9 Fuzzy Multiobjective LP

9.1 Introduction

This chapter continues Chapters 7 and 8. However now we have multiple objective functions we wish to maximize. We first discuss the general multiobjective fully fuzzified linear program in the next section. Then we study an example problem in Section 9.3. We have previously obtained an approximate fuzzy solution to this type of problem using an evolutionary algorithm [2]. In Section 9.4 we will apply our fuzzy Monte Carlo method to the problem to generate another approximate solution. Unfortunately, we will be unable to compare our Monte Carlo solution to our previous solution because we now are forced to use a different method of evaluating fuzzy inequalities.

Fuzzy multiobjective linear programming has also (along with fuzzy linear programming) become a large area of research. A few recent references to this topic are the papers ([1],[3],[6],[7],[9],[10],[16],[17]) and books (or articles in these books) ([4],[5],[8],[11]-[15]).

9.2 Multiobjective Fully Fuzzified Linear Programming

We are interested in the following problem

$$\max\left(\overline{Z} = (\overline{Z}_1, \ldots, \overline{Z}_K)\right) \tag{9.1}$$

where

$$\overline{Z}_k = \sum_{j=1}^{n} \overline{C}_{kj}\overline{X}_j, 1 \le k \le K, \tag{9.2}$$

subject to

$$\sum_{j=1}^{n} \overline{A}_{ij}\overline{X}_j \le \overline{B}_i, 1 \le i \le m, \tag{9.3}$$

$$\overline{X}_j \ge 0, \text{for all } j. \tag{9.4}$$

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In the above problem $\overline{C}_{kj}$, $\overline{A}_{ij}$, and $\overline{B}_i$ are all triangular fuzzy numbers. The $\overline{X}_i$ are triangular shaped fuzzy numbers. Let us rewrite it using matrix notation

$$\max\left(\overline{Z} = (\overline{Z}_1, \ldots, \overline{Z}_K)\right) \tag{9.5}$$
$$\overline{Z}_k = \overline{C}_k\,\overline{X}, 1 \leq k \leq K \tag{9.6}$$
$$\overline{A}\,\overline{X} \leq \overline{B}, \overline{X} \geq 0, \tag{9.7}$$

for $\overline{C}_k = (\overline{C}_{k1}, \ldots, \overline{C}_{kn})$, $1 \leq k \leq K$, $\overline{X}^t = (\overline{X}_1, \ldots, \overline{X}_n)$, $\overline{B}^t = (\overline{B}_1 \ldots, \overline{B}_m)$ and $\overline{A} = [\overline{A}_{ij}]$ a $m \times n$ matrix of fuzzy numbers.

To completely define the problem in equations (9.5)-(9.7) we must do two things: (1) define what we mean by max $\overline{Z}$, or finding the maximum of a vector of triangular shaped fuzzy numbers; and (2) explain what is meant by $\overline{A}\,\overline{X} \leq \overline{B}$. In the previous publications (see [2]) we handled max/min $\overline{Z}$ basically as discussed in Section 2.5 of Chapter 2. Also, in those publications we used both Kerre's Method (Section 2.6.2) and Chen's Method (Section 2.6.3) to evaluate $\leq$ and $\geq$ between fuzzy numbers. In our fuzzy Monte Carlo method we want to use both Kerre's Method and Chen's Method to evaluate $\leq$, $\geq$, $>$ and $<$ between fuzzy numbers but we will explain that we need to use another method.

9.3 Example Problem

This example is adapted from an example in ([18], p. 217). The crisp problem is

$$\max\left(z_1 = 5x_1 + 3x_2, z_2 = 2x_1 + 8x_2\right) \tag{9.8}$$
$$\text{subject to: } x_1 + 4x_2 \leq 100 \tag{9.9}$$
$$3x_1 + 2x_2 \leq 150 \tag{9.10}$$
$$5x_1 + 3x_2 \geq 200 \tag{9.11}$$
$$2x_1 + 8x_2 \geq 75 \tag{9.12}$$
$$x_1, x_2 \geq 0. \tag{9.13}$$

The feasible set is shown $\mathcal{F}$ in Figure 9.1, which is the bounded region with vertices A,B,C and D, and the undominated set is the line segment from A to B. The last constraint is inactive and does not define any part of the boundary of the feasible set. Undominated crisp vectors are discussed in Section 2.7 in Chapter 2. The general solution to this type of optimization problem is the set of undominated vectors $z = (z_1, z_2)$.

