Chapter 5
C_1^k-Subdivision Algorithms

In the last chapter, we have defined a C_0^k-subdivision algorithm as a pair (A, G) consisting of a subdivision matrix A and a C^k-system G of generating rings. The conditions given in Definition 4.27/80 guarantee that the generated splines are consistent at the center. Such algorithms are easy to construct, but of course, they do not live up to the demands arising in applications, where smoothness is required also at extraordinary knots. In this chapter, we consider subdivision algorithms in more detail with the goal to find conditions for normal continuity and single-sheetedness. First, in Sect. 5.1/84, we define ‘generic’ sets of initial data $\mathbf{Q}$. Restriction to generic data is necessary to exclude degenerate configurations which, even for impeccable algorithms, yield non-smooth surfaces. Section 5.2/84 defines standard algorithms. This class of algorithms, which is predominant in applications, is characterized by a double positive subdominant eigenvalue. Here, the characteristic ring, which is a planar ring built from the subdominant eigenfunctions, plays a key role in the analysis. With a careful generalization of terms, Sect. 5.3/89 yields a complete classification of all C_1^k-subdivision algorithms. Because we will mostly focus on standard algorithms throughout the book, this part, which is quite technical, may be skipped on a first reading. In Sect. 5.4/95, we consider shift invariant algorithms. Shift invariant algorithms have the property that the shape of the generated splines is independent of the starting point which we choose for labeling the segments $\mathbf{x}_j, j \in \mathbb{Z}_n$. The subdivision matrix of shift invariant algorithms is block-circulant and can be transformed to block-diagonal form by means of the Discrete Fourier Transform. This process is of major importance in applications, as well as for the further development of the theory. Typically, subdivision algorithms are not only shift invariant, but also indifferent with respect to a reversal of orientation of segment labels. Such symmetric algorithms are discussed in Sect. 5.5/103. We show that symmetric algorithms necessarily have a pair of real subdominant eigenvalues, justifying our focus on such schemes. Further, we specify easy-to-verify conditions for the characteristic ring which guarantee normal continuity and single-sheetedness of the generated spline surfaces.

5.1 Generic Initial Data

Since degenerate cases are unavoidable in any linear setting, we cannot expect a subdivision algorithm to generate geometrically smooth spline surfaces for all choices of initial data. In the extreme, subdivision will not even generate a surface: if all coefficients $\mathbf{q}_0 = \cdots = \mathbf{q}_{\bar\ell}$ coincide, it generates a sequence of rings that are all shrunk to a single point. The following definition allows us to discard degenerate constellations of coefficients so that we can focus on situations that have practical meaning.

Definition 5.1 (Generic initial data). A vector $\mathbf{Q} = [\mathbf{q}_0; \ldots; \mathbf{q}_{\bar\ell}]$ of initial data $\mathbf{q}_\ell \in \mathbb{R}^3$, and equally the corresponding vector $\mathbf{P} = V^{-1}\mathbf{Q}$ of eigencoefficients $\mathbf{p}_\ell \in \mathbb{C}^3$, is called *generic*, if any triple of eigencoefficients has full rank,

$$\det(\mathbf{p}_{r_1}^{i_1}, \mathbf{p}_{r_2}^{i_2}, \mathbf{p}_{r_3}^{i_3}) \neq 0, \quad (r_1, i_1) \neq (r_2, i_2) \neq (r_3, i_3) \neq (r_1, i_1).$$

Imposing conditions on all triples of eigencoefficients is more than needs to be required in the following. For instance, in the next section on standard algorithms, it is sufficient to assume that the eigencoefficients $\mathbf{p}_1, \mathbf{p}_2$ are linearly independent. However, since the set of non-generic initial data as introduced above has measure zero in $\mathbb{R}^{(\bar\ell+1)\times 3}$, anyway, we choose the simple, more stringent form of the requirement that will cover all cases of interest.

To classify subdivision algorithms, we regard smoothness of the generated surfaces for generic initial data only.

Definition 5.2 (C_r^k-subdivision algorithm). A C_0^k-subdivision algorithm is called

- C_r^k, respectively
- *normal continuous*, respectively
- *single-sheeted*,

if it generates spline surfaces that are

- C_r^k in the sense of Definition 3.12/51, respectively
- normal continuous in the sense of Definition 3.11/51, respectively
- single-sheeted in the sense of Definition 3.11/51

for any generic vector $\mathbf{Q}$ of initial data.

5.2 Standard Algorithms

Most subdivision algorithms of practical relevance have a double subdominant eigenvalue that is real. As will be explained in Sect. 5.5/103, double subdominant eigenvalues arise from symmetry properties of the algorithms.

Definition 5.3 (Standard algorithm, subdominant eigenvalue λ). A C_0^k-subdivision algorithm (A, G) is called a *standard algorithm*, if the subdivision matrix A has eigenvalues according to

$$1 > \lambda_1 = \lambda_2 > |\lambda_3|, \quad \ell_1 = \ell_2 = 0.$$

Moreover,

$$\lambda := \lambda_1 = \lambda_2$$

is called the *subdominant eigenvalue* of A.

This definition means that the second largest eigenvalue λ of the subdivision matrix is positive and has geometric and algebraic multiplicity 2. According to (4.26/74) and (4.29/74), with $w_1^{\mathrm{t}}, w_2^{\mathrm{t}}$ denoting the second and third row of the matrix V^{-1} of left eigenvectors, the equations

$$Av_i = \lambda v_i, \quad f_i = Gv_i, \quad \mathbf{p}_i = w_i^{\mathrm{t}}\mathbf{Q}, \quad i \in \{1,2\}, \tag{5.1}$$

characterize the corresponding pairs of *subdominant eigenvectors, eigenrings, and eigencoefficients*, respectively. With $f_0 = Gv_0 = 1$, we obtain the asymptotic expansion

$$\mathbf{x}^m = FJ^m\mathbf{P} \stackrel{*}{=} \mathbf{p}_0 + \lambda^m(f_1\mathbf{p}_1 + f_2\mathbf{p}_2) \tag{5.2}$$

for the sequence of rings generated by a standard algorithm. That is, first order behavior of $\mathbf{x}^m$ is completely determined by the user-given data $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ and the eigenrings f_1, f_2, which depend only on the algorithm. Together, f_1 and f_2 form a planar ring whose properties are crucial for understanding first order differentiability properties of subdivision surfaces.

Definition 5.4 (Characteristic ring $\boldsymbol{\psi}$ and spline $\boldsymbol{\chi}$, standard). Let (A, G) be a standard algorithm with Jordan decomposition $A = VJV^{-1}$ of A according to (4.22/73) and subdominant eigenrings f_1, f_2 according to (5.1/85). The planar ring

$$\boldsymbol{\psi} := [f_1, f_2] = F[v_1, v_2] \in C^k(\mathbf{S}_n^0, \mathbb{R}^2, G)$$

is called the *characteristic ring* of the algorithm corresponding to V. Accordingly, with the subdominant eigensplines e_1, e_2 of Definition 4.24/78,

$$\boldsymbol{\chi} := [e_1, e_2] = B[v_1, v_2] \in C_0^k(\mathbf{S}_n, \mathbb{R}^2) \tag{5.3}$$

is called the *characteristic spline*.

Since $A^m[v_1, v_2] = \lambda^m[v_1, v_2]$, the rings of the characteristic spline are scaled copies of the characteristic ring,

$$\boldsymbol{\chi}^m = \lambda^m\boldsymbol{\psi}. \tag{5.4}$$

For standard algorithms, the characteristic spline $\boldsymbol{\chi}$ inherits from equation (4.17/69) the scaling property

$$\boldsymbol{\chi}(2^{-m}\mathbf{s}) = \lambda^m\boldsymbol{\chi}(\mathbf{s}), \quad \mathbf{s} \in \mathbf{S}_n,\ m \in \mathbb{N}_0. \tag{5.5}$$

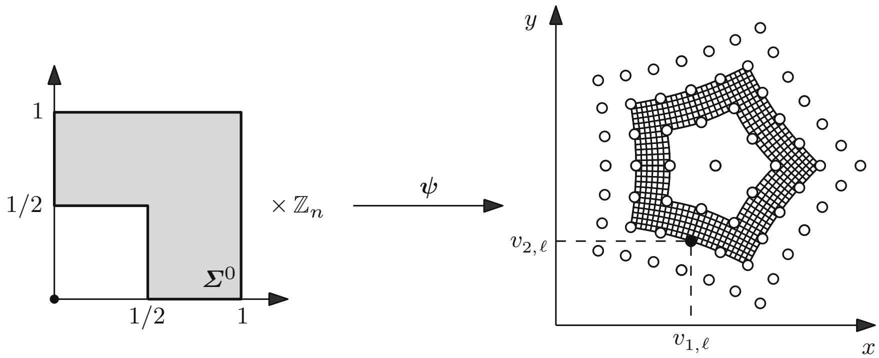

Fig. 5.1 Illustration of Definition 5.4/85: Characteristic ring ψ and its coefficients $\circ$, which are given by the components of the subdominant eigenvectors v_1, v_2.

The coefficients of the characteristic ring are points in $\mathbb{R}^2$ given by the rows of the matrix $[v_1, v_2]$ of subdominant eigenvectors v_1, v_2 (cf. Fig. 5.1/86):

$$\psi = G[v_1, v_2].$$

These eigenvectors are not uniquely defined, and hence also the matrix V used for Jordan decomposition is ambiguous. However, any two admissible pairs are related by a regular (2×2)-matrix L according to $[\tilde{v}_1, \tilde{v}_2] = [v_1, v_2]L$. The corresponding characteristic rings satisfy $\tilde{\psi} = \psi L$. That is, ψ and $\tilde{\psi}$ are related by a regular linear map. By this relation, the set of all possible characteristic rings becomes an equivalence class. The basic properties of characteristic rings that are employed in the sequel, namely regularity and induced winding numbers, are invariant under that relation. In this regard, any choice of V is as good as any other. Therefore, we omit the suffix "corresponding to V" when talking about characteristic rings or characteristic splines (Fig. 5.2/86).

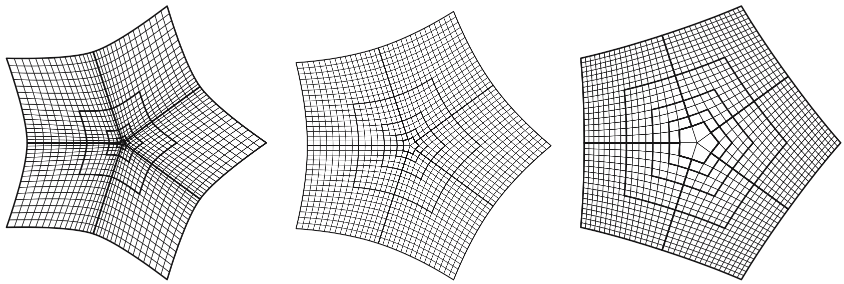

Fig. 5.2 Illustration of Definition 5.4/85: Characteristic spline χ of a standard algorithm for subdominant eigenvalues (*left*) $\lambda = 3/8$, (*middle*) $\lambda = 1/2$, and (*right*) $\lambda = 5/8$.

Now, (5.2/85) reads

$$\mathbf{x}^m \stackrel{*}{=} \mathbf{p}_0 + \lambda^m \boldsymbol{\psi}[\mathbf{p}_1; \mathbf{p}_2]. \tag{5.6}$$

In order to compute normal vectors, we use the operator ${}^{\times}\!\mathcal{D}$, as introduced in (2.1/17). By (2.2/17),

$${}^{\times}\!\mathcal{D}\mathbf{x}^m = D_1\mathbf{x}^m \times D_2\mathbf{x}^m \stackrel{*}{=} \lambda^{2m}\, {}^{\times}\!\mathcal{D}\boldsymbol{\psi}\,(\mathbf{p}_1 \times \mathbf{p}_2), \tag{5.7}$$

where we recall that ${}^{\times}\!\mathcal{D}\boldsymbol{\psi} = D_1 f_1 D_2 f_2 - D_2 f_1 D_1 f_2$ is the Jacobian determinant of the characteristic ring.

Definition 5.5 (Regularity of $\boldsymbol{\psi}$). The characteristic ring $\boldsymbol{\psi}$ is called *regular*, if ${}^{\times}\!\mathcal{D}\boldsymbol{\psi}$ has no zeros.

The following theorem shows that regularity of the characteristic ring is sufficient for a standard algorithm to be normal continuous, and, moreover, discards algorithms with ${}^{\times}\!\mathcal{D}\boldsymbol{\psi}$ changing sign.

Theorem 5.6 (Regularity of $\boldsymbol{\psi}$ and normal continuity, standard). A standard algorithm with characteristic ring $\boldsymbol{\psi}$ is

- normal continuous with central normal

  $$\mathbf{n}^{\mathrm{c}} = \operatorname{sign}({}^{\times}\!\mathcal{D}\boldsymbol{\psi})\,\frac{\mathbf{p}_1 \times \mathbf{p}_2}{\|\mathbf{p}_1 \times \mathbf{p}_2\|},$$

  if $\boldsymbol{\psi}$ is regular;
- not normal continuous, if ${}^{\times}\!\mathcal{D}\boldsymbol{\psi}$ changes sign.

Proof. We assume generic initial data, hence $\mathbf{p}_1 \times \mathbf{p}_2 \neq \mathbf{0}$, for both parts of the proof. First, let us assume that $\boldsymbol{\psi}$ is regular. Since, by Theorem 4.7/64, ${}^{\times}\!\mathcal{D}\boldsymbol{\psi}$ is continuous on the compact domain $\mathbf{S}_n^0$, the absence of zeros implies that $\operatorname{sign}({}^{\times}\!\mathcal{D}\boldsymbol{\psi})$ is continuous, and that $1/|{}^{\times}\!\mathcal{D}\boldsymbol{\psi}|$ is bounded. Hence, we obtain

$$\frac{{}^{\times}\!\mathcal{D}\mathbf{x}^m}{\lambda^2\,|{}^{\times}\!\mathcal{D}\boldsymbol{\psi}|} \stackrel{*}{=} \operatorname{sign}({}^{\times}\!\mathcal{D}\boldsymbol{\psi})\,(\mathbf{p}_1 \times \mathbf{p}_2) \neq \mathbf{0}$$

and see that $\mathbf{x}^m$ is regular for almost all m. Further, the normal vectors $\mathbf{n}^m$ are convergent according to

$$\mathbf{n}^m = \frac{{}^{\times}\!\mathcal{D}\mathbf{x}^m}{\|{}^{\times}\!\mathcal{D}\mathbf{x}^m\|} \stackrel{*}{=} \operatorname{sign}({}^{\times}\!\mathcal{D}\boldsymbol{\psi})\,\frac{\mathbf{p}_1 \times \mathbf{p}_2}{\|\mathbf{p}_1 \times \mathbf{p}_2\|} = \mathbf{n}^{\mathrm{c}}.$$

Theorem 4.7/64 implies normal continuity, as stated. Second, let us assume that ${}^{\times}\!\mathcal{D}\boldsymbol{\psi}(\mathbf{s}_1)\,{}^{\times}\!\mathcal{D}\boldsymbol{\psi}(\mathbf{s}_2) < 0$ for some arguments $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{S}_n^0$. Here, we obtain

$$\mathbf{n}^m(\mathbf{s}_i) \stackrel{*}{=} \operatorname{sign}({}^{\times}\!\mathcal{D}\boldsymbol{\psi}(\mathbf{s}_i))\,\frac{\mathbf{p}_1 \times \mathbf{p}_2}{\|\mathbf{p}_1 \times \mathbf{p}_2\|}, \quad i \in \{1, 2\},$$

and see that $\mathbf{n}^m$ cannot converge to a constant limit since $\|\mathbf{n}^m(\mathbf{s}_1) - \mathbf{n}^m(\mathbf{s}_2)\| \stackrel{*}{=} 2$. □

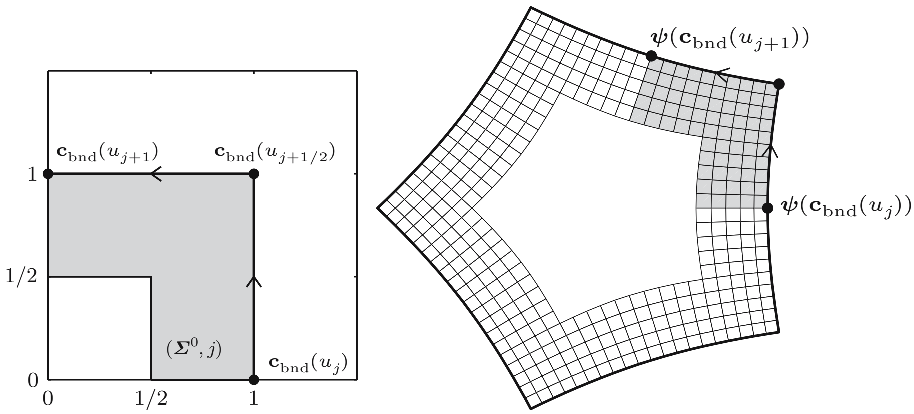

Fig. 5.3 Illustration of Definition 5.7/88: (*left*) Curve $\mathbf{c}_{\mathrm{bnd}}$ in $\mathbf{S}_n^0$ and (*right*) curve $\boldsymbol{\psi} \circ \mathbf{c}_{\mathrm{bnd}}$ in $\mathbb{R}^2$.

Theorem 5.6/87 covers all but the case where ${}^{\times}\!D\boldsymbol{\psi}$ has zeros without changing sign. Here, the behavior of ${}^{\times}\!D\mathbf{x}^m$ depends on higher order eigencoefficients and cannot be determined a priori.

Now, the issue of single-sheetedness has to be addressed, and again, properties of the characteristic ring are crucial. We consider the curve $\mathbf{c}_{\mathrm{bnd}} : U = [0,1] \to \mathbf{S}_n^0$ in the domain of $\boldsymbol{\psi}$ which parametrizes the outer boundary: With $u_j := j/n$, let

$$\mathbf{c}_{\mathrm{bnd}}(t) := \begin{cases} \big(1, 2n(u - u_j), j\big) & \text{if } u_j \le u \le u_{j+1/2} \\ \big(2n(u_{j+1} - u), 2, j\big) & \text{if } u_{j+1/2} \le u \le u_{j+1}, \end{cases}$$

see Fig. 5.3/88 *left*. As shown in Example 3.10/50, its winding number is $\nu(\mathbf{c}_{\mathrm{bnd}}, \mathbf{0}) = 1$.

Definition 5.7 (Winding number of $\boldsymbol{\psi}$). The *winding number* of the characteristic ring $\boldsymbol{\psi} \in C^k(\mathbf{S}_n^0, \mathbb{R}^2)$ is defined as

$$\nu(\boldsymbol{\psi}) := \nu(\boldsymbol{\psi} \circ \mathbf{c}_{\mathrm{bnd}}, \mathbf{0}),$$

see Fig. 5.3/88. We say that $\boldsymbol{\psi}$ is *uni-cyclic* if $|\nu(\boldsymbol{\psi})| = 1$.

We are now able to prove an easy-to-verify criterion for the single-sheetedness of subdivision algorithms in terms of the winding number of the characteristic ring.

Theorem 5.8 (Winding number of $\boldsymbol{\psi}$ and single-sheetedness, standard). Consider a standard algorithm (A, G) with a regular characteristic ring $\boldsymbol{\psi} \in C^k(\mathbf{S}_n^0, \mathbb{R}^2)$. Then the following assertions are equivalent:

- (A, G) is a C_1^k-subdivision algorithm.
- The characteristic ring $\boldsymbol{\psi}$ is uni-cyclic.
- The characteristic ring $\boldsymbol{\psi}$ is injective.

Proof. First, we prove equivalence of the first and the second assertion. We consider the rings $\boldsymbol{\xi}_*^m$ of the tangential component $\boldsymbol{\xi}_*$ of $\mathbf{x}_*$ according to (4.11/64). By (5.6/87),

$$\boldsymbol{\xi}_*^m = (\mathbf{x}^m - \mathbf{x}^c) \cdot \mathbf{T}^c \stackrel{*}{=} \lambda^m \boldsymbol{\psi}[\mathbf{p}_1; \mathbf{p}_2] \cdot \mathbf{T}^c.$$

For generic initial data, the (2×2)-matrix $L := [\mathbf{p}_1; \mathbf{p}_2] \cdot \mathbf{T}^c$ is invertible. On one hand, by (2.5/17),

$$^{\times}D\boldsymbol{\xi}_*^m \stackrel{*}{=} \lambda^{2m}\, {}^{\times}D\boldsymbol{\psi} \det L.$$

This implies that $\boldsymbol{\xi}_*$ and hence also $\mathbf{x}$ is locally almost regular. On the other hand, let

$$\tilde{\boldsymbol{\xi}}^m := \lambda^{-m} \boldsymbol{\xi}_*^m L^{-1},$$

then $\tilde{\boldsymbol{\xi}}^m \stackrel{*}{=} \boldsymbol{\psi}$. By continuity and affine invariance of the winding number,

$$\lim_{m\to\infty} \nu(\boldsymbol{\xi}_*^m \circ \mathbf{c}_{\mathrm{bnd}}, 0) = \lim_{m\to\infty} \nu(\tilde{\boldsymbol{\xi}}^m \circ \mathbf{c}_{\mathrm{bnd}}, 0) = \nu(\boldsymbol{\psi}).$$

Combining the two observations, we see that there exists $m_0 \in \mathbb{N}_0$ such that $\boldsymbol{\xi}_*$ is m_0-almost regular and $\nu(\boldsymbol{\xi}_*^{m_0} \circ \mathbf{c}_{\mathrm{bnd}}, 0) = \nu(\boldsymbol{\psi})$. Hence, by Theorem 4.8/64, $\boldsymbol{\xi}_*$ is single-sheeted if and only if $|\nu(\boldsymbol{\psi})| = 1$, i.e., if ν is uni-cyclic.

Second, we prove equivalence of the second and the third assertion. We define the spline surface $\mathbf{x} := [\boldsymbol{\chi}, 0]$. Using (5.4/85), we have $\|{}^{\times}D\mathbf{x}^m\| = |{}^{\times}D\boldsymbol{\chi}^m| = \lambda^{2m}|{}^{\times}D\boldsymbol{\psi}|$, showing that $\mathbf{x}$ is almost regular. Further, $\mathbf{x}$ is normal continuous with $\mathbf{n}^c = [0, 0, 1]$ and $\mathbf{x}^c = \mathbf{0}$. Hence, $\boldsymbol{\xi}_* = \boldsymbol{\chi}$, and

$$\nu(\boldsymbol{\xi}_* \circ \mathbf{c}_{\mathrm{bnd}}, 0) = \nu(\boldsymbol{\chi} \circ \mathbf{c}_{\mathrm{bnd}}, 0) = \nu(\boldsymbol{\psi} \circ \mathbf{c}_{\mathrm{bnd}}, 0) = \nu(\boldsymbol{\psi}).$$

If $\boldsymbol{\psi}$ is uni-cyclic, then $\mathbf{x}$ is single-sheeted by Theorem 3.15/53. This implies that $\boldsymbol{\chi}$ and hence also $\boldsymbol{\psi} = \boldsymbol{\chi}^0$ is injective. Conversely, if $\boldsymbol{\psi}$ is injective, then the curve $\boldsymbol{\psi} \circ \mathbf{c}_{\mathrm{bnd}}$ is injective and can be deformed continuously to a circle with winding number ± 1. □

In applications, it is *much* easier to check if $\boldsymbol{\psi}$ is uni-cyclic than to consider global injectivity. Since the conditions given above are sufficient and (almost) necessary for generating C_1^k-surfaces, we conclude with the following definition:

Definition 5.9 (Standard C_1^k-algorithm). A standard algorithm is called a *standard C_1^k-algorithm*, if its characteristic ring $\boldsymbol{\psi}$ is regular with ${}^{\times}D\boldsymbol{\psi} > 0$ and uni-cyclic.

Assuming positivity of ${}^{\times}D\boldsymbol{\psi}$ is not restrictive. If ${}^{\times}D\boldsymbol{\psi} = {}^{\times}D(G[v_1, v_2]) < 0$ then interchanging v_1 and v_2 readily yields the desired sign.

5.3 General Algorithms

Standard algorithms cover most cases of practical relevance. Yet, there are legitimate algorithms, for example "simplest subdivision" in Sect. 6.3/120, that have a different eigenstructure-structure; and, certainly, identifying and characterizing more classes of C_1^k-subdivision algorithms is of interest in its own right. This section shows how a careful extension of the concepts used in the standard case yields results of very similar flavor and identical wording also in more general settings.

Specifically, we give six families of possible subdivision matrices that, in principle, are suitable to generate C_1^k-surfaces. We focus on showing that membership in each family implies smoothness; completeness is proven in [Rei99].

Let us consider the sequence $\{\mathbf{x}^m\}_m$ of rings according to (4.26/74) forming the spline $\mathbf{x}$. By (2.3/17), the cross product of partial derivatives has the form

$$^{\times}D\mathbf{x}^m = \sum_i a_i^m h_i \mathbf{c}_i. \tag{5.8}$$

As specified later, the $\{a_i^m\}_m$ form decaying sequences of scaling factors, the $h_i = h_i(\mathbf{s})$ are real-valued rings, and the $\mathbf{c}_i$ are cross products of pairs of eigencoefficients. If the above sum has a single dominant term, i.e.,

$$^{\times}D\mathbf{x}^m \stackrel{*}{=} a_1^m h_1 \mathbf{c}_1, \tag{5.9}$$

and if in addition $a_1^m h_1(\mathbf{s})$ has constant sign $s = \pm 1$ for all $(m, \mathbf{s}) \in \mathbb{N} \times \mathbf{S}_n^0$, then normalization yields normal continuity according to

$$\mathbf{n}^m \stackrel{*}{=} s\,\mathbf{c}_1/\|\mathbf{c}_1\|.$$

It is easy to see that alternating behavior of the sequence with elements a_1^m, or sign changes of h_1 destroy convergence.

Now, we consider the case of a multiple dominant term in (5.8/90). For simplicity, we assume that it is double, and write

$$^{\times}D\mathbf{x}^m \stackrel{*}{=} |a_1^m|\big(s_1^m h_1 \mathbf{c}_1 + s_2^m h_2 \mathbf{c}_2\big), \quad |s_1^m| = |s_2^m| = 1.$$

Choosing a subsequence (for simplicity we reuse the index m) such that $s_1^m \to s_1$, $s_2^m \to s_2$, we obtain

$$\mathbf{n}^m \stackrel{*}{=} \frac{s_1 h_1 \mathbf{c}_1 + s_2 h_2 \mathbf{c}_2}{\|s_1 h_1 \mathbf{c}_1 + s_2 h_2 \mathbf{c}_2\|}.$$

This expression can only converge to a constant limit either if the vectors $\mathbf{c}_1, \mathbf{c}_2$ or the functions h_1, h_2 are linearly dependent. The simple argument is left to the reader. The first case is possible only for non-generic data, while the second one corresponds to the exceptional situation that two functions, which are not interrelated by deeper principles, happen to be linearly dependent.

Consequently, we will search algorithms for which the sum in (5.8/90) has a single dominant term. Using (4.26/74) and (4.27/74), we obtain

$$\begin{aligned}\mathbf{x}^m = \mathbf{p}_0 &+ \lambda_1^{m,\ell_1} f_1^0 \mathbf{p}_1^{\ell_1} + \lambda_1^{m,\ell_1-1}(f_1^0 \mathbf{p}_1^{\ell_1-1} + f_1^1 \mathbf{p}_1^{\ell_1}) \\ &+ \lambda_1^{m,\ell_1-2}(f_1^0 \mathbf{p}_1^{\ell_1-2} + f_1^1 \mathbf{p}_1^{\ell_1-1} + f_1^2 \mathbf{p}_1^{\ell_1}) \\ &+ \lambda_2^{m,\ell_2} f_2^0 \mathbf{p}_2^{\ell_2} + \mathbf{r}^m,\end{aligned}$$

where the remainder term satisfies, with the notation for asymptotics of sequences in Sect. 4.5/71,

$$\mathbf{r}^m \preccurlyeq \lambda_1^{m,\ell_1-3} + \lambda_2^{m,\ell_2-1} + \lambda_3^{m,\ell_3}.$$

The cross product of partial derivatives can be computed with the aid of (2.3/17). We obtain

$$^{\times}D\mathbf{x}^m = \tilde{\mathbf{n}}_1^m + \tilde{\mathbf{n}}_2^m + \tilde{\mathbf{r}}^m \tag{5.10}$$

with

$$\begin{aligned}
\tilde{\mathbf{n}}_1^m &:= \lambda_1^{m,\ell_1}\lambda_2^{m,\ell_2}\, {}^{\times}D[f_1^0, f_2^0](\mathbf{p}_1^{\ell_1} \times \mathbf{p}_2^{\ell_2})\\
\tilde{\mathbf{n}}_2^m &:= \big((\lambda_1^{m,\ell_1-1})^2 - \lambda_1^{m,\ell_1}\lambda_1^{m,\ell_1-2}\big)\, {}^{\times}D[f_1^0, f_1^1](\mathbf{p}_1^{\ell_1} \times \mathbf{p}_1^{\ell_1-1})
\end{aligned}$$

and a remainder term $\tilde{\mathbf{r}}^m$ which can be bounded by

$$\tilde{\mathbf{r}}^m \preccurlyeq \lambda_1^{m,\ell_1}(\lambda_1^{m,\ell_1-3} + \lambda_2^{m,\ell_2-1} + \lambda_3^{m,\ell_3}) + \lambda_1^{m,\ell_1-1}(\lambda_1^{m,\ell_1-2} + \lambda_2^{m,\ell_2}).$$

Recalling our convention that $\lambda^{m,\ell} = 0$ for $\ell < 0$, we find $\tilde{\mathbf{n}}_2^m = 0$ if $\ell_1 = 0$, and

$$\tilde{\mathbf{n}}_2^m \stackrel{*}{=} \ell_1^{-1}\, (\lambda_1^{m,\ell_1-1})^2\, {}^{\times}D[f_1^0, f_1^1](\mathbf{p}_1^{\ell_1} \times \mathbf{p}_1^{\ell_1-1}) \quad \text{if} \quad \ell_1 > 0.$$

The order of decay of the three summands is easily determined using (4.19/72),

$$\begin{aligned}
\tilde{\mathbf{n}}_1^m &\sim (\lambda_1\lambda_2)^m\, m^{\ell_1+\ell_2}\\
\tilde{\mathbf{n}}_2^m &\sim \begin{cases} 0 & \text{if } \ell_1 = 0\\ \lambda_1^{2m}\, m^{2\ell_1-2} & \text{if } \ell_1 > 0 \end{cases}\\
\tilde{\mathbf{r}}^m &\preccurlyeq \begin{cases} (\lambda_1\lambda_3)^m\, m^{\ell_3} & \text{if } \ell_1 = 0\\ \lambda_1^{2m}\, m^{2\ell_1-3} + (\lambda_1\lambda_2)^m\, m^{\ell_1+\ell_2-1} + (\lambda_1\lambda_3)^m\, m^{\ell_1+\ell_3} & \text{if } \ell_1 > 0. \end{cases}
\end{aligned}$$

Now, we are prepared to determine a list of cases where in the representation (5.10/91) either $\tilde{\mathbf{n}}_1^m$ or $\tilde{\mathbf{n}}_2^m$ is the strictly dominant term. In the following,

$$\mathcal{A} := \{A : \lambda_1 \neq 0\}$$

denotes the set of all subdivision matrices according to Definition 4.27/80 excluding the trivial case $\lambda_1 = 0$. We distinguish the following cases:

Case 1: $(\lambda_1, \ell_1) \sim (\lambda_2, \ell_2)$, i.e., there is a multiple subdominant eigenvalue. In this case, $\tilde{\mathbf{n}}_2^m \prec \tilde{\mathbf{n}}_1^m$, and $\tilde{\mathbf{r}}^m \prec \tilde{\mathbf{n}}_2^m$ if $(\lambda_2, \ell_2) \succ (\lambda_3, \ell_3)$. We distinguish two sub-cases:

1-1: $\lambda_1 \in \mathbb{R}$. Here, λ_2 is also real. If $\lambda_1 = -\lambda_2$, then $\tilde{\mathbf{n}}_2^m$ is alternating and the algorithm cannot be normal continuous. The case $\lambda_1 = \lambda_2$ does not lead to such problems, and the corresponding class of subdivision matrices is denoted by

$$\mathcal{A}_1^1 := \{A \in \mathcal{A} : (\lambda_1, \ell_1) = (\lambda_2, \ell_2) \succ (\lambda_3, \ell_3),\ \lambda_1 \in \mathbb{R}\}.$$

We note that standard algorithms, as introduced in the last section, belong to this class with $\ell_1 = \ell_2 = 0$.

1-2: $\lambda_1 \notin \mathbb{R}$. The complex sub-case yields the class

$$\mathcal{A}_1^2 := \{A \in \mathcal{A} : (\lambda_1, \ell_1) = (\overline{\lambda_2}, \ell_2) \succ (\lambda_3, \ell_3),\ \lambda_1 \notin \mathbb{R}\}.$$

Case 2: $|\lambda_1| = |\lambda_2|$, $\ell_1 > \ell_2$, i.e., we have equal modulus, but differing multiplicities of the first and second eigenvalue. We distinguish three sub-cases:

2-1: $\ell_1 = \ell_2 + 1$. Here, $\tilde{\mathbf{n}}_2^m \prec \tilde{\mathbf{n}}_1^m$, and $\tilde{\mathbf{r}}^m \prec \tilde{\mathbf{n}}_1^m$ if $(\lambda_2, \ell_2) \succ (\lambda_3, \ell_3)$. In that case, λ_1 and λ_2 are both real, and their product has to be positive to avoid alternating behavior of $\tilde{\mathbf{n}}_1^m$. We obtain the class

$$\mathcal{A}_2^1 := \{A \in \mathcal{A} : \lambda_1 = \lambda_2,\ \ell_1 = \ell_2 + 1,\ (\lambda_2, \ell_2) \succ (\lambda_3, \ell_3)\}.$$

2-2: $\ell_1 > \ell_2 + 2$. Here, $\tilde{\mathbf{r}}^m \preccurlyeq \tilde{\mathbf{n}}_1^m \prec \tilde{\mathbf{n}}_2^m$, and we denote

$$\mathcal{A}_2^2 := \{A \in \mathcal{A} : |\lambda_1| = |\lambda_2|,\ \ell_1 > \ell_2 + 2\}.$$

2-3: $\ell_1 = \ell_2 + 2$. Here, $\tilde{\mathbf{n}}_1^m \sim \tilde{\mathbf{n}}_2^m$ implies decay at equal rates of both terms, and normal continuity cannot be expected by the argument similar to the one ruling out multiple dominant terms in (5.8/90).

Case 3: $|\lambda_1| > |\lambda_2|$. We distinguish two sub-cases:

3-1: $\ell_1 = 0$. Here, $\tilde{\mathbf{n}}_2^m \prec \tilde{\mathbf{n}}_1^m$, and $\tilde{\mathbf{r}}^m \prec \tilde{\mathbf{n}}_1^m$ if $(\lambda_2, \ell_2) \succ (\lambda_3, \ell_3)$. Further, the sign of λ_1 and λ_2 has to be equal to avoid alternating behavior. This sub-case is denoted by

$$\mathcal{A}_3^2 := \{A \in \mathcal{A} : |\lambda_1| > |\lambda_2|,\ \ell_1 = 0,\ \lambda_1\lambda_2 > 0,\ (\lambda_2, \ell_2) \succ (\lambda_3, \ell_3)\}.$$

3-2: $\ell_1 \geq 1$. Here, $\tilde{\mathbf{n}}_1^m \prec \tilde{\mathbf{n}}_2^m$ and also $\tilde{\mathbf{r}}^m \prec \tilde{\mathbf{n}}_2^m$. We denote

$$\mathcal{A}_3^1 := \{A \in \mathcal{A} : |\lambda_1| > |\lambda_2|,\ \ell_1 \geq 1\}.$$

Summarizing, $\tilde{\mathbf{n}}_1^m$ is dominant if the subdivision matrix lies in $\mathcal{A}_1^1, \mathcal{A}_1^2, \mathcal{A}_2^1$ or $\mathcal{A}_3^1$, and $\tilde{\mathbf{n}}_2^m$ is dominant if it lies in $\mathcal{A}_2^2$ or $\mathcal{A}_3^2$.

The, off-hand heuristic, partition of cases into six families will turn out to simplify the analysis that we start by extending the definition of the characteristic ring.

Definition 5.10 (Characteristic ring, general). Let (A, G) be a subdivision algorithm with subdivision matrix $A \in \mathcal{A}_p^q$, $p \in \{1,2,3\}$, $q \in \{1,2\}$. We define the *characteristic ring* $\psi \in C^k(\mathbf{S}_n^0, \mathbb{R}^2)$ by

$$\psi := \begin{cases} [f_1^0, f_2^0] & \text{if } A \in \mathcal{A}_1^1 \cup \mathcal{A}_2^1 \cup \mathcal{A}_3^1 \\ [\operatorname{Re} f_1^0, \operatorname{Im} f_1^0] & \text{if } A \in \mathcal{A}_1^2 \\ [f_1^0, f_1^1] & \text{if } A \in \mathcal{A}_2^2 \cup \mathcal{A}_3^2, \end{cases}$$

and the (2×3)-matrix $\mathbf{P}^*$ of eigencoefficients by

$$\mathbf{P}^* := [\mathbf{p}_1^*; \mathbf{p}_2^*] := \begin{cases} [\mathbf{p}_1^{\ell_1}; \mathbf{p}_2^{\ell_2}] & \text{if } A \in \mathcal{A}_1^1 \cup \mathcal{A}_2^1 \cup \mathcal{A}_3^1 \\ [\operatorname{Re} \mathbf{p}_1^{\ell_1}; -\operatorname{Im} \mathbf{p}_1^{\ell_1}] & \text{if } A \in \mathcal{A}_1^2 \\ [\mathbf{p}_1^{\ell_1}; \mathbf{p}_1^{\ell_1 - 1}] & \text{if } A \in \mathcal{A}_2^2 \cup \mathcal{A}_3^2. \end{cases}$$

ψ is called *regular*, if its Jacobian determinant ${}^{\times}D\psi$ has no zeros.

The following result on normal continuity is completely analogous to Theorem 5.6/87 and the conclusion is verbatim the same.

Theorem 5.11 (Regularity of ψ and normal continuity, general). *A subdivision algorithm (A, G) with $A \in \mathcal{A}_p^q$ and characteristic ring ψ according to the preceding definition is*

- *normal continuous with limit*

$$\mathbf{n}^c = \operatorname{sign}({}^{\times}D\psi) \frac{\mathbf{p}_1^* \times \mathbf{p}_2^*}{\|\mathbf{p}_1^* \times \mathbf{p}_2^*\|},$$

  if the characteristic ring is regular,
- *not normal continuous; if ${}^{\times}D\psi$ changes sign.*

Proof. With the scaling factor

$$a_m := \begin{cases} \lambda_1^{m,\ell_1} \lambda_2^{m,\ell_2} & \text{if } A \in \mathcal{A}_1^1 \cup \mathcal{A}_2^1 \cup \mathcal{A}_3^1 \cup \mathcal{A}_1^2 \\ \ell_1^{-1} (\lambda_1^{m,\ell_1 - 1})^2 & \text{if } A \in \mathcal{A}_2^2 \cup \mathcal{A}_3^2, \end{cases}$$

the cross product of the partial derivatives of the rings is

$${}^{\times}D\mathbf{x}^m \stackrel{*}{=} a_m \, {}^{\times}D\psi \, (\mathbf{p}_1^* \times \mathbf{p}_2^*).$$

In the complex case $A \in \mathcal{A}_1^2$ the relations $f_1^0 = \overline{f}_2^0$ and $\mathbf{p}_1^{\ell_1} = \overline{\mathbf{p}}_2^{\ell_2} = \mathbf{p}_1^* - i\mathbf{p}_2^*$ are used to obtain the real representation. Now, the proof proceeds exactly as for Theorem 5.6/87. □

The wording of the theorem below in the general case is verbatim the same as in the standard case, i.e., as for Theorem 5.8/88 on C_k^1-regularity.

Theorem 5.12 (Winding number of ψ and single-sheetedness, general). *Consider a subdivision algorithm with matrix $A \in A_p^q$ with a regular characteristic ring $\psi \in C^k(\mathbf{S}_n^0, \mathbb{R}^2)$. Then the following assertions are equivalent:*

- *(A, G) is a C_1^k-subdivision algorithm.*
- *The characteristic ring ψ is uni-cyclic.*
- *The characteristic ring ψ is injective.*

Proof. In all six cases, we will specify sequences $\{\Lambda^m\}$ of invertible (2×2)-matrices such that the rings can be written as

$$\mathbf{x}^m = \mathbf{p}_0 + \boldsymbol{\psi}\Lambda^m \mathbf{P}^* + \mathbf{r}^m$$

with a suitable remainder term $\mathbf{r}^m$. Let us denote the LQ-decomposition of $\mathbf{P}^*$ by $\mathbf{P}^* = L\mathbf{T}$. That is, L is a lower triangular matrix, and $\mathbf{T}$ consists of two orthonormal row-vectors spanning the same plane as the rows of $\mathbf{P}^*$. For generic data, $\mathbf{P}^*$ has full rank, and hence L is invertible. The projection to the tangent plane at the center is

$$\boldsymbol{\xi}^m = (\mathbf{x}^m - \mathbf{p}_0)\cdot\mathbf{T} = \boldsymbol{\psi}\Lambda^m L + \mathbf{r}^m\cdot\mathbf{T}.$$

We define $\tilde{\boldsymbol{\xi}}^m := \boldsymbol{\xi}^m L^{-1}(\Lambda^m)^{-1}$ and obtain

$$\tilde{\boldsymbol{\xi}}^m = \boldsymbol{\psi} + \boldsymbol{\rho}^m, \quad \boldsymbol{\rho}^m := (\mathbf{r}^m\cdot\mathbf{T})L^{-1}(\Lambda^m)^{-1}.$$

We will show that in all cases the remainder term satisfies $\boldsymbol{\rho}^m \prec 1$, i.e., it converges to 0. Thus, $\tilde{\boldsymbol{\xi}}^m \stackrel{*}{=} \boldsymbol{\psi}$, and all the rest of the conclusion proceeds exactly as in Theorem 5.8/88. It remains to add the details for the six cases. Throughout, we omit the subscript of the first eigenvalue, $(\lambda,\ell) := (\lambda_1,\ell_1)$.

Case 1-1: For $(\lambda_2,\ell_2) = (\lambda,\ell)$, the leading terms are

$$\mathbf{x}^m \stackrel{*}{=} \mathbf{p}_0 + \lambda^{m,\ell} f_1^0 \mathbf{p}_1^\ell + \lambda^{m,\ell} f_2^0 \mathbf{p}_2^\ell.$$

With $\boldsymbol{\psi} = [f_1^0, f_2^0]$, $\mathbf{P}^* = [\mathbf{p}_1^\ell; \mathbf{p}_2^\ell]$, and

$$\Lambda^m := \begin{bmatrix} \lambda^{m,\ell} & 0 \\ 0 & \lambda^{m,\ell} \end{bmatrix}, \quad (\Lambda^m)^{-1} \preccurlyeq 1/\lambda^{m,\ell},$$

the remainder terms satisfy

$$\mathbf{r}^m \prec \lambda^{m,\ell}, \quad \boldsymbol{\rho}^m \sim \mathbf{r}^m (\Lambda^m)^{-1} \prec 1.$$

Case 1-2: For $(\lambda_2,\ell_2) = (\overline{\lambda},\ell)$, we have

$$\mathbf{x}^m \stackrel{*}{=} \mathbf{p}_0 + 2\,\mathrm{Re}(\lambda^{m,\ell} f_1^0 \mathbf{p}_1^\ell)$$
$$\boldsymbol{\psi} = [\mathrm{Re}\, f_1^0, \mathrm{Im}\, f_1^0], \quad \mathbf{P}^* = [\mathrm{Re}\,\mathbf{p}_1^\ell; -\mathrm{Im}\,\mathbf{p}_1^\ell]$$
$$\Lambda^m := 2\begin{bmatrix} \mathrm{Re}\,\lambda^{m,\ell} & \mathrm{Im}\,\lambda^{m,\ell} \\ -\mathrm{Im}\,\lambda^{m,\ell} & \mathrm{Re}\,\lambda^{m,\ell} \end{bmatrix}, \quad (\Lambda^m)^{-1} = \frac{1}{2}(\Lambda^m)^{\mathrm{t}}/|\lambda^{m,\ell}|^2 \preccurlyeq 1/\lambda^{m,\ell}$$
$$\mathbf{r}^m \prec \lambda^{m,\ell}, \quad \boldsymbol{\rho}^m \sim \mathbf{r}^m(\Lambda^m)^{-1} \prec 1$$

Case 2-1: For $(\lambda_2,\ell_2) = (\lambda,\ell-1)$, the leading terms of $\mathbf{x}^m$ are

$$\mathbf{x}^m \stackrel{*}{=} \mathbf{p}_0 + \lambda^{m,\ell} f_1^0 \mathbf{p}_1^\ell + \lambda^{m,\ell-1}(f_2^0 \mathbf{p}_2^{\ell-1} + f_1^0 \mathbf{p}_1^{\ell-1} + f_1^1 \mathbf{p}_1^\ell),$$

and the characteristic ring is $\boldsymbol{\psi} = [f_1^0, f_2^0]$. With a vector $\mathbf{n}^{\mathrm{c}}$ perpendicular to $\mathbf{P}^* = [\mathbf{p}_1^\ell; \mathbf{p}_2^{\ell-1}]$ we decompose $\mathbf{p}_1^{\ell-1} = [a, b]\mathbf{P}^* + c\mathbf{n}^{\mathrm{c}}$. Setting

$$\Lambda^m := \begin{bmatrix} \lambda^{m,\ell} & b\lambda^{m,\ell-1} \\ 0 & \lambda^{m,\ell-1} \end{bmatrix}, \quad (\Lambda^m)^{-1} \sim (\lambda^{m,\ell})^{-1} \begin{bmatrix} 1 & -b \\ 0 & m \end{bmatrix},$$

we obtain the remainder term $\mathbf{r}^m \stackrel{*}{=} \lambda^{m,\ell-1}(cf_1^0\mathbf{n}^{\mathrm{c}} + af_1^0\mathbf{p}_1^\ell + f_1^1\mathbf{p}_1^\ell)$. Using $\mathbf{n}^{\mathrm{c}}\mathbf{P}^* = 0$ and $\mathbf{p}_1^\ell\mathbf{P}^* \sim [1, 0]$, we find

$$\boldsymbol{\rho}^m \sim [1/m, 0] \begin{bmatrix} 1 & -b \\ 0 & m \end{bmatrix} \prec 1.$$

Case 2-2: For $|\lambda_2| = |\lambda|$ and $\ell_2 < \ell - 2$, we have

$$\begin{gathered} \mathbf{x}^m \stackrel{*}{=} \mathbf{p}_0 + \lambda^{m,\ell} f_1^0 \mathbf{p}_1^\ell + \lambda^{m,\ell-1}(f_1^0\mathbf{p}_1^{\ell-1} + f_1^1\mathbf{p}_1^\ell) \\ \boldsymbol{\psi} = [f_1^0, f_1^1], \quad \mathbf{P}^* = [\mathbf{p}_1^\ell; \mathbf{p}_1^{\ell-1}] \\ \Lambda^m := \begin{bmatrix} \lambda^{m,\ell} & \lambda^{m,\ell-1} \\ 0 & \lambda^{m,\ell-1} \end{bmatrix}, \quad (\Lambda^m)^{-1} \preccurlyeq (\lambda^{m,\ell-1})^{-1} \\ \mathbf{r}^m \prec \lambda^{m,\ell-1}, \quad \boldsymbol{\rho} \sim \mathbf{r}^m(\Lambda^m)^{-1} \prec 1. \end{gathered}$$

Case 3-1: For $|\lambda_2| < |\lambda_1|$ and $\ell = 0$, we have

$$\begin{gathered} \mathbf{x}^m \stackrel{*}{=} \mathbf{p}_0 + \lambda^{m,\ell} f_1^0 \mathbf{p}_1^\ell + \lambda_2^{m,\ell_2} f_2^0 \mathbf{p}_2^{\ell_2} \\ \boldsymbol{\psi} = [f_1^0, f_2^0], \quad \mathbf{P}^* = [\mathbf{p}_1^\ell; \mathbf{p}_2^{\ell_2}] \\ \Lambda^m := \begin{bmatrix} \lambda^{m,\ell} & 0 \\ 0 & \lambda_2^{m,\ell_2} \end{bmatrix}, \quad (\Lambda^m)^{-1} \preccurlyeq 1/\lambda_2^{m,\ell_2} \\ \mathbf{r}^m \prec \lambda_2^{m,\ell_2}, \quad \boldsymbol{\rho}^m \sim \mathbf{r}^m(\Lambda^m)^{-1} \prec 1. \end{gathered}$$

Case 3-2: For $|\lambda_2| < |\lambda_1|$ and $\ell > 0$, we have

$$\begin{gathered} \mathbf{x}^m \stackrel{*}{=} \mathbf{p}_0 + \lambda^{m,\ell} f_1^0 \mathbf{p}_1^\ell + \lambda^{m,\ell-1}(f_1^0\mathbf{p}_1^{\ell-1} + f_1^1\mathbf{p}_1^\ell) \\ \boldsymbol{\psi} = [f_1^0, f_1^1], \quad \mathbf{P}^* = [\mathbf{p}_1^\ell; \mathbf{p}_1^{\ell-1}] \\ \Lambda^m := \begin{bmatrix} \lambda^{m,\ell} & \lambda^{m,\ell-1} \\ 0 & \lambda^{m,\ell-1} \end{bmatrix}, \quad (\Lambda^m)^{-1} \preccurlyeq (\lambda^{m,\ell-1})^{-1} \\ \mathbf{r}^m \prec \lambda^{m,\ell-1}, \quad \boldsymbol{\rho} \sim \mathbf{r}^m(\Lambda^m)^{-1} \prec 1. \end{gathered}$$

□

5.4 Shift Invariant Algorithms

A subdivision algorithm is shift invariant if the *shape* generated by the sequence of rings remains unchanged regardless which segment is labeled first when numbering them. Most subdivision algorithms currently in use have this property. It allows analyzing the spectrum of the subdivision matrix with the help of the *Discrete Fourier Transform (DFT)*. We will show that shift invariance is possible only for subdivision matrices with a pair of – either real or complex conjugate – subdominant Jordan blocks. Further, the characteristic ring is symmetric in the sense that neighboring segments are related by a $2\pi/n$-rotation.

Corresponding to the partition of a ring $\mathbf{x}^m = GA^m\mathbf{Q}$ into segments $\mathbf{x}_j^m = \mathbf{x}^m(\cdot, j), j \in \mathbb{Z}_n$, the coefficients $\mathbf{Q}$ can typically be partitioned into n blocks $\mathbf{Q} = [\mathbf{Q}_0; \ldots; \mathbf{Q}_{n-1}]$, where all blocks[1] $\mathbf{Q}_j$ have equal size $\tilde{\ell} := (\bar{\ell}+1)/n$. This grouping of coefficients into blocks with equal structure is a natural process; by contrast, assigning the label $j = 0$ to one of these blocks is a random choice, unless the blocks are intentionally treated differently. We expect from a shift invariant algorithm that this choice determines the labelling of segments, but not their shape. To make this precise, let us consider two possible representations $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$ of a given set of initial data, differing only by the labelling of blocks, i.e., $\tilde{\mathbf{Q}}_j = \mathbf{Q}_{j-i}$ for some $i \in \mathbb{Z}_n$. Then the corresponding rings $\mathbf{x}^m$ and $\tilde{\mathbf{x}}^m$ should have segments related by an equal shift of labels, i.e., $\tilde{\mathbf{x}}_j^m = \mathbf{x}_{j-i}^m$. Let us investigate the consequences of this requirement. With $\mathbb{1}$ the identity matrix of size $\tilde{\ell}$, let

$$S := \begin{bmatrix} 0 & 0 & \cdots & 0 & \mathbb{1} \\ \mathbb{1} & 0 & \cdots & 0 & 0 \\ 0 & \mathbb{1} & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & \mathbb{1} & 0 \end{bmatrix}, \quad \mathbb{1} := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

denote the n-block *shift matrix*. Then $\tilde{\mathbf{Q}} = S^i\mathbf{Q}$, and shift invariance formally reads

$$\tilde{\mathbf{x}}_j^m = G(\cdot, j)A^m S^i \mathbf{Q} = G(\cdot, j-i)A^m\mathbf{Q} = \mathbf{x}_{j-i}^m. \tag{5.11}$$

For $m = 0$, we obtain $G(\cdot, j)S^i = G(\cdot, j-i)$, and hence, for arbitrary m, $G(\cdot, j)A^m S^i = G(\cdot, j)S^i A^m$. Disregarding possible linear dependence of the generating system G, the latter equality suggests that A^m and S^i commute. These considerations give rise to the following definition:

Definition 5.13 (Shift invariance). A subdivision algorithm (A, G) is called *shift invariant*, if the generating system satisfies

$$G(\cdot, j)S = G(\cdot, j-1), \quad j \in \mathbb{Z}_n,$$

and if A and S commute,

$$AS = SA.$$

[1] The partition of vectors of coefficients and functions into n similar blocks must not be confused with the partition into Jordan blocks, see Sect. 4.6/72.

The two conditions imply $G(\cdot,j)S^i = G(\cdot,j-i)$ and $A^m S^i = S^i A^m$ for all $j,i \in \mathbb{Z}_n$ and $m \in \mathbb{N}_0$. Following (5.11/96), we obtain

$$\tilde{\mathbf{x}}_j^m = \mathbf{x}_{j-i}^m \quad \text{if} \quad \tilde{\mathbf{Q}}_j = \mathbf{Q}_{j-i}, \quad i,j \in \mathbb{Z}_n,$$

for a shift invariant algorithm, as intended.

According to the partitioning of the coefficients, the subdivision matrix of a shift-invariant algorithm can be represented by $(n \times n)$ blocks $A_{j,i}$ of size $(\tilde{\ell} \times \tilde{\ell})$,

$$A = \begin{bmatrix} A_{0,0} & \cdots & A_{0,n-1} \\ \vdots & & \vdots \\ A_{n-1,0} & \cdots & A_{n-1,n-1} \end{bmatrix}.$$

If A and S commute, we obtain for the blocks

$$(AS)_{j,i} = A_{j,i+1} = A_{j+1,i} = (SA)_{j,i}, \quad i,j \in \mathbb{Z}_n.$$

Hence, the matrix A is completely determined by the blocks $A_j := A_{j,0}$ of the first column via $A_j = A_{j+i,i}$. We say that A is *block-circulant* and write

$$A = \operatorname{circ}(A_0, \ldots, A_{n-1}) := \begin{bmatrix} A_0 & A_{n-1} & \cdots & A_1 \\ A_1 & A_0 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_{n-2} & \cdots & A_0 \end{bmatrix}.$$

The given conditions for shift invariance are more general than might appear at first sight. This is best explained by example.

