2 Fuzzy Sets

2.1 Introduction

In this chapter we have collected together the basic ideas from fuzzy sets and fuzzy functions needed for the book. Any reader familiar with fuzzy sets, fuzzy numbers, the extension principle, α-cuts, interval arithmetic, and fuzzy functions may go on and have a look at Sections 2.5-2.7. In Section 2.5 we present a method that we have used in the past of maximizing/minimizing a fuzzy number $\overline{Z}$ which represents the value of some objective function in a fuzzy optimization problem. In Section 2.6 we are concerned with ordering a finite set of fuzzy numbers from smallest to largest to be used in our fuzzy Monte Carlo studies. Basically, given two fuzzy numbers $\overline{M}$ and $\overline{N}$, we need a method of deciding which of the following three possibilities is true: $\overline{M} < \overline{N}$, $\overline{M} \approx \overline{N}$, $\overline{M} > \overline{N}$. Three methods are discussed in Section 2.6. Section 2.7 discusses dominated and undominated fuzzy vectors needed in Chapter 9. Fuzzy vectors are vectors made up of fuzzy numbers. A good general reference for fuzzy sets and fuzzy logic is [4] and [19].

Our notation specifying a fuzzy set is to place a "bar" over a letter. So $\overline{A}$, $\overline{B}, \ldots$, $\overline{X}$, $\overline{Y}, \ldots$, $\overline{\alpha}$, $\overline{\beta}, \ldots$, will all denote fuzzy sets.

2.2 Fuzzy Sets

If Ω is some set, then a fuzzy subset $\overline{A}$ of Ω is defined by its membership function, written $\overline{A}(x)$, which produces values in $[0, 1]$ for all x in Ω. So, $\overline{A}(x)$ is a function mapping Ω into $[0, 1]$. If $\overline{A}(x_0) = 1$, then we say x_0 belongs to $\overline{A}$, if $\overline{A}(x_1) = 0$ we say x_1 does not belong to $\overline{A}$, and if $\overline{A}(x_2) = 0.6$ we say the membership value of x_2 in $\overline{A}$ is 0.6. When $\overline{A}(x)$ is always equal to one or zero we obtain a crisp (non–fuzzy) subset of Ω. For all fuzzy sets $\overline{B}$, $\overline{C}, \ldots$ we use $\overline{B}(x)$, $\overline{C}(x), \ldots$ for the value of their membership functions at x. Most of the fuzzy sets we will be using will be fuzzy numbers.

The term "crisp" will mean not fuzzy. A crisp set is a regular set. A crisp number is just a real number. A crisp matrix (vector) has real numbers as its components. A crisp function maps real numbers (or real vectors) into real numbers.

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A crisp solution to a problem is a solution involving crisp sets, crisp numbers, crisp functions, etc.

2.2.1 Fuzzy Numbers

A general definition of a fuzzy number may be found in ([4],[19]), however our fuzzy numbers will be almost always triangular (shaped), or trapezoidal (shaped), fuzzy numbers. A triangular fuzzy number (TFN) $\overline{N}$ is defined by three numbers $a < b < c$ where the base of the triangle is the interval $[a, c]$ and its vertex is at $x = b$. Triangular fuzzy numbers will be written as $\overline{N} = (a/b/c)$. A triangular fuzzy number $\overline{N} = (1.2/2/2.4)$ is shown in Figure 2.1. We see that $\overline{N}(2) = 1$, $\overline{N}(1.6) = 0.5$, etc.

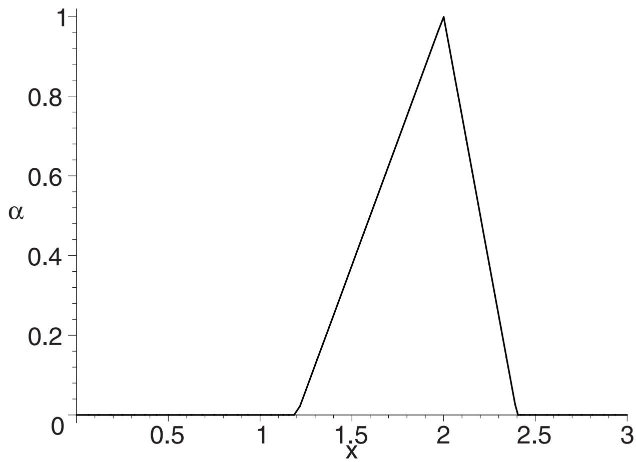

Fig. 2.1. Triangular Fuzzy Number $\overline{N}$

A trapezoidal fuzzy number $\overline{M}$ is defined by four numbers $a < b < c < d$ where the base of the trapezoid is the interval $[a, d]$ and its top (where the membership equals one) is over $[b, c]$. We write $\overline{M} = (a/b, c/d)$ for trapezoidal fuzzy numbers. Figure 2.2 shows $\overline{M} = (1.2/2, 2.4/2.7)$.

A triangular shaped fuzzy number $\overline{P}$ is given in Figure 2.3. $\overline{P}$ is only partially specified by the three numbers 1.2, 2, 2.4 since the graph on $[1.2, 2]$, and $[2, 2.4]$, is not a straight line segment. To be a triangular shaped fuzzy number we require the graph to be continuous and: (1) monotonically increasing on $[1.2, 2]$; and (2) monotonically decreasing on $[2, 2.4]$. For triangular shaped fuzzy number $\overline{P}$ we use the notation $\overline{P} \approx (1.2/2/2.4)$ to show that it is partially defined by the three numbers 1.2, 2, and 2.4. If $\overline{P} \approx (1.2/2/2.4)$ we know its base is on the interval $[1.2, 2.4]$ with vertex (membership value one) at $x = 2$. Similarly we define trapezoidal shaped fuzzy number $\overline{Q} \approx (1.2/2, 2.4/2.7)$ whose base is $[1.2, 2.7]$ and top is over the interval $[2, 2.4]$. The graph of $\overline{Q}$ is similar to $\overline{M}$ in Figure 2.2 but it has continuous curves for its sides.

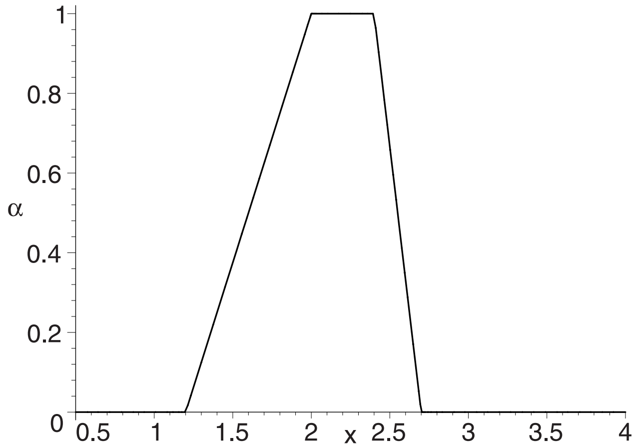

Fig. 2.2. Trapezoidal Fuzzy Number $\overline{M}$

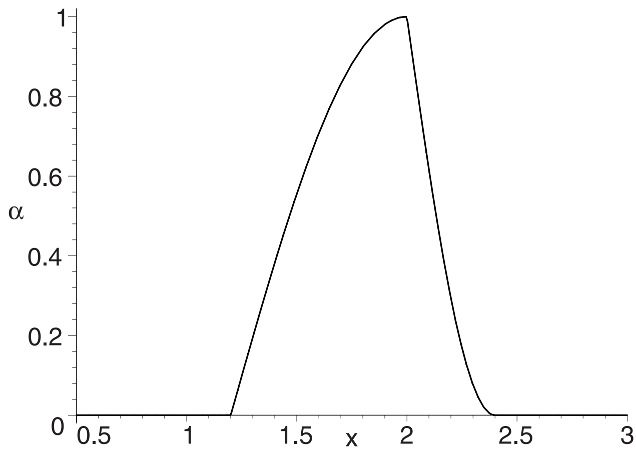

Fig. 2.3. Triangular Shaped Fuzzy Number $\overline{P}$

We will use special triangular shaped fuzzy numbers where their sides are defined by quadratic functions. These will be called quadratic fuzzy numbers (coded QBGFNs) and they are defined in Chapter 4.

Although we will be using triangular, trapezoidal(shaped) and quadratic fuzzy numbers throughout the book, many results can be extended to more general fuzzy numbers, but we shall be content to work with only these special fuzzy numbers.

2.2.2 Alpha-Cuts

Alpha-cuts are slices through a fuzzy set producing regular (nonfuzzy) sets. If $\overline{A}$ is a fuzzy subset of some set Ω, then an α-cut of $\overline{A}$, written $\overline{A}[\alpha]$ is defined as

$$\overline{A}[\alpha] = \{x \in \Omega | \overline{A}(x) \geq \alpha\}, \tag{2.1}$$

for all α, $0 < \alpha \leq 1$. The $\alpha = 0$ cut, or $\overline{A}[0]$, must be defined separately.

Let $\overline{N}$ be the fuzzy number in Figure 2.1. Then $\overline{N}[0] = [1.2, 2.4]$. Notice that using equation (2.1) to define $\overline{N}[0]$ would give $\overline{N}[0]$ = all the real numbers. Similarly, $\overline{M}[0] = [1.2, 2.7]$ from Figure 2.2 and in Figure 2.3 $\overline{P}[0] = [1.2, 2.4]$. For any fuzzy set $\overline{A}$, $\overline{A}[0]$ is called the support, or base, of $\overline{A}$. Many authors call the support of a fuzzy number the open interval (a, b) like the support of $\overline{N}$ in Figure 2.1 would then be $(1.2, 2.4)$. However in this book we use the closed interval $[a, b]$ for the support (base) of the fuzzy number.

The core of a fuzzy number is the set of values where the membership value equals one. If $\overline{N} = (a/b/c)$, or $\overline{N} \approx (a/b/c)$, then the core of $\overline{N}$ is the single point $x = b$. However, if $\overline{M} = (a/b, c/d)$, or $\overline{M} \approx (a/b, c/d)$, then the core of $\overline{M} = [b, c]$.

For any fuzzy number $\overline{Q}$ we know that $\overline{Q}[\alpha]$ is a closed, bounded, interval for $0 \leq \alpha \leq 1$. We will write this as

$$\overline{Q}[\alpha] = [q_1(\alpha), q_2(\alpha)], \tag{2.2}$$

where $q_1(\alpha)$ $(q_2(\alpha))$ will be an increasing (decreasing) function of α with $q_1(1) \leq q_2(1)$. If $\overline{Q}$ is a triangular shaped or a trapezoidal shaped fuzzy number then: (1) $q_1(\alpha)$ will be a continuous, monotonically increasing function of α in $[0, 1]$; (2) $q_2(\alpha)$ will be a continuous, monotonically decreasing function of α, $0 \leq \alpha \leq 1$; and (3) $q_1(1) = q_2(1)$ ($q_1(1) < q_2(1)$ for trapezoids). We sometimes check monotone increasing (decreasing) by showing that $dq_1(\alpha)/d\alpha > 0$ ($dq_2(\alpha)/d\alpha < 0$) holds.

For the $\overline{N}$ in Figure 2.1 we obtain $\overline{N}[\alpha] = [n_1(\alpha), n_2(\alpha)]$, $n_1(\alpha) = 1.2 + 0.8\alpha$ and $n_2(\alpha) = 2.4 - 0.4\alpha$, $0 \leq \alpha \leq 1$. Similarly, $\overline{M}$ in Figure 2.2 has $\overline{M}[\alpha] = [m_1(\alpha), m_2(\alpha)]$, $m_1(\alpha) = 1.2 + 0.8\alpha$ and $m_2(\alpha) = 2.7 - 0.3\alpha$, $0 \leq \alpha \leq 1$. The equations for $n_i(\alpha)$ and $m_i(\alpha)$ are backwards. With the y–axis vertical and the x–axis horizontal the equation $n_1(\alpha) = 1.2 + 0.8\alpha$ means $x = 1.2 + 0.8y$, $0 \leq y \leq 1$. That is, the straight line segment from $(1.2, 0)$ to $(2, 1)$ in Figure 2.1 is given as x a function of y whereas it is usually stated as y a function of x. This is how it will be done for all α-cuts of fuzzy numbers.

2.2.3 Inequalities

Let $\overline{N} = (a/b/c)$. We write $\overline{N} \geq \delta$, δ some real number, if $a \geq \delta$, $\overline{N} > \delta$ when $a > \delta$, $\overline{N} \leq \delta$ for $c \leq \delta$ and $\overline{N} < \delta$ if $c < \delta$. We use the same notation for triangular shaped and trapezoidal (shaped) fuzzy numbers whose support is the interval $[a, c]$.

If $\overline{A}$ and $\overline{B}$ are two fuzzy subsets of a set Ω, then $\overline{A} \leq \overline{B}$ means $\overline{A}(x) \leq \overline{B}(x)$ for all x in Ω, or $\overline{A}$ is a fuzzy subset of $\overline{B}$. $\overline{A} < \overline{B}$ holds when $\overline{A}(x) < \overline{B}(x)$, for all x. There is a potential problem with the symbol $\leq$. In some places in the book, for example see Section 2.6, $\overline{M} \leq \overline{N}$, for fuzzy numbers $\overline{M}$ and $\overline{N}$, means that $\overline{M}$ is less than or equal to $\overline{N}$. It should be clear on how we use "$\leq$" as to which meaning is correct.

2.2.4 Discrete Fuzzy Sets

Let $\overline{A}$ be a fuzzy subset of Ω. If $\overline{A}(x)$ is not zero only at a finite number of x values in Ω, then $\overline{A}$ is called a discrete fuzzy set. Suppose $\overline{A}(x)$ is not zero only at x_1, x_2, x_3 and x_4 in Ω. Then we write the fuzzy set as

$$\overline{A} = \{\frac{\mu_1}{x_1}, \cdots, \frac{\mu_4}{x_4}\}, \tag{2.3}$$

where the μ_i are the membership values. That is, $\overline{A}(x_i) = \mu_i$, $1 \leq i \leq 4$, and $\overline{A}(x) = 0$ otherwise. We can have discrete fuzzy subsets of any space Ω. Notice that α-cuts of discrete fuzzy sets of $\mathbb{R}$, the set of real numbers, do not produce closed, bounded, intervals.

2.3 Fuzzy Arithmetic

If $\overline{A}$ and $\overline{B}$ are two fuzzy numbers we will need to add, subtract, multiply and divide them. There are two basic methods of computing $\overline{A}+\overline{B}$, $\overline{A}-\overline{B}$, etc. which are: (1) extension principle; and (2) α-cuts and interval arithmetic.

2.3.1 Extension Principle

Let $\overline{A}$ and $\overline{B}$ be two fuzzy numbers. If $\overline{A}+\overline{B} = \overline{C}$, then the membership function for $\overline{C}$ is defined as

$$\overline{C}(z) = \sup_{x,y}\{\min(\overline{A}(x), \overline{B}(y))|x + y = z\}. \tag{2.4}$$

If we set $\overline{C} = \overline{A} - \overline{B}$, then

$$\overline{C}(z) = \sup_{x,y}\{\min(\overline{A}(x), \overline{B}(y))|x - y = z\}. \tag{2.5}$$

Similarly, $\overline{C} = \overline{A} \cdot \overline{B}$, then

$$\overline{C}(z) = \sup_{x,y}\{\min(\overline{A}(x), \overline{B}(y))|x \cdot y = z\}, \tag{2.6}$$

and if $\overline{C} = \overline{A}/\overline{B}$,

$$\overline{C}(z) = \sup_{x,y}\{\min(\overline{A}(x), \overline{B}(y))|x/y = z\}. \tag{2.7}$$

In all cases $\overline{C}$ is also a fuzzy number [19]. We assume that zero does not belong to the support of $\overline{B}$ in $\overline{C} = \overline{A}/\overline{B}$. If $\overline{A}$ and $\overline{B}$ are triangular (trapezoidal) fuzzy numbers then so are $\overline{A} + \overline{B}$ and $\overline{A} - \overline{B}$, but $\overline{A} \cdot \overline{B}$ and $\overline{A}/\overline{B}$ will be triangular (trapezoidal) shaped fuzzy numbers.

We should mention something about the operator "sup" in equations (2.4) – (2.7). If Ω is a set of real numbers bounded above (there is a M so that $x \leq M$, for all x in Ω), then $\sup(\Omega)$ = the least upper bound for Ω. If Ω has a maximum member, then $\sup(\Omega) = \max(\Omega)$. For example, if $\Omega = [0,1)$, $\sup(\Omega) = 1$ but if $\Omega = [0,1]$, then $\sup(\Omega) = \max(\Omega) = 1$. The dual operator to "sup" is "inf". If Ω is bounded below (there is a M so that $M \leq x$ for all $x \in \Omega$), then $\inf(\Omega)$ = the greatest lower bound. For example, for $\Omega = (0,1]$ $\inf(\Omega) = 0$ but if $\Omega = [0,1]$, then $\inf(\Omega) = \min(\Omega) = 0$.

Obviously, given $\overline{A}$ and $\overline{B}$, equations (2.4) – (2.7) appear quite complicated to compute $\overline{A} + \overline{B}$, $\overline{A} - \overline{B}$, etc. So, we now present an equivalent procedure based on α-cuts and interval arithmetic. First, we present the basics of interval arithmetic.

2.3.2 Interval Arithmetic

We only give a brief introduction to interval arithmetic. For more information the reader is referred to ([21],[22]). Let $[a_1, b_1]$ and $[a_2, b_2]$ be two closed, bounded, intervals of real numbers. If $\circ$ denotes addition, subtraction, multiplication, or division, then $[a_1, b_1] \circ [a_2, b_2] = [\alpha, \beta]$ where

$$[\alpha, \beta] = \{a \circ b | a_1 \leq a \leq b_1, a_2 \leq b \leq b_2\}. \tag{2.8}$$

If $*$ is division, we must assume that zero does not belong to $[a_2, b_2]$. We may simplify equation (2.8) as follows:

$$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]\,, \tag{2.9}$$

$$[a_1, b_1] - [a_2, b_2] = [a_1 - b_2, b_1 - a_2]\,, \tag{2.10}$$

$$[a_1, b_1] \,/\, [a_2, b_2] = [a_1, b_1] \cdot \left[\frac{1}{b_2}, \frac{1}{a_2}\right], \tag{2.11}$$

and

$$[a_1, b_1] \cdot [a_2, b_2] = [\alpha, \beta], \tag{2.12}$$

where

$$\alpha = \min\{a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2\}, \tag{2.13}$$

$$\beta = \max\{a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2\}. \tag{2.14}$$

Multiplication and division may be further simplified if we know that $a_1 > 0$ and $b_2 < 0$, or $b_1 > 0$ and $b_2 < 0$, etc. For example, if $a_1 \geq 0$ and $a_2 \geq 0$, then

$$[a_1, b_1] \cdot [a_2, b_2] = [a_1 a_2, b_1 b_2], \tag{2.15}$$

and if $b_1 < 0$ but $a_2 \geq 0$, we see that

$$[a_1, b_1] \cdot [a_2, b_2] = [a_1 b_2, a_2 b_1]. \tag{2.16}$$

Also, assuming $b_1 < 0$ and $b_2 < 0$ we get

$$[a_1, b_1] \cdot [a_2, b_2] = [b_1 b_2, a_1 a_2], \tag{2.17}$$

but $a_1 \geq 0$, $b_2 < 0$ produces

$$[a_1, b_1] \cdot [a_2, b_2] = [a_2 b_1, b_2 a_1]. \tag{2.18}$$

2.3.3 Fuzzy Arithmetic

Again we have two fuzzy numbers $\overline{A}$ and $\overline{B}$. We know α-cuts are closed, bounded, intervals so let $\overline{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$, $\overline{B}[\alpha] = [b_1(\alpha), b_2(\alpha)]$. Then if $\overline{C} = \overline{A} + \overline{B}$ we have

$$\overline{C}[\alpha] = \overline{A}[\alpha] + \overline{B}[\alpha]. \tag{2.19}$$

We add the intervals using equation (2.9). Setting $\overline{C} = \overline{A} - \overline{B}$ we get

$$\overline{C}[\alpha] = \overline{A}[\alpha] - \overline{B}[\alpha], \tag{2.20}$$

for all α in $[0, 1]$. Also

$$\overline{C}[\alpha] = \overline{A}[\alpha] \cdot \overline{B}[\alpha], \tag{2.21}$$

for $\overline{C} = \overline{A} \cdot \overline{B}$ and

$$\overline{C}[\alpha] = \overline{A}[\alpha] / \overline{B}[\alpha], \tag{2.22}$$

when $\overline{C} = \overline{A}/\overline{B}$, provided that zero does not belong to $\overline{B}[\alpha]$ for all α. This method is equivalent to the extension principle method of fuzzy arithmetic [19]. Obviously, this procedure, of α-cuts plus interval arithmetic, is more user (and computer) friendly.

Example 2.3.3.1

Let $\overline{A} = (-3/-2/-1)$ and $\overline{B} = (4/5/6)$. We determine $\overline{A} \cdot \overline{B}$ using α-cuts and interval arithmetic. We compute $\overline{A}[\alpha] = [-3+\alpha, -1-\alpha]$ and $\overline{B}[\alpha] = [4+\alpha, 6-\alpha]$. So, if $\overline{C} = \overline{A} \cdot \overline{B}$ we obtain $\overline{C}[\alpha] = [(\alpha-3)(6-\alpha), (-1-\alpha)(4+\alpha)]$, $0 \leq \alpha \leq 1$. The graph of $\overline{C}$ is shown in Figure 2.4.

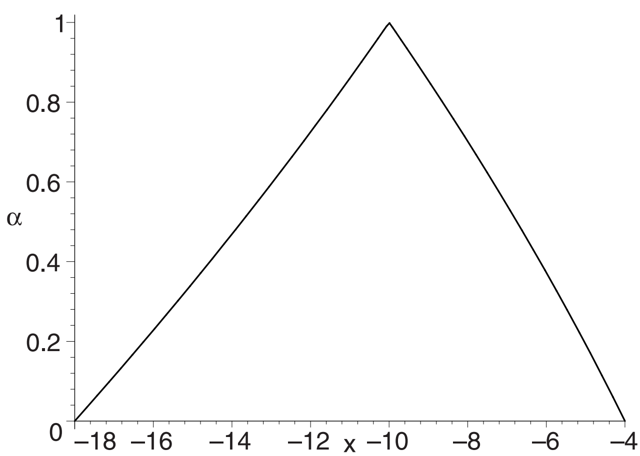

Fig. 2.4. The Fuzzy Number $\overline{C} = \overline{A} \cdot \overline{B}$

2.4 Fuzzy Functions

In this book a fuzzy function is a mapping from fuzzy numbers into fuzzy numbers. We write $H(\overline{X}) = \overline{Z}$ for a fuzzy function with one independent variable $\overline{X}$. Usually $\overline{X}$ will be a triangular (trapezoidal) fuzzy number and then we usually obtain $\overline{Z}$ as a triangular (trapezoidal) shaped fuzzy number. For two independent variables we have $H(\overline{X}, \overline{Y}) = \overline{Z}$.

Where do these fuzzy functions come from? They are usually extensions of real–valued functions. Let $h : [a, b] \to \mathbb{R}$. This notation means $z = h(x)$ for x in $[a, b]$ and z a real number. One extends $h : [a, b] \to \mathbb{R}$ to $H(\overline{X}) = \overline{Z}$ in two ways: (1) the extension principle; or (2) using α-cuts and interval arithmetic.

2.4.1 Extension Principle

Any $h : [a, b] \to \mathbb{R}$ may be extended to $H(\overline{X}) = \overline{Z}$ as follows

$$\overline{Z}(z) = \sup_x \left\{ \overline{X}(x) \mid h(x) = z,\ a \le x \le b \right\}. \tag{2.23}$$

Equation (2.23) defines the membership function of $\overline{Z}$ for any triangular (trapezoidal) fuzzy number $\overline{X}$ in $[a, b]$.

If h is continuous, then we have a way to find α-cuts of $\overline{Z}$. Let $\overline{Z}[\alpha] = [z_1(\alpha), z_2(\alpha)]$. Then [8]

$$z_1(\alpha) = \min\{\ h(x) \mid x \in \overline{X}[\alpha]\ \}, \tag{2.24}$$

$$z_2(\alpha) = \max\{\ h(x) \mid x \in \overline{X}[\alpha]\ \}, \tag{2.25}$$

for $0 \le \alpha \le 1$.

If we have two independent variables, then let $z = h(x, y)$ for x in $[a_1, b_1]$, y in $[a_2, b_2]$. We extend h to $H(\overline{X}, \overline{Y}) = \overline{Z}$ as

$$\overline{Z}(z) = \sup_{x,y} \left\{ \min\left(\overline{X}(x), \overline{Y}(y)\right) \mid h(x, y) = z \right\}, \tag{2.26}$$

for $\overline{X}$ ($\overline{Y}$) a triangular or trapezoidal fuzzy number in $[a_1, b_1]$ ($[a_2, b_2]$). For α-cuts of $\overline{Z}$, assuming h is continuous, we have

$$z_1(\alpha) = \min\{\ h(x, y) \mid x \in \overline{X}[\alpha],\ y \in \overline{Y}[\alpha]\ \}, \tag{2.27}$$

$$z_2(\alpha) = \max\{\ h(x, y) \mid x \in \overline{X}[\alpha],\ y \in \overline{Y}[\alpha]\ \}, \tag{2.28}$$

$0 \le \alpha \le 1$.

Applications

Let $f(x_1, ..., x_n; \theta_1, ..., \theta_m)$ be a continuous function. Then

$$I[\alpha] = \{ f(x_1, ..., x_n; \theta_1, ..., \theta_m) |\quad \mathbf{S}\quad \}, \tag{2.29}$$

for $\alpha \in [0, 1]$ and $\mathbf{S}$ is the statement "$\theta_i \in \overline{\theta}_i[\alpha]$, $1 \le i \le m$", for fuzzy numbers $\overline{\theta}_i$, $1 \le i \le m$, defines an interval $I[\alpha]$. The endpoints of $I[\alpha]$ may be found as in equations (2.24),(2.25) and (2.27),(2.28). $I[\alpha]$ gives the α-cuts of $f(x_1, ..., x_n; \overline{\theta}_i, ..., \overline{\theta}_m)$.

2.4.2 Alpha-Cuts and Interval Arithmetic

All the functions we usually use in engineering and science have a computer algorithm which, using a finite number of additions, subtractions, multiplications and divisions, can evaluate the function to required accuracy [7]. Such functions can be extended, using α-cuts and interval arithmetic, to fuzzy functions. Let $h : [a, b] \rightarrow \mathbb{R}$ be such a function. Then its extension $H(\overline{X}) = \overline{Z}$, $\overline{X}$ in $[a, b]$ is done, via interval arithmetic, in computing $h(\overline{X}[\alpha]) = \overline{Z}[\alpha]$, α in $[0, 1]$. We input the interval $\overline{X}[\alpha]$, perform the arithmetic operations needed to evaluate h on this interval, and obtain the interval $\overline{Z}[\alpha]$. Then we put these α-cuts together to obtain the value $\overline{Z}$. The extension to more independent variables is straightforward.

For example, consider the fuzzy function

$$\overline{Z} = H(\overline{X}) = \frac{\overline{A}\;\overline{X} + \overline{B}}{\overline{C}\;\overline{X} + \overline{D}}, \tag{2.30}$$

for triangular fuzzy numbers $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ and triangular fuzzy number $\overline{X}$ in $[0, 10]$. We assume that $\overline{C} \geq 0$, $\overline{D} > 0$ so that $\overline{C}\;\overline{X} + \overline{D} > 0$. This would be the extension of

$$h(x_1, x_2, x_3, x_4, x) = \frac{x_1 x + x_2}{x_3 x + x_4}. \tag{2.31}$$

We would substitute the intervals $\overline{A}[\alpha]$ for x_1, $\overline{B}[\alpha]$ for x_2, $\overline{C}[\alpha]$ for x_3, $\overline{D}[\alpha]$ for x_4 and $\overline{X}[\alpha]$ for x, do interval arithmetic, to obtain interval $\overline{Z}[\alpha]$ for $\overline{Z}$. Alternatively, the fuzzy function

$$\overline{Z} = H(\overline{X}) = \frac{2\overline{X} + 10}{3\overline{X} + 4}, \tag{2.32}$$

would be the extension of

$$h(x) = \frac{2x + 10}{3x + 4}. \tag{2.33}$$

2.4.3 Differences

Let $h : [a, b] \rightarrow \mathbb{R}$. Just for this subsection let us write $\overline{Z}^* = H(\overline{X})$ for the extension principle method of extending h to H for $\overline{X}$ in $[a, b]$. We denote $\overline{Z} = H(\overline{X})$ for the α-cut and interval arithmetic extension of h.

We know that $\overline{Z}$ can be different from $\overline{Z}^*$. But for basic fuzzy arithmetic in Section 2.3 the two methods give the same results. In the example below we show that for $h(x) = x(1-x)$, x in $[0, 1]$, we can get $\overline{Z}^* \neq \overline{Z}$ for some $\overline{X}$ in $[0, 1]$. What is known ([8],[21]) is that for usual functions in science and engineering $\overline{Z}^* \leq \overline{Z}$. Otherwise, there is no known necessary and sufficient conditions on h so that $\overline{Z}^* = \overline{Z}$ for all $\overline{X}$ in $[a, b]$. See also [20].

There is nothing wrong in using α-cuts and interval arithmetic to evaluate fuzzy functions. Surely, it is user, and computer friendly. However, we should be aware that whenever we use α-cuts plus interval arithmetic to compute

$\overline{Z} = H(\overline{X})$ we may be getting something larger than that obtained from the extension principle. The same results hold for functions of two or more independent variables.

Example 2.4.3.1

The example is the simple fuzzy expression

$$\overline{Z} = (1 - \overline{X})\ \overline{X}, \tag{2.34}$$

for $\overline{X}$ a triangular fuzzy number in $[0, 1]$. Let $\overline{X}[\alpha] = [x_1(\alpha), x_2(\alpha)]$. Using interval arithmetic we obtain

$$z_1(\alpha) = (1 - x_2(\alpha))x_1(\alpha), \tag{2.35}$$

$$z_2(\alpha) = (1 - x_1(\alpha))x_2(\alpha), \tag{2.36}$$

for $\overline{Z}[\alpha] = [z_1(\alpha), z_2(\alpha)]$, α in $[0, 1]$.

The extension principle extends the regular equation $z = (1 - x)x$, $0 \leq x \leq 1$, to fuzzy numbers as follows

$$\overline{Z}^*(z) = \sup_x \left\{ \overline{X}(x) | (1 - x)x = z,\ 0 \leq x \leq 1 \right\}. \tag{2.37}$$

Let $\overline{Z}^*[\alpha] = [z_1^*(\alpha), z_2^*(\alpha)]$. Then

$$z_1^*(\alpha) = \min\{(1 - x)x | x \in \overline{X}[\alpha]\}, \tag{2.38}$$

$$z_2^*(\alpha) = \max\{(1 - x)x | x \in \overline{X}[\alpha]\}, \tag{2.39}$$

for all $0 \leq \alpha \leq 1$. Now let $\overline{X} = (0/0.25/0.5)$, then $x_1(\alpha) = 0.25\alpha$ and $x_2(\alpha) = 0.50 - 0.25\alpha$. Equations (2.35) and (2.36) give $\overline{Z}[0.50] = [5/64, 21/64]$ but equations (2.38) and (2.39) produce $\overline{Z}^*[0.50] = [7/64, 15/64]$. Therefore, $\overline{Z}^* \neq \overline{Z}$. We do know that if each fuzzy number appears only once in the fuzzy expression, the two methods produce the same results ([8],[21]). However, if a fuzzy number is used more than once, as in equation (2.34), the two procedures can give different results.

2.5 Min/Max of a Fuzzy Number

In some fuzzy optimization problems we will want to determine the values of some decision variables $y = (x_1, ..., x_n)$ that will minimize (or maximize) a fuzzy function $\overline{E}(y)$. For each value of y we obtain a fuzzy number $\overline{E}(y)$. We have employed the method described below in previous publications and we will not use it in this book. We have included it so that the reader may understand our previous solution method when we compare it to our new fuzzy Monte Carlo procedure.

We can not minimize a fuzzy number so what we are going to do, which we have done before ([6],[9]-[13]), is first change $min\overline{E}(y)$ into a multiobjective problem

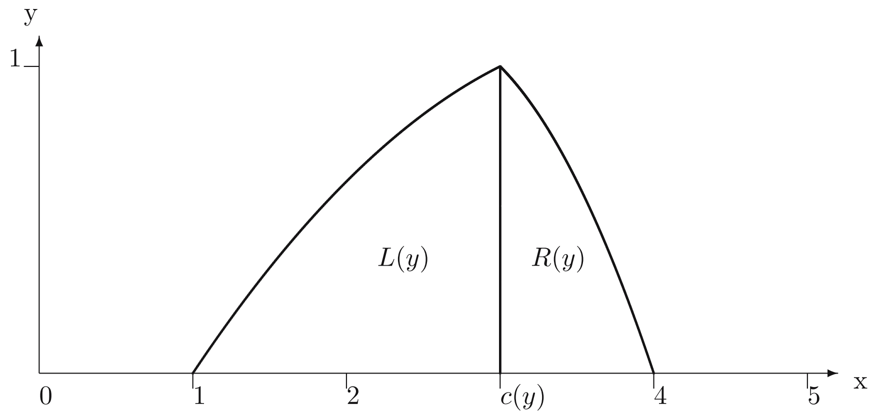

Fig. 2.5. Computations for the Minimum of a Fuzzy Number

and then translate the multiobjective problem into a single objective problem. This strategy is adopted from the finance literature where they had the problem of minimizing a random variable X whose values are constrained by a probability density function $g(x)$. They considered the multiobjective problem: (1) minimize the expected value of X; (2) minimize the variance of X; and (3) minimize the skewness of X to the right of the expected value. For our problem let: (1) $c(y)$ be the center of the core of $\overline{E}(y)$, the core of a fuzzy number is the interval where the membership function equals one, for each y; (2) $L(y)$ be the area under the graph of the membership function to the left of $c(y)$; and (3) $R(y)$ be the area under the graph of the membership function to the right of $c(y)$. See Figure 2.5. For $min\overline{E}(y)$ we substitute: (1) $min[c(y)]$; (2) $maxL(y)$, or maximize the possibility of obtaining values less than $c(y)$; and (3) $minR(y)$, or minimize the possibility of obtaining values greater then $c(y)$. So for $min\overline{E}(y)$ we have

$$V = (maxL(y), min[c(y)], minR(y)). \tag{2.40}$$

First let M be a sufficiently large positive number so that $maxL(y)$ is equivalent to $minL^*(y)$ where $L^*(y) = M - L(y)$. The multiobjective problem become

$$minV' = (minL^*(y), min[c(y)], minR(y)). \tag{2.41}$$

In a multiobjective optimization problem a solution is a value of the decision variable y that produces an undominated vector V'. Let $\mathcal{V}$ be the set of all vectors V' obtained for all possible values of the decision variable y. Vector $v_a = (v_{a1}, v_{a2}, v_{a3})$ dominates vector $v_b = (v_{b1}, v_{b2}, v_{b3})$, both in $\mathcal{V}$, if $v_{ai} \leq v_{bi}$, $1 \leq i \leq 3$, with one of the $\leq$ a strict inequality $<$. A vector $v \in \mathcal{V}$ is undominated if no $w \in \mathcal{V}$ dominates v. The set of undominated vectors in $\mathcal{V}$ is considered the general solution and the problem is to find values of the decision variables that produce undominated V'. The above definition of undominated was for a *min* problem, obvious changes need to be made for a *max* problem.

One way to explore the undominated set is to change the multiobjective problem into a single objective. The single objective problem is

$$min(\lambda_1[M - L(y)] + \lambda_2 c(y) + \lambda_3 R(y)), \tag{2.42}$$

where $\lambda_i > 0$, $1 \leq i \leq 3$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$. You will get different undominated solutions by choosing different values of $\lambda_i > 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$. It is known that solutions to this problem are undominated, but for some problems it will be unable to generate all undominated solutions [17]. The decision maker is to choose the values of the weights λ_i for the three minimization goals. Usually one picks different values for the λ_i to explore the solution set and then lets the decision maker choose an optimal y^* from this set of solutions.

2.6 Ordering Fuzzy Numbers

Given a finite set of fuzzy numbers $\overline{A}_1, ..., \overline{A}_n$ we would like to order them from smallest to largest. For a finite set of real numbers there is no problem in ordering them from smallest to largest. However, in the fuzzy case there is no universally accepted way to do this. There are probably more than 40 methods proposed in the literature of defining $\overline{M} \leq \overline{N}$, for two fuzzy numbers $\overline{M}$ and $\overline{N}$. Here the symbol $\leq$ means "less than or equal" and not "a fuzzy subset of". A few key references on this topic are ([1],[14]-[16],[18],[23],[24]) where the interested reader can look up many of these methods and see their comparisons.

In this section we will present three methods of defining $\overline{M} < \overline{N}$, $\overline{M} \approx \overline{N}$ and $\overline{M} \leq \overline{N}$ for two fuzzy numbers $\overline{M}$ and $\overline{N}$ which we will be using in this book.

2.6.1 Buckley's Method

For this book we have named this procedure Buckley's Method because we have used it before ([2],[3]). But note that different definitions of $\leq$ between fuzzy numbers can give different orderings. We first define $<$ between two fuzzy numbers $\overline{M}$ and $\overline{N}$. Define

$$v(\overline{M} \leq \overline{N}) = max\{min(\overline{M}(x), \overline{N}(y)) | x \leq y\}, \tag{2.43}$$

which measures how much $\overline{M}$ is less than or equal to $\overline{N}$. We write $\overline{N} < \overline{M}$ if $v(\overline{N} \leq \overline{M}) = 1$ but $v(\overline{M} \leq \overline{N}) < \eta$, where η is some fixed fraction in $(0,1]$. In this book we will usually use $\eta = 0.8$ or $\eta = 0.9$. Then $\overline{N} < \overline{M}$ if $v(\overline{N} \leq \overline{M}) = 1$ and $v(\overline{M} \leq \overline{N}) < 0.8$. We then define $\overline{M} \approx \overline{N}$ when both $\overline{N} < \overline{M}$ and $\overline{M} < \overline{N}$ are false. $\overline{M} \leq \overline{N}$ means $\overline{M} < \overline{N}$ or $\overline{M} \approx \overline{N}$. Now this $\approx$ may not be transitive. If $\overline{N} \approx \overline{M}$ and $\overline{M} \approx \overline{O}$ implies that $\overline{N} \approx \overline{O}$, then $\approx$ is transitive. However, it can happen that $\overline{N} \approx \overline{M}$ and $\overline{M} \approx \overline{O}$ but $\overline{N} < \overline{O}$ because $\overline{M}$ lies a little to the right of $\overline{N}$ and $\overline{O}$ lies a little to the right of $\overline{M}$ but $\overline{O}$ lies sufficiently far to the right of $\overline{N}$ that we obtain $\overline{N} < \overline{O}$.

But this ordering is still useful in partitioning the set of fuzzy numbers $\overline{A}_i$, $1 \leq i \leq n$, up into disjoint sets $H_1, ..., H_K$ where ([2],[3]): (1) given any $\overline{A}_i$ and

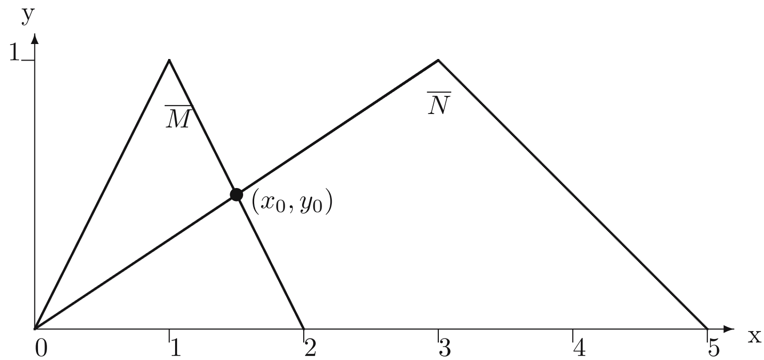

Fig. 2.6. Determining $v(\overline{N} \leq \overline{M})$

$\overline{A}_j$ in H_k, $1 \leq k \leq K$, then $\overline{A}_i \approx \overline{A}_j$; and (2) given $\overline{A_i} \in H_i$ and $i < j$, there is a $\overline{A}_j \in H_j$ with $\overline{A}_i < \overline{A}_j$. We say a fuzzy number $\overline{A}_i$ is dominated if there is another fuzzy number $\overline{A}_j$ so that $\overline{A}_i < \overline{A}_j$. So H_K will be all the undominated $\overline{A}_i$. Now H_K is nonempty and if it does not contain all the fuzzy numbers we then define H_{K-1} to be all the undominated fuzzy numbers after we delete all those in H_K. We continue this way to the last set H_1. Then the highest ranked fuzzy numbers lie in H_K, the second highest ranked fuzzy numbers are in H_{K-1}, etc. This result is easily seen if you graph all the fuzzy numbers on the same axis then those in H_K will be clustered together farthest to the right, proceeding from the H_K cluster to the left the next cluster will be those in H_{K-1}, etc.