We now fully fuzzify the crisp problem using triangular fuzzy numbers whose vertex (core) value is the corresponding number in the crisp problem. The variables $\overline{X}_i$ will be triangular shaped fuzzy numbers. The multiobjective fully fuzzified linear program is

$$\max\left(\overline{Z} = (\overline{Z}_1, \overline{Z}_2)\right) \tag{9.14}$$
$$\overline{Z}_1 = (4/5/6)\overline{X}_1 + (2/3/4)\overline{X}_2 \tag{9.15}$$
$$\overline{Z}_2 = (1/2/3)\overline{X}_1 + (6/8/10)\overline{X}_2 \tag{9.16}$$

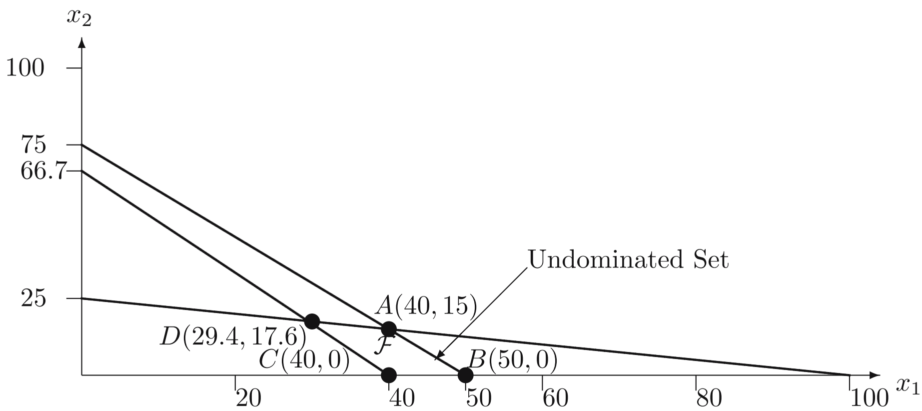

Fig. 9.1. Feasible Set in the Example problem

subject to:

$$(0/1/2)\overline{X}_1 + (3/4/5)\overline{X}_2 \leq (95/100/105) \tag{9.17}$$

$$(2/3/4)\overline{X}_1 + (1/2/3)\overline{X}_2 \leq (140/150/160) \tag{9.18}$$

$$(3/5/7)\overline{X}_1 + (2/3/4)\overline{X}_2 \geq (180/200/220) \tag{9.19}$$

$$(1/2/3)\overline{X}_1 + (6/8/10)\overline{X}_2 \geq (70/75/80) \tag{9.20}$$

$$\overline{X}_1, \overline{X}_2 \geq 0. \tag{9.21}$$

9.4 Fuzzy Monte Carlo Method

We will need to find intervals $I_i = [0, M_i]$, $i = 1, 2$, as explained in Section 6.3 in Chapter 6, for the $\overline{X}_i$. We randomly generate $\overline{X}_i \in [0, M_i]$, $i = 1, 2$, and form the random fuzzy vector $\overline{V} = (\overline{X}_1, \overline{X}_2)$. Since we form fuzzy vector $\overline{V} = (\overline{X}_1, \overline{X}_2)$, and each $\overline{X}_i$ consumes 5 crisp numbers, we choose our stream of Sobol quasi-random numbers which had been generated 10 at a time to get the two pairs of 5. We test to see if $\overline{V}$ is feasible, or the $\overline{X}_i$ satisfy the constraints. Assuming that $\overline{V}$ is feasible we compute $\overline{Z}_1 = \overline{C}_{11}\overline{X}_1 + \overline{C}_{12}\overline{X}_2$ and $\overline{Z}_2 = \overline{C}_{21}\overline{X}_1 + \overline{C}_{22}\overline{X}_2$. Now we combine $\overline{Z}_1$ and $\overline{Z}_2$ into one fuzzy number $\overline{Z}$ for ranking (finding the max). Let

$$\overline{Z} = \lambda\overline{Z}_1 + (1 - \lambda)\overline{Z}_2, \tag{9.22}$$

for $0 < \lambda < 1$. The decision maker(s) will choose various values for λ and then solve the fuzzy optimization problem max $\overline{Z}$. Let the current value of $\overline{Z} = \overline{Z}_0$ from feasible fuzzy vector $\overline{V}$. If $\overline{Z}^*$ is the best (max) value of $\overline{Z}$ up to now, then we replace $\overline{Z}^*$ with $\overline{Z}_0$ if $\overline{Z}^* < \overline{Z}_0$, otherwise we discard $\overline{Z}_0$.

