3

Theoretical Prices of European Interest-Rate Derivatives

3.1 Overview

In this section, we want to give a representative selection of different interest-rate contracts for which the pricing framework used in this thesis is able to produce semi closed-form solutions[48]. In doing this we distinguish, for didactical purposes, between contracts based on the short rate $r(\mathbf{x}_t)$ and contracts based on a simple yield $Y(\mathbf{x}_t, t, T)$ over a specified time period τ. These yields to maturity are often referred to as simple compound rates, e.g. LIBOR rates, and denote the constant compounding of wealth over a fixed period of time τ, which is related to a zero bond with corresponding time to maturity.

Definition 3.1.1 (Simply-Compounded Yield to Maturity). *The simple yield to maturity $Y(\mathbf{x}_t, t, T)$ of a zero bond $P(\mathbf{x}_t, t, T)$, maturing after the time period τ, is defined through the equality*

$$\frac{1}{1 + \tau Y(\mathbf{x}_t, t, T)} = P(\mathbf{x}_t, t, T). \tag{3.1}$$

Therefore the simple yield to maturity can be derived as

$$Y(\mathbf{x}_t, t, T) = \frac{P(\mathbf{x}_t, t, T)^{-1} - 1}{\tau} = \frac{1 - P(\mathbf{x}_t, t, T)}{\tau P(\mathbf{x}_t, t, T)}. \tag{3.2}$$

In the following sections, we generally distinguish in the derivation of theoretical prices of contingent claims between contracts based on the instantaneous interest rate $r(\mathbf{x}_t)$ and contracts depending on the simple yield

[48] A comprehensive summary of different valuation formulae of fixed-income securities is given, e.g. Brigo and Mercurio (2001) and Musiela and Rutkowski (2005).

$Y(\mathbf{x}_t, t, T)$. Moreover, we differentiate between contracts with unconditional and conditional exercise rights. This distinction is introduced because of the different mathematical derivation of the particular model prices. For contracts with unconditional exercise, we obtain pricing formulae, which bear strong resemblance to moment-generating functions of the particular underlying state process whereas contracts with conditional exercise rights, i.e. option contracts, need an explicit integration due to the natural exercise boundary. All derivative prices for which we derive the corresponding pricing formulae are European-style derivatives, meaning that the exercise can only be performed at maturity T.

3.2 Derivatives with Unconditional Payoff Functions

This derivatives class is characterized by the trivial exercise of the contract at maturity. This means that the contract is always exercised, no matter if the holder suffers a loss or make a profit as consequence of the exercise. Although trivially exercised, a zero-coupon bond is a special case of this class since it pays at maturity a predefined *riskless* quantity of monetary units.

Definition 3.2.1 (Zero-Coupon Bond). *A zero-coupon bond maturing at time T guarantees its holder the payment of one monetary unit at maturity. The value of this contract at $t < T$ is then denoted as $P(\mathbf{x}_t, t, T)$, which is the expected value of the discounted terminal condition $G(x_T) = 1$. This can be formally expressed as,*

$$P(\mathbf{x}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\right] \tag{3.3}$$

It is easily seen that the payoff function $G(\mathbf{x}_T)$ used in equation (3.3) is independent both of the time variable and the state variables in $\mathbf{x}_T$. Using the formal definition in equation (3.3), a zero-coupon bond, or as shorthand a zero bond, is just the present value of one monetary unit paid at time T. Hence, we are able to interpret $P(\mathbf{x}_t, t, T)$ as the expected discount factor relevant for the time period t up to T. Due to this intuitive interpretation, these contracts are often used in calibrating interest-rate models to empirical data sets.

A slightly more elaborated contract is given by the combination of *certain* payments at different times. We denote this contract then as a coupon-bearing bond.

Definition 3.2.2 (Coupon-Bearing Bond). *A coupon-bearing bond guarantees its holder a number of A deterministic payments $c_a \in \mathbf{c}$ at specific coupon dates $T_a \in \mathbf{T}$ for $a = 1, \ldots, A$. Typically, at maturity T_A, a nominal face value C is included in c_A in addition to the ordinary coupon. The present value of a coupon bond $CB(\mathbf{x}_t, \mathbf{c}, t, \mathbf{T})$ is then given as*

$$CB(\mathbf{x}_t, \mathbf{c}, t, \mathbf{T}) = \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\, \mathrm{d}s} c_a \right] = \sum_{a=1}^{A} P\left(\mathbf{x}_t, t, T_a\right) c_a . \tag{3.4}$$

Obviously, a coupon-bearing bond, or as shorthand a coupon bond, is just the cumulation of payments c_a discounted with the particular zero-bond prices $P\left(\mathbf{x}_t, t, T_a\right)$.

If a firm is requiring a hedge position for a risk exposure in the form of a future payment of interest, due to an uncertain floating interest rate, we are able to conclude a forward-rate agreement.

Definition 3.2.3 (Forward-Rate Agreement). *A forward-rate agreement concluded in time t guarantees its holder the right to exchange his variable interest payments to a fixed rate K, scaled upon a notional principal Nom. The contract is sold in t. The interest payments exchanged relate then to the time period, say $[T, \hat{T}]$ with $t < T < \hat{T}$. We distinguish the cases, where the forward-rate agreement refers to the short rate $r\left(\mathbf{x}_t\right)$ and to the yield $Y\left(\mathbf{x}_t, t, T\right)$. Hence, for a contract based on the short rate, the relevant time interval is then $[T, \hat{T}] = [T, T + dT]$. The price of this contract is given as*

$$\begin{aligned} &FRA_r(\mathbf{x}_t, K, Nom, t, T) \\ &\quad = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{T} r_s\, \mathrm{d}s} \left(K - r\left(\mathbf{x}_T\right)\right) \right] Nom \\ &\quad = \left(K\, P\left(\mathbf{x}_t, t, T\right) - \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{T} r_s\, \mathrm{d}s} r\left(\mathbf{x}_T\right) \right] \right) Nom. \end{aligned} \tag{3.5}$$

The price for a forward-rate agreement over a discrete time period of length $\hat{\tau} = \hat{T} - T$*, written on a yield* $Y\left(\mathbf{x}_T, T, \hat{T}\right)$ *and paid in arrears, can be represented as*[49]

$$\begin{aligned} FRA_Y(\mathbf{x}_t, K, Nom, t, T, \hat{T}) &= \hat{\tau}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - Y\left(\mathbf{x}_T, T, \hat{T}\right)\right)\right] Nom \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\left(\hat{\tau}K - P\left(\mathbf{x}_T, T, \hat{T}\right)^{-1} + 1\right)\right] Nom \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T} r(\mathbf{x}_s)\,\mathrm{d}s}\left(P\left(\mathbf{x}_T, T, \hat{T}\right)(\hat{\tau}K + 1) - 1\right)\right] Nom \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T} r(\mathbf{x}_s)\,\mathrm{d}s}\left(P\left(\mathbf{x}_T, T, \hat{T}\right) - \tilde{K}\right)\right] \frac{Nom}{\tilde{K}} \\ &= \left(P\left(\mathbf{x}_t, t, \hat{T}\right) - \tilde{K}P\left(\mathbf{x}_t, t, T\right)\right)\frac{Nom}{\tilde{K}}, \end{aligned} \tag{3.6}$$

with $\tilde{K} = \frac{1}{\hat{\tau}K+1}$.

To give a more illustrative example, we consider a firm, which has to make a future payment subject to an uncertain, floating rate of interest. Reducing the immanent interest-rate risk exposure, this firm wants to transform this payment into a certain cash-flow, locked at a fixed rate K. This can be achieved by contracting a forward-rate agreement, therefore exchanging the floating interest rate to the fixed rate K. Thus, the firm is, in its future calculation, independent of the evolution of the term structure.

[49] Here we use the fact that the exponential-affine model exhibits the Markov ability. Thus, the expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\right] = P\left(\mathbf{x}_t, t, \hat{T}\right)$ can be represented as the iterated expectation $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T} r(\mathbf{x}_s)\,\mathrm{d}s}\mathbb{E}^{\mathbb{Q}_T}\left[e^{-\int_T^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\right]\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T} r(\mathbf{x}_s)\,\mathrm{d}s}P\left(\mathbf{x}_T, T, \hat{T}\right)\right]$, where the inner expectation is made with respect to time T.

Another point, we want to mention is the special strike value $K = K_{FRA}$ for which the yield-based forward-rate agreement becomes a fair zero value at time t. This value is commonly referred to as the forward rate and corresponds then to the simply-compounded rate

$$\begin{aligned} K_{FRA} &= \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\hat{T}} r(\mathbf{x}_s)\,\mathrm{d}s}\left(P\left(\mathbf{x}_T, T, \hat{T}\right)^{-1} - 1\right)\right]}{\hat{\tau} P\left(\mathbf{x}_t, t, \hat{T}\right)} \\ &= \frac{P\left(\mathbf{x}_t, t, T\right) - P\left(\mathbf{x}_t, t, \hat{T}\right)}{\hat{\tau} P\left(\mathbf{x}_t, t, \hat{T}\right)} \\ &= \frac{1}{\hat{\tau}}\left(\frac{P\left(\mathbf{x}_t, t, T\right)}{P\left(\mathbf{x}_t, t, \hat{T}\right)} - 1\right). \end{aligned} \tag{3.7}$$

Most of the time a firm does not want to insure itself against a floating interest payment for only one time period. For example, the firm has to serve a debt contract, which is linked to a LIBOR interest rate. In this case, the firm possibly wants to reduce its risk exposure due to the floating interest accrues over time and it is desired to make an exchange of interest payments for several successive time periods, where in each period the payment for the relevant floating rate is exchanged with a fixed rate K. This task can be achieved buying a receiver swap contract.

Definition 3.2.4 (Swap). *A forward-starting interest-rate receiver swap is defined as a portfolio of forward-rate agreements for different time periods $T_{a+1} - T_a$ with $T_a \in \mathbf{T}$ and $t < T_a$ for $a = 1, \ldots, A$ on the same strike rate K. The payments of the contract are made at dates $T_2, \ldots, T_A$, whereas the contract is said to reset the floating rate at dates $T_1, \ldots, T_{A-1}$.*

Due to the instantaneous character of the floating rate based swap contract, the payment and reset dates coincide. Hence, the swap contract in this case, with nominal principal Nom and A payment dates contained in the vector $\mathbf{T}$, can be represented as

$$
\begin{aligned}
SWA_r(\mathbf{x}_t, K, Nom, t, \mathbf{T}) \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{a=1}^{A} e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - r\left(\mathbf{x}_{T_a}\right)\right)\right] Nom \\
&= Nom \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - r\left(\mathbf{x}_{T_a}\right)\right)\right] \\
&= Nom \left(K \sum_{a=1}^{A} P\left(\mathbf{x}_t, t, T_a\right)\right. \\
&\qquad \left. - \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} r\left(\mathbf{x}_{T_a}\right)\right]\right).
\end{aligned}
\tag{3.8}
$$

The equivalent representation for a swap contract, exchanging a yield-based floating rate at $A-1$ payment dates paid in-arrears is then

$$
\begin{aligned}
SWA_Y(\mathbf{x}_t, K, Nom, t, \mathbf{T}) \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{a=1}^{A-1} e^{-\int_t^{T_{a+1}} r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - Y\left(\mathbf{x}_{T_a}, T_a, T_{a+1}\right)\right) \hat{\tau}_{a+1}\right] Nom \\
&= Nom \times \\
&\quad \sum_{a=1}^{A-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left((K\hat{\tau}_{a+1} + 1) P\left(\mathbf{x}_{T_a}, T_a, T_{a+1}\right) - 1\right)\right] \\
&= Nom \sum_{a=1}^{A-1} \big((K\hat{\tau}_{a+1} + 1)\, P\left(\mathbf{x}_t, t, T_{a+1}\right) - P\left(\mathbf{x}_t, t, T_a\right)\big) \\
&= Nom \left(P\left(\mathbf{x}_t, t, T_A\right) - P\left(\mathbf{x}_t, t, T_1\right)\right. \\
&\qquad \left. + K \sum_{a=1}^{A-1} \hat{\tau}_{a+1} P\left(\mathbf{x}_t, t, T_{a+1}\right)\right),
\end{aligned}
\tag{3.9}
$$

with $\hat{\tau}_{a+1} = T_{a+1} - T_a$.

