15 Fuzzy Two-Person Zero-Sum Games

15.1 Introduction

In this chapter we use fuzzy Monte Carlo methods to get approximate optimal fuzzy mixed strategies for fuzzy two-person zero-sum games. In the next section we briefly review the results for crisp two-person zero-sum games. Then in Section 15.3 we fuzzify the games and define optimal fuzzy values for the players and optimal fuzzy mixed strategies. In the fourth section we introduce our fuzzy Monte Carlo method and use it on an example problem to generate approximate solutions. The last section contains our conclusions and suggestions for future research. Our fuzzy Monte Carlo method will be programmed in MATLAB [6]. This chapter is based on [1].

15.2 Two-Person Zero-Sum Games

There are two players named Player I and Player II. A is a $m \times n$ matrix of real numbers a_{ij}. Player I has pure strategies $i = 1, 2, 3, ..., m$, the row labels, and Player II has pure strategies $j = 1, 2, 3, ..., n$, the labels for the columns. If Player I chooses pure strategy i and Player II chooses pure strategy j, then the payoff from Player II to Player I is a_{ij} when $a_{ij} > 0$. If $a_{ij} < 0$, then the payoff is $-a_{ij}$ from Player I to Player II.

Sometimes the games have optimal strategies for both players in pure strategies. This is when the game has a saddle point. Suppose a_{ij} is both the maximum entry in its column and the minimum entry in its row. We have a saddle point and the pure strategy i for Player I and pure strategy j for Player II are optimal strategies for both players. So assume that the game has no saddle points. We now consider mixed strategies.

A mixed strategy for Player I is a probability vector $x = (x_1, ..., x_m)$, $x_i \in [0, 1]$ all i, and $\sum_{i=1}^{m} x_i = 1$. A mixed strategy for Player II is also a probability vector $y = (y_1, ..., y_n)$, $y_j \in [0, 1]$ all j, $\sum_{j=1}^{n} y_j = 1$. Player I chooses pure strategy i with probability x_i and Player II will choose pure strategy j with probability y_j. The expected payoff to Player I will be

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$$E = xAy^t, \tag{15.1}$$

where y^t is the transpose of row vector y.

Let X (Y) be the set of all mixed strategies for Player I (II). For a fixed $x \in X$ let

$$v(x) = min\{xAy^t | y \in Y\}, \tag{15.2}$$

and

$$v_I = max\{v(x) | x \in X\}. \tag{15.3}$$

The value of the game for Player I is v_I and a mixed strategy x^* in X solving equation (15.3) is an optimal mixed strategy for Player I. For a fixed $y \in Y$ define

$$v(y) = max\{xAy^t | x \in X\}, \tag{15.4}$$

and

$$v_{II} = min\{v(y) | y \in Y\}. \tag{15.5}$$

The value of the game for Player II is v_{II} and a mixed strategy y^* in Y solving equation (15.5) is an optimal mixed strategy for Player II. The minimax theorem says that $v_I = v_{II}$.

The details on two-person zero-sum games are in many books and two references are ([8],[10]). We now consider the probability vectors and the payoff matrix becoming fuzzy.

15.3 Fuzzy Two-Person Zero-Sum Games

There have been some papers/chapters in books, about fuzzy two-person zero-sum games which consider fuzzy payoffs, and sometimes fuzzy goals for the fuzzy payoffs ([2],[4],[5],[7],[9],[11],[12]), but not with fuzzy mixed strategies. We will allow both fuzzy payoffs and fuzzy mixed strategies.

We first fuzzify the payoff matrix $\overline{A} = (\overline{a}_{ij})$ where the $\overline{a}_{ij}$ are trapezoidal fuzzy numbers, or real numbers. Some of the payoffs can be real numbers but we still write all of them as fuzzy numbers. For example, if $a_{23} = 20$ we write $\overline{a}_{23} = 20$. The fact that a $\overline{a}_{ij}$ is fuzzy represents any uncertainty in the exact value of the payment.