We discussed weak and strong domination between fuzzy vectors in Section 2.7. We concluded that if we use Buckley's Method (Section 2.6.1) of evaluating inequalities between fuzzy numbers, then solutions $\overline{V} = (\overline{X}_1, \overline{X}_2)$ to $max\overline{Z}$ in equation (9.22) are strongly undominated solutions to the multiobjective fully fuzzified linear program. Let us explain in more detail what this means.

Pick and fix a value for the $\lambda > 0$. Let $\overline{V} = (\overline{X}_1, \overline{X}_2)$ be feasible (satisfy the constraints in equations (9.17)-(9.21) using Buckley's Method) and maximize $\overline{Z}$ in equation (9.22). Then $\overline{V}$ is strongly undominated which means that no other feasible fuzzy vector $\overline{W}$ can strongly dominate $\overline{V}$. This result is not true using Kerre's Method or Chen's Method (Section 2.7). The general solution to the multiobjective fully fuzzified linear program will be the set of strongly undominated fuzzy vectors. This means that we do not use Kerre's Method or Chen's Method to evaluate fuzzy inequalities for this problem.

Looking at the undominated set in Figure 9.1 we chose the interval $[20, 70]$ for $\overline{X}_1$ and $[0, 50]$ for $\overline{X}_2$. The $\overline{X}_i$ will be Bézier (quadratic) fuzzy numbers (QBGFNs in Chapter 4). So we now randomly generate a sequence $\overline{V}_k = (\overline{X}_{1k}, \overline{X}_{2k})$ with $\overline{X}_{1k} \in [20, 70]$ and $\overline{X}_{2k} \in [0, 50]$ all k. Using our Sobol quasi-random number generator we produce sequences of random vectors $v_{1k} = (x_{1k1}, ..., x_{1k5})$, $v_{2k} = (x_{2k1}, ..., x_{2k5})$, $k = 1, 2, 3, ...$ The sequence v_{1k} is used to get the sequence of quadratic fuzzy numbers $\overline{X}_{1k}$, recall that we only require vectors of length five for these fuzzy numbers (see Chapter 4), and the other sequence v_{2k} constructs the sequence of quadratic fuzzy numbers $\overline{X}_{2k}$, $k = 1, 2, 3, ...$ If $\overline{V}_k$ is feasible we compute

$$\overline{Z}_{1k} = \overline{C}_{11}\overline{X}_{1k} + \overline{C}_{12}\overline{X}_{2k}, \tag{9.23}$$

and

$$\overline{Z}_{2k} = \overline{C}_{21}\overline{X}_{1k} + \overline{C}_{22}\overline{X}_{2k}, \tag{9.24}$$

for $k = 1, 2, 3, ..., N$, where N is the predetermined total number of iterations.

Assume that the decision maker(s) believe that the two goals are equally important so they picked $\lambda = 0.5$. We want to find a k value, and hence a $\overline{V}_k$, to solve

$$max\{\overline{Z}_k | k = 1, 2, 3, ..., N\}, \tag{9.25}$$

where

$$\overline{Z}_k = 0.5\overline{Z}_{1k} + 0.5\overline{Z}_{2k}, \tag{9.26}$$

$k = 1, 2, 3, ..., N$.