Example 5.14 (Catmull–Clark algorithm in circulant form). In its standard form, the Catmull–Clark algorithm uses $\bar{\ell} + 1 = 12n + 1$ coefficients, arranged as shown in Fig. 6.3/111 to describe a ring. There is one central coefficient $\tilde{\mathbf{q}}_0$, and n blocks $\tilde{\mathbf{Q}}_0, \ldots, \tilde{\mathbf{Q}}_{n-1}$ with always 12 elements. The corresponding subdivision matrix and the generating system have the structure

$$\tilde{A} := \begin{bmatrix} \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_1 & \cdots & \tilde{a}_1 \\ \tilde{a}_2 & \tilde{A}_0 & \tilde{A}_{n-1} & \cdots & \tilde{A}_1 \\ \tilde{a}_2 & \tilde{A}_1 & \tilde{A}_0 & \cdots & \tilde{A}_2 \\ \vdots & & \ddots & \ddots & \\ \tilde{a}_2 & \tilde{A}_{n-1} & \tilde{A}_{n-2} & \cdots & \tilde{A}_0 \end{bmatrix}, \quad \tilde{\mathbf{Q}} := \begin{bmatrix} \tilde{\mathbf{q}}_0 \\ \tilde{\mathbf{Q}}_0 \\ \tilde{\mathbf{Q}}_1 \\ \vdots \\ \tilde{\mathbf{Q}}_{n-1} \end{bmatrix}, \quad \tilde{G} := [\tilde{g}_0, \tilde{G}_1, \ldots, \tilde{G}_{n-1}],$$

where $\tilde{a}_0$ is a real number, $\tilde{a}_1$ is row-vector, and $\tilde{a}_2$ is a column vector with always 12 elements. Of course, one can adapt the notion of shift invariance to cover also such situations, but we want to show now that this is actually not necessary if we slightly modify the structure of the coefficients. The trick is to artificially extend each block $\tilde{\mathbf{Q}}_j$ by a copy $\mathbf{q}_j := \tilde{\mathbf{q}}_0$ of the central coefficient to obtain the arrangement

$$\mathbf{Q} := [\mathbf{Q}_0; \ldots; \mathbf{Q}_{n-1}], \quad \mathbf{Q}_j := [\mathbf{q}_j; \tilde{\mathbf{Q}}_j]$$

with $13n$ coefficients, see Fig. 6.3/111. Accordingly, the subdivision matrix yields the desired circulant structure,

$$A := \operatorname{circ}(A_0, \ldots, A_{n-1}), \quad A_j := \begin{bmatrix} \tilde{a}_0/n & \tilde{a}_1 \\ \tilde{a}_2/n & \tilde{A}_j \end{bmatrix}.$$

Division by n is applied to ensure that also the rows of A sum to 1. Further, all points $\mathbf{q}_0^m = \cdots = \mathbf{q}_{n-1}^m$ remain equal throughout the iteration. The new system of generating rings is

$$G := [G_0, \ldots, G_{n-1}], \quad G_j := [\tilde{g}_0/n, \ \tilde{G}_j],$$

where division by n retains partition of unity. It is easily shown that the original algorithm and its variant are equivalent in the sense that

$$GA^m\mathbf{Q} = \tilde{G}\tilde{A}^m\tilde{\mathbf{Q}}$$

for any choice of initial data. Unlike the original generating rings, the new system G is linearly dependent. However, no ineffective eigenvectors are introduced. □

The key tool for handling circulant matrices is the *Discrete Fourier Transform (DFT)*. We denote the imaginary unit and the primitive n-th root of unity by

$$\mathbf{i} := \sqrt{-1}, \quad w_n := c_n + \mathbf{i}s_n := \exp(2\pi\mathbf{i}/n). \tag{5.12}$$

With $\mathbb{1}$ the identity matrix of size $\tilde{\ell}$ as above, we define the *Fourier block matrix* $\mathcal{W}$ by

$$\mathcal{W} := (w_n^{-ji}\mathbb{1})_{j,i\in\mathbb{Z}_n} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \cdots & \mathbb{1} \\ \mathbb{1} & w_n^{-1}\mathbb{1} & w_n^{-2}\mathbb{1} & \cdots & w_n^{1}\mathbb{1} \\ \mathbb{1} & w_n^{-2}\mathbb{1} & w_n^{-4}\mathbb{1} & \cdots & w_n^{2}\mathbb{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{1} & w_n^{1}\mathbb{1} & w_n^{2}\mathbb{1} & \cdots & w_n^{-1}\mathbb{1} \end{bmatrix}.$$

It is easily verified by inspection that the inverse of $\mathcal{W}$ is given by

$$\mathcal{W}^{-1} = \frac{1}{n}\,(w_n^{+ji}\mathbb{1})_{j,i\in\mathbb{Z}_n} = \frac{1}{n}\,\overline{\mathcal{W}}.$$

In particular, the i-th block column of $\mathcal{W}^{-1}$ is

$$\mathcal{W}_i^{-1} := \frac{1}{n} \begin{bmatrix} \mathbb{1} \\ w_n^{i}\mathbb{1} \\ \vdots \\ w_n^{(n-1)i}\mathbb{1} \end{bmatrix}. \tag{5.13}$$

The DFT of the matrix A is defined by $\hat{A} := \mathcal{W} A \mathcal{W}^{-1}$, and a standard computation shows that

$$\hat{A} = \operatorname{diag}(\hat{A}_0, \ldots, \hat{A}_{n-1}) = \begin{bmatrix} \hat{A}_0 & & 0 \\ & \ddots & \\ 0 & & \hat{A}_{n-1} \end{bmatrix} \tag{5.14}$$

is block-diagonal with entries obtained by applying the Fourier matrix to the first block column of A,

$$\begin{bmatrix} \hat{A}_0 \\ \vdots \\ \hat{A}_{n-1} \end{bmatrix} = \mathcal{W} \begin{bmatrix} A_0 \\ \vdots \\ A_{n-1} \end{bmatrix}, \quad \text{that is} \quad \hat{A}_i = \sum_{j \in \mathbb{Z}_n} w_n^{-ji} A_j.$$

By definition, A and $\hat{A}$ are similar, and in particular, they have equal eigenvalues. More precisely, the Jordan decompositions of A and $\hat{A}$ are related by

$$A = VJV^{-1}, \quad \hat{A} = \hat{V} J \hat{V}^{-1}, \quad \hat{V} = \mathcal{W} V.$$

Since $\hat{A}$ is block-diagonal, its Jordan decomposition is obtained from the respective decompositions of the blocks,

$$\hat{V} = \operatorname{diag}(\hat{V}_0, \ldots, \hat{V}_{n-1}), \quad J = \operatorname{diag}(\hat{J}_0, \ldots, \hat{J}_{n-1}), \quad \hat{A}_i = \hat{V}_i \hat{J}_i \hat{V}_i^{-1}.$$

This means that with the help of the DFT, Jordan decomposition of the subdivision matrix A, which typically is quite large, boils down to decomposing the n much smaller blocks $\hat{A}_0, \ldots, \hat{A}_{n-1}$ individually. More specifically, let $\hat{v}$ be a (generalized) eigenvector of $\hat{A}_i$. Then $\hat{v}$ is the i-th block of a column of $\hat{V}$, all other blocks of this column are zero. Hence, using the Kronecker symbol $\delta_{j,i}$ and $V = \mathcal{W}^{-1}\hat{V}$, the corresponding (generalized) eigenvector v of A is

$$v = \mathcal{W}^{-1} \begin{bmatrix} \delta_{0,i}\hat{v} \\ \delta_{1,i}\hat{v} \\ \vdots \\ \delta_{n-1,i}\hat{v} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} w_n^0 \hat{v} \\ w_n^i \hat{v} \\ \vdots \\ w_n^{(n-1)i} \hat{v} \end{bmatrix},$$

or, with (5.13/98), briefly

$$v = \mathcal{W}_i^{-1} \hat{v}. \tag{5.15}$$

Moreover, v is always an eigenvector of S to the eigenvalue w_n^{-i}, i.e.,

$$Sv = w_n^{-i} v. \tag{5.16}$$

This implies for the segments of the corresponding eigenring $f := Gv$

$$f_j = G(\cdot, j)v = w_n^i G(\cdot, j) S v = w_n^i G(\cdot, j-1) v = w_n^i f_{j-1}. \tag{5.17}$$

The observation that every Jordan block of A corresponds to a Jordan block of one of the diagonal blocks leads to the following

Definition 5.15 (Fourier index). For a complex number λ, the set of all indices i with the property that λ is eigenvalue of $\hat{A}_i$ is called the *Fourier index* of λ and denoted by

$$\mathcal{F}(\lambda) := \{i \in \mathbb{Z}_n : \lambda \text{ is eigenvalue of } \hat{A}_i\}.$$

Equally, the *Fourier index* of a Jordan block J, see (4.20/72), is

$$\mathcal{F}(J) := \{i \in \mathbb{Z}_n : J \text{ is Jordan block of } \hat{A}_i\}.$$

It is easily shown that the unique dominant eigenvalue $\lambda_0 = 1$ of a subdivision matrix has the Fourier index

$$\mathcal{F}(1) = \{0\}.$$

Since A is real, the blocks of $\hat{A}$ and also their Jordan decompositions come in complex conjugate pairs,

$$\hat{A}_{n-i} = \overline{\hat{A}_i}, \quad \hat{V}_{n-i} = \overline{\hat{V}}_i, \quad \hat{J}_i = \overline{\hat{J}}_i.$$

In particular, if J is a Jordan block of $\hat{A}_i$, then $\overline{J}$ is a Jordan block of $\hat{A}_{n-i}$,

$$i \in \mathcal{F}(J) \quad \Leftrightarrow \quad n - i \in \mathcal{F}(\overline{J}). \tag{5.18}$$

Together with (5.17/99), this pairing allows us to discard shift invariant subdivision algorithms without a pair of real or complex subdominant Jordan blocks.

Theorem 5.16 (Shift invariant algorithms). *Consider a shift invariant subdivision algorithm (A, G) with $A \in \mathcal{A}_p^q$ according to Sect. (5.3/89) and a regular characteristic ring ψ. Then (A, G) can be a C_1^k-algorithm only if $A \in \mathcal{A}_1^1 \cup \mathcal{A}_1^2$.*

Proof. The excluded cases $A \in \mathcal{A}_p^q, p \geq 2$, are characterized by the fact that the first eigenvalue dominates the second one, $(\lambda_1, \ell_1) \succ (\lambda_2, \ell_2)$. λ_1 has to be real, since otherwise there would exist a similar, but different, eigenvalue $(\overline{\lambda}_1, \ell_1)$. Since the Jordan block J_1 corresponding to (λ_1, ℓ_1) appears only once, its Fourier index contains exactly one element, $\mathcal{F}(J_1) = \{i_1\}$. However, by (5.18/100), $n - i_1$ is also in the Fourier index of $\overline{J}_1 = J_1$, what implies $i_1 = n - i_1 \mod n$. This condition has at most two solutions. Either $i_1 = 0$ or, if n is even, $i_1 = n/2$. In both cases, $2i_1 = 0 \mod n$. Hence, by (5.17/99), any eigenring f_1^i corresponding to J_1 has coinciding segments $f_1^i(\cdot, 2) = f_1^i(\cdot, 0)$. Now, we show that in all excluded cases the characteristic ring ψ is *not* injective, and hence, in view of Theorem 5.12/93, the algorithm is not C_1^k.

If $A \in \mathcal{A}_2^2 \cup \mathcal{A}_3^2$, then both components of the characteristic ring correspond to the first Jordan block, $\psi = [f_1^0, f_1^1]$. Hence, $\psi(\cdot, 2) = \psi(\cdot, 0)$, and ψ is not injective.

If $A \in \mathcal{A}_2^1 \cup \mathcal{A}_3^1$, then the second eigenvalue dominates the third one, $(\lambda_2, \ell_2) \succ (\lambda_3, \ell_3)$. By the same arguments as above, J_2 is real, the single element i_2 of the Fourier index $\mathcal{F}(J_2)$ satisfies $2i_2 = 0 \mod n$, and the segments $f_2^i(\cdot, 2) = f_2^i(\cdot, 0)$ of the eigenring f_2^i coincide. Hence, also in this case, the characteristic ring $\psi = [f_1^0, f_2^0]$ is not injective. $\square$

We now focus on the two remaining classes of algorithms. If $A \in \mathcal{A}_1^1$, then we have a double subdominant Jordan block $J_1 = J_2$ with Fourier index $\mathcal{F}(J_1) = \mathcal{F}(J_2) = \{i, n-i\}$. If $A \in \mathcal{A}_1^2$, then we have a complex conjugate pair of subdominant Jordan blocks $J_1 = \overline{J_2}$ with Fourier indices $\mathcal{F}(J_1) = \{i\}$ and $\mathcal{F}(J_2) = \{n-i\}$. In both cases, we call

$$\mathcal{F}_{\text{sub}} := \{i, n-i\}$$

the *subdominant Fourier index* of the algorithm. With $\hat{v}$ the eigenvector of $\hat{A}_i$ to λ_1 and $v = \mathcal{W}_i^{-1}\hat{v}$, the two subdominant eigenrings $f = Gv_1$ and $\overline{f} = G\overline{v}$ are complex-valued. For $A \in \mathcal{A}_1^2$, λ_1 is complex, and this is just the situation that we expect. We set $v_1^0 := v$, $v_2^0 := \overline{v}$ to obtain $f_1^0 = f$, $f_2^0 = \overline{f}$ and the characteristic ring $\psi := [\operatorname{Re} f_1^0, \operatorname{Im} f_1^0] = [\operatorname{Re} f, \operatorname{Im} f]$. For $A \in \mathcal{A}_1^1$, λ is real, and $v_1^0 := \operatorname{Re} v$, $v_2^0 := \operatorname{Im} v$ are real eigenvectors of A. Hence, $f_1^0 = \operatorname{Re} f$, $f_2^0 = \operatorname{Im} f$ are real subdominant eigenring, and again, the characteristic ring is $\psi := [f_1^0, f_2^0] = [\operatorname{Re} f, \operatorname{Im} f]$. Thus, the case distinction made in Definition 5.10/92 is resolved using the complex-valued eigenring f.