How do we get these trapezoidal fuzzy numbers. We could employ expert opinion if we do not have any historical/statistical data to estimate these parameters. Suppose a_{34} is an uncertain payoff value in the fuzzy matrix $\overline{A}$. First assume we have only one expert and he/she is to estimate the value of some a_{ij}. We can solicit this estimate from the expert as is done in estimating job times in project scheduling ([10], Chapter 13). Let $a =$ the "pessimistic" value of a_{ij}, or the smallest possible value, let $d =$ the "optimistic" value of a_{ij}, or the highest possible value, and let $[b, c]$ be the interval of the most likely values of a_{ij}. We then ask the expert to give values for a, b, c, d and we construct the trapezoidal fuzzy number $\overline{a}_{34} = (a/b, c/d)$ for a_{34}. If we have a group of experts all to estimate the value of some a_{ij} we would average their response.

We define a saddle point for the fuzzy games the same way it was defined for the crisp game. Adopt the definitions of $<$, $\approx$ and $>$ between fuzzy numbers discussed in Section 15.4.2. We say the fuzzy game has a saddle point at $\overline{a}_{34}$ if $\overline{a}_{34} \geq \overline{a}_{i4}$, $i = 1, 2, \ldots, n$ and $\overline{a}_{34} \leq \overline{a}_{3j}$, $j = 1, 2, \ldots, m$. A $\leq$ ($\geq$) between fuzzy numbers means $<$ or $\approx$ ($>$ or $\approx$). If $\overline{a}_{34}$ is a saddle point the players have optimal strategies in pure strategies: Player I chooses the third row and Player II picks the fourth column. So assume that the fuzzy game does not have a saddle point.

Next the probability vectors for mixed strategies become fuzzy probabilities [3]. A fuzzy mixed strategy for Player I is $\overline{x} = (\overline{x}_1, ..., \overline{x}_m)$ where $\overline{x}_i \in [0, 1]$ is a triangular fuzzy number or a real number, all i, and there are $x_i \in \overline{x}_i[1]$ so that $\sum_{i=1}^{m} x_i = 1$. The last constraint says that for any $\alpha \in [0, 1]$ we can find $x_i \in \overline{x}_i[\alpha]$, $1 \leq i \leq m$, so that $x_1 + ... + x_m = 1$. Similarly we define a fuzzy mixed strategy $\overline{y}$ for Player II. For example if $m = 4$ we could have $\overline{x} = (0.2, \overline{x}_2, \overline{x}_3, 0.3)$ where $\overline{x}_1 = 0.2$, $\overline{x}_2 = (0/0.1/0.2)$, $\overline{x}_3 = (0.3/0.4/0.5)$ and $\overline{x}_4 = 0.3$. In this example we could use $(0.2, 0.2, 0.3, 0.3)$ when $\alpha = 0$ for a crisp mixed strategy. Let $\mathcal{X}$ ($\mathcal{Y}$) be all fuzzy mixed strategies for Player I (Player II). We wish to define, and find, optimal $\overline{x}^* \in \mathcal{X}$ ($\overline{y}^* \in \mathcal{Y}$) for Player I (Player II). We could consider trapezoidal fuzzy numbers for the $\overline{x}_i$ in $\overline{x}$ but we will use triangular fuzzy numbers in this chapter.

The fuzzy expected payoff $\overline{E}$ from the fuzzy game is determined by its α-cuts [3]

$$\overline{E}(\overline{x}, \overline{y})[\alpha] = \{\sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j |\ \mathcal{S}\ \}, \tag{15.6}$$

where $\mathcal{S}$ denotes the statement " $\overline{x} \in \mathcal{X}$, $x_i \in \overline{x}_i[\alpha]$ all i, $x_1 + ... + x_m = 1$, $\overline{y} \in \mathcal{Y}$, $y_j \in \overline{y}_j[\alpha]$ all j, $y_1 + ... + y_n = 1$ and $a_{ij} \in \overline{a}_{ij}[\alpha]$ all i, j ". This is how we will compute with fuzzy probabilities: for any α-cut we always choose only crisp probability distributions [3]. We may find these $\alpha-$cuts as follows

$$e(\overline{x}, \overline{y})_1(\alpha) = min\{\sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j |\ \mathcal{S}\ \}, \tag{15.7}$$

and

$$e(\overline{x}, \overline{y})_2(\alpha) = max\{\sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j |\ \mathcal{S}\ \}, \tag{15.8}$$