In contrast to the total number of A swap payments in equation (3.8), where these payments refer merely to specific time dates, for the yield-based swap contracts we have to consider $A-1$ time periods, which explains the resulting summation term in equation (3.9). Subsequently, a swap contract can be interpreted as the sum of successive forward-rate agreements.

Similar to forward-rate agreements we are able to introduce the terminology of a special strike K_S, which makes the yield-based swap contract a fair zero valued contract. This special strike is then denoted as the swap rate and can be represented in the case of a yield-based swap as

$$\begin{aligned} K_S &= \frac{\sum_{a=1}^{A-1}\left(P\left(\mathbf{x}_t, t, T_a\right) - P\left(\mathbf{x}_t, t, T_{a+1}\right)\right)}{\sum_{a=1}^{A-1}\hat{\tau}_{a+1} P\left(\mathbf{x}_t, t, T_{a+1}\right)} \\ &= \frac{P\left(\mathbf{x}_t, t, T_1\right) - P\left(\mathbf{x}_t, t, T_A\right)}{\sum_{a=1}^{A-1}\hat{\tau}_{a+1} P\left(\mathbf{x}_t, t, T_{a+1}\right)} . \end{aligned} \tag{3.10}$$

The last contract with unconditional exercise right which we include in the pricing methodology used is an Asian-type average-rate contract based on the floating rate $r\left(\mathbf{x}_t\right)$. These contracts do not belong to the class of traded derivatives in any exchange. However, this type of interest-rate derivative seems to be quite popular in over-the-counter markets[50]. Asian contracts belong to the field of path-dependent derivatives. Thus, the payoff consists not only of the terminal value of the underlying rate at maturity but of the complete sample path over the averaging period.

Definition 3.2.5 (Unconditional Average-Rate Contract). *An unconditional average-rate agreement concluded in time t guarantees its holder the right at maturity T to exchange the continuously measured average of the floating rate $r\left(\mathbf{x}_t\right)$ over the period $T-t$ against a fixed strike rate K. The value of this difference is then scaled by a nominal principal Nom. Hence, the price of this contract is given as*

$$\begin{aligned} &UARC_r\left(\mathbf{x}_t, K, Nom, t, T\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - \frac{1}{T-t}\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s\right)\right] Nom \\ &= Nom\left(P\left(\mathbf{x}_t, t, T\right) K - \frac{1}{\tau}\,\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s\right]\right) . \end{aligned} \tag{3.11}$$

Consequently, in contrast to the forward-rate agreement according to equation (3.5), where the sole expectation of $r(\mathbf{x}_T)$ played the major part, we are

[50] See Ju (1997).

interested in the discounted expectation of the integral of $r(\mathbf{x}_t)$ over the time to maturity at this point.

3.3 Derivatives with Conditional Payoff Functions

In the last subsection, we considered the pricing formulae for contracts with unconditional exercise at maturity under the risk-neutral measure $\mathbb{Q}$. Obviously, these contracts can be expressed e.g. in terms of zero bonds and some constants. In this section we want to derive general pricing formulae for contracts with conditional or optional exercise rights at maturity. These derivatives contracts are therefore often referred to as option contracts. Basically, we are interested in calculating the particular option prices with underlying contracts of the form (3.5), (3.6), and (3.9) with optional exercise rights. Basically, the particular pricing formulae can be separated into zero bond and coupon-bond options, respectively, can be seen as a portfolio of several zero-bond options in case of a yield-based swap contract. Hence, we begin the introduction with option contracts written on a zero bond.