There is an easy way to determine if $\overline{M} < \overline{N}$, or $\overline{M} \approx \overline{N}$, for many fuzzy numbers. This will be all we need in randomness tests and Monte Carlo studies. First, it is easy to see that if the core of $\overline{N}$ lies completely to the right of the core of $\overline{M}$, then $v(\overline{M} \leq \overline{N}) = 1$. Also, if the core of $\overline{M}$ and the core of $\overline{N}$ overlap, then $\overline{M} \approx \overline{N}$. Now assume that the core of $\overline{N}$ lies to the right of the core of $\overline{M}$, as shown in Figure 2.6 for triangular fuzzy numbers, and we wish to compute $v(\overline{N} \leq \overline{M})$. The value of this expression is simply y_0 in Figure 2.6. In general, for triangular (shaped), and trapezoidal (shaped), fuzzy numbers $v(\overline{N} \leq \overline{M})$ is the height of their intersection when the core of $\overline{N}$ lies to the right of the core of $\overline{M}$.

2.6.2 Kerre's Method

We first need to present the fuzzy max (written $\overline{\max}$) of two fuzzy numbers. If $\overline{O} = \overline{\max}(\overline{M}, \overline{N})$, then

$$\overline{O}(z) = \sup\left\{\min(\overline{M}(x), \overline{N}(y)) | \max(x, y) = z\right\}. \tag{2.44}$$

The authors in [19] give a detailed study of the properties of $\overline{\max}$ and $\overline{\min}$ (fuzzy min).

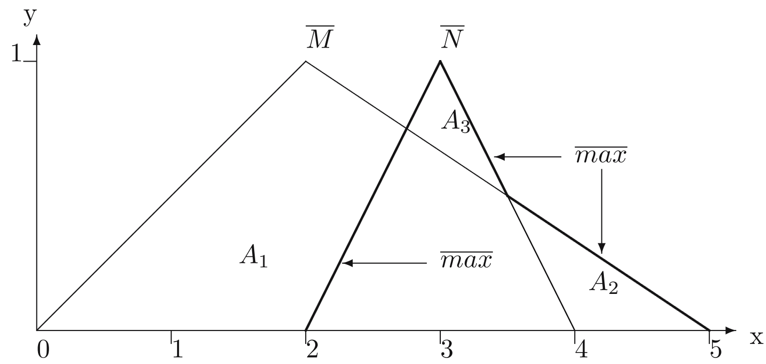

Fig. 2.7. Fuzzy Max

Next we define the Hamming distance between $\overline{M}$ and $\overline{N}$. The Hamming distance, $d(\overline{M}, \overline{N})$, is defined as

$$d(\overline{M}, \overline{N}) = \int_{-\infty}^{\infty} |\overline{M}(x) - \overline{N}(x)| dx. \tag{2.45}$$

Clearly, d is a metric (distance measure) on the space of continuous fuzzy numbers (those whose membership function is continuous).

Then we say $\overline{M} < \overline{N}$ is true whenever

$$d(\overline{N}, \overline{\max}(\overline{M}, \overline{N})) < d(\overline{M}, \overline{\max}(\overline{M}, \overline{N})). \tag{2.46}$$

This is simply a fuzzification of $x < y$ if and only if $\max(x, y) = y$ for real $x \neq y$. We write $\overline{M} \approx \overline{N}$ if you get equality in equation (2.46) and $\overline{M} \leq \overline{N}$ means $\overline{M} < \overline{N}$ or $\overline{M} \approx \overline{N}$. A numerical example showing $\overline{M} \leq \overline{N}$ by this method is in ([19], p. 407 - 408). We call this procedure for evaluating fuzzy inequalities Kerre's method [15].

Figure 2.7 shows the fuzzy max of two fuzzy numbers. We see that d $(\overline{M}, \overline{\max}(\overline{M}, \overline{N}))$ is the area of regions A_1 plus A_3 and $d(\overline{N}, \overline{\max}(\overline{M}, \overline{N}))$ is the area of region A_2. It appears that the area of region A_2 is less than the area of regions A_1 plus A_3 so $\overline{M} < \overline{N}$.

We point out from [23] that Kerre's $\leq$ is transitive.

2.6.3 Chen's Method

A third method of ranking fuzzy numbers we focus on was presented by Chen in [15]. A score is computed for each fuzzy number which is needed for ranking. The fuzzy set with the highest score is the largest fuzzy number. In order to rank triangular shaped fuzzy numbers $\overline{N} \approx (n_1/n_2/n_3)$ and $\overline{M} \approx (m_1/m_2/m_3)$ Chen defined a fuzzy max and a fuzzy min where the supports of fuzzy max and min is $[x_{\min}, x_{\max}]$ where

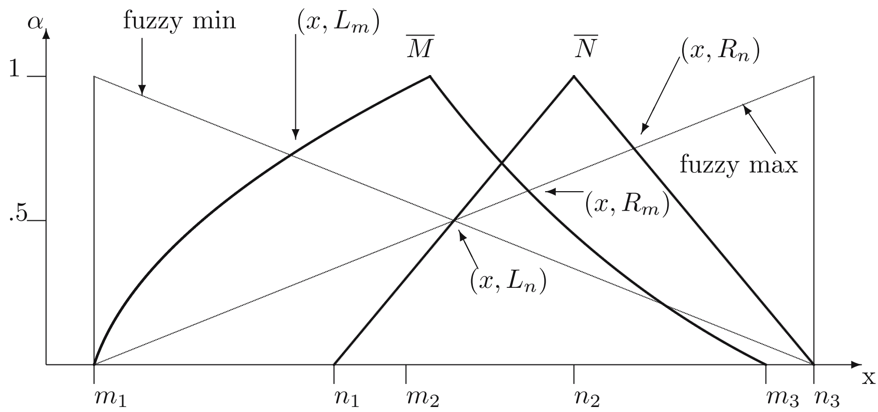

Fig. 2.8. Ranking Fuzzy Numbers Based on Chen's Method

$$x_{\min} = \min(n_1, m_1), \tag{2.47}$$
$$x_{\max} = \max(n_3, m_3). \tag{2.48}$$

Fuzzy min and fuzzy max are triangular fuzzy numbers with membership degree one at the left and the right limit of the support, respectively (see Figure 2.8). The membership functions are

$$\mu_{\min}(x) = \begin{cases} \frac{x - x_{\max}}{x_{\min} - x_{\max}} & : \ x_{\min} \leq x \leq x_{\max}, \\ 0 & : \ \text{otherwise} \end{cases} \tag{2.49}$$

$$\mu_{\max}(x) = \begin{cases} \frac{x - x_{\min}}{x_{\max} - x_{\min}} & : \ x_{\min} \leq x \leq x_{\max}, \\ 0 & : \ \text{otherwise.} \end{cases} \tag{2.50}$$

The intersection points between fuzzy max and $\overline{M}$ and $\overline{N}$ as well as the intersection points between fuzzy min and $\overline{M}$ and $\overline{N}$ are needed for computing the final scores. We compute

$$\mu_R(\overline{M}) = \sup_x (min(\mu_{\max}(x), \overline{M}(x))), \tag{2.51}$$

and

$$\mu_L(\overline{M}) = \sup_x (min(\mu_{\min}(x), \overline{M}(x))), \tag{2.52}$$

where $\mu_R(\overline{M})$ indicates the max of the intersection point between fuzzy max and $\overline{M}$ and $\mu_L(\overline{M})$ stands for the left score which is given by the max intersection point with fuzzy min. The larger $\mu_R(\overline{M})$ is, the higher $\overline{M}$ should be ranked. On the other hand a high value of $\mu_L(\overline{M})$ and $\overline{M}$ is close to the fuzzy min, and therefore should be ranked lower. By combining both scores we get the final rating

$$\mu_T(\overline{M}) = \frac{1}{2} \left(\mu_R(\overline{M}) + (1 - \mu_L(\overline{M}) \right). \tag{2.53}$$

Similarly, we get $\mu_T(\overline{N})$. We then say that $\overline{M} < \overline{N}$ is true if $\mu_T(\overline{M}) < \mu_T(\overline{N})$. In Figure 2.8 we used the notation $L_m = \mu_L(\overline{M})$, $R_m = \mu_R(\overline{M})$, $L_n = \mu_L(\overline{N})$ and $R_n = \mu_R(\overline{N})$. The labeling of L_m, R_m, L_n and R_n in Figure 2.8 may be a little misleading. These numbers are the y coordinates of the point indicated in the figure.