where $\overline{E}(\overline{x}, \overline{y})[\alpha] = [e(\overline{x}, \overline{y})_1(\alpha), e(\overline{x}, \overline{y})_2(\alpha)]$. We will need to solve this optimization problem using the Optimization Toolbox in MATLAB [6]. Also, this is the way we compute with fuzzy probabilities. We use "complete" crisp probabilities selected from the fuzzy numbers $\overline{x}_i$, $1 \leq i \leq m$ ($\overline{y}_j$, $1 \leq j \leq n$). Equation (15.6) defines the $\alpha-$cuts of trapezoidal shaped fuzzy number $\overline{E}(\overline{x}, \overline{y})$. Notice that $\overline{E}(\overline{x}, \overline{y})$ was not evaluated using the extension principle nor by α-cuts and interval arithmetic.

Now we define optimal fuzzy mixed strategies and consider a fuzzy minimax theorem. $\mathcal{X}$ ($\mathcal{Y}$) is the set of fuzzy mixed strategies to be used by Player I (Player II). These could be finite, or some other infinite restricted set of fuzzy probabilities, and are not necessarily all possible fuzzy probabilities. Let us first assume that $\mathcal{X}$ ($\mathcal{Y}$) is all fuzzy mixed strategies for Player I (Player II). We will use some method of defining $\leq$, $<$ and $\approx$ between fuzzy numbers, and in choosing the maximum/minimum of a (finite) set of fuzzy numbers, which will be defined and discussed in the next section. For each $\overline{x} \in \mathcal{X}$ define the fuzzy function

$$\overline{V}(\overline{x}) = min\{\overline{E}(\overline{x}, \overline{y}) \mid \overline{y} \in \mathcal{Y}\}. \tag{15.9}$$

Then the fuzzy value for the game for Player I is

$$\overline{V}_I = max\{\overline{V}(\overline{x}) \mid \overline{x} \in \mathcal{X}\}. \tag{15.10}$$

But we are unable to determine the fuzzy function $\overline{V}(\overline{x}) = \overline{z}$, $\overline{z}$ a trapezoidal shaped fuzzy number. For this reason we will employ our fuzzy Monte Carlo method to approximate $\overline{V}_I$ ($\overline{V}_{II}$) for Player I (Player II). This means that we will restrict $\mathcal{X}$ and $\mathcal{Y}$ to be finite sets of fuzzy mixed strategies.

In order for the notation here to match that in the next section on our Monte Carlo method we will now assume that $\mathcal{X}$ and $\mathcal{Y}$ are finite. Let $\mathcal{X} = \{\overline{x}_i \mid i = 1, \ldots, N\}$ and $\mathcal{Y} = \{\overline{y}_j \mid j = 1, \ldots, N\}$. We have changed our notation where now $\overline{x}_i$ ($\overline{y}_j$) is the whole fuzzy mixed strategy for Player I (II) and not a component of a fuzzy mixed strategy. That is, now $\overline{x}_i = (\overline{x}_{i1}, \ldots, \overline{x}_{im})$, $i = 1, \ldots, N$, and a similar expression for $\overline{y}_j$. Pick and fix $\overline{x}_i \in \mathcal{X}$. Let

$$\overline{E}(\overline{x}_i, \overline{y}_j) = \overline{U}_{ij}, \tag{15.11}$$

for $j = 1, \ldots, N$. Next compute

$$\overline{U}_i = min\{\overline{U}_{ij} \mid j = 1, \ldots, N\}. \tag{15.12}$$

We do this for each $i = 1, 2, \ldots, N$.

Next we determine

$$\overline{U}^* = max\{\overline{U}_i \mid i = 1, \ldots, N\}. \tag{15.13}$$

Now $\overline{U}^*$ will equal $\overline{U}_i$ for some i. If $\overline{U}^* = \overline{U}_{143}$, then set $\overline{x}^* = \overline{x}_{143}$. The fuzzy value for the game for Player I is $\overline{V}_I = \overline{U}^*$ and his/her optimal fuzzy mixed strategy is $\overline{x}^*$. Now we do similar calculations for Player II.

