Total Latency in Singleton Congestion Games*

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Abstract. We provide a collection of new upper and lower bounds on the *price of anarchy* for *singleton congestion games*. In our study, we distinguish between restricted and unrestricted strategy sets, between weighted and unweighted player weights, and between linear and polynomial latency functions.

1 Introduction

Congestion games [19] and variants thereof [17] have long been used to model non-cooperative resource sharing among selfish players. Examples include traffic behavior in road or communication networks or competition among firms for production processes. In this work, we study *singleton congestion games* where each player's *strategy* consists only of a single resource. A sample application for these modified games is load balancing [3].

The focal point of our work is determining the *price of anarchy* [15], a measure of the extent to which competition approximates the global objective, e.g., the minimum total travel time (latency) in the case of road networks. Typically, the price of anarchy is the worst-case ratio between the value of an objective function in some state where no *player* can *unilaterally* improve its situation, and that of some optimum. As such, the price of anarchy represents a rendezvous of *Nash equilibrium* [18], a concept fundamental to Game Theory, with *approximation*, an omnipresent concept in Theoretical Computer Science today.

1.1 Preliminaries and Model

Notation. For all $d \in \mathbb{N}$, let $[d] := \{1, \ldots, d\}$ and $[d]_0 := [d] \cup \{0\}$. For a vector $\mathsf{v} = (v_1, \ldots, v_n)$, let $(\mathsf{v}_{-\mathsf{i}}, v'_i) := (v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_n)$. Moreover, we denote by B_d the d-th Bell Number and by Φ_d a natural generalization of the golden ratio such that Φ_d is the (only) positive real solution to $(x+1)^d = x^{d+1}$.

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Instance. A (weighted) *singleton congestion game* is a tuple $\Gamma = \big(n, m, (w_i)_{i\in[n]}, (S_i)_{i\in[n]}, (f_e)_{e\in[m]}\big)$. Here, n is the number of *players* and m is the number of *resources*. For every player $i \in [n]$, $w_i \in \mathbb{R}_{>0}$ is its *weight* (w.l.o.g., $w_i = 1$ if Γ is *unweighted*) and $S_i \subseteq [m]$ its *pure strategy set*. Denote by $W := \sum_{i\in[n]} w_i$ the *total weight* of the players. Strategy sets are *unrestricted* if $S_i = [m]$ for all $i \in [n]$ and *restricted* otherwise. Denote $S := S_1 \times \ldots \times S_n$. For every resource $e \in [m]$, the *latency function* $f_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defines the *latency* on resource e. We consider *polynomial latency functions* with maximum degree d and non-negative coefficients, i.e., for each $e \in [m]$, the latency function is of the form $f_e(x) = \sum_{j=0}^{d} a_{e,j} \cdot x^j$ with $a_{e,j} \geq 0$ for all $j \in [d]_0$. For the special case of *affine latency functions*, we let $a_e := a_{e,1}$ and $b_e := a_{e,0}$, i.e., for any $e \in [m]$ we have $f_e(x) = a_e \cdot x + b_e$. Affine latency functions are *linear* if $b_e = 0$ for all $e \in [m]$.

Strategies and Strategy Profiles. A *mixed* strategy $P_i = (P_{i,e})_{e\in S_i}$ of player $i \in [n]$ is a probability distribution over S_i. For a pair of pure and mixed *strategy profiles* $\boldsymbol{s} = (s_1, \ldots, s_n)$ and $\mathbf{P} = (P_1, \ldots, P_n)$, we denote by $\mathbf{P}(\boldsymbol{s}) := \prod_{i\in[n]} P_{i,s_i}$ the probability that the players choose $\boldsymbol{s}$. Throughout the paper, we identify any pure strategy (profile) with the respective degenerate mixed strategy (profile).

