10

Non-Affine Term-Structure Models and Short-Rate Models with Stochastic Jump Intensity

10.1 Overview

Although the model setup proposed in this thesis is of the exponential-affine class, we can also extend the framework to allow for certain non-affine models and models with state-dependent jump intensities $\boldsymbol{\lambda}^{\mathbb{Q}}(\mathbf{x}_t)$. Moreover, option prices under these more sophisticated model dynamics can be priced in our numerical scheme without greater effort, due to an exponential separable structure of the governing characteristic function. However, working with a non-affine model, we have to abandon jump components for those particular non-affine factors. A stochastic jump intensity in the general exponential-affine model framework is introduced in Duffie, Pan and Singleton (2000). Consequently, the jump transform is no longer independent of the coefficient function $a(z,\tau)$, and therefore a complicated system of ODEs has to be determined numerically anyway. Since both approaches need to establish further restrictions, they are only discussed as possibilities for extending and modifying the base model, respectively.

10.2 Quadratic Gaussian Models

Non-Affine exponential separable models are characterized by a non-affine structure of the factors in the relevant moment-generating function, as well as the general characteristic function, while preserving the separability of coefficient functions for different powers of the particular factors included in the model. Thus, the essential system of ODEs can be derived. Prominent representatives of this model class are in an equity context the stochastic volatility model of Schöbel and Zhu (1999), which is a generalized version of the Stein and Stein (1991) model. In case of interest rates, we have, e.g. the Double Square-Root model of Longstaff (1989), the quadratic Gaussian model approach of Beaglehole and Tenney (1991)[203], and the general linear-quadratic jump-diffusion model of Cheng and Scaillet (2004)[204].

Although the quadratic Gaussian and the Double Square-Root model seem quite attractive to implement, it is impossible to compute theoretical model prices within the Fourier-based pricing framework if jumps are incorporated, while Monte-Carlo pricing approaches might still work. This stems from the fact that in equation (2.39), for the nth jump J_{mn} in the non-affine factor $x_t^{(m)}$, there would be a corresponding term $(x_t^{(m)} + J_{mn})^2$ resulting in a mixed expression. Hence, the exponential separation approach will no longer be available in deriving the general characteristic function. Since none of the non-affine interest-rate models are capable of exhibiting any jump component we completely ignored these models in our base setup according to Section 2.1.

The one-factor quadratic Gaussian approach models the short rate under the risk-neutral measure, as the square of some factor x_t governed by an Ornstein-Uhlenbeck process according to equation (8.5). In order to price interest-rate derivatives for this particular process, we need to have the general characteristic function to consider both the state variable x_t and its square x_t^2. Thus, for the squared Gaussian interest-rate model we use the following form of the general characteristic function

$$\psi\left(\mathbf{y}_t, z, 0, \mathbf{w}, g_0, \mathbf{g}, \tau\right) = e^{a(z,\tau)+\mathbf{b}(z,\tau)'\mathbf{y}_t + \imath z g_0},$$

with

$$\mathbf{y}_t = \begin{pmatrix} x_t \\ x_t^2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For convenience, we use again the time-dependent coefficient functions[205]

[203] Ahn, Dittmar and Gallant (2002) give a good overview of general multi-dimensional linear-quadratic Gaussian interest-rate models.

[204] Linear-quadratic in this context means all factors contained in the state vector $\mathbf{x}_t$ are allowed to enter the interest rate both in a linear and quadratic fashion.

[205] The constant parameter g_0 is represented by the term $\bar{A}$.

$$a(z,\tau) = A(z,\tau),$$

and

$$\mathbf{b}(z,\tau) = \begin{pmatrix} B(z,\tau) \\ C(z,\tau) \end{pmatrix} + \imath z \begin{pmatrix} \bar{B} \\ \bar{C} \end{pmatrix}.$$

Inserting the above characteristic function in equation (2.33) and applying the separation approach result again in a system of coupled ODEs[206]

$$\begin{aligned} A(z,\tau)_\tau =& \kappa\theta(B(z,\tau) + \imath z\bar{B}) + \sigma^2(C(z,\tau) + \imath z\bar{C}) \\ &+ 2\sigma^2(B(z,\tau) + \imath z\bar{B})^2, \\ B(z,\tau)_\tau =& (B(z,\tau) + \imath z\bar{B})(\sigma^2(C(z,\tau) + \imath z\bar{C}) - \kappa) \\ &+ 2\kappa\theta(C(z,\tau) + \imath z\bar{C}), \\ C(z,\tau)_\tau =& -2\kappa(C(z,\tau) + \imath z\bar{C}) + 2\sigma^2(C(z,\tau) + \imath z\bar{C})^2 - 1, \end{aligned}$$

which can be solved successively. The advantage of this modeling approach lies in its tractability while describing a more elaborated interest-rate behavior. Additionally, the short rate in this approach is always positive, compared to possible negative short rates using the Vasicek model. In the Double Square-Root model according to Longstaff (1989), we encounter a very similar situation, since we are able to transform the model into a quadratic Gaussian model and vice versa but with additional restrictions on the parameter set[207].

Cheng and Scaillet (2004) introduce a linear-quadratic jump-diffusion model. Here, the diffusion part of some random variable, for example the short rate $r(\mathbf{x}_t)$ or the payoff characteristic function $g(\mathbf{x}_t)$, is built similarly to the multivariate quadratic Gaussian model in Beaglehole and Tenney (1991), as the sum of linear and quadratic terms of the state vector $\mathbf{x}_t$ containing correlated Ornstein-Uhlenbeck processes. To gain a closed-form solution for the general characteristic function, additional jump parts only occur in the affine terms of $\mathbf{x}_t$. Therefore, we can think of this interest-rate model as a simple combination of an additive multivariate Ornstein-Uhlenbeck model augmented with jump components and an additive multivariate quadratic Gaussian model.

[206] Although the vector $\mathbf{y}_t$ occurs in the characteristic function, derivatives remain still to be taken with respect to the unique state variables which is in this one-dimensional model just the factor x_t.

[207] See Beaglehole and Tenney (1992), pp. 346-347.

10.3 Stochastic Jump Intensity

Another possibility for extending the base model setup stated in Section 2.1 is to implement stochastic jump intensities. Duffie, Pan and Singleton (2000) introduced, with their affine jump-diffusion model, a vector of stochastic jump intensities where the stochastic component is affine in the state variable $\mathbf{x}_t$. Thus, they implement stochastic intensities without overly aggravating their solution technique. Defining the vector of jump intensities as[208]

$$\boldsymbol{\lambda}^{\mathbb{Q}}(\mathbf{x}_t) = \boldsymbol{\lambda}_0^{\mathbb{Q}} + \boldsymbol{\lambda}_1^{\mathbb{Q}} \mathbf{x}_t,$$

with $(\boldsymbol{\lambda}_0^{\mathbb{Q}}, \boldsymbol{\lambda}_1^{\mathbb{Q}}) \in \mathbb{R}^M \times \mathbb{R}^{M \times M}$, we therefore get a slightly modified system of ODEs for the vector coefficient $\mathbf{b}(z, \tau)$ compared to equation (2.41), which is

$$\begin{aligned} \mathbf{b}(z,\tau)_\tau = & - \mathbf{w} + \boldsymbol{\mu}_1^{\mathbb{Q}\prime} \mathbf{b}(z,\tau) + \frac{1}{2}\, \mathbf{b}(z,\tau)' \boldsymbol{\Sigma}_1 \mathbf{b}(z,\tau) \\ & + \boldsymbol{\lambda}_1^{\mathbb{Q}} \mathbb{E}_{\mathbf{J}} \left[\boldsymbol{\psi}^*(z, w_0, \mathbf{w}, g_0, \mathbf{g}, \mathbf{J}, \tau) - 1 \right]. \end{aligned}$$

Obviously, in implementing this type of jump intensity, values of the coefficient vector $\mathbf{b}(z, \tau)$ must be determined numerically due to the complicated structure of the relevant ODE. Subsequently, the same statement holds also for the coefficient function $a(z, \tau)$, which depends on $\mathbf{b}(z, \tau)$. Although this type of jump specification enriches the modeling capabilities of the short-rate dynamics, it is infrequently implemented in interest-rate models because of the numerical difficulties mentioned above. However, our FRFT algorithm presented in Chapter 6 can be easily modified to handle this type of stochastic jump intensity.

[208] To stay conform with our base model setup in equation (2.1), we suggest to include N Poisson processes with stochastic intensities.