Pick and fix $\overline{y}_j$ in $\mathcal{Y}$. Let

$$\overline{E}(\overline{x}_i, \overline{y}_j) = \overline{V}_{ij}, \tag{15.14}$$

for $i = 1, \ldots, N$. Next compute

$$\overline{V}_j^* = max\{\overline{V}_{ij} \mid i = 1, \ldots, N\}. \tag{15.15}$$

We do this for each $j = 1, 2, \ldots, N$.

Next we determine

$$\overline{V}^* = min\{\overline{V}_j \mid j = 1, \ldots, N\}. \tag{15.16}$$

Now $\overline{V}^*$ will equal $\overline{V}_j$ for some j. If $\overline{V}^* = \overline{V}_{643}$, then set $\overline{y}^* = \overline{y}_{643}$. The fuzzy value for the game for Player II is $\overline{V}_{II} = \overline{V}^*$ and his/her optimal fuzzy mixed strategy is $\overline{y}^*$.

Now compare $\overline{U}^*$ and $\overline{V}^*$ both are trapezoidal shaped fuzzy numbers. Can they be equal? This would be the fuzzy minimax theorem. We will investigate this possibility using our fuzzy Monte Carlo method discussed in detail in the next section. However we do have the following result. We assume that we are using definitions of $<$, $\leq$ and $\approx$ between fuzzy numbers (see Section 15.4.2) so that: (1) we may find a unique solution to $max\{\overline{A}_\tau \mid \tau \in \Upsilon\}$ and to $min\{\overline{A}_\tau \mid \tau \in \Upsilon\}$ for the $\overline{A}_\tau$ fuzzy numbers and Υ an index set; and (2) given two fuzzy numbers $\overline{M}$ and $\overline{N}$ one and only one of the following is true $\overline{M} < \overline{N}$, $\overline{M} \approx \overline{N}$, $\overline{M} > \overline{N}$.

Theorem 15.1. $\overline{V}_I \leq \overline{V}_{II}$.

Proof
For $\overline{y} \in \mathcal{Y}$ consider

$$max\{\overline{E}(\overline{x}, \overline{y}) \mid \overline{x} \in \mathcal{X}\ \} = \overline{V}(\overline{y}). \tag{15.17}$$

This defines a mapping from each $\overline{y} \in \mathcal{Y}$ to an $\overline{x}^*$ in $\mathcal{X}$ so that $\overline{E}(\overline{x}^*, \overline{y}) = \overline{V}(\overline{y})$. We write $\overline{x}^* = f(\overline{y})$ so that $\overline{E}(f(\overline{y}), \overline{y}) = \overline{V}(\overline{y})$. Then

$$\overline{E}(\overline{x}, \overline{y}) \leq \overline{E}(f(\overline{y}), \overline{y}), \tag{15.18}$$

for all $\overline{x}$ and all $\overline{y}$. It follows that

$$min\{\overline{E}(\overline{x}, \overline{y}) \mid \overline{y} \in \mathcal{Y}\ \} \leq min\{\overline{E}(f(\overline{y}), \overline{y}) \mid \overline{y} \in \mathcal{Y}\ \}, \tag{15.19}$$

for all $\overline{x}$. But the right side of equation (15.19) is $\overline{V}_{II}$. Hence

$$min\{\overline{E}(\overline{x}, \overline{y}) \mid \overline{y} \in \mathcal{Y}\ \} \leq \overline{V}_{II}, \tag{15.20}$$

for all $\overline{x}$. Now take the max on $\overline{x} \in \mathcal{X}$ of the left side of equation (15.20) and the result follows. ■

A complete fuzzy Monte Carlo study would generate N fuzzy mixed strategies $\overline{x} \in \mathcal{X}$ and N fuzzy mixed strategies $\overline{y} \in \mathcal{Y}$, for $l = 1, 2, 3, \ldots, L$. Each study would produce fuzzy values $\overline{V}_I^{(l)}$, $\overline{V}_{II}^{(l)}$, and optimal fuzzy mixed strategies $\overline{x}^*_{(l)}$, $\overline{y}^*_{(l)}$, $l = 1, 2, \ldots, L$. Then we would compare these fuzzy values to choose our final approximations to the fuzzy value of the game for the players and their optimal fuzzy mixed strategies. However, because of the long computer time to accomplish each fuzzy Monte Carlo study, we will be able to do only one of them in the example in Section 15.4.3.