Definition 3.3.1 (Zero Bond Option). *We define a zero-bond call (put) option as a contract giving its holder the right, not the obligation, to buy (sell) a zero bond $P\left(\mathbf{x}_t, t, \hat{T}\right)$ for a strike price K at time T. The remaining time to maturity of this zero bond at the exercise date of the option is then given as $\hat{\tau}$. Formally, the price of a zero-bond call can be obtained as*

$$\begin{aligned} &ZBC\left(\mathbf{x}_t, K, t, T, \hat{T}\right) \\ &\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \max\left(P\left(\mathbf{x}_T, T, \hat{T}\right) - K, 0\right)\right] \\ &\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \left(P\left(\mathbf{x}_T, T, \hat{T}\right) - K\right)^+\right], \end{aligned} \tag{3.12}$$

whereas a zero-bond put option can be calculated as

$$ZBP\left(\mathbf{x}_t, K, t, T, \hat{T}\right) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - P\left(\mathbf{x}_T, T, \hat{T}\right)\right)^+\right]. \tag{3.13}$$

Zero bond options can be used to price two contracts commonly used to hedge interest-rate risk. Namely, we want to introduce cap and floor contracts. In this terminology, a cap contract is meant to hedge upside interest-rate risk exposure. This is often required for a firm which holds some debt position with interest payments on a floating rate base and fears that future interest rates are rising. So it wants the interest rate capped at some fixed level, in order to limit its risk position due to this fixed rate. In contrast to the above introduced forward-rate agreement or swap, a firm can now both participate on advantageously low interest rates and simultaneously cap its interest payments against high rates. The opposite effect can be observed, if an institution or firm has outstanding loans based on a floating rate. In this case the firm is interested in limiting the downside risk, since low floating rates correspond to low interest payments. The contract with the desired properties is then a floor, where interest payments are exchanged under an agreed fixed rate.

Definition 3.3.2 (Cap and Floor Contract). *A cap (floor) contract is defined as a portfolio of caplets (floorlets) for different time periods $T_{a+1} - T_a$ with $T_a \in \mathbf{T}$ and $t < T_a$ for $a = 1, \dots, A$ on the same strike rate K. The payments of the contract are made at dates $T_2, \dots, T_A$, whereas the contract is said to reset the floating rate at dates $T_1, \dots, T_{A-1}$.*

Due to the short rate, the character of the floating rate based swap contract, the payment and reset dates coincide. Hence, the model price of a caplet with nominal principal Nom and A payment dates contained within the vector $\mathbf{T}$, is then given by

$$CPL_r\left(\mathbf{x}_t, K, Nom, t, T_a\right) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s}\left(r\left(\mathbf{x}_{T_a}\right) - K\right)^+\right] Nom. \quad (3.14)$$

The price of a cap contract, as a simple summation of caplets for different times $T_a \in \mathbf{T}$, can then be represented as

$$\begin{aligned} &CAP_r\left(\mathbf{x}_t, K, Nom, t, \mathbf{T}\right) \\ &\quad = \sum_{a=1}^{A} CPL_r(\mathbf{x}_t, K, Nom, t, T_a) \\ &\quad = Nom \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s}\left(r\left(\mathbf{x}_{T_a}\right) - K\right)^+\right]. \end{aligned} \quad (3.15)$$

Subsequently, we have for a floor the pricing formula

$$FLR_r\left(\mathbf{x}_t, K, Nom, t, \mathbf{T}\right) = Nom \sum_{a=1}^{A} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - r\left(\mathbf{x}_{T_a}\right)\right)^+\right]. \tag{3.16}$$

The particular yield-based cap and floor options, exchanging, if exercised, arbitrary yields with a fixed rate K at $A-1$ payment dates, are given by