Definition 5.17 (Characteristic ring, complex). Let (A, G), $A \in \mathcal{A}_1^1 \cup \mathcal{A}_1^2$, be a shift invariant C_0^k-subdivision algorithm with subdominant Fourier index $\mathcal{F}_{\text{sub}} = \{i, n-i\}$ and a subdominant eigenvector

$$v := \mathcal{W}_i^{-1}\hat{v}, \quad \hat{A}_i\hat{v} = \lambda_1\hat{v}. \tag{5.19}$$

Then the *characteristic ring in complex form* of the algorithm is defined as the complex-valued ring

$$f = Gv \in C^k(\mathbf{S}_n^0, \mathbb{C}, G).$$

If clear from the context, the suffix "in complex form" is omitted.

As explained above, f is just the complexification of the formerly defined real characteristic ring,

$$\psi = [\operatorname{Re} f, \operatorname{Im} f].$$

Due to the relation (5.17/99), the complex version is sometimes more convenient for analytical purposes than the real form. For instance, it is helpful when proving the following theorem on the Fourier index of the subdominant eigenvalue. Its claim is illustrated by Fig. 5.4/102. On the left hand side, it shows the characteristic ring of the standard Doo–Sabin algorithm for $n = 5$ with weights according to (6.15/116). Here, the subdominant eigenvalue $\lambda = 1/2$ has the correct Fourier index $\mathcal{F}_{\text{sub}} = \{1, 4\}$. On the right hand side, the modified weights $a = [1, 0, 1, 1, 0]/3$ are used, which yield the subdominant eigenvalue $\lambda = (1 + \sqrt{5})/6 \approx 0.54$ with the inappropriate Fourier index $\mathcal{F}_{\text{sub}} = \{2, 3\}$.

Theorem 5.18 (Winding number of ψ and Fourier index). *Let (A, G) be a shift invariant subdivision algorithm with $A \in \mathcal{A}_1^1 \cup \mathcal{A}_1^2$. If the characteristic ring f is uni-cyclic then the subdominant Fourier index is $\mathcal{F}_{\text{sub}} = \{1, n-1\}$.*

Proof. Following Definition 5.7/88, we define the curve $z := f \circ \mathbf{c}_{\text{bnd}}$, which parametrizes the outer boundary of the image of the complex characteristic ring f. Let

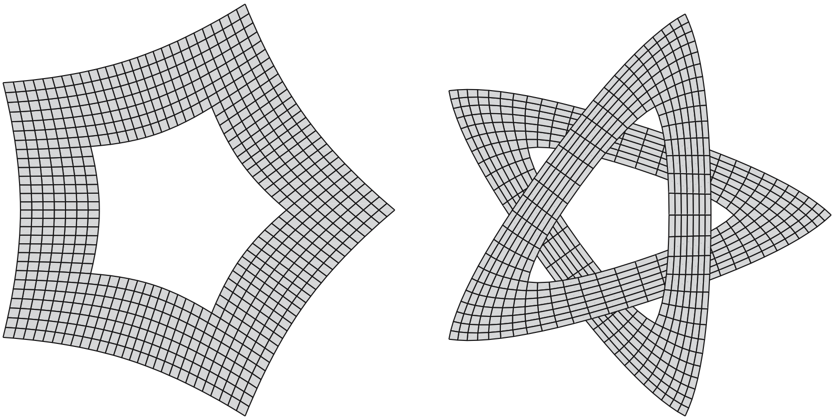

Fig. 5.4 Illustration of Theorem 5.18/101: Characteristic ring $\boldsymbol{\psi}$ of an algorithm for $n = 5$ using (*left*) standard Doo–Sabin weights so that the Fourier index is $\mathcal{F}(\lambda) = \{1, 4\}$ and (*right*) intentionally modified weights so that $\mathcal{F}(\lambda) = \{2, 3\}$. The figure shows and Theorem 5.18/101 proves that $\boldsymbol{\psi}$ is *not* uni-cyclic in the latter case.

us assume that $\mathcal{F}_{\text{sub}} = \{i, n-i\}$, then (5.17/99) implies

$$\frac{z'(u+j/n)}{z(u+j/n)} = \frac{w_n^{ij} z'(u)}{w_n^{ij} z(u)}, \quad u \in [0, 1/n]$$

for all $j \in \mathbb{Z}_n$. We obtain

$$2\pi \mathbf{i}\, \nu(\boldsymbol{\psi}) = \int_0^1 \frac{z'(u)}{z(u)}\, du = n \int_0^{1/n} \frac{z'(u)}{z(u)}\, du = n \ln \frac{z(1/n)}{z(0)},$$

where the imaginary part of the logarithm is only determined up to an integer multiple of 2π. By consistency of neighboring segments according to (4.9/62) and by (5.17/99),

$$z(1/n) = f(0,1,0) = f(1,0,1) = w_n^i z(0). \tag{5.20}$$

Hence, for some $\ell \in \mathbb{Z}$,

$$2\pi\, \nu(f) = n(2\pi i/n + 2\pi\ell),$$

implying that

$$1 = |\nu(f)| = |i + \ell n|.$$

The only solutions to this equation are given by $|i| = 1 \mod n$, as stated. □

Summarizing, a shift invariant C_1^k-algorithm must have a double subdominant eigenvalue, either real or complex, corresponding to the Fourier blocks 1 and $n-1$. The following definition removes some of the ambiguities in choosing the characteristic ring by fixing the index $i = 1$ in (5.19/101) and requiring $f(1,1,0)$ to be real and positive.

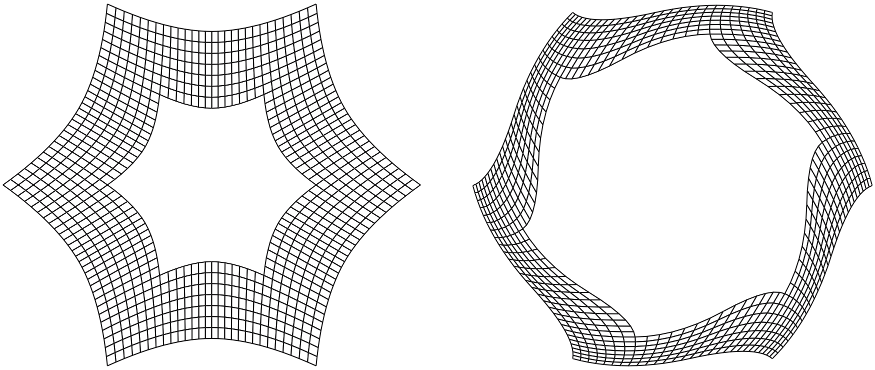

Fig. 5.5 Illustration of Example 5.20/103: Characteristic ring of algorithms for $n = 6$ using (*left*) standard Doo–Sabin weights and (*right*) asymmetric weights.

Definition 5.19 (Characteristic ring, normalized). The characteristic ring $f = Gv$ of a shift-invariant subdivision algorithm (A, G) is called *normalized*, if

$$v = \mathcal{W}_1^{-1}\hat{v}, \quad \hat{A}_1\hat{v} = \lambda_1\hat{v}$$

and the value

$$f(1,1,0) \in \mathbb{R}_{>0}$$

is a positive real number.

It is easily shown that normalization is always possible for a C_1^k-algorithm. In this case, by Theorem 5.12/93, f is injective. Further, $\mathcal{F}_{\text{sub}} = \{1, n-1\}$, and a subdominant eigenvector v can be defined as above. By (5.17/99), the characteristic ring $f = Gv$ satisfies

$$f_j = w_n^j f_0, \quad j \in \mathbb{Z}_n. \tag{5.21}$$

Hence, because f is injective, we have $f(1,1,1) = w_n f(1,1,0) \neq f(1,1,0)$ implying that $f(1,1,0) \neq 0$. Now, the rescaled eigenvector $\tilde{v} := rv$ yields the normalized complex characteristic ring $\tilde{f} = G\tilde{v}$ if we set, e.g., $r := 1/f(1,1,0)$.

5.5 Symmetric Algorithms

Now, we consider subdivision algorithms that are not only invariant under shift but also invariant under reversal of orientation when labelling the initial data. We call the reversal operation ‘flipping’. The following example illustrates lack of flip invariance:

Example 5.20 (Flip symmetry). On the left hand side, Fig. 5.5/103 shows the characteristic ring of the standard Doo–Sabin algorithm with weights according to (6.15/116) and subdominant eigenvalue $\lambda = 1/2$. On the right hand side, asymmetric weights $a = [4, 1, 0, 0, 0, 3]/8$ are used. These yield the complex subdominant eigenvalue $\lambda = (6 + \sqrt{3}i)/8 \approx 0.75 + \mathbf{i}0.22$ and the characteristic ring is not symmetric with respect to the x-axis. □

Orientation reversal of coefficient labels can be expressed by means of a square matrix R, the *flip matrix*. Analogous to shift invariance, flip invariance requires that A and R commute and that the rings $\mathbf{x}^m$ and $\tilde{\mathbf{x}}^m$ corresponding to $\mathbf{Q}$ and $\tilde{\mathbf{Q}} = R\mathbf{Q}$, respectively, differ only by a flip of orientation.

Definition 5.21 (Symmetry). A subdivision algorithm (A, G) is called *flip invariant*, if the system of generating rings satisfies

$$G(s, t, j) = G(t, s, -j)R, \quad (s, t, j) \in \mathbf{S}^0_n,$$

for some matrix R commuting with A,

$$AR = RA.$$

The algorithm is called *symmetric*, if it is both shift and flip invariant.

We observe that if the generating rings in G are linearly independent, then R must be an involution, $R = R^{-1}$.

The spectrum of the asymmetric case in Example 5.20/103 included a complex subdominant eigenvalue. This case is ruled out by symmetry.