15.4 Fuzzy Monte Carlo

Assume that the fuzzy payoff matrix $\overline{A}$ is given. We first need to do two things: (1) describe how to get random sequences of fuzzy mixed strategies $\overline{x}_k$ and $\overline{y}_k$,

$k = 1, \ldots, N$; and (2) how we will determine the maximum/minimum of a finite set of fuzzy numbers. We first consider random sequences of fuzzy mixed strategies and then the max/min of a finite set of fuzzy numbers (equations (15.12),(15.13),(15.15),(15.16)). Then we can outline our fuzzy Monte Carlo method for producing an approximate solution to the problem discussed above and consider an example problem.

15.4.1 Random Sequences of Fuzzy Mixed Strategies

To obtain a random sequence $\overline{x}_k = (\overline{x}_{k1}, \ldots, \overline{x}_{km})$, $k = 1, 2, ..., N$, where each $\overline{x}_{kj}$ is a triangular fuzzy number in $[0, 1]$ and $\sum_{j=1}^{m} \overline{x}_{kj}[1] = 1$, we first randomly generate crisp vectors $v_k = (a_{k1}, ..., a_{k,3m})$ with all the a_{ki} in $[0, 1]$, $k = 1, 2, .., N$. We obtain the sequence v_k using our Sobol quasi-random number generator discussed in Chapter 3. We choose the first three numbers in v_k and order them from smallest to largest. Assume that $a_{k3} < a_{k1} < a_{k2}$. Then the first triangular fuzzy number $\overline{z}_{k1} = (a_{k3}/a_{k1}/a_{k2})$. Continue with the next three numbers in v_k, making $\overline{z}_{k2}$, etc. Assume $\overline{z}_{kj} = (z_{kj1}/z_{kj2}/z_{kj3})$, all k and j. Let $L_k = \sum_{j=1}^{m} z_{kj2}$. Then the final $\overline{x}_k$ is $\overline{x}_{kj} = (1/L_k)z_{kj}$ all k and j. If $\overline{x}_{kj} = (x_{kj1}/x_{kj2}/x_{kj3})$ we now have $\sum_{j=1}^{m} x_{kj2} = 1$. We construct the random sequence of fuzzy mixed strategies $\overline{y}_k$ for Player II the same way.

15.4.2 Max/Min of Fuzzy Numbers

Given a finite set of fuzzy numbers $\overline{U}_1, ..., \overline{U}_N$ we want to find the maximum and the minimum. For a finite set of real numbers there is no problem in ordering them from smallest to largest. However, in the fuzzy case there is no universally accepted way to do this. There are probably more than 50 methods proposed in the literature of defining $\overline{M} \le \overline{N}$, for two fuzzy numbers $\overline{M}$ and $\overline{N}$.

Here we will use only Buckley's Method presented in Section 2.6.1. We will now use $\eta = 0.9$ in Buckley's Method to help reduce the number of fuzzy numbers that could be considered approximately equal for the maximum/minimum of a set of fuzzy numbers. But note that different definitions of $\le$ between fuzzy numbers can give different orderings and therefore different final answers to the fuzzy game theory problem.

Now apply this to $\overline{U}_{ij}$, $j = 1, \ldots, N$, in equation (15.12). We will find the minimum sequentially. Suppose we are at stage $j = T - 1$ and the current minimum of $\overline{U}_{ij}$, $1 \le j \le T - 1$, is $\overline{S}$. The next step computes $\overline{U}_{iT} = \overline{R}$. There are three possibilities: (1) if $\overline{S} < \overline{R}$, then min remains $\overline{S}$ go on to the next step; (2) if $\overline{S} > \overline{R}$, then the new min is $\overline{R}$; and (3) if $\overline{S} \approx \overline{R}$, there are three more cases. Let $\overline{S} \approx (s_1/s_2, s_3/s_4)$ and $\overline{R} \approx (r_1/r_2, r_3/r_4)$ since they will be trapezoidal shaped fuzzy numbers. The next three cases are: (1) if $s_2 < r_2$, then min remains $\overline{S}$; (2) if $s_2 > r_2$, the min is now $\overline{R}$; and (3) if $s_2 = r_2$, there are three more cases. At this point $\overline{S} \approx \overline{R}$ and $s_2 = r_2$. The three new cases are: (1) if $s_3 < r_3$, the min remains $\overline{S}$; (2) if $s_3 > r_3$, then the min is $\overline{R}$; and (3) if $s_3 = r_3$, there are three more cases. We are at $\overline{S} \approx \overline{R}$, $s_2 = r_2$ and $s_3 = r_3$. The next three cases are: (1) if $s_1 < r_1$, then the min remains $\overline{S}$; (2) if $s_1 > r_1$, then the min is $\overline{R}$; and (3) if $s_1 = r_1$, then we have a final three cases to consider. We are at $\overline{S} \approx \overline{R}$, $s_2 = r_2$, $s_3 = r_3$ and $s_1 = r_1$. The final three cases are: (1) if $s_4 < r_4$, then the minimum remains $\overline{S}$; (2) if $s_4 > r_4$, then the minimum is $\overline{R}$; and (3) if $s_4 = r_4$, then randomly delete one of them and the other is the min. We do this so that there will be one and only one minimum. It is clear what changes are needed for a maximum.