$$\begin{aligned} CAP_Y\left(\mathbf{x}_t, K, Nom, t, \mathbf{T}\right) &= \sum_{a=1}^{A-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left(\tilde{K}_a - P\left(\mathbf{x}_{T_a}, T_a, T_{a+1}\right)\right)^+\right] \frac{Nom}{\tilde{K}_a} \\ &= \sum_{a=1}^{A-1} ZBP\left(\mathbf{x}_t, \tilde{K}_a, t, T_a, T_{a+1}\right) \frac{Nom}{\tilde{K}_a}, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} FLR_Y\left(\mathbf{x}_t, K, Nom, t, \mathbf{T}\right) &= \sum_{a=1}^{A-1} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_a} r(\mathbf{x}_s)\,\mathrm{d}s} \left(P\left(\mathbf{x}_{T_a}, T_a, T_{a+1}\right) - \tilde{K}_a\right)^+\right] \frac{Nom}{\tilde{K}_a} \\ &= \sum_{a=1}^{A-1} ZBC\left(\mathbf{x}_t, \tilde{K}_a, t, T_a, T_{a+1}\right) \frac{Nom}{\tilde{K}_a}, \end{aligned} \tag{3.18}$$

with $\tilde{K}_a = \frac{1}{\hat{\tau}_{a+1} K + 1}$.

Definition 3.3.2 shows that a cap or floor contract is just the summation of their legs, the caplets and floorlets, respectively. Especially for the more realistic case of yield-based contracts, we can identify the similarity to zero-bond options, since contract prices can be obtained as the summation of these options.

The yield-based options are said to be at the money if the modified strike rate $\bar{K}_a$ is equal to equation (3.10). A cap is therefore in the money if the modified strike rate is less than K_S, and for $\bar{K}_a > K_S$ it is out of the money. The opposite results hold for a floor contract. Furthermore, we can conclude that holding a cap contract long and a floor contract short, both with the same contract specifications, we are able to replicate a swap contract. This can be easily justified comparing the payoff of such a portfolio given for a yield $Y(\hat{\tau}_{a+1})$, which is then

$$\begin{aligned}(Y(\mathbf{x}_{T_a}, T_a, T_{a+1}) - K)^+ - (K - Y(\mathbf{x}_{T_a}, T_a, T_{a+1}))^+ \\ = Y(\mathbf{x}_{T_a}, T_a, T_{a+1}) - K,\end{aligned} \tag{3.19}$$

and the corresponding swap payment. Taking the discounted expectation of the sum of terms in equation (3.19) for all periods, we have the equivalent swap contract.

A more challenging contract in calculating model prices is a coupon-bond option. This option is only exercised if the coupon-bond price at maturity exceeds the strike K. Hence, we have to apply the maximum operator to the discounted sum of all outstanding coupon payments and the strike price. This is in contrast to the other option contracts mentioned above, where we applied the maximum operator to each term of the sum separately.

Definition 3.3.3 (Coupon-Bond Option). *A coupon-bond call (put) option is defined as the right but not the obligation to buy (sell) a coupon bond $CB(\mathbf{x}_T, \mathbf{c}, t, \mathbf{T})$ with payment dates $T_a \in \mathbf{T}$, with $T_a > T$ for $a = 1, \ldots, A$ and strike price K. The price of a coupon-bond call option is given by*

$$\begin{aligned}CBC(\mathbf{x}_t, \mathbf{c}, K, t, T, \mathbf{T}) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(CB(\mathbf{x}_T, \mathbf{c}, T, \mathbf{T}) - K\right)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(\sum_{a=1}^{A} P(\mathbf{x}_T, T, T_a)\, c_a - K\right)^+\right],\end{aligned} \tag{3.20}$$

and the corresponding coupon-bond put option is given by

$$\begin{aligned}CBP(\mathbf{x}_t, \mathbf{c}, K, t, T, \mathbf{T}) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - CB(\mathbf{x}_T, \mathbf{c}, T, \mathbf{T})\right)^+\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s}\left(K - \sum_{a=1}^{A} P(\mathbf{x}_T, T, T_a) c_a\right)^+\right].\end{aligned} \tag{3.21}$$

Since the maximum operator is not distributive with respect to sums, the term inside the maximum operator in equation (3.20) and (3.21) cannot be decomposed easily without making further assumptions. Another popular option we want to discuss is an option on a swap contract or as shorthand often referred to as a swaption. With a swaption one can choose at the maturity of the option if it is advantageous to enter the underlying swap contract or otherwise leave the option unexercised.