Theorem 5.22 (Symmetry requires real subdominant eigenvalues). *The symmetric subdivision algorithm (A, G) can be C^k_1 only if $A \in \mathcal{A}^1_1$, i.e., the subdominant Jordan block is double and real,*

$$(\lambda, \ell) := (\lambda_1, \ell_1) = (\lambda_2, \ell_2) \succ (\lambda_2, \ell_3), \quad \lambda \in \mathbb{R}.$$

Proof. According to Theorem 5.16/100, $A \in \mathcal{A}^1_1$ or $A \in \mathcal{A}^2_1$, where we recall form Sect. 5.3/89 that the class $\mathcal{A}^2_1$ contains algorithms with a pair of complex conjugate subdominant eigenvalues. We assume $A \in \mathcal{A}^2_1$ and derive a contradiction:

From $AR = RA$ and $Av = \lambda_1 v$, we conclude $ARv = RAv = \lambda_1 Rv$, i.e., Rv is either 0 or an eigenvector of A to λ_1. Since the eigenvector to λ_1 is unique up to scaling, $Rv = av$ for some $a \in \mathbb{C}$. Using the definition of flip invariance, we obtain

$$f(1, 1, 0) = G(1, 1, 0)v = G(1, 1, 0)Rv = aG(1, 1, 0)v = af(1, 1, 0).$$

As explained in the sequel of Definition 5.19/103, we may assume that f is injective if (A, G) is a C^k_1-algorithm. In particular, we have $f(1, 1, 0) \neq 0$ so that $a = 1$. Further, by (5.21/103),

$$f(1, 0, 0) = G(1, 0, 0)v = G(1, 0, 0)Rv = G(0, 1, 0)v = f(0, 1, 0)$$

contradicting injectivity of f. □

This theorem explains why most subdivision algorithms of practical importance are standard algorithms according to Definition 5.3/84. Shift and flip invariance necessarily lead to a double subdominant Jordan block $J_1 = J_2 = J(\lambda, \ell)$ and, typically, this block is reduced to the trivial case $\ell = 0$, where the Jordan block is a singleton λ. An algorithm with non-trivial Jordan blocks is given in Sect. 6.3/120.

Consider the characteristic spline $h = Bv$ in complex form, where v is the subdominant eigenvector according to Definition 5.17/101. The m-th ring of h is

$$h^m = GA^m v = \lambda^m f.$$

That is, h is built from complex multiples of the characteristic ring. In the real case, applying the factor λ^m simply amounts to scaling, while in the complex case $\lambda = |\lambda| \exp(\mathbf{i}\phi)$. Hence, there is an additional rotation, $\boldsymbol{\xi}^m = |\lambda|^m \exp(\mathbf{i}m\phi) f$. This rotation is illustrated by Fig. 5.5/103 (*right*).

The following theorem establishes an additional symmetry property for the characteristic ring of a symmetric subdivision algorithm.

Theorem 5.23 (Symmetry of the characteristic ring). *Let $f = Gv$ be the normalized characteristic ring of a symmetric subdivision algorithm (A, G) with $A \in \mathcal{A}_1^1$. Then*

$$f(s,t,j) = \overline{f(t,s,-j)}, \quad (s,t,j) \in \mathbf{S}_n^0.$$

Proof. Here, the subdominant eigenvalue $\lambda := \lambda_1 = \lambda_2$ is double. As before, one can show that Rv is 0 or an eigenvector of A to λ. Hence, $Rv = av + b\overline{v}$ for some constants $a, b \in \mathbb{C}$ and by (5.16/99), $Sv = w_n^{-1} v$ and $S\overline{v} = w_n \overline{v}$. This implies $S^j R S^j v = a w_n^{-2j} v + b\overline{v}$. Let us assume without loss of generality that $f(1,1,0) = 1$. By symmetry, we obtain

$$1 = G(1,1,0)v = G(1,1,0)S^j R S^j v = a w_n^{-2j} G(1,1,0)v + bG(1,1,0)\overline{v}$$

for any $j \in \mathbb{Z}_n$. Since G is real, it follows $G(1,1,0)\overline{v} = G(1,1,0)v = 1$, and

$$1 = a w_n^{-2j} + b, \quad j \in \mathbb{Z}.$$

This implies $a = 0, b = 1$ and $Rv = \overline{v}$. Hence,

$$f(s,t,j) = G(s,t,j)v = G(t,s,-j)Rv = G(t,s,-j)\overline{v} = \overline{f(t,s,-j)}.$$

□

In case of symmetry, C_1^k-subdivision algorithms can be detected using significantly simplified criteria, which involve only properties of the upper half of the segment f_0 of the characteristic ring. In particular, the appropriate winding number $\nu(f) = 1$ can be proven by showing that one arc of the outer boundary of f_0 does not intersect the non-positive part of the real axis.

Theorem 5.24 (Conditions for symmetric C_1^k-algorithms). *Let (A, G) be a symmetric C_0^k-subdivision algorithm with $A \in \mathcal{A}_1^1$ and $\mathcal{F}(\lambda) = \{1, n-1\}$, and assume* *that the characteristic ring f is normalized. Then f is regular if and only if the first segment f_0 satisfies*

$$^{\times}\!D f_0(s,t) \neq 0 \quad \textit{for all} \quad (s,t) \in \boldsymbol{\Sigma}^0 \textit{ with } s \leq t.$$

Further, if f is regular, then (A,G) is a C_1^k-subdivision algorithm if and only if all real points on the curve $c(u) := f_0(u,1), u \in [0,1]$, are positive, i.e.,

$$c(u) \in \mathbb{R} \quad \Rightarrow \quad c(u) > 0.$$

Proof. By Theorem 5.23/105, $^{\times}\!D f_0(s,t) = {}^{\times}\!D f_0(t,s)$. Further, by (5.21/103), $^{\times}\!D f_j(s,t) = {}^{\times}\!D f_0(s,t)$, what proves the first part of the theorem.

To prove the second part, let us assume that $c(u_*) = f_0(u_*,1)$ is a negative real number. Then $u_* \neq 1$ because normalization requires $f_0(1,1) > 0$. By Theorem 5.23/105, $f(u_*,1,0) = f(1,u_*,0)$, showing that f is not injective. Hence, by Theorem 5.12/93, the algorithm is not C_1^k. Conversely, let the condition given in the theorem be satisfied. Following Definition 5.7/88, the winding number of f is

$$\nu(f) := \nu(f \circ \mathbf{c}_{\mathrm{bnd}}, \mathbf{0}).$$

The curves c and $z := f \circ \mathbf{c}_{\mathrm{bnd}}$ are related as follows: Let $u_j := j/n$. The curves c and $\bar{c}$ combine to the outer boundary of the segment f_0,

$$z_0(u) := \begin{cases} \bar{c}(2nu) & \text{if } u_0 \leq u < u_1/2 \\ c(2-2nu) & \text{if } u_1/2 \leq u \leq u_1, \end{cases}$$

and the segments of z are rotated copies of z_0,

$$z(u) = w_n^j z_0(u - u_j), \quad u_j \leq u \leq u_{j+1}.$$

Now, we apply Lemma 2.20/36. The disjoint half-lines are given by $h_j := -w_n^{j-1}$. Further, by (5.20/102) and Theorem 5.23/105,

$$c_0(u_1) = w_n c_0(u_0) = \overline{c_0}(u_0).$$

Hence, $\arg(z_1) = -\arg(z_0) = \pi/n$, and therefore

$$\arg(z_j/h_j) - \arg(z_{j-1}/h_j) = (1+1/n)\pi - (1-1/n)\pi = 2\pi/n.$$

Finally, we obtain the winding number

$$\nu(f) = \nu(z,0) = \frac{1}{2\pi}\sum_{j=1}^{n} 2\pi/n = 1,$$

showing that f is uni-cyclic. By Theorem (5.12/93), (A,G) is a C_1^k-algorithm. □

In some cases, an even simpler sufficient condition is applicable:

Theorem 5.25 (More conditions for symmetric C_1^k-algorithms). *Let (A,G) be a symmetric C_0^k-subdivision algorithm with $A \in \mathcal{A}_1^1$ and $\mathcal{F}(\lambda) = \{1, n-1\}$, and assume that the characteristic ring f is normalized. Then (A,G) is a C_1^k-subdivision algorithm if both components of $D_2 f_0$ are positive,*

$$\mathrm{Re}(D_2 f_0) > 0, \quad \mathrm{Im}(D_2 f_0) > 0. \tag{5.22}$$

Proof. Symmetry implies $\mathrm{Re}(D_1 f_0(s,t)) = \mathrm{Re}(D_2 f_0(t,s)) > 0$ and $\mathrm{Im}(D_1 f_0 (s,t)) = -\mathrm{Im}(D_2 f_0(t,s)) < 0$. Hence,

$$^{\times}\!D f_0 = \mathrm{Re}(D_1 f_0)\,\mathrm{Im}(D_2 f_0) - \mathrm{Im}(D_1 f_0)\,\mathrm{Re}(D_2 f_0) > 0,$$

showing that hat f_0 is regular. Further,

$$\int_u^1 D_1 f_0(\tau, 1)\, dt = f_0(1,1) - f_0(u,1) = f_0(1,1) - c(u).$$

$f_0(1,1)$ is real, and the imaginary part of the integrand is negative so that

$$\mathrm{Im}(c(u)) > 0 \quad \text{for} \quad u \in (0,1].$$

Hence, $c(1) = 1$ is the only real point in the image of c, and the argument is complete. □

Bibliographic Notes

1. Early attempts at verifying normal continuity of subdivision surfaces [DS78, BS86, BS88] were based on considering discrete normals derived from the control net and did not take into account the properties of the generating rings. The incompleteness of the attempts was exposed in [Rei93].

2. The concept of the characteristic ring was introduced by Reif in [Rei93, Rei95c] under the name 'characteristic map'. Earlier, Ball and Storry [BS86] introduced the related notion of *natural configuration* for the geometric layout of the control points of a characteristic ring.

3. The relevance of the Discrete Fourier Transform for the analysis of circulant subdivision matrices was already recognized by Doo and Sabin [DS78], and the approach was used by Ball and Storry [BS86, BS88]. The symmetry properties of characteristic rings and their relation to its Fourier index were highlighted in [PR98]. Further results can be found in [Kob98b].

4. The analysis of general algorithms of Sect. 5.3/89 was first derived in Reif's habilitation [Rei99] and was confirmed by Zorin's results [Zor00a]. Both sources list all possible leading eigenvalues compatible with C_1^k-algorithms. Given the tedium and complexity of the computation this provides welcome agreement.

5. The development of shift-invariant and symmetric algorithms in Sects. 5.4/95 and 5.5/103 follows [PR98]. As pointed out in that paper, shift and flip invariance of a C_1^k-algorithm imply a double subdominant Jordan block.

6. For a long time, it was taken for granted that a double subdominant eigenvalue $\lambda < 1$ were necessary for normal continuity. So it came as a surprise when the analysis in [PR97] revealed an algebraically eightfold, yet innocent, normal continuity preserving, subdominant eigenvalue λ for Simplest subdivision with $n = 3$, see also Sect. 6.3/120.

7. Reif [Rei93, Rei95c] established that regularity and injectivity of the characteristic ring are sufficient for smoothness. Necessity was proven in [PR98].

8. Regularity tests for the characteristic ring can be based on properties of the regular spline. For example, if the directional derivative can be expressed as a difference of spline coefficients of generating splines then the convex hull property can be used to confine directional derivatives to cones whose non-intersection establishes regularity. This geometric approach has been used, for example, by Umlauf and Ginkel [Uml99, Uml04, GU07a]. Zorin [Zor97, Zor00a] proposes to test regularity and injectivity via interval arithmetic. Injectivity for polynomial patches can also be checked by a technique of Goodman and Unsworth [GU96].

9. The simple condition of Theorem 5.24/105 for injectivity of the characteristic ring was not discovered before [RP05]. However, the proof given there is pedestrian. The idea of using concepts form algebraic topology is due to Zorin [Zor00b].

10. Conditions for asymmetric algorithms, as depicted in Fig. 5.5/103 *right*, were first discussed in [Rei95b].