15.4.3 Fuzzy Monte Carlo Solution Method

The basic program was written in MATLAB [6]. We must decide on N, the number of random fuzzy mixed strategies for each player and the number of α-cuts we need to determine for all the fuzzy numbers. For the α-cuts we need $\alpha = 0, 1$ and $\alpha = 0.9$ for the comparison of two fuzzy numbers (Section 15.4.2). We will use one more α-cut between zero and 0.9 so the α-cuts will be $\alpha = 0, 0.4, 0.9, 1$. After generating the random fuzzy mixed strategies $\overline{x}_k$ and $\overline{y}_k$, $k = 1, \ldots, N$, we need to evaluate equation (15.11) N^2 times. Also, each one is done four times for the α-cuts. So to get to equation (15.13) we compute equation (15.6) $4N^2$ times. For $\overline{y}^*$ also $4N^2$ times. A total of $8N^2$. With $N = 1000$ that equals $8,000,000$. That seems like the absolute max for our computer. So for the example below we will pick the smallest $\overline{A}$, a 2×2 fuzzy payoff matrix.

Before we consider the example let us look more closely at our MATLAB program. Let $\overline{E}(\overline{x}_i, \overline{y}_j) = \overline{W}_{ij}$. We previously called this $\overline{U}_{ij}$ in equation (15.11) and then $\overline{V}_{ij}$ in equation (15.14). Now we call it just $\overline{W}_{ij}$, a trapezoidal shaped fuzzy number evaluated at α-cuts, $\alpha = 0, 0.4, 0.9, 1$. We compute these α-cuts as in equations (15.7) and (15.8) using the Optimization Toolbox in MATLAB. Now form a $N \times N$ matrix rows labeled $\overline{x}_i$, columns labeled $\overline{y}_j$, whose ij^{th} element is the α-cuts of $\overline{W}_{ij}$. For each row $\overline{x}_i$ scan the row $j = 1, \ldots, N$ for the minimum $\overline{U}_i$ producing column vector $(\overline{U}_1, \ldots, \overline{U}_N)^t$. Now scan this column vector for the maximum $\overline{V}_I$. Next for each column $\overline{y}_j$ scan the row $i = 1, \ldots, N$ for the maximum $\overline{V}_j$ producing the row vector $(\overline{V}_1, \ldots, \overline{V}_N)$. Scan this row vector for its minimum $\overline{V}_{II}$.

Example 15.4.3.1

The fuzzy payoff matrix $\overline{A}$ will be 2×2 with trapezoidal fuzzy numbers $\overline{a}_{11} = (0/1, 2/3)$, $\overline{a}_{12} = 0$, $\overline{a}_{21} = (-2/-1, 0/1)$ and $\overline{a}_{22} = (1/2, 3/4)$. Now we want to estimate the fuzzy values of the game using $\overline{A}$ and random fuzzy mixed strategies for both players. The results are in Table 15.1 and Figure 15.1. In Figure 15.1 we approximated the trapezoidal shaped fuzzy numbers $\overline{V}_I$ and $\overline{V}_{II}$ by trapezoidal fuzzy numbers using only the support and core.