We write $\overline{M} \approx \overline{N}$ when $\mu_T(\overline{M}) = \mu_T(\overline{N})$ and as usual $\overline{M} \leq \overline{N}$ means $\overline{M} < \overline{N}$ or $\overline{M} \approx \overline{N}$.

We point out from [23] that Chen's $\leq$ is transitive.

2.6.4 Breaking Ties

We first adopt some method of deciding on $\leq$, $<$ and $\approx$ between fuzzy numbers. Assume we will use Buckley's Method. Sometimes in a fuzzy optimization problem, assume a max problem, we may get too many ties for maximum. Suppose we wish to $max \overline{Z} = f(\overline{X}_1, ..., \overline{X}_n)$ where the $\overline{X}_i$ are triangular fuzzy numbers and $\overline{Z}$ is a triangular shaped fuzzy number. Using our fuzzy Monte Carlo method we will generate a sequence $\overline{Z}_j$, $j = 1, 2, 3, ...$ Let H_K be the highest ranked fuzzy numbers in the sequence (Section 2.6.1). But H_K could contain 10, or 20, or 100 fuzzy numbers. Given $\overline{Z}_a$ and $\overline{Z}_b$ in H_K we know that $\overline{Z}_a \approx \overline{Z}_b$. What we can now do is rank the fuzzy numbers in H_K by their vertices. Let $\overline{Z}_a \approx (z_{a1}/z_{a2}/z_{a3})$ and $\overline{Z}_b \approx (z_{b1}/z_{b2}/z_{b3})$. We say $\overline{Z}_a < \overline{Z}_b$ if $z_{a2} < z_{b2}$, $\overline{Z}_a > \overline{Z}_b$ if $z_{a2} > z_{b2}$, and $\overline{Z}_a \approx \overline{Z}_b$ if $z_{a2} = z_{b2}$. The resulting highest ranked fuzzy numbers H_K^* should be more manageable. If we require a unique solution and we still have "ties" then we use the left (right) end points of the support. For example, if $\overline{Z}_a \approx \overline{Z}_b$ and: (1) $z_{b2} = z_{a2}$ but $z_{b1} < z_{a1}$ we say $\overline{Z}_b < \overline{Z}_a$; (2) $z_{b2} = z_{a2}$, $z_{b1} = z_{a1}$ and $z_{b3} < z_{a3}$ we say $\overline{Z}_b < \overline{Z}_a$; (3) $z_{b2} = z_{a2}$, $z_{b1} = z_{a1}$, $z_{b3} = z_{a3}$ we randomly discard one of them and declare the other the max (or min).

2.7 Undominated Fuzzy Vectors

We will first review the concept of undominated for crisp vectors. Consider a multiobjective optimization problem

$$max\ v = (v_1 = f_1(x), ..., v_m = f_m(x)), \tag{2.54}$$

where $x = (x_1, ..., x_n)$ is in the feasible set $\mathcal{F}$. Usually the x_i are non-negative. The optimization problem has constraints on the variables x_i and $\mathcal{F}$ is all x which satisfy these constraints. There will be certain changes for a min problem.

Let $\mathcal{V}$ be all vectors v from equation (2.54) obtained using all the $x \in \mathcal{F}$. Given $v_a = (v_{a1}, ..., v_{am})$ and $v_b = (v_{b1}, ..., v_{bm})$ in $\mathcal{V}$ we say v_a dominates v_b if $v_{ai} \geq v_{bi}$ all i with at least one of the $\geq$ is equal to $>$. The solution set $\mathcal{S}$ to the multiobjective max problem is all undominated $v \in \mathcal{V}$. The decision maker(s), depending on their preferences, would now choose certain $v \in \mathcal{S}$ as solutions to the optimization problem.

A way to generate undominated solutions is to consider the single objective optimization problem

$$max(\ \lambda_1 f_1(x) + ... + \lambda_m f_m(x)\ ), \tag{2.55}$$

for the $\lambda_i \in (0,1)$ all i and $\lambda_1 + ... + \lambda_m = 1$. It is known that all solutions are undominated but in certain problems we may not be able to obtain all undominated solutions by varying the values of the λ_i [17].

Now, as in Chapter 9, we consider a fuzzy multiobjective optimization problem

$$max\ \overline{V} = (\overline{V}_1 = f_1(\overline{X}), ..., \overline{V}_m = f_m(\overline{X})), \tag{2.56}$$

where $\overline{X} = (\overline{X}_1, ..., \overline{X}_n)$ is in the feasible set $\overline{\mathcal{F}}$. Assume we are using one of the three methods discussed above for evaluating $\leq$, $<$ and $\approx$ between fuzzy numbers. Usually the $\overline{X}_i \geq 0$ all i.

Let $\overline{\mathcal{V}}$ be all vectors $\overline{V}$ from equation (2.56) obtained using all the $\overline{X} \in \overline{\mathcal{F}}$. Given $\overline{V}_a = (\overline{V}_{a1}, ..., \overline{V}_{am})$ and $\overline{V}_b = (\overline{V}_{b1}, ..., \overline{V}_{bm})$ in $\overline{\mathcal{V}}$ we say $\overline{V}_a$ weakly dominates $\overline{V}_b$ if $\overline{V}_{ai} \geq \overline{V}_{bi}$ all i with at least one of the $\geq$ equal to $>$. We will call this definition of dominance "weak dominance". We will say $\overline{V}_a$ strongly dominates $\overline{V}_b$ if $\overline{V}_{ai} > \overline{V}_{bi}$ all i. We will employ both definitions of dominance. The solution set $\overline{\mathcal{S}}$ to the fuzzy multiobjective max problem is all (weakly, strongly) undominated $\overline{V} \in \overline{\mathcal{V}}$. Of course, we would like to show that this undominated set is nonempty.

Next we change the fuzzy multiobjective optimization problem into a single objective

$$max(\ \lambda_1 \overline{V}_1 + ... + \lambda_m \overline{V}_m\ ), \tag{2.57}$$

for $\lambda_i > 0$ all i and $\lambda_1 + ... + \lambda_m = 1$. We would now like to argue that any solution to equation (2.57) is (weakly, strongly) undominated. The argument depends on what definition for $\leq$, $<$ and $\approx$ you are using between fuzzy numbers. All that is needed is that if $\overline{V}_a$ (weakly, strongly) dominates $\overline{V}_b$ then

$$\sum_{i=1}^{m} \lambda_i \overline{V}_{ai} > \sum_{i=1}^{m} \lambda_i \overline{V}_{bi}. \tag{2.58}$$

If this is true one can easily obtain the desired result. Let us now prove this result for the special case of $m = 2$ and then we consider this problem for the three methods of defining $<$, $\leq$ and $\approx$ between fuzzy numbers discussed above.

We still consider the max problem with obvious changes for the min problem. Suppose the solution to

$$max(\ \lambda_1 \overline{X}_1 + \lambda_2 \overline{X}_2\ ) \tag{2.59}$$

is $\overline{X}_1^*$ and $\overline{X}_2^*$ for given (and fixed) $0 < \lambda_i < 1$, $i = 1, 2$ and $\lambda_1 + \lambda_2 = 1$. Assume $\overline{V}^* = (\overline{X}_1^*, \overline{X}_2^*)$ is not (weakly, strongly) undominated but $\overline{W} \in \overline{\mathcal{F}}$ (weakly, strongly) dominates $\overline{V}^*$. We now consider the two cases of weak and strong domination.

First we assume weak domination. So $\overline{X}_1^* < \overline{W}_1$ and $\overline{X}_2^* \le \overline{W}_2$. Assume we are using a method of defining $\le$, $<$ and $\approx$ between fuzzy numbers so that the following two results are true.

$$\lambda_1 \overline{X}_1^* < \lambda_1 \overline{W}_1, \ \lambda_2 \overline{X}_2^* \le \lambda_2 \overline{W}_2, \tag{2.60}$$

$$\lambda_1 \overline{X}_1^* + \lambda_2 \overline{X}_2^* < \lambda_1 \overline{W}_1 + \lambda_2 \overline{W}_2. \tag{2.61}$$

Then $\overline{V}^*$ is not the optimal solution. A contradiction. So if $\le$, $<$ and $\approx$ has the properties in equations (2.60) and (2.61) we get that the optimization problem in equation (2.59) only produces weakly undominated solutions.