Load and Private Cost. Denote by $\delta_e(\mathbf{P}) = \sum_{i\in[n]} P_{i,e} \cdot w_i$ the (expected) *load* on resource $e \in [m]$ under profile $\mathbf{P}$. The *private cost* of a player $i \in [n]$ is $\mathsf{PC}_i(\mathbf{P}) := \sum_{\boldsymbol{s}\in S} \mathbf{P}(\boldsymbol{s}) \cdot f_{s_i}(\delta_{s_i}(\boldsymbol{s}))$.

Nash Equilibria. A profile $\mathbf{P}$ is a *Nash equilibrium* if no player $i \in [n]$ could unilaterally improve its private cost; i.e., $\mathsf{PC}_i(\mathbf{P}) \leq \mathsf{PC}_i(\mathbf{P}_{-i}, e)$ for all $i \in [n]$ and $e \in S_i$. Depending on the profile, we distinguish *pure* and *mixed* Nash equilibria. $\mathcal{NE}(\Gamma)$ and $\mathcal{NE}_{\mathsf{pure}}(\Gamma)$ are the sets of all mixed (resp. pure) Nash equilibria.

Social Cost. *Social cost* $\mathsf{SC}(\Gamma, \mathbf{P})$ is defined as the (expected) *total latency* [20], i.e., $\mathsf{SC}(\Gamma, \mathbf{P}) := \sum_{\boldsymbol{s}\in S} \mathbf{P}(\boldsymbol{s}) \sum_{e\in[m]} \delta_e(\boldsymbol{s}) \cdot f_e(\delta_e(\boldsymbol{s})) = \sum_{i\in[n]} w_i \cdot \mathsf{PC}_i(\mathbf{P})$. The *optimum total latency* is $\mathsf{OPT}(\Gamma) := \min_{\boldsymbol{s}\in S} \mathsf{SC}(\Gamma, \boldsymbol{s})$.

Price of Anarchy. Let $\mathcal{G}$ be a class of weighted singleton congestion games. The *mixed price of anarchy* is defined as $\mathsf{PoA}(\mathcal{G}) := \sup_{\Gamma\in\mathcal{G}, \mathbf{P}\in\mathcal{NE}(\Gamma)} \frac{\mathsf{SC}(\Gamma,\mathbf{P})}{\mathsf{OPT}(\Gamma)}$. For the definition of the *pure price of anarchy* $\mathsf{PoA}_{\mathsf{pure}}$ replace $\mathcal{NE}$ with $\mathcal{NE}_{\mathsf{pure}}$.

1.2 Previous Work and Our Contribution

The price of anarchy was first introduced and studied by Koutsoupias and Papadimitriou [15] for weighted singleton congestion games with unrestricted strategy sets and linear latency functions, yet social cost defined as the expected maximum latency on a resource. Their setting became known as the *KP-model* and initiated a sequence of papers determining the price of anarchy both for the KP-model and generalizations thereof; see, e.g., [14,9,12,10,13,6].

For general (weighted) congestion games and social cost defined as the total latency, exact values for the price of anarchy have been given in [2,5,1]. In particular, Aland et al. [1] proved that for identical players the price of anarchy for polynomial latency functions (of maximum degree d and with non-negative

Table 1. Lower/upper bounds on the price of anarchy for singleton congestion games. Terms $o(1)$ are in m.

			$\mathsf{PoA}_{\mathsf{pure}}$		PoA	
	$f_e(x) =$	player	LB	UP	LB	UP
unrestricted strategies	x	ident.	1		$2 - \frac{1}{m}$ [16]	
	x	arb.	$\frac{9}{8}$ [16]		$2 - \frac{1}{m}$ [16,12]	
	$a_e x$	ident.	$\frac{4}{3}$ [16]		$2 - \frac{1}{m}$ (T.1)	
	$a_e x$	arb.	$2 - o(1)$ (T.2)	$1 + \Phi$ [2]	2.036 (T.3)	$1 + \Phi$ [2]
	x^d	ident.	1		$B_{d+1} - o(1)$ [11]	B_{d+1} [11]
	$\sum_{j=0}^{d} a_{e,j} x^j$	arb.	$B_{d+1} - o(1)$ (T.2)	Φ_d^{d+1} [1]		
restricted strategies	x	ident.	2.012 [21]	2.012 [3]		
	$a_e x$	ident.	$\frac{5}{2} - o(1)$ [3]	$\frac{5}{2}$ [21]	$\frac{5}{2} - o(1)$ [3]	$\frac{5}{2}$ [4]
	$\sum_{j=0}^{d} a_{e,j} x^j$	ident.	$\Upsilon(d) - o(1)$ (T.5)	$\Upsilon(d)$ [1]	$\Upsilon(d) - o(1)$ (T.5)	$\Upsilon(d)$ [1]
	$a_e x$	arb.	$1 + \Phi - o(1)$ [3]	$1 + \Phi$ [2]	$1 + \Phi - o(1)$ [3]	$1 + \Phi$ [2]
	$\sum_{j=0}^{d} a_{e,j} x^j$	arb.	$\Phi_d^{d+1} - o(1)$ (T.4)	Φ_d^{d+1} [1]	$\Phi_d^{d+1} - o(1)$ (T.4)	Φ_d^{d+1} [1]

coefficients) is exactly $\Upsilon(d) := \frac{(\lambda+1)^{2d+1} - \lambda^{d+1}(\lambda+2)^d}{(\lambda+1)^{d+1} - (\lambda+2)^d + (\lambda+1)^d - \lambda^{d+1}}$, where $\lambda = \lfloor \Phi_d \rfloor$. For weighted players the price of anarchy increases slightly to Φ_d^{d+1} [1].

Finally, singleton congestion games with social cost defined as the total latency have been studied in [3,11,16,21]; see Table 1 for a comparison. Since such games always possess a pure Nash equilibrium (if latency functions are non-decreasing [8]), also the *pure* price of anarchy is of interest. In this work, we prove a collection of new bounds on the price of anarchy for multiple interesting classes of singleton congestion games, as shown (and highlighted by a gray background) in Table 1. Surprisingly, the upper bounds from [1] – proved for general congestion games with polynomial latency functions – are already exact for the case of singleton strategy sets and pure Nash equilibria.

2 Unrestricted Strategy Sets

Proposition 1. *Let Γ be a weighted singleton congestion game with unrestricted strategy sets, affine latency functions and associated Nash equilibrium $\mathbf{P}$. Then, for all nonempty subsets $\mathcal{M} \subseteq [m]$, $\mathsf{SC}(\Gamma, \mathbf{P}) \leq \sum_{i \in [n]} w_i \cdot \frac{W + (|\mathcal{M}|-1) w_i + \sum_{j \in \mathcal{M}} \frac{b_j}{a_j}}{\sum_{j \in \mathcal{M}} \frac{1}{a_j}}$.*

Proposition 2. *Let Γ be a weighted singleton congestion game with unrestricted strategy sets and affine latency functions. Let $\boldsymbol{s}$ be an associated pure strategy profile with optimum total latency and let $\mathcal{M} = \{e : \delta_e(\boldsymbol{s}) > 0\}$. Define $X = \{\boldsymbol{x} \in \mathbb{R}_{\geq 0}^{\mathcal{M}} : \sum_{j \in \mathcal{M}} x_j = W\}$ and let $\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x} \in X} \{\sum_{j \in \mathcal{M}} x_j \cdot f_j(x_j)\}$. Denote $\mathcal{M}^* = \{j \in \mathcal{M} : x_j^* > 0\}$. Then, $\mathsf{OPT}(\Gamma) = \mathsf{SC}(\Gamma, \boldsymbol{s}) \geq \frac{W^2 + \frac{W}{2} \cdot \sum_{j \in \mathcal{M}^*} \frac{b_j}{a_j}}{\sum_{j \in \mathcal{M}^*} \frac{1}{a_j}}$.*

We are now equipped with all tools to prove the following upper bounds:

Theorem 1. *Let $\mathcal{G}_a$ be the class of unweighted singleton congestion games with at most m resources, unrestricted strategy sets and affine latency functions and $\mathcal{G}_b$ be the subset of $\mathcal{G}_a$ with linear latency functions. Then (a) $\mathsf{PoA}(\mathcal{G}_a) < 2$ and (b) $\mathsf{PoA}(\mathcal{G}_b) \leq 2 - \frac{1}{m}$.*

Theorem 2. *Let $\mathcal{G}$ be the class of weighted singleton congestion games with unrestricted strategy sets and polynomial latency functions of maximum degree d. Then $\mathsf{PoA}_{\mathsf{pure}}(\mathcal{G}) \geq B_{d+1}$.*

Proof. For some parameter $k \in \mathbb{N}$ define the following weighted singleton congestion game $\Gamma(k)$ with unrestricted strategy sets and polynomial latency functions:

– There are $k+1$ disjoint sets $\mathcal{M}_0, \ldots, \mathcal{M}_k$ of resources. Set $\mathcal{M}_j, j \in [k]_0$, consists of $|\mathcal{M}_j| = 2^{k-j} \cdot \frac{k!}{j!}$ resources sharing the polynomial latency function $f_e(x) = 2^{-jd} \cdot x^d$ for all resources $e \in \mathcal{M}_j$.
– There are k disjoint sets of players $\mathcal{N}_1, \ldots, \mathcal{N}_k$. Set $\mathcal{N}_j$, $j \in [k]$, consists of $|\mathcal{N}_j| = |\mathcal{M}_{j-1}| = 2^{k-(j-1)} \cdot \frac{k!}{(j-1)!}$ players with weight $w_i = 2^{j-1}$ for all players $i \in \mathcal{N}_j$.

Observe that $|\mathcal{M}_j| = 2^{k-j} \cdot \frac{k!}{j!} = 2^{k-(j+1)} \cdot \frac{k!}{(j+1)!} \cdot 2(j+1) = |\mathcal{M}_{j+1}| \cdot 2(j+1)$.

Let $\boldsymbol{s}$ be a pure strategy profile that assigns exactly $2j$ players from $\mathcal{N}_j$ to each resource in $\mathcal{M}_j$ for $j \in [k]_0$. Then, for all resources $e \in \mathcal{M}_j$, $j \in [k]$, we have $\delta_e(\boldsymbol{s}) = 2j \cdot 2^{j-1} = j \cdot 2^j$ and $f_e(\delta_e(\boldsymbol{s})) = 2^{-jd} \cdot (j \cdot 2^j)^d = j^d$. It is now easy to check that $\boldsymbol{s}$ is a Nash equilibrium for $\Gamma(k)$ with $\mathsf{SC}(\Gamma(k), \boldsymbol{s}) = 2^k \cdot k! \cdot \sum_{j \in [k]_0} \frac{j^{d+1}}{j!}$. Now let $\boldsymbol{s}^*$ be a strategy profile that assigns each player $\mathcal{N}_j$ to a separate resource in $\mathcal{M}_{j-1}$. Then, for all resources $e \in \mathcal{M}_j$, $j \in [k-1]_0$, we have $\delta_e(\boldsymbol{s}^*) = 2^j$ and $f_e(\delta_e(\boldsymbol{s}^*)) = 2^{-jd} \cdot (2^j)^d = 1$. So $\mathsf{SC}(\Gamma(k), \boldsymbol{s}^*) = 2^k \cdot k! \sum_{j \in [k-1]_0} \frac{1}{j!}$. Hence, $\mathsf{PoA}_{\mathsf{pure}}(\mathcal{G}) \geq \lim_{k \to \infty} \frac{\mathsf{SC}(\Gamma(k), \boldsymbol{s})}{\mathsf{SC}(\Gamma(k), \boldsymbol{s}^*)} = \frac{\sum_{j=1}^{\infty} \frac{j^{d+1}}{j!}}{\sum_{j=0}^{\infty} \frac{1}{j!}} = \frac{1}{\mathsf{e}} \sum_{j=1}^{\infty} \frac{j^{d+1}}{j!} = B_{d+1}$. □

