10 Solving Fuzzy Equations

10.1 Introduction

We start in the next Section 10.2 with looking at possible solutions to the simple fuzzy linear equation $\overline{A} \cdot \overline{X} + \overline{B} = \overline{C}$. We discuss three different types of solution which we have studied before in solving fuzzy equations. Then we present a fourth type of solution, based on our fuzzy Monte Carlo method, in Section 10.2.2. This new solution is based on random fuzzy numbers. In Section 10.3 we look at only "classical" solutions to the fuzzy quadratic equation and apply our fuzzy Monte Carlo method to obtain new solutions. Then in Section 10.4 we consider the fuzzy matrix equation $\overline{A} \cdot \overline{X} = \overline{B}$ and a number of solution types for $\overline{X}$ and then another solution based on fuzzy Monte Carlo techniques. The last section contains a brief summary and our conclusions.

In this chapter $\overline{M} \leq \overline{N}$ will mean that $\overline{M}$ is a fuzzy subset of $\overline{N}$ (Section 2.2.3) and not that $\overline{M}$ is less than or equal to $\overline{N}$. Solving fuzzy equations has always been an active area of research. Some recent references on this topic are ([1]-[4],[16]-[18],[21],[22]).

10.2 $\overline{A}\,\overline{X} + \overline{B} = \overline{C}$

$\overline{A}$, $\overline{B}$ and $\overline{C}$ will be triangular fuzzy numbers so let $\overline{A} = (a_1/a_2/a_3)$, $\overline{B} = (b_1/b_2/b_3)$ and $\overline{C} = (c_1/c_2/c_3)$. $\overline{X}$, if it exists, will be a triangular shaped fuzzy number so let $\overline{X} \approx (x_1/x_2/x_3)$. In the crisp equation

$$ax + b = c, \tag{10.1}$$

we immediately obtain $X = (c - b)/a$, if $a \neq 0$. We used the important facts $b - b = 0$ and $(1/a)a = 1$ from real numbers to get the solution.

We try this same approach with the fuzzy equation

$$\overline{A}\,\overline{X} + \overline{B} = \overline{C}, \tag{10.2}$$

we get

$$(1/\overline{A})(\overline{A}\,\overline{X} + (\overline{B} - \overline{B})) = (1/\overline{A})(\overline{C} - \overline{B}). \tag{10.3}$$

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But the left side of the equation (10.3) does not equal $\overline{X}$ since $\overline{B} - \overline{B} \neq 0$ and $(1/\overline{A})(\overline{A}) \neq 1$. For example, if $\overline{B} = (1/2/3)$, then $\overline{B} - \overline{B} = (-2/0/2)$ not zero. Also, if $\overline{A} = (1/2/3)$, $(1/\overline{A})(\overline{A}) \approx (\frac{1}{3}/1/3)$, a triangular shaped fuzzy number, not $(1/1/1)$.

This shows a major problem in solving fuzzy equations: some basic operations we used to solve crisp equations do not hold for fuzzy equations. Actually, this comes as no great surprise because this also happens in probability theory. If X is a random variable with positive variance, then $X - X \neq 0$ and $X/X \neq 1$ since both $X - X$ and X/X will have positive variance.

We now introduce our first solution method, called the classical method, producing solution $\overline{X}_c$ (when it exists). This procedure employs α-cuts and interval arithmetic (Section 2.4.2) to solve for $\overline{X}_c$. Let $\overline{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$, $\overline{B}[\alpha] = [b_1(\alpha), b_2(\alpha)]$, $\overline{C}[\alpha] = [c_1(\alpha), c_2(\alpha)]$ and $\overline{X}_c[\alpha] = [x_1(\alpha), x_2(\alpha)]$, $0 \leq \alpha \leq 1$. Substitute these into equation (10.2) producing

$$[a_1(\alpha), a_2(\alpha)][x_1(\alpha), x_2(\alpha)] + [b_1(\alpha), b_2(\alpha)] = [c_1(\alpha), c_2(\alpha)]. \tag{10.4}$$

We now use interval arithmetic (Section 2.3.2) to solve equation (10.4) for $x_1(\alpha)$ and $x_2(\alpha)$. We say that this method defines solution $\overline{X}_c$ when $[x_1(\alpha), x_2(\alpha)]$ defines the α-cuts of a fuzzy number. For the $x_1(\alpha)$, $x_2(\alpha)$ to specify a fuzzy number we need:

1. $x_1(\alpha)$ monotonically increasing, $0 \leq \alpha \leq 1$;
2. $x_2(\alpha)$ monotonically decreasing, $0 \leq \alpha \leq 1$; and
3. $x_1(1) \leq x_2(1)$.

We did not mention anything about the $x_i(\alpha)$ being continuous because throughout this chapter $x_1(\alpha)$, $x_2(\alpha)$ will be continuous.

Example 10.2.1

Let $\overline{A} = (1/2/3)$, $\overline{B} = (-3/-2/-1)$ and $\overline{C} = (3/4/5)$. Then $\overline{A}[\alpha] = [1+\alpha, 3-\alpha]$, $\overline{B}[\alpha] = [-3+\alpha, -1-\alpha]$, $\overline{C}[\alpha] = [3+\alpha, 5-\alpha]$. Since $\overline{A} > 0$ and $\overline{C} > 0$, we must have $\overline{X}_c > 0$, and equation (10.4) gives

$$[a_1(\alpha)x_1(\alpha) + b_1(\alpha), a_2(\alpha)x_2(\alpha) + b_2(\alpha)] = [c_1(\alpha), c_2(\alpha)], \tag{10.5}$$

or

$$x_1(\alpha) = \frac{6}{1+\alpha}, \tag{10.6}$$

$$x_2(\alpha) = \frac{6}{3-\alpha}, \tag{10.7}$$

after substituting for $a_1(\alpha), \ldots, c_2(\alpha)$ and solving for $x_i(\alpha)$. But $x_1(\alpha)$ is decreasing and $x_2(\alpha)$ is increasing. So, $\overline{X}_c$ does not exist.

Example 10.2.2

Now set $\overline{A} = (8/9/10)$, $\overline{B} = (-3/-2/-1)$ and $\overline{C} = (3/5/7)$. So $\overline{A}[\alpha] = [8+\alpha, 10-\alpha]$, $\overline{B}[\alpha] = [-3+\alpha, -1-\alpha]$, $\overline{C}[\alpha] = [3+2\alpha, 7-2\alpha]$. Again we must have $\overline{X}_c > 0$ so we obtain

$$x_1(\alpha) = \frac{6+\alpha}{8+\alpha}, \tag{10.8}$$

$$x_2(\alpha) = \frac{8-\alpha}{10-\alpha}. \tag{10.9}$$

We see that $x_1(\alpha)$ is increasing (its derivative is positive), $x_2(\alpha)$ is decreasing (derivative is negