a weaker assumptions as was shown in [7].
4. It is an open question whether any large cardinal hypothesis implies (directly, not consistently) the existence of a precipitous ideal on $\aleph_1$. In view of 1, a kind of "almost" precipitousness follows from $\aleph_1$-Erdős cardinal.
5. We do not know whether $\aleph_1$-Erdős cardinal is needed for the conclusion of 1. Note only that it is easy to show that $\aleph_1$ must be a weakly compact limit of weakly compact cardinals in L (just the tree property and a generic elementary embedding). Also, if $\aleph_1 = \aleph_1^{\mathcal{K}}$ then at least $0^\sharp$ exists.
6. We do not know if the analog of the theorem holds once $\aleph_1$ is replaced by a bigger cardinal.

2 The Game

Let λ be an $\aleph_1$-Erdős cardinal. Fix some $\tau < \lambda$.

Consider the following game $\mathcal{G}_\tau$:

Player I starts by picking a stationary subset A_0 of $\aleph_1$. Player II chooses a function $f_1 : A_0 \to \tau$ and either a partition $\langle B_n | n < \omega \rangle$ of A_0 into at most countably many pieces or a sequence $\langle B_\alpha | \alpha < \aleph_1 \rangle$ of disjoint subsets of $\aleph_1$ so that

$$\nabla_{\alpha<\omega_1} B_\alpha \supseteq A_0.$$

The first player then supposed to respond by picking an ordinal $\alpha_2 < \lambda$ and a stationary set A_2 which is a subset of A_0 and of one of B_n's or B_α's.

At the next stage the second player supplies again a function $f_3 : A_2 \to \tau$ and either a partition $\langle B_n | n < \omega\rangle$ of A_2 into at most countably many pieces or a sequence $\langle B_\alpha | \alpha < \aleph_1\rangle$ of disjoint subsets of $\aleph_1$ so that

$$\nabla_{\alpha<\omega_1} B_\alpha \supseteq A_2.$$

The first player then supposed to respond by picking a stationary set A_4 which is a subset of A_2 and of one of B_n's or B_α's on which everywhere f_1 is either above f_3 or equal f_3 or below f_3. In addition he picks an ordinal $\alpha_4 < \lambda$ such that

$$\alpha_2 < \alpha_4 \text{ iff } f_1 \restriction A_4 < f_3 \restriction A_4.$$

Intuitively, α_{2n} pretends to represent f_{2n-1} in a generic ultrapower.

Continue further in the same fashion.

Player I wins if the game continues infinitely many moves. Otherwise Player II wins. Clearly it is a determined game.

Let us argue that the second player cannot have a winning strategy.

Lemma 3. *For each $\tau < \lambda$ Player II does not have a winning strategy in the game $\mathcal{G}_\tau$.*

Proof. Suppose otherwise. Let σ be a strategy of two. Find a set $X \subset \lambda$ of cardinality $\aleph_1$ such that σ does not depend on ordinals picked from X. In order to get such X let us consider a structure

$$\mathfrak{A} = \langle H(\lambda), \in, \lambda, \tau, \mathcal{P}(\aleph_1), \mathcal{G}, \sigma\rangle.$$

Let X be a set of $\aleph_1$ indiscernibles for $\mathfrak{A}$.

Pick now a countable elementary submodel M of $H(\chi)$ for $\chi > \lambda$ big enough with $\sigma, X \in M$. Let $\alpha = M \cap \omega_1$. Let us produce an infinite play in which the second player uses σ. This will give us the desired contradiction.

Consider the set $S = \{f(\alpha) | f \in M, f \text{ is a partial function from } \omega_1 \text{ to } \tau\}$. Obviously, S is countable. Hence we can fix an order preserving function $\pi : S \to X$.

Let one start with $A_0 = \omega_1$. Consider $\sigma(A_0)$. Clearly, $\sigma(A_0) \in M$. It consists of a function $f_1 : A_0 \to \tau$ and, say a sequence $\langle B_\xi | \xi < \aleph_1\rangle$ of disjoint subsets of $\aleph_1$ so that

$$\nabla_{\xi<\omega_1} B_\xi \supseteq A_0.$$

Now, $\alpha \in A_0$, hence there is $\xi^* < \alpha$ such that $\alpha \in B_{\xi^*}$. Then $B_{\alpha^*} \in M$, as $M \supseteq \alpha$. Hence, $A_0 \cap B_{\xi^*} \in M$ and $\alpha \in A_0 \cap B_{\xi^*}$. Let $A_2 = A_0 \cap B_{\xi^*}$. Pick $\alpha_2 = \pi(f_1(\alpha))$.

Consider now the answer of two which plays according to σ. It does not depend on α_2, hence it is in M. Let it be a function $f_3 : A_2 \to \tau$ and, say a sequence $\langle B_\xi | \xi < \aleph_1\rangle$ of disjoint subsets of $\aleph_1$ so that

$$\nabla_{\xi<\omega_1} B_\xi \supseteq A_2.$$

As above find $\xi^* < \alpha$ such that $\alpha \in B_{\xi^*}$. Then $B_{\alpha^*} \in M$, as $M \supseteq \alpha$. Hence, $A_2 \cap B_{\xi^*} \in M$ and $\alpha \in A_2 \cap B_{\xi^*}$. Let $A_2' = A_2 \cap B_{\xi^*}$. Split it into three sets $C_<, C_=, C_>$ such that

$$C_< = \{\nu \in A_2' | f_3(\nu) < f_1(\nu)\},$$

$$C_= = \{\nu \in A_2' | f_3(\nu) = f_1(\nu)\},$$

$$C_> = \{\nu \in A_2' | f_3(\nu) > f_1(\nu)\}.$$

Clearly, α belongs to only one of them, say to $C_<$. Set then $A_4 = C_<$. Then, clearly, $A_4 \in M$, it is stationary and $f_3(\alpha) < f_1(\alpha)$. Set $\alpha_4 = \pi(f_3(\alpha))$.

Continue further in the same fashion. □

It follows that the first player has a winning strategy.

3 The Construction of an Ideal

Let $\tau < \aleph_3$. We like to construct an ideal on $\aleph_1$ with a generic ultrapower wellfounded up to the image of τ.

Fix a winning strategy σ for Player I in the game $\mathcal{G}_\tau$.

Set $I = \{X \subseteq \omega_1 \mid \sigma \text{ never picks } X\}$.

Lemma 4. *I is a normal proper ideal over ω_1.*

Proof. Let us show for example the ω_1-completeness. Thus let that $\langle B_n | n < \omega\rangle$ be a partition of a set $A \in I^+$. Consider a game according to σ in which A appears as a move of the player one. Let two to answer by $\langle B_n | n < \omega\rangle$ (and arbitrary function). Then the answer of one according to σ will be a subset of one of B_n's. But this means that this B_n is I-positive. □

Fix a sequence $\langle h_\alpha | \alpha < \aleph_2\rangle$ of the first $\aleph_2$ canonical functions from ω_1 to ω_1.

We would like to have a function that represents $(\aleph_2)^V$ in a generic ultrapower. If there exists the $\aleph_2$nd canonical function then it will be as desired. Here we do not assume its existence, but rather a weaker property that there is $f : \omega_1 \to \omega_1$ with $\|f\| = \omega_2$. Clearly, such f is above each $h_\alpha, \alpha < \omega_2$ (modulo the nonstationary ideal). The problem is that there may be many such f's without the least one. The way to overcome this will be to find an ideal $J \supseteq I$ which has have the J-least function above all canonical functions.

Proceed as follows. Set

$$S = \{f \in {}^{\omega_1}\omega_1 \mid \|f\| \geq \omega_2\}.$$

Basically we let Player II to play functions in S and Player I to respond using the strategy σ. Find a function $h \in S$, a finite play $\boldsymbol{t} = \langle t_1, ..., t_n\rangle$ and an ordinal η such that

1. $\boldsymbol{t}$ was played according σ
2. h was picked by Player II at his last move t_{n-1}
3. Player I responded with η
4. there is no continuation of $\boldsymbol{t}$, with Player I using σ, in which a response to a function from S less than η.

Note that such $\eta \geq \omega_2$, since otherwise Player II can easily win by playing h_η at the very next move. Then Player I should respond respond by some $\eta_1 < \eta$ on which II respond by h_{η_1} etc.

Also note that such h is not necessary unique, but any other function attached to η which appears further in the game will be equal to h on the corresponding set.

Set now

$$J = \{X \subseteq \omega_1 \mid X \text{ is never picked by } \sigma \text{ in the continuation of } \boldsymbol{t}\}.$$

The proof of the next lemma repeats those of Lemma 4.

Lemma 5. *J is a normal proper ideal over ω_1 extending I.*

Lemma 6. *Generic ultrapowers by J are wellfounded at least up to $(\omega_2)^V + 1$. Moreover $(\omega_2)^V$ is represented by h.*

Proof. Just note, that by the choice of h and the definition of J, the only functions that are below h on a J-positive set are the canonical functions $h_\alpha, \alpha < \omega_2$. □

Assume without loss of generality that for each $\alpha < \aleph_2$ we have $h_\alpha(\nu) < h(\nu)$, for each $\nu < \omega_1$. Also fix for each $\nu < \omega_1$ a function $H_\nu : \omega \to_{onto} h(\nu)$.

Let

$$A_{n\alpha} = \{\nu < \omega_1 \mid H_\nu(n) = h_\alpha(\nu)\}.$$

Lemma 7. *Let $X \in J^+$. Then for each $n < \omega$ there is $\alpha < \omega_2$ such that $X \cap A_{n\alpha} \in J^+$.*

Proof. By 6 a generic ultrapower with J is wellfounded up to $\omega_2^V + 1$ and ω_2^V is represented by h.

Let $G \subseteq J^+$ be a generic ultrafilter with $X \in G$ and $j : V \to M_G = V \cap {}^{\omega_1 >}V/G$ be the corresponding elementary embedding. We may assume that the ordinals of M up to $[h]_G$ are just ω_2^V. Consider $H = [\langle H_\nu | \nu < \omega_1^V]_G$. Then, $H : \omega \to_{onto} \omega_2^V$ in M_G. So, for some $\alpha < \omega_2^V$ we have $H(n) = \alpha$. But then $X \cap A_{n\alpha} \in G$ and be are done. □

The following lemma is similar.

Lemma 8. *Let $X \in J^+$. Then for each $m < \omega$ there is $n > m$ so that $|\{\alpha < \omega_2 \mid X \cap A_{n\alpha} \in J^+\}| = \aleph_2$.*

Proof. Just otherwise X or its extension will force that the range of H (as in 7) will be bounded in ω_2^V. □

Now we will use an argument similar to those of [3] in order to extend J to an ideal with the desired property.

Let $\langle f_\alpha \mid \alpha < \aleph_2\rangle$ be an enumeration of the set of all functions from ω_1 to τ (recall that τ is a fixed ordinal less than $\aleph_3$ and $2^{\aleph_1} = \aleph_2$). Fix an enumeration $\langle X_\alpha \mid \alpha < \aleph_2\rangle$ of J-positive sets.

By 8 there is $n < \omega$ such that

$$|\{\alpha < \omega_2 \mid A_{n\alpha} \in J^+\}| = \aleph_2.$$

Suppose for simplicity that $n = 0$. Let

$$\langle A_{0\tau(\xi)} \mid \xi < \omega_2 \rangle$$

be a one to one enumeration of this set.

We construct by induction a sequence of ordinals $\langle \xi_{0\alpha} | \alpha < \omega_2 \rangle$ and a sequence of J positive sets $\langle C_{0\alpha} | \alpha < \omega_2 \rangle$. Let $\alpha < \omega_2$. If there is $\xi < \omega_2$ such that $\xi \neq \xi_{0\beta}$ for each $\beta < \alpha$ and $X_\alpha \cap A_{0\tau(\xi)} \in J^+$, then let $\xi_{0\alpha}$ be the least such ξ. We would like now to attach an ordinal to the function f_α. So let us play the game $\mathcal{G}$ (which continues $\boldsymbol{t}$)where the player one uses the strategy σ until the stage at which the player one plays $X_\alpha \cap A_{0\tau(\xi)}$. All the previous move do not matter much here, but we fix some such play. Let the player two respond by $X_\alpha \cap A_{0\tau(\xi)}$ and f_α. The strategy σ provides then the answer of the player one. It consists of a subset $C_{0\alpha}$ of $X_\alpha \cap A_{0\tau(\xi)}$ and an ordinal $\eta_{0\alpha}$.

Let

$I_{0\alpha} = \{X \subseteq \omega_1 \mid \sigma$ never picks X in all possible continuations of the play started above.$\}$

If there is no such ξ then

$$X_\alpha \subseteq \nabla_{\varepsilon<\omega_1} A_{0\tau(\xi_{\beta_\varepsilon})},$$

where $\langle \beta_\varepsilon | \varepsilon < \omega_1 \rangle$ is an enumeration of α. Let then $\xi_{0\alpha}$ be the least ordinal above all $\xi_{0\beta}$ with $\beta < \alpha$. Replace X_α be $\aleph_1$ and then proceed with it as above.

Set $I_0 = \bigcap \{I_{0\alpha} | \alpha < \aleph_2\}$. Then I_0 is a normal ideal over $\aleph_1$, since each of $I_{0\alpha}$ is such.

The next lemma follows from the construction above.

Lemma 9. *For each $X \in J^+$ we have $X \in I_{0\alpha}$, for some $\alpha < \aleph_2$ or $X \subseteq \{\nu < \omega_1 | \quad \exists \beta < \nu \quad \nu \in A_{0\zeta_\beta}\}$ mod J, for some sequence $\langle \zeta_\beta | \beta < \omega_1 \rangle$ of ordinals below ω_2.*

As in [3] we can now deduce the following:

Lemma 10. *Let $X \subseteq \omega_1$. Then $X \in I_0$ iff $X \subseteq \{\nu < \omega_1 | \quad \exists \beta < \nu) \quad \nu \in Y_\beta\}$ mod J, for some sequence $\langle Y_\beta | \beta < \omega_1 \rangle$ such that for some sequence $\langle \alpha_\beta | \beta < \omega_1 \rangle$ of ordinals below ω_2 we have $Y_\beta \subseteq A_{0\tau(\xi_{\alpha_\beta})}$ and $Y_\beta \in I_{0\alpha_\beta}$.*

Let now $n = 1$. Fix some $\gamma < \omega_2$. We apply 8 to find the least $n_\gamma \geq 1$ such that the set

$$|\{\alpha < \omega_2 | A_{n_\gamma \alpha} \in I_{0\gamma}^+\}| = \aleph_2.$$

Let

$$\langle A_{n_\gamma \tau(\xi)} | \xi < \omega_2 \rangle$$

be a one to one enumeration of this set. For each $\xi < \omega_2$ we would like to attach an ordinal to a restriction of f_ξ to an $I_{0\gamma}$ positive subset of $A_{n_\gamma \tau(\xi)}$.

Proceed as above. Define recursively sequences $\langle \xi_{\langle 0\gamma,1\alpha\rangle} | \alpha < \omega_2\rangle$ and $\langle C_{\langle 0\gamma,1\alpha\rangle} | \alpha < \omega_2\rangle$.

At stage α consider the α-th set X_α in $I_{0\gamma}$. If there is $\xi < \omega_2$ such that $\xi \neq \xi_{\langle 0\gamma,1\beta\rangle}$, for each $\beta < \alpha$ and $X_\alpha \cap A_{n_\gamma\tau(\xi)} \in I^+_{0\gamma}$, then let $\xi_{\langle 0\gamma,1\alpha\rangle}$ be the least such ξ. We would like to shrink $I_{0\gamma}$ below $X_\alpha \cap A_{n_\gamma\tau(\xi_{\langle 0\gamma,1\alpha\rangle})}$ in order to decide an ordinal which will correspond to f_α. As above we fix a play according to σ which is a continuation of the previous play (the one from the definition of $I_{0\gamma}$ reaching $X_\alpha \cap A_{n_\gamma\tau(\xi_{\langle 0\gamma,1\alpha\rangle})}$. Let the second player plays at his next move $X_\alpha \cap A_{n_\gamma\tau(\xi_{\langle 0\gamma,1\alpha\rangle})}$ and f_α. Apply the strategy σ. It supplies an $I_{0\gamma}$ positive subset $C_{\langle 0\gamma,1\alpha\rangle}$ of $X_\alpha \cap A_{n_\gamma\tau(\xi_{\langle 0\gamma,1\alpha\rangle})}$ and an ordinal $\eta_{0\gamma,1\alpha}$. This will be the ordinal corresponding to $f_\alpha \restriction C_{\langle 0\gamma,1\alpha\rangle}$.