Theorem 3. *Let $\mathcal{G}$ be the class of weighted singleton congestion games with unrestricted strategy sets and linear latency functions. Then $\mathsf{PoA}(\mathcal{G}) > 2.036$.*

Proof. For $w \in \mathbb{R}_{>0}$, define the singleton congestion game $\Gamma(w)$ with 5 players of weights $w_1 = w$ and $w_i = 1$ for $i \in \{2, \ldots, 5\}$ and 5 resources with latency functions $f_1(x) = \frac{w}{w+4} \cdot x$ and $f_e(x) = x$ for $e \in \{2, \ldots, 5\}$.

Let $\boldsymbol{s} := (i)_{i=1}^n \in S$ and let $\mathbf{P}$ be the mixed strategy profile where $P_{1,1} = p$, $P_{1,e} = \frac{1-p}{4}$ for $e \in \{2, \ldots, 5\}$, and $P_{i,1} = 1$ for $i \in \{2, \ldots, 5\}$. It is easy to check that $\mathbf{P}$ is a Nash equilibrium for $p \leq \frac{w^2-8w+16}{5w^2+4w}$. Since $\mathsf{SC}(\Gamma(w), \mathbf{P}) = p\frac{4w^2}{w+4} + \frac{16w}{w+4} + w^2$ is monotonically increasing in p, choose $p = \frac{w^2-8w+16}{5w^2+4w}$. Clearly, $\mathsf{PoA}(\mathcal{G}) \geq \frac{\mathsf{SC}(\Gamma(w), \mathbf{P})}{\mathsf{SC}(\Gamma(w), \boldsymbol{s})}$. Setting $w = 3.258$ yields the claimed lower bound. □

3 Restricted Strategy Sets

Theorem 4. *Let $\mathcal{G}$ be the class of weighted singleton congestion games with restricted strategy sets and polynomial latency functions of maximum degree d. Then $\mathsf{PoA}(\mathcal{G}) = \mathsf{PoA}_{\mathsf{pure}}(\mathcal{G}) = \Phi_d^{d+1}$.*

Proof. Due to [1], we only need to show the lower bound. For $n \in \mathbb{N}$, define the singleton congestion game $\Gamma(n)$ with n players and $n+1$ resources. The weight of player $i \in [n]$ is $w_i = \Phi_d^i$ and the latency functions are $f_{n+1}(x) = \Phi_d^{-(d+1)\cdot(n-1)} \cdot x^d$ for resource $n+1$ and $f_e(x) = \Phi_d^{-(d+1)\cdot e} \cdot x^d$ for resources $e \in [n]$. Each player $i \in [n]$ only has two available resources: $S_i = \{i, i+1\}$.

Let $\boldsymbol{s} := (i)_{i=1}^n \in S$. One can verify that $\boldsymbol{s}$ is a Nash Equilibrium and $\mathsf{SC}(\Gamma(n), \boldsymbol{s}) = n$. Now let $\boldsymbol{s}^* := (i+1)_{i=1}^n \in S$. Then, $\mathsf{SC}(\Gamma(n), \boldsymbol{s}^*) = (n-1) \cdot \frac{1}{\Phi_d^{d+1}} + 1$, so $\sup_{n\in\mathbb{N}} \left\{ \frac{\mathsf{SC}(\Gamma(n),\boldsymbol{s})}{\mathsf{SC}(\Gamma(n),\boldsymbol{s}^*)} \right\} = \Phi_d^{d+1}$. □

Theorem 5. *Let $\mathcal{G}$ be the class of unweighted singleton congestion games with restricted strategy sets and polynomial latency functions of maximum degree d. Then $\mathit{PoA}(\mathcal{G}) = \mathit{PoA}_{\mathsf{pure}}(\mathcal{G}) = \Upsilon(d)$.*

Proof (Sketch). For $k \in \mathbb{N}$, define an unweighted singleton congestion game $\Gamma(k)$. We borrow the representation introduced by [7] which makes use of an "interaction graph" $G = (N, A)$: Resources correspond to nodes and players correspond to arcs. Every player has exactly two strategies, namely choosing one or the other of its adjacent nodes.