Table 9.1. Monte Carlo Solution to the Fuzzy Linear Program, Buckley's Method, N=100,000

$max\overline{Z}$	$\overline{X}_i$ (QBGFN from Sobol)
$\approx (76.25/188.57/337.92)$	$\overline{X}_1 = (39.75, 40.54, 45.21, -0.49, 0.55)$
	$\overline{X}_2 = (8.26, 8.49, 8.55, 0.69, 0.68)$

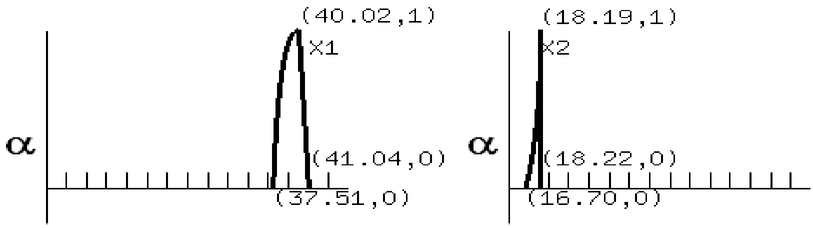

Fig. 9.2. $\overline{X}_1, \overline{X}_2$ using Buckley's Inequality, Example Problem

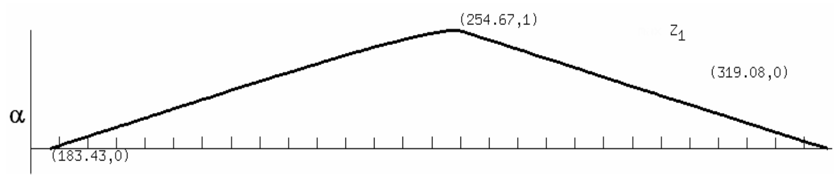

Fig. 9.3. $\overline{Z}_1$ using Buckley's Inequality, Example Problem

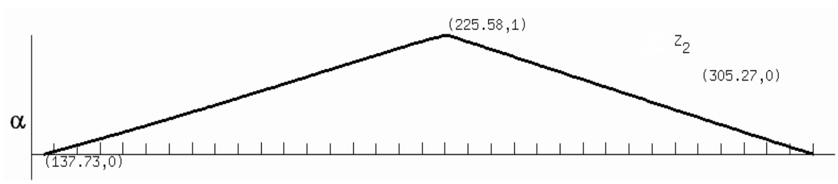

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Fig. 9.4. $\overline{Z}_2$ using Buckley's Inequality, Example Problem

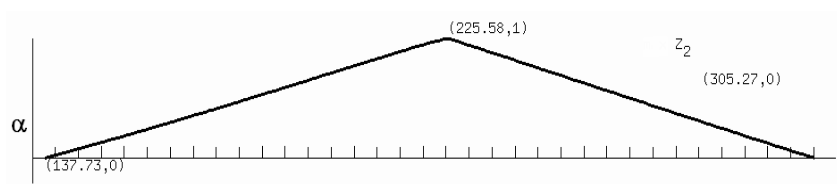

Fig. 9.5. $max\overline{Z}$ using Buckley's Inequality, Example Problem

With $N = 100,000$ the results of the fuzzy Monte Carlo method are shown in Table 9.1. All the fuzzy numbers in Table 9.1 are triangular shaped fuzzy numbers. $\overline{X}_1$ and $\overline{X}_2$ are QBGFNs. The notation we use for these fuzzy numbers was explained in Section 4.3.2. We define a QBGFN as (a, b, c, d, e) where: (1) the support is the interval $[a, c]$; (2) the vertex is at $x = b$; (3) the three numbers a, d, b define the quadratic function for the left side of the fuzzy number; and (4) the three numbers b, e, c specify the quadratic function for the right side of the fuzzy number. Since $max\overline{Z}$ is not necessarily, or likely to be a QBGFN, we only give the support and core for $max\overline{Z}$.

Figure 9.2 shows the "optimal" $\overline{X}_i$ and Figure 9.5 shows the value of the objective function $\overline{Z}$ using $\lambda = 0.5$, corresponding to the values of the $\overline{X}_i$ given in Figure 9.2.

9.5 Compare Solutions

For this multiobjective linear programming problem we have no crisp solution.

All of these software efforts were performed on Windows-based PCs. For the fuzzy Monte Carlo optimizations, several computers were used, mostly Dell Optiplex GX270's, 3.0GHz, 1GB RAM. This particular Monte Carlo simulation was performed on a Dell Inspiron 8200, 1.8GHz, 1GB RAM.

