Deterministic Leader Election in Anonymous Sensor Networks Without Common Coordinated System

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Abstract. We address the Leader Election (LE) problem in networks of anonymous sensors sharing no kind of common coordinate system. The contribution of this paper is twofold: First, assuming n anonymous sensors agreeing on a common *handedness* (*chirality*) of their own coordinate system, we provide a complete characterization on the sensors positions to deterministically elect a leader. Our result holds for any $n > 1$, even if the n sensors have unlimited visibility and regardless of their capabilities, unbounded memory, mobility, and communication settings. Second, we show that this statement also holds assuming sensors without chirality provided that n is odd.

Keywords: Distributed Leader Election, Sense of Direction, Chirality, Sensor Networks.

1 Introduction

In distributed settings, many problems that are hard to solve otherwise become easier to solve with a *leader* to coordinate the system. The problem of electing a leader among a set of computing units is then one of the fundamental tasks in distributed systems. The *Leader Election* (LE) Problem consists in moving the system from an initial configuration were all entities are in the same state into a final configuration were all entities are in the same state, except one, the leader. The leader election problem is covered in depth in many books related to distributed systems, *e.g.,* [13,16].

The distributed systems considered in this paper are *sensor networks*. Sensor networks are dense wireless networks that are used to collect (to sense) environmental data such as temperature, sound, vibration, pressure, motion, etc. The data are either simply sent toward some data collectors or used as an input to perform some basic cooperative tasks. Wireless Sensor Networks (WSN) are emerging distributed systems providing diverse services to numerous applications in industries, manufacturing, security, environment and habitat monitoring, healfcare, traffic control, etc. WSN aim for being composed of a large quantity of sensors as small, inexpensive, and low-powered as possible. Thus, the interest has shifted towards the design of distributed protocols for very weak sensors, *i.e.,* sensors requiring very limited capabilities, *e.g., uniformity* (or, *homogeneity* — all the sensors follow the same program —, *anonymity* — the sensors are *a priori* indistinguishable —, *disorientation* — the sensors share no kind of coordinate system nor common sense of direction.

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However, in weak distributed environments, many tasks have no solution. In particular, in uniform anonymous general networks, the impossibility of breaking a possibly symmetry in the initial configuration makes the leader election unsolvable deterministically [1]. In this paper, we investigate the leader election problem with sensors having minimal capabilities, *i.e.,* they are anonymous, uniform and disoriented. We come up with the following question: "*Given a set of such weak sensors scattered on the plane, what are the (minimal) geometric conditions to be able to deterministically agree on a single sensor?*"

Related Works. Similar questions are addressed in [2,9,10]. In the former, the authors address the problem of *Localization* in sensor networks. This problem is to reconstruct the positions of a set of sensors with a *Limited Visibility*, *i.e.,* sensors which are able to locate of other sensors within a certain distance $v > 0$. The authors show that no polynomial-time algorithm can solve this problem in general. In [9,10], the authors address the *Pattern Formation* problem for sensors having the additional capability of *mobility*. Such mobile sensors are often referred to as *robots* or *agents*. The Pattern Formation problem consists in the design of protocols allowing autonomous mobile robots to form a specific class of patterns, *e.g.,* [18,9,10,4,11,6,7,8]. In [9], the authors discuss whether the pattern formation problem can be solved or not according to the capabilities the robots are supposed to have. They consider the ability to agree on the direction and orientation of one axis of their coordinate system (North) (Sense of Direction) and a common *handedness* (*Chirality*). Assuming sense of direction, chirality, and *Unlimited Visibility* — each robot is able to locate all the robots —,, they show that the robots can form any arbitrary pattern. Then, they show that with the lack of chirality, the problem can solved in general with an odd number of robots only. With the lack of both sense of direction and chirality, the pattern formation problem is unsolvable in general.

In [10], the authors show the fundammental relationship between the Pattern Formation problem and the Leader Election problem. They show that under sense of Direction and chirality, the Leader Election problem can be solved by constructing a total order over the coordinates of all the agents. With sense of direction and lack of chirality, the Leader Election is solvable if and only if the number of robots is odd. Informally, the results in [9,10] comes from the fact that starting from some symmetric configurations, no robot can be distinguished if the number of robots is even. In other words, they show that even if the robots have sense of direction and unlimited visibility, the lack of chirality prevents from breaking symmetry in a deterministic way.