The interaction graph is a tree which is constructed as follows: At the root there is a complete $(d+1)$-ary tree with $k+1$ levels. Each leaf of this tree is then the root of a complete d-ary tree the leafs of which are again the root of a complete $(d-1)$-ary tree; and so on. This recursive definition stops with the unary trees. For an example of this construction, see Figure 1.

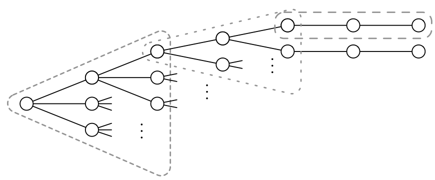

Fig. 1. The game graph for $d = k = 2$

Altogether, the game graph consists of $(d+1) \cdot k + 1$ levels. We let level 0 denote the root level. Thus, clearly, the nodes on level $i \cdot k$, where $i \in [d]_0$, are the root of a complete $(d+1-i)$-ary subtree (as indicated by the hatched shapes).

For any resource on level $(d+1-i)\cdot k+j$, where $i \in [d+1]$ and $j \in [k-1]_0$, let the latency function be $f_{i,j} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $f_{i,j}(x) := \left[\prod_{l=i+1}^{d+1} \frac{l}{l+1}\right]^{d\cdot(k-1)} \cdot \left(\frac{i}{i+1}\right)^{dj} \cdot x^d$. The resources on level $(d+1)\cdot k$ have the same latency function $f_{0,0} := f_{1,k-1}$ as those on level $(d+1)\cdot k - 1$.

Let $\boldsymbol{s}$ denote the strategy profile in $\Gamma(k)$ where each player uses the resource which is closer to the root. Similarly, let $\boldsymbol{s}^*$ be the profile where players us the resources farther away from the root. One can verify that $\boldsymbol{s}$ is a Nash equilibrium and the quotient $\frac{\mathsf{SC}(\Gamma(k),\boldsymbol{s})}{\mathsf{SC}(\Gamma(k),\boldsymbol{s}^*)}$ can be written in the form $\frac{\sum_{i=0}^{d+1} \beta_i \cdot \alpha_i^{k-1}}{\sum_{i=0}^{d+1} \gamma_i \cdot \alpha_i^{k-1}}$ where $\beta_i, \gamma_i \in \mathbb{Q}$, $\alpha_0 = 1$, and $\alpha_i = \prod_{l=i}^{d+1} \frac{l^{d+1}}{(l+1)^d} = \frac{i^{d+1}}{(d+2)^d} \cdot \prod_{l=i+1}^{d+1} l$ for all $i \in [d+1]$.

Now let $\lambda := \lfloor \Phi_d \rfloor$. Then, $(\lambda+1)^d > \lambda^{d+1}$ but $(\lambda+2)^d < (\lambda+1)^{d+1}$, so $\lambda \in [d]$. It holds that $\alpha_{\lambda+1} > \alpha_i$ for all $i \in [d+1]_0 \setminus \{\lambda+1\}$ because, for all $i \in [d]$, $\alpha_{i+1} > \alpha_i$ if and only if $(i+1)^d > i^{d+1}$ and $\alpha_1 = \frac{(d+1)!}{(d+2)^d} < 1$

and $\alpha_{d+1} = \frac{(d+1)^{d+1}}{(d+2)^d} > 1$. Using standard calculus we therefore get $\lim_{k\to\infty} \frac{\mathsf{SC}(\Gamma(k),\boldsymbol{s})}{\mathsf{SC}(\Gamma(k),\boldsymbol{s}^*)} = \frac{\beta_{\lambda+1}}{\gamma_{\lambda+1}} = \frac{(\lambda+1)^{2d+1}-\lambda^{d+1}\cdot(\lambda+2)^d}{(\lambda+1)^{d+1}-(\lambda+2)^d+(\lambda+1)^d-\lambda^{d+1}}$. □

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