The fuzzy Monte Carlo optimization found $25,464$ feasible sets in a stream of 100,000 ($\overline{X}_{1k}$,$\overline{X}_{2k}$); 36 of them triggered new maximums. Elapsed time for the run was 16:15:57, but the last minimum was found after $11,332$ feasible sets at 07:26:06 (iteration $44,528$ of $100,000$) into the execution.

One may compare this solution is that given in [1], obtained by using an Evolutionary Algorithm to arrive at a solution. The results of that Evolutionary Algorithm method, using Kerre's method, are shown in Figures 9.6, 9.7, 9.8, and 9.9. The fuzzy numbers in Figure 9.6 are triangular fuzzy numbers. The fuzzy numbers in Figures 9.7, 9.8, and 9.9 are triangular shaped fuzzy numbers. One will need to review [1] to understand their evolutionary algorithm method. We compare our Monte Carlo solution using Sobol quasi-random numbers and Buckley's method with the Evolutionary Algorithm solution from [1] and find that the $max\overline{Z}$ from our Monte Carlo solution is greater than the $max\overline{Z}$ from the Evolutionary Algorithm solution (regardless which of Buckley's method, Kerre's method, or Chen's method is used to compare the maximums).

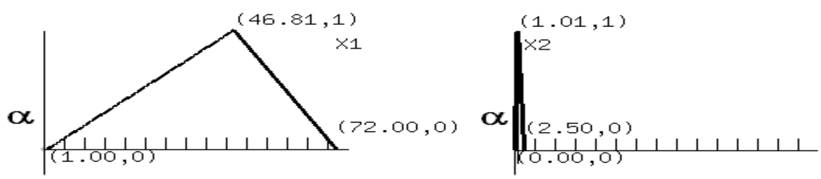

Fig. 9.6. Evolutionary Algorithm, $\overline{X}_1, \overline{X}_2$ using Kerre's $\leq$

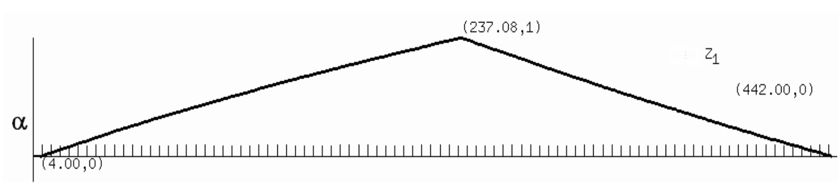

Fig. 9.7. Evolutionary Algorithm, $\overline{Z}_1$ using Kerre's $\leq$

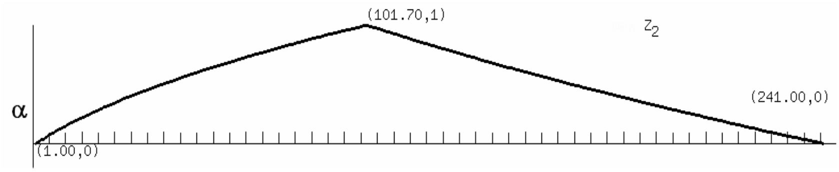

Fig. 9.8. Evolutionary Algorithm, $\overline{Z}_2$ using Kerre's $\leq$

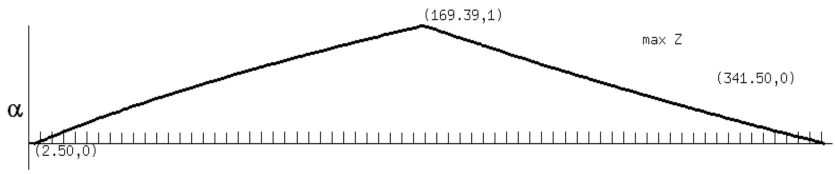

Fig. 9.9. Evolutionary Algorithm, $max\overline{Z}$ using Kerre's $\leq$

Next we attempt to compare our Monte Carlo results using Chen's method with an Evolutionary Algorithm solution given in [1]. Unfortunately, we determine a discrepancy in those Evolutionary Algorithm results. That Evolutionary Algorithm solution fails to satisfy the second and third constraint equations. Thus we were not able to use them.

We see by comparing these fuzzy Monte Carlo solutions with an Evolutionary Algorithm solution, the fuzzy Monte Carlo solution finds a greater fuzzy maximum.

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