Contribution. In this paper, we address the leader election problem under very weak assumptions: the sensors share no kind of common coordinate system. More precisely, they are not required to share any unit measure, common orientation or direction. However, even under such an assumption, they can agree on a common handedness or not.

The contribution of this paper is twofold. Assuming a set of n anonymous sensors with chirality, we first provide a complete characterization (necessary and sufficient conditions) on the sensors positions to deterministically elect a leader. Our result holds for any $n > 1$, even if the sensors have unlimited visibility and regardless of their capabilities, unbounded memory, mobility, and communication settings. The sufficient condition is shown by providing a deterministic algorithm electing a leader.

The proof is based on the ability for the sensors to construct a *Lyndon word* from the sensors' positions as an input. A Lyndon word is a non-empty word strictly smaller in the lexicographic order than any of its suffixes, except itself and the empty word. Lyndon words have been widely studied in the combinatorics of words area [12]. However, only a few papers consider Lyndon words addressing issues in other areas than word algebra, *e.g.,* [3,5,17,7]. In [7], we already shown the power of Lyndon words to built an efficient and simple deterministic protocol to form a regular n-gon. However, the results in [7] hold for a prime number n of robots only.

The second fold of our contribution addresses the lack of chirality. We show that our characterization still holds if and only if the number of sensors is odd. Again, we give a deterministic algorithm that shows the sufficient condition.

In the next section (Section 2), we formally describe the distributed model and the words considered in this paper. Both results are presented in Section 3. Finally, we conclude this paper in Section 4.

2 Preliminaries

In this section, we define the distributed system considered in this paper. Next, we review some formal definitions and basic results on words and Lyndon words

2.1 Model

Consider a set of n *sensors* (or *agents*, *robots*) arbitrarily scattered on the plane such that no two sensors are located at the same position. The sensors are *uniform* and *anonymous*, i.e, they all execute the same program using no local parameter (such that an identity) allowing to differentiate any of them. However, we assume that each sensor is a computational unit having the ability to determine the positions of the n sensors within an infinite decimal precision. We assume no kind of communication medium. Each sensor has its own local x-y Cartesian coordinate system defined by two coordinate axes (x and y), together with their *orientations*, identified as the positive and negative sides of the axes.

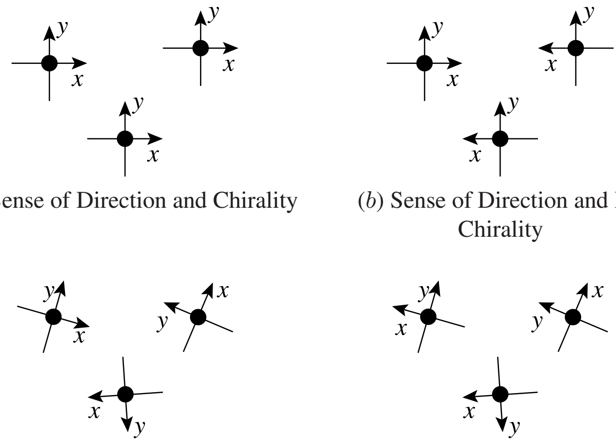
In this paper, we discuss the influence of *Sense of Direction* and *Chirality* in a sensor network.

Definition 1 (Sense of Direction). *A set of n sensors has sense of direction if the n sensors agree on a common direction of one axis (x or y) and its orientation. The sense of direction is said to be* partial *if the agreement relates to the direction only* —i.e., *they are not required to agree on the orientation.*

In Figure 1, the sensors have sense of direction in the cases (a) and (b), whereas they have no sense of direction in the cases (c) and (d).

Given an x-y Cartesian coordinate system, the *handedness* is the way in which the orientation of the y axis (respectively, the x axis) is inferred according to the orientation of the x axis (resp., the y axis).

Definition 2 (Chirality). *A set of n sensors has chirality if the n sensors share the same handedness.*



(a) Sense of Direction and Chirality (b) Sense of Direction and No Chirality

(c) No Sense of Direction and Chirality (d) No Sense of Direction and No Chirality

Fig. 1. Four examples showing the relationship between Sense of Direction and Chirality

In Figure 1, the sensors have chirality in the cases (a) and (c), whereas they have no chirality in the cases (b) and (d).