Next we look at strong domination. So $\overline{X}_1^* < \overline{W}_1$ and $\overline{X}_2^* < \overline{W}_2$. Assume we are using a method of defining $\le$, $<$ and $\approx$ between fuzzy numbers so that the following two results are true.

$$\lambda_1 \overline{X}_1^* < \lambda_1 \overline{W}_1, \ \lambda_2 \overline{X}_2^* < \lambda_2 \overline{W}_2, \tag{2.62}$$

$$\lambda_1 \overline{X}_1^* + \lambda_2 \overline{X}_2^* < \lambda_1 \overline{W}_1 + \lambda_2 \overline{W}_2. \tag{2.63}$$

Then $\overline{V}^*$ is not the optimal solution. A contradiction. So if $\le$, $<$ and $\approx$ has the properties in equations (2.62) and (2.63) we get that the optimization problem in equation (2.59) only produces strongly undominated solutions.

2.7.1 Buckley's Method

We will use strong domination and show equation (2.63) is true for Buckley's Method. Let $\overline{X}_i^* \approx (x_{i1}/x_{i2}/x_{i3})$ and $\overline{W}_i \approx (w_{i1}/w_{i2}/w_{i3})$ for $i = 1, 2$. Assume that $x_{12} < w_{12}$ and $x_{22} < w_{22}$ as in Figure 2.6 and we are using $\eta = 0.8$. Then $v(\overline{X}_i^*, \overline{W}_i) = \eta_i < 0.8$ and $v(\overline{W}_i, \overline{X}_i^*) = 1$ for $i = 1, 2$. Assume that $\eta_1 \le \eta_2$. Now let $\overline{X}_i^*[\eta_2] = [x_{i1}(\eta_2), x_{i2}(\eta_2)]$ and $\overline{W}_i[\eta_2] = [w_{i1}(\eta_2), w_{i2}(\eta_2)]$ for $i = 1, 2$.

We know that $x_{22}(\eta_2) = w_{21}(\eta_2)$ and $x_{12}(\eta_2) \le w_{11}(\eta_2)$. Now let $\overline{X}^* = \lambda_1 \overline{X}_1^* + \lambda_2 \overline{X}_2^*$ and $\overline{W} = \lambda_1 \overline{W}_1 + \lambda_2 \overline{W}_2$. We see that

$$\overline{X}^*[\eta_2] = [\lambda_1 x_{11}(\eta_2) + \lambda_2 x_{21}(\eta_2), \lambda_1 x_{12}(\eta_2) + \lambda_2 x_{22}(\eta_2)], \tag{2.64}$$

and

$$\overline{W}[\eta_2] = [\lambda_1 w_{11}(\eta_2) + \lambda_2 w_{21}(\eta_2), \lambda_1 w_{12}(\eta_2) + \lambda_2 w_{22}(\eta_2)]. \tag{2.65}$$

Therefore $\overline{X}^* < \overline{W}$ since

$$\lambda_1 x_{12}(\eta_2) + \lambda_2 x_{22}(\eta_2) < \lambda_1 w_{11}(\eta_2) + \lambda_2 w_{21}(\eta_2). \tag{2.66}$$

Now we can explain why we did now use weak domination. Because we can have $\overline{X}_1^* \approx \overline{W}_1$, so $\overline{X}_1^* \le \overline{W}_1$ is true, and $\overline{X}_2^* < \overline{W}_2$ but for certain values of the λ_i we get $\overline{X}^* \approx \overline{W}$ so $\overline{X}^* < \overline{W}$ is not true.

2.7.2 Kerre's Method

First assume that we are using weak dominance. It was shown in [23] that if $\overline{X}_1^* < \overline{W}_1$ and $\overline{X}_2^* \leq \overline{W}_2$, then $\overline{X}_1^* + \overline{X}_2^* < \overline{W}_1 + \overline{W}_2$ may not be true. For this reason we will not use Kerre's Method in fuzzy multiobjective optimization problems in Chapter 9.

2.7.3 Chen's Method

First assume that we are using weak dominance. It was shown in [23] that if $\overline{X}_1^* < \overline{W}_1$ and $\overline{X}_2^* \leq \overline{W}_2$, then $\overline{X}_1^* + \overline{X}_2^* < \overline{W}_1 + \overline{W}_2$ may not be true. For this reason we will not use Chen's Method in fuzzy multiobjective optimization problems in Chapter 9.

References

1. Bortolon, G., Degani, R.: A Review of Some Methods for Ranking Fuzzy Subsets. Fuzzy Sets and Systems 15, 1–19 (1985)
2. Buckley, J.J.: Ranking Alternatives Using Fuzzy Numbers. Fuzzy Sets and Systems 15, 21–31 (1985)
3. Buckley, J.J.: Fuzzy Hierarchical Analysis. Fuzzy Sets and Systems 17, 233–247 (1985)
4. Buckley, J.J., Eslami, E.: Introduction to Fuzzy Logic and Fuzzy Sets. Physica-Verlag, Heidelberg (2002)
5. Buckley, J.J., Feuring, T.: Fuzzy and Neural: Interactions and Applications. Physica-Verlag, Heidelberg (1999)
6. Buckley, J.J., Feuring, T.: Evolutionary Algorithm Solutions to Fuzzy Problems: Fuzzy Linear Programming. Fuzzy Sets and Systems 109, 35–53 (2000)
7. Buckley, J.J., Hayashi, Y.: Can Neural Nets be Universal Approximators for Fuzzy Functions? Fuzzy Sets and Systems 101, 323–330 (1999)
8. Buckley, J.J., Qu, Y.: On Using α-cuts to Evaluate Fuzzy Equations. Fuzzy Sets and Systems 38, 309–312 (1990)
9. Buckley, J.J., Eslami, E., Feuring, T.: Fuzzy Mathematics in Economics and Engineering. Physica-Verlag, Heidelberg (2002)
10. Buckley, J.J., Feuring, T., Hayashi, Y.: Solving Fuzzy Problems in Operations Research. J. Advanced Computational Intelligence 3, 171–176 (1999)
11. Buckley, J.J., Feuring, T., Hayashi, Y.: Multi-Objective Fully Fuzzified Linear Programming. Int. J. Uncertainty, Fuzziness and Knowledge Based Systems 9, 605–622 (2001)
12. Buckley, J.J., Feuring, T., Hayashi, Y.: Fuzzy Queuing Theory Revisited. Int. J. Uncertainty, Fuzziness and Knowledge Based Systems 9, 527–538 (2001)
13. Buckley, J.J., Feuring, T., Hayashi, Y.: Solving Fuzzy Problems in Operations Research: Inventory Control. Soft Computing 7, 121–129 (2002)
14. Chang, P.T., Lee, E.S.: Fuzzy Arithmetic and Comparison of Fuzzy Numbers. In: Delgado, M., Kacprzyk, J., Verdegay, J.L., Vila, M.A. (eds.) Fuzzy Optimization: Recent Advances, pp. 69–81. Physica-Verlag, Heidelberg (1994)
15. Chen, S.J., Hwang, C.L.: Fuzzy Multiple Attribute Decision Making. Springer, Heidelberg (1992)

16. Dubois, D., Kerre, E., Mesiar, R., Prade, H.: Fuzzy Interval Analysis. In: Dubois, D., Prade, H. (eds.) Fundamentals of Fuzzy Sets, The Handbook of Fuzzy Sets, pp. 483–581. Kluwer Acad. Publ., Dordrecht (2000)
17. Geoffrion, A.M.: Proper Efficiency and the Theory of Vector Maximization. J. Math. Analysis and Appl. 22, 618–630 (1968)
18. Gonzalez, A., Vila, M.A.: Dominance Relations on Fuzzy Numbers. Information Sciences 64, 1–16 (1992)
19. Klir, G.J., Yuan, B.: Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall, Upper Saddle River, N.J. (1995)
20. Kreinovich, V., Longpre, L., Buckley, J.J.: Are There Easy-to-Check Necessary and Sufficient Conditions for Straightforward Interval Computations to be Exact? Reliable Computing 9, 349–358 (2003)
21. Moore, R.E.: Methods and Applications of Interval Analysis, SIAM Studies in Applied Mathematics, Philadelphia (1979)
22. Neumaier, A.: Interval Methods for Systems of Equations. Cambridge University Press, Cambridge, U.K. (1990)
23. Wang, X., Kerre, E.E.: Reasonable Properties for the Ordering of Fuzzy Quantities (I). Fuzzy Sets and Systems 118, 375–385 (2001)
24. Wang, X., Kerre, E.E.: Reasonable Properties for the Ordering of Fuzzy Quantities (II). Fuzzy Sets and Systems 118, 387–405 (2001)