Definition 3.3.4 (Swaption). *We define a forward-starting swaption as a contract conferring the right, but not the obligation to enter a forward starting receiver swap at maturity T. The particular underlying receiver swap contract is defined according to definition 3.2.4, with $T_1 \geq T$. Formally, the yield-based forward-starting receiver swaption for an underlying swap with $A-1$ payment periods is given as*

$$\begin{aligned} &SWP_Y\left(\mathbf{x}_t, K, Nom, t, T, \mathbf{T}\right) \\ &\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \left(SWA_Y(\mathbf{x}_T, K, Nom, T, \mathbf{T})\right)^+ \right] \\ &\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \left(K \left(\sum_{a=1}^{A-1} P(\mathbf{x}_T, T, T_{a+1})\hat{\tau}_{a+1} \right) \right.\right. \\ &\qquad \left.\left. + P(\mathbf{x}_T, T, T_A) - P(\mathbf{x}_T, T, T_1) \right)^+ \right] Nom. \end{aligned} \tag{3.22}$$

Typically, the swaption maturity coincides with the first reset date of the underlying swap contract. Thus, a yield-based receiver swaption with $T_1 = T$, can be equivalently represented as a coupon-bond call option

$$SWP_Y\left(\mathbf{x}_t, K, Nom, t, T_1, \mathbf{T}^*\right) = CBC\left(\mathbf{x}_t, \mathbf{c}_{SWP}, 1, t, T_1, \mathbf{T}^*\right), \tag{3.23}$$

with

$$\mathbf{c}_{SWP} = \begin{pmatrix} K\hat{\tau}_2 \\ K\hat{\tau}_3 \\ \vdots \\ 1 + K\hat{\tau}_A \end{pmatrix} \times Nom,$$

and new time dates

$$\mathbf{T}^* = \begin{pmatrix} T_2 \\ T_3 \\ \vdots \\ T_A \end{pmatrix}.$$

Subsequently, we reduce the valuation problem of a swaption to the calculation of an equivalent coupon-bond option with strike one, a coupon vector $\mathbf{c}_{SWP}$ and a vector with payment dates $\mathbf{T}^*$.

According to the unconditional contract defined in equation (3.11), we are also able to price an average-rate option contract. The definition of the model price of an average-rate option is given below.

Definition 3.3.5 (Average-Rate Option). *An average-rate cap option gives its holder the right, but not the obligation to exchange at expiration a fixed strike rate K, over the period $T-t$, against the continuously measured average of the short rate $r(\mathbf{x}_t)$. Formally, the price of an average-rate cap option can be obtained as*

$$\begin{aligned} & ARC_r(\mathbf{x}_t, K, Nom, t, T) \\ =& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \left(\frac{1}{\tau} \int_t^T r(\mathbf{x}_s)\,\mathrm{d}s - K \right)^+ \right] Nom. \end{aligned} \tag{3.24}$$

Consequently, we have for an average-rate floor the pricing formula

$$\begin{aligned} & ARF_r(\mathbf{x}_t, K, Nom, t, T) \\ =& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_s)\,\mathrm{d}s} \left(K - \frac{1}{\tau} \int_t^T r(\mathbf{x}_s)\,\mathrm{d}s \right)^+ \right] Nom. \end{aligned} \tag{3.25}$$

Asian options show the advantageous ability to exhibit reduced risk positions in comparison to ordinary options because of the time-averaging of the underlying price process. Moreover, asian option contracts are more robust against price manipulations since the option payoff includes the sample path over a finite time period. These options are not standard instruments traded on exchanges. However, they are popular over-the-counter contracts used by banks and corporations to hedge their interest-rate risk over a time period[51].

For all theoretical option prices presented in this section, we give in Section 5.3 the corresponding pricing formulae which have to be used in a numerical

[51] See, for example, Ju (1997).

scheme. Thus, we distinguish between the calculation of a portfolio of options, e.g. used for the pricing of cap and floor contracts and as a special case for zero-bond options, respectively, and the computation of options on a portfolio which is the case for coupon-bond options and swaption contracts. This is done because only in case of a one-factor interest-rate process semi closed-form solutions for swaptions and coupon bonds can be calculated.