2.2 Words and Lyndon Words

Let an ordered alphabet A be a finite set of letters. Denote $\prec$ an order on A. A non empty *word* w over A is a finite sequence of letters $a_1, \ldots, a_i, \ldots, a_l$, $l > 0$. The *concatenation* of two words u and v, denoted $u \circ v$ or simply uv, is equal to the word $a_1, \ldots, a_i, \ldots, a_k, b_1, \ldots, b_j, \ldots, b_l$ such that $u = a_1, \ldots, a_i, \ldots, a_k$ and $v = b_1, \ldots, b_j, \ldots, b_l$. Let ϵ be the *empty word* such that for every word w, $w\epsilon = \epsilon w = w$. The *length* of a word w, denoted by $|w|$, is equal to the number of letters of w—$|\epsilon| = 0$.

A word u is *lexicographically* smaller than or equal to a word v, denoted $u \preceq v$, iff there exists either a word w such that $v = uw$ or three words r, s, t and two letters a, b such that $u = ras$, $v = rbt$, and $a \prec b$.

Let k and j be two positive integers. The k^{th} *power* of a word w is the word denoted s^k such that $s^0 = \epsilon$, and $s^k = s^{k-1}s$. A word u is said to be *primitive* if and only if $u = v^k \Rightarrow k = 1$. Otherwise ($u = v^k$ and $k > 1$), u is said to be *strictly periodic*. The *reversal* of a word $w = a_1 a_2 \cdots a_n$ is the word $\tilde{w} = a_n \cdots a_1$. The j^{th} *rotation* of a word w, notation $R_j(w)$, is defined by:

$$R_j(w) \stackrel{\text{def}}{=} \begin{cases} \epsilon & \text{if } w = \epsilon \\ a_j, \ldots, a_l, a_1, \ldots, a_{j-1} & \text{otherwise } (w = a_1, \ldots, a_l,\ l \geq 1) \end{cases}$$

Note that $R_1(w) = w$.

Lemma 1 ([12]). *Let w and $R_j(w)$ be a word and a rotation of w, respectively. The word w is primitive if and only if $R_j(w)$ is primitive.*

A word w is said to be *minimal* if and only if $\forall j \in 1, \ldots, l$, $w \preceq R_j(w)$.

Definition 3 (Lyndon Word). *A word w ($|w| > 0$) is a Lyndon word if and only if w is nonempty, primitive and minimal, i.e., $w \neq \epsilon$ and $\forall j \in 2, \ldots, |w|,\ w \prec R_j(w)$.*

For instance, if $A = \{a, b\}$, then a, b, ab, aab, abb are Lyndon words, whereas aba, and $abab$ are not— aba is not minimal ($aab \preceq aba$) and $abab$ is not primitive ($abab = (ab)^2$).

3 Leader Election

The *leader election* problem considered in this paper is stated as follows: Given the positions of n sensors in the plane, the n sensors are able to deterministically agree on the same position L called the leader.

3.1 Leader Election with Chirality

In this subsection, we assume a sensor networks having the property of chirality. A *configuration* π of the sensor network is a set of positions $p_1, \ldots, p_n$ ($n > 1$) occupied by the sensors. Given a configuration π, SEC denotes the *smallest enclosing circle* of the positions in π. The center of SEC is denoted $\mathtt{O}$. In any configuration π, SEC is unique and can be computed in linear time [14,19]. It passes either through two of the positions that are on the same diameter (opposite positions), or through at least three of the postions in π. Note that if $n = 2$, then SEC passes both sensors and no sensor can be located inside SEC, in particular at $\mathtt{O}$. Since the sensors have the ability of chirality, they are able to agree on a common orientation of SEC, denoted $\circlearrowright$.

Given a smallest enclosing circle SEC, the radii are the line segments from the center $\mathtt{O}$ of SEC to the boundary of SEC. Let $\mathcal{R}$ be the finite set of radii such that a radius r belongs to $\mathcal{R}$ iff at least one sensor is located on r but $\mathtt{O}$. Denote $\sharp\mathcal{R}$ the number of radii in $\mathcal{R}$. In the sequel, we will abuse language by considering radii in $\mathcal{R}$ only. Given two distinct positions p_1 and p_2 located on the same radius r ($\in \mathcal{R}$), $d(p_1, p_2)$ denotes the Euclidean distance between p_1 and p_2.