Let $I_{\langle 0\gamma,1\alpha\rangle} = \{X \subseteq \omega_1 \mid \sigma$ never picks X in all possible continuations of the play started above.$\}$

If there is no such ξ then let $\xi_{\langle 0\gamma,1\alpha\rangle}$ be the least ordinal above all $\xi_{\langle 0\gamma,1\beta\rangle}$ for $\beta < \alpha$. Take ω_1 instead of X_α and run the construction above.

Set $I_1 = \bigcap\{I_{0\gamma,1\alpha\rangle} \mid \gamma, \alpha < \aleph_2\}$. Then I_1 is a normal ideal over $\aleph_1$, since each of $I_{\langle 0\gamma,1\alpha\rangle}$ is such.

Continue similar and define I_s and I_n for each $n < \omega$ and $s \in [\omega \times \omega_2]^{<\omega}$. Let F_s and F_n be the corresponding dual filters. Finally set

$$I_\omega = \text{ the closure under } \omega \text{ unions of } \bigcup_{n<\omega} I_n.$$

Let F_ω be the corresponding dual filter.

The following lemmas of [3] transfer directly to the preset context.

Lemma 11. *$F \subseteq F_0 \subseteq ... \subseteq F_n \subseteq ... \subseteq F_\omega$ and $I \subseteq J \subseteq I_0 \subseteq ... \subseteq I_n \subseteq ... \subseteq I_\omega$.*

Lemma 12

$$F_\omega = \left\{ X \subseteq \omega_1 \middle| \exists \langle X_n | n < \omega\rangle \forall n < \omega X_n \in F_n \quad X = \bigcap_{n<\omega} X_n \right\}$$

and

$$I_\omega = \left\{ X \subseteq \omega_1 | \exists \langle X_n \middle| n < \omega\rangle \forall n < \omega X_n \in I_n \quad X = \bigcup_{n<\omega} X_n \right\}$$

Lemma 13. *I_ω is a proper ω_1-complete filter over ω_1.*

Lemma 14. *If $\langle Y_\beta | \beta < \omega_1\rangle$ is a sequence of sets in I_ω then the set*

$$Y = \{\nu < \kappa | \quad \exists \beta < \nu \quad \nu \in Y_\beta\}$$

is in I_ω as well and hence I_ω is normal.

Lemma 15. *A set X is in I^+_ω iff $X \in F_s$, for some $s \in [\omega \times \omega_2]^{<\omega}$.*

Now we are ready to show the desired result.

Theorem 16. *Let G be a generic subset of I_ω^+ and $j_G : V \to M_G = V \cap {}^{\omega_1}V/G$ be the corresponding elementary embedding. Then M_G is wellfounded at least up to $j_G(\tau)$.*

Proof. Suppose that $\langle \dot{g_n} | n < \omega\rangle$ is a sequence of I_ω^+-names of old (in V) functions from $\omega_1 \to \tau$.

Let $G \subseteq I_\omega^+$ be a generic ultrafilter. Pick a set $X_0 \in G$ and a function

$$g_0 : \omega_1 \longrightarrow \tau$$

in V such that

$$X_0 \Vdash_{I_\omega^+} \dot{g}_0 = \check{g}_0.$$

Let $\alpha_0 < (\omega_2)^V$ be so that $f_{\alpha_0} = g_0$.

Apply Lemma 15 to X_0 . There is a sequence s_0 with F_{s_0} defined and so that $X_0 \in F_{s_0}$. Recall now the definition of the filters $F_{s_0 {}^\frown \langle |s_0| \alpha\rangle}$ which extend F_{s_0} at the very next stage of the construction. There will be $\beta_0 < \kappa^+$ and $n_0 > |s_0|$ such that $A_{n_0 \tau(\alpha_0)} \in F_{s_0 {}^\frown \langle |s_0| \beta_0\rangle}$. Denote by η_0 the the ordinal attached to f_{α_0} at the level of s_0 in the construction of $F^+_{s_0 {}^\frown \langle |s_0| \beta_0\rangle}$. By shrinking if necessary we can assume that $A_{n_0 \tau(\alpha_0)} \cap X_0 \in F_t$ implies that the sequence $s_0{}^\frown\langle |s_0| \beta_0\rangle$ is an extension of the sequence t or vice verse. Without loss of generality we can assume that $A_{n_0 \tau(\alpha_0)} \cap X_0 \in G$, just otherwise replace X_0 by arbitrary positive subset and use density.