We see that the intersection of the core of $\overline{V}_I$ with the core of $\overline{V}_{II}$ is non-empty. Then from our definition of $<$ and $\approx$ between fuzzy numbers at the beginning of Section 15.4.2 we get $\overline{V}_I \approx \overline{V}_{II}$. However, if we also use the rest of Section 15.4.2 where we "fine tuned" $<$ between fuzzy numbers to obtain a unique max/min of a set of fuzzy numbers, we have $\overline{V}_I < \overline{V}_{II}$. Our Monte Carlo study showed $\overline{V}_I \leq \overline{V}_{II}$ which is the theorem in Section 15.3.

Table 15.1. Optimal Results from the Fuzzy Monte Carlo Method

Player I	Player II
$\overline{x}_1^* = (0.0398/0.7491/1.3658)$	$\overline{y}_1^* = (0.5247/0.5842/0.6245)$
$\overline{x}_2^* = (0.1432/0.2509/2.5463)$	$\overline{y}_2^* = (0.3454/0.4158/0.5953)$
$\overline{V}_I \approx (-1.8958/0.4983, 1.4671/2.6918)$	$\overline{V}_{II} \approx (-0.4976/0.7383, 1.7183/2.9469)$

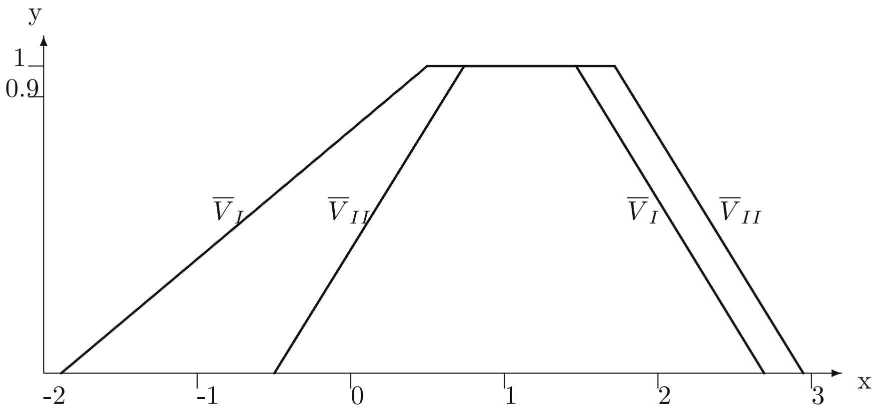

Fig. 15.1. Fuzzy Values $\overline{V}_I$ and $\overline{V}_{II}$ for the Players from the Monte Carlo Method

15.5 Conclusions and Future Research

In this chapter we considered a two-person zero-sum game with fuzzy payoffs and fuzzy mixed strategies for both players. We defined the fuzzy value of the game for both players ($\overline{V}_I$,$\overline{V}_{II}$) and also defined an optimal fuzzy mixed strategy for both players. We showed that $\overline{V}_I \leq \overline{V}_{II}$. We then employed our fuzzy Monte Carlo method to produce approximate solutions, to an example fuzzy game with no (fuzzy) saddle point, for the fuzzy values $\overline{V}_I$ for Player I and $\overline{V}_{II}$ for Player II; and also approximate solutions for the optimal fuzzy mixed strategies for both players. We then looked at $\overline{V}_I$ and $\overline{V}_{II}$ to see if there could be a Minimax theorem ($\overline{V}_I = \overline{V}_{II}$) for this fuzzy game. All our Monte Carlo study showed was $\overline{V}_I \leq \overline{V}_{II}$ which was the theorem in Section 15.3. So, it remains an open question will $\overline{V}_I = \overline{V}_{II}$ for these fuzzy games?

For all our Monte Carlo calculations, we used a Dell Optiplex GX 250 with a dual core and a 64-bit pentium D 2.8 GHz processor running on Windows XP. The computer time for $N = 100,000$ random fuzzy mixed strategies for both players was approximately 68 hours.

It would be nice to try $1,000,000$ random fuzzy mixed strategies for both players, but the computing time would be too excessive. However, if we could run the fuzzy Monte Carlo program with $N = 100,000$ random fuzzy mixed strategies for each player simultaneously on ten separate machines and then combine the results, we could go to $1,000,000$. The MATLAB program is available from the authors.

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