Definition 4 (Radius Word). *Let $p_1, \ldots, p_k$ be the respective positions of k robots ($k \geq 1$) located on the same radius $r \in \mathcal{R}$. Let w_r be the word such that*

$$w_r \stackrel{\text{def}}{=} \begin{cases} 0 \text{ if there exists one sensor at } \mathtt{O} \\ a_1, \ldots, a_k \text{ with } a_1 = d(\mathtt{O}, p_1) \text{ and } \forall i \in [2, k], a_i = d(p_{i-1}, p_i)\ , \text{ otherwise} \end{cases}$$

Note that all the distances are computed by each sensor with respect to its own coordinate system, *i.e., proportionally* to its own measure unit. Let RW be the set of radius words built over $\mathcal{R}$, computed by any sensor s. The lexicographic order $\preceq$ on RW is naturally built over the natural order $<$ on the set of real numbers.

Remark 1. If there exists one sensor on $\mathtt{O}$ ($n > 2$), then for every radius $r \in \mathcal{R}$, $w_r = 0$.

Let r be a radius in $\mathcal{R}$. The successor of r, denoted by $Succ(r, \circlearrowright)$, is the next radius in $\mathcal{R}$, according to $\circlearrowright$. The i^{th} successor of r, denoted by $Succ_i(r, \circlearrowright)$, is the radius such that $Succ_0(r, \circlearrowright) = r$, and $Succ_i(r, \circlearrowright) = Succ(Succ_{i-1}(r, \circlearrowright), \circlearrowright)$. Given r and its successor $r' = Succ(r, \circlearrowright)$, $\sphericalangle(r\mathtt{O}r')$ denotes the angle between r and r'.

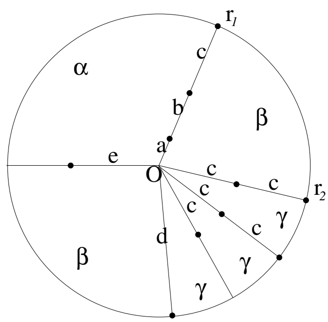


Fig. 2. Computation of Configuration words — the sensors are the black bullets

Definition 5 (Configuration Word Set)
Given an orientation $\circlearrowright$, let $CW^{\circlearrowright}$ be the set of configuration words, computed by any sensor s, build over $\mathcal{R}$ such that for each radius $r \in \mathcal{R}$, the associated configuration word $W(r)$ is equal to $(0,0)$ if $w_r = 0$, otherwise $W(r)$ is equal to the word $a_1, \ldots, a_k$ such that $k = \sharp\mathcal{R}$ and $\forall i \in [1,k],\ a_i = (Succ_{i-1}(r,\circlearrowright), \sphericalangle(Succ_{i-1}(r,\circlearrowright)$ $\mathtt{O} Succ_i(r,\circlearrowright)))$.

Remark 2. The three following propositions are equivalent:

1. There exists one sensor on $\mathtt{O}$
2. For every radius $r \in \mathcal{R}$, $W(r) = (0,0)$
3. $CW^{\circlearrowright} = \{(0,0)\}$

In Figure 2, if $\circlearrowright$ is the clockwise orientation, then: $W(r_1) = (abc,\beta)(c^2,\gamma)^2(c,\gamma)$ $(d,\beta)(e,\alpha)$ and $W(r_2) = (c^2,\gamma)^2(c,\gamma)(d,\beta)(e,\alpha)(abc,\beta)$.

If $\circlearrowright$ is the counterclockwise orientation, then: $W(r_1) = (abc,\alpha)(e,\beta)(d,\gamma)\ (c,\gamma)$ $(c^2,\gamma)(c^2,\beta)$ and $W(r_2) = (c^2,\beta)(abc,\alpha)(e,\beta)(d,\gamma)(c,\gamma)(c^2,\gamma)$.

Remark 3. Let $W(r_1)$ and $W(r_2)$ be two words in $CW^{\circlearrowright}$, r_1 and r_2 belong to $\mathcal{R}$. Then, $W(r_1)$ (respectively, $W(r_2)$) is a rotation of $W(r_2)$ (resp. $W(r_1)$) — refer to Figure 2.

Let $A_{CW^{\circlearrowright}}$ be the set of letters over $CW^{\circlearrowright}$. Let (u,x) and (v,y) be any two letters in $A_{CW^{\circlearrowright}}$. Define the order $\lessdot$ over $A_{CW^{\circlearrowright}}$ as follows:

$$(u,x) \lessdot (v,y) \iff \begin{cases} u \precneqq v \\ \text{or} \\ u = v \text{ and } x < y \end{cases}$$

The lexicographic $\preceq$ order over $CW^{\circlearrowright}$ is naturally built over $\lessdot$.