Continue now below $A_{n_0 \tau(\alpha_0)} \cap X_0$ and pick $X_1 \in G$ such that for some function

$$g_1 : \kappa \longrightarrow \tau$$

in V we have

$$X_1 \Vdash_{F_\omega^+} \dot{g}_1 = \check{g}_1.$$

Let $g_1 = f_{\alpha_1}$. Again, by 15, there is a sequence s_1 extending s_0 with F_{s_1} defined and so that $X_1 \in F_{s_1}$. Recall now the definition of the filters $F_{s_1 {}^\frown \langle |s_1| \alpha\rangle}$ which extend F_{s_1} at the very next stage of the construction. There will be $\beta_1 < \kappa^+$ and $n_1 > |s_1|$ such that $A_{n_1 \tau(\alpha_1)} \in F_{s_1 {}^\frown \langle |s_1| \beta_1\rangle}$. Denote by η_1 the the ordinal attached to f_{α_1} at the level of s_1 in the construction of $F^+_{s_1 {}^\frown \langle |s_1| \beta_1\rangle}$. By shrinking if necessary we can assume that $A_{n_1 \tau(\alpha_1)} \cap X_1 \in F_t$ implies that the sequence $s_1{}^\frown\langle |s_1| \beta_1\rangle$ is an extension of the sequence t or vice verse. Without loss of generality we can assume that $A_{n_1 \tau(\alpha_1)} \cap X_1 \in G$, just otherwise replace X_1 by arbitrary positive subset and use density.

Continue the process for each $n < \omega$. There will be $k < m < \omega$ with $\rho_k \leq \rho_m$. Then the set

$$\{\nu \in X_m \cap A_{n_m \alpha_m} | f_{\alpha_k}(\nu) \leq f_{\alpha_m}(\nu)\} \in F_{s_m {}^\frown \langle |s_m| \beta_m\rangle}.$$

But $X_m \cap A_{n_m \alpha_m} \in G$ as well. Then,

$$\{\nu \in X_m \cap A_{n_m \alpha_m} | f_{\alpha_k}(\nu) \leq f_{\alpha_m}(\nu)\} \in G,$$

just no elements of G can be outside of $X_m \cap A_{n_m \alpha_m}$ (mod $F \subseteq F_\omega$)since all of them are in F_t's for sequences t which are subsequences of s_n, for some $n < \omega$. □

Actually the argument provides a bit more information. Thus the following holds:

Theorem 17. *Assume that $2^{\aleph_1} = \aleph_2$ and $\|f\| = \omega_2$, for some $f : \omega_1 \to \omega_1$. Suppose that Player I has a winning strategy in the game $\mathcal{G}_\tau$, for some $\tau < \aleph_3$, then there is a normal ideal on $\aleph_1$ with a generic ultrapower wellfounded up to the image of τ.*

Proof. Note that the construction of I_ω above relays only on the strategy for the player one in the game $\mathcal{G}_\tau$. □

The opposite direction is true as well:

Theorem 18. *Suppose that J is a normal ideal on $\aleph_1$ with a generic ultrapower well founded up to the image of τ (for some ordinal τ), then Player I has a winning strategy in the game $\mathcal{G}_\tau$.*

Proof. Just start with ω_1 or any J-positive set. At a stage $2n-1 (n > 0)$ the second player responds with a function $f : A_{2n-2} \to \tau$ and, say, a sequence $\langle B_\alpha | \alpha < \aleph_1 \rangle$ such that

$$\triangledown_{\alpha<\omega_1} B_\alpha \supseteq A_{2n-2}.$$

Then one of B_α's should have the intersection with A_{2n-2} in J^+ (J is normal and we assume that $A_{2n-2} \in J^+$). Pick the least α such that $A_{2n-2} \cap B_\alpha \in J^+$. Shrink then $A_{2n-2} \cap B_\alpha$ to a set deciding the value of $[f]_{\dot{G}}$ in the generic ultrapower. Let A_{2n} be such a set.

The above defines a winning strategy for the player one in the game $\mathcal{G}_\tau$. □

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