Remark 4. Each sensor having its own unit measure, given $r \in \mathcal{R}$, the word $W(r)$ computed by any sensor s can be different to the one computed by another sensor s'. However, all the distances are computed by each sensor proportionally to its own measure unit. So, if $W(r) \preceq W(r')$ for one sensor s, then $W(r) \preceq W(r')$ for every sensor s'. In particular, if $W(r)$ is a Lyndon word for one sensor s, then $W(r)$ is a Lyndon word for every sensor s'.

Lemma 2. *If there exists two distinct radii r_1 and r_2 in $\mathcal{R}$ such that both $W(r_1)$ and $W(r_2)$ are Lyndon words, then $CW^{\circlearrowright} = \{(0,0)\}$.*

Proof. Assume by contradiction that there exists two distinct radii r_1 and r_2 such that both $W(r_1)$ and $W(r_2)$ are Lyndon words and $CW^{\circlearrowright} \neq \{(0,0)\}$. By Remark 2, there exists no sensor located at O. By Remark 3, $W(r_1)$ (respectively, $W(r_2)$) is a rotation of $W(r_2)$ (resp. $W(r_1)$). So, by Definition 3, $W(r_1) \prec W(r_2)$ and $W(r_2) \prec W(r_1)$. A contradiction.

Lemma 3. *If there exists $r \in \mathcal{R}$ such that $W(r)$ is a Lyndon word, then the n sensors are able to determiniscally agree on the same sensor L.*

Proof. Directly follows from Lemma 2 and Remark 4: If there is a sensor s located on O, then the n sensors are able to agree on $L = s$. Otherwise, there exists a single $r \in \mathcal{R}$ such that $W(r)$ is a Lyndon word. In that case, all the sensors are able to agree on the sensor on r which is the nearest one from O.

Lemma 4. *If there exists no radius $r \in \mathcal{R}$ such that $W(r)$ is a Lyndon word, then there exists no deterministic algorithm allowing the n sensors to agree on the same sensor L.*

Proof. Assume by contradiction that no radius $r \in \mathcal{R}$ exists such that $W(r)$ is a Lyndon word and there exists an algorithm A allowing the n sensors to deterministically agree on the same sensor L. Let $minW$ be a word in $CW^{\circlearrowright}$ such that $\forall r \in \mathcal{R}$, $minW \preceq W(r)$. That is, $minW$ is minimal. Assume first that $minW$ is primitive. Then, $minW$ is a Lyndon word which contradicts the assumption. So, $minW$ is a strictly periodic word (there exists u and $k > 1$ such that $minSC = u^k$) and, from Lemma 1, we deduce that for all $r \in \mathcal{R}$, $W(r)$ is also strictly periodic. Thus, for every $r \in \mathcal{R}$, there exists at least one radius $r' \in \mathcal{R}$ such that $r \neq r'$ and $W(r) = W(r')$. So, for every radius $r \in \mathcal{R}$, there are $k > 1$ radii in $\mathcal{R}$ on which the sensors can have the same view of π. It is the case if the sensors have the same measure unit and their y axis meet the

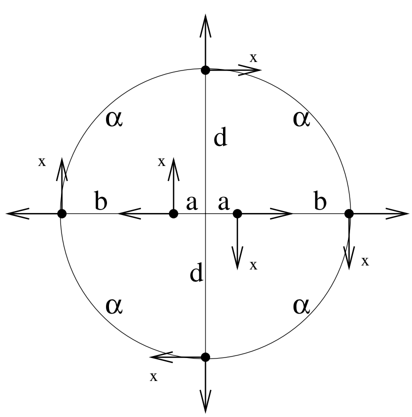


Fig. 3. A counter example showing Lemma 4

radius on which they are located — refer to Figure 3. In that case, A cannot allow the n sensors to deterministically agree on the same sensor L.

The following theorem follows from Lemmas 3 and 4:

Theorem 1. *Given a configuration π of any number $n \geq 2$ sensors with chirality scattered on the plane, the n sensors are able to deterministically agree on the same sensor L if and only if there exists a radius $r \in \mathcal{R}$ such that $W(r)$ is a Lyndon Word.*

3.2 Leader Election Without Chirality

Without chirality, the sensors are not able to agree on a common orientation of SEC. Define $\circlearrowright$ (respectively, $\circlearrowleft$) the clockwise (resp., counterclockwise) orientation. Obviously, with respect to their handedness, some of the n sensors choose to orient SEC according to $\circlearrowright$, whereas some other to $\circlearrowleft$. In this subsection, we use same definition of radius word (Definition 4) as in Subsection 3.1. Since the sensors have no chirality, for each radius $r \in \mathcal{R}$, there are two configuration words w.r.t. the orientation of SEC, denoted by $W(r)^{\circlearrowright}$ and $W(r)^{\circlearrowleft}$. Let CW be the set of all the configuration words, computed by any sensor s, in both clockwise and counterclockwise orientations.

We now show that the statement of Theorem 1 also holds assuming no chirality if n is odd.

Lemma 5. *Given an orientation $\circ$ of SEC in $\{\circlearrowright, \circlearrowleft\}$, if there exists two distinct radii r_1 and r_2 in $\mathcal{R}$ such that both $W(r_1)^\circ$ and $W(r_2)^\circ$ are Lyndon words, then $CW^\circ = \{(0,0)\}$.*

Proof. The proof is similar to that of Lemma 2.

Let $\mathcal{R}_L$ be the subset of radius $r \in \mathcal{R}$ such that $W(r)$ is a Lyndon word in the clockwise or in the counterclockwise orientation. Denote $\sharp\mathcal{R}_L$ the number of radii in $\mathcal{R}_L$.

Lemma 6. *If $\sharp\mathcal{R}_L > 2$, then for any orientation $\circ$ of SEC in $\{\circlearrowright, \circlearrowleft\}$, $\forall r \in \mathcal{R}$, $W(r)^\circ = (0,0)$.*

Proof. Assume by contradiction that $\sharp\mathcal{R}_L > 2$ and there exists $\circ \in \{\circlearrowright, \circlearrowleft\}$ and $r \in \mathcal{R}$ such that $W(r)^\circ \neq (0,0)$. Since $\sharp\mathcal{R}_L > 2$, there exists at least two distinct radii r_1 and r_2 such that either $W(r_1)^{\circlearrowright}$ and $W(r_2)^{\circlearrowright}$ are Lyndon words or $W(r_1)^{\circlearrowleft}$ and $W(r_2)^{\circlearrowleft}$ are Lyndon words. Without loss of generality, assume that $W(r_1)^{\circlearrowright}$ and $W(r_2)^{\circlearrowright}$ are Lyndon words. By, Lemma 5, $CW^{\circlearrowright} = \{(0,0)\}$. By Remark 2, $\forall r \in \mathcal{R}$, $W(r)^{\circlearrowright} = (0,0)$ and $W(r)^{\circlearrowleft} = (0,0)$. Hence, there exists no $r \in \mathcal{R}$ such that $W(r)^\circ \neq (0,0)$. A contradiction.

Lemma 7. *If n is odd and $\sharp\mathcal{R}_L \geq 1$, then the n sensors are able to determiniscally agree on the same sensor L.*

Proof. Since n is odd, $n \geq 3$. From Lemma 6, there are three cases to consider:

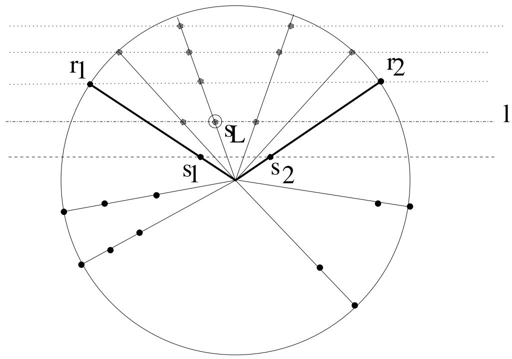


Fig. 4. An example showing the construction in the proof of Lemma 7

1. $\sharp\mathcal{R}_L > 2$. Then, $\forall r \in \mathcal{R}$ and for any orientation $\circ$ of SEC in $\{\circlearrowright, \circlearrowleft\}$, $W(r)^\circ = (0,0)$. Thus, there exists one sensor s located at O. The n sensors are then able to agree on s.
2. $\mathcal{R}_L = \{r\}$. If $W(r)$ the leader is the nearest sensor to O, on r.
3. $\mathcal{R}_L = \{r_1, r_2\}$. Again, there are two subcases :
   (a) $W(r_1)^\circlearrowright = (0,0)$. The leader is the sensor at the center of SEC.
   (b) $W(r_1)^\circlearrowright \neq (0,0)$. From Lemma 5 again, if $W(r_1)^\circlearrowright$ (respectively $W(r_1)^\circlearrowleft$) is a Lyndon word, then $W(r_2)^\circlearrowleft$ (resp. $W(r_2)^\circlearrowright$) is a Lyndon word. Without loss of generality, assume that $W(r_1)^\circlearrowright$ and $W(r_2)^\circlearrowleft$ are Lyndon words. We have two subsubcases :
      i. $W(r_1)^\circlearrowright \neq W(r_2)^\circlearrowleft$. Without loss of generality again, assume that $W(r_1)^\circlearrowright \prec W(r_2)^\circlearrowleft$. The leader is the nearest sensor to O on r_1.
      ii. $W(r_1)^\circlearrowright = W(r_2)^\circlearrowleft$. In that case, note that r_1 and r_2 divide SEC into two parts, π_1 and π_2, where n_1 and n_2 are the number of robots inside π_1 and π_2, respectively. Since $W(r_1)^\circlearrowright = W(r_2)^\circlearrowleft$, the number x of sensors located on r_1 is equal to the number of robots located on r_2. So, the total number of sensors located on r_1 and r_2 is equal to $2x$ (because there is no sensor at the center of SEC). Thus, $n_1 + n_2 = n - 2x$ because no sensors located on r_1 and r_2 is in π_1 or π_2. Since $n - 2x$ is odd (n is odd), there exists one part of SEC with an even number of robots, and one part of SEC with an odd number of sensors. Without loss of generality, assume that n_1 is odd. Let s_1 and s_2 be the nearest sensors to O on r_1 and r_2, respectively. Consider P the set of lines passing through the sensors in π_1 which are parallel to the line (s_1, s_2) — refer to Figure 4. Since n_1 is odd, there exists at least one line in P with an odd number of sensors located on it. Among those lines, choose the unique line which is the nearest from both O and the line (s_1, s_2). Denote this line by l and the number of sensors located on it in π_1 by n_l. Therefore, the leader is the unique sensor s_L which is the median sensor among the sensor on l and π_1, i.e., the $(\lfloor \frac{n_l}{2} \rfloor + 1)^{\text{th}}$ sensor starting indifferently from the left or the right of l in π_1.

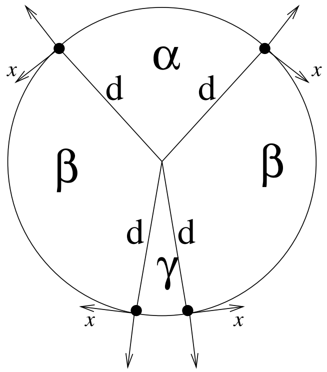


Fig. 5. A counter example showing that the statement of Theorem 2 does not hold if n is even

Lemma 8. *If there exists no radius $r \in \mathcal{R}$ such that either $W(r)^{\circlearrowright}$ or $W(r)^{\circlearrowleft}$ is a Lyndon word, then there exists no algorithm allowing the n sensors to deterministically agree on the same sensor L.*

Proof. The proof is similar as to that of Lemma 4.

The following theorem follows from Lemmas 7 and 8:

Theorem 2. *Given a configuration π of any number $n \geq 2$ sensors without chirality scattered on the plane, the n sensors are able to determiniscally agree on the same sensor L if and only if n is odd and there exists a radius $r \in \mathcal{R}$ such that $W(r)$ is a Lyndon Word.*

Note that the equivalence does not work with an even number of sensors. A counter example is shown in Figure 5. For any orientation in $\{\circlearrowright, \circlearrowleft\}$, there exists one Lyndon word equal to $(d, \alpha)(d, \beta)(d, \gamma)(d, \beta)$. However, the symetry of the configuration does not allow to choose any sensor as a leader.

4 Conclusion

We studied the leader election problem in networks of anonymous sensors sharing no kind of common coordinate system. Assuming anonymous sensors with chirality, we used properties of Lyndon words to give a complete characterization on the sensors positions to deterministically elect a leader for any number $n > 1$ of sensors. We also showed that our characterization still holds with sensors without chirality if and only if the number of sensors is odd.

Our future work will concentrate to find a similar characterization for an even number of sensors without chirality. A more general problem is to find the minimal geometrical conditions to determiniscally solve other collaborative tasks in mobile sensor networks such as pattern formation for which we know that no solution exists in general if the sensors do not agree on a sense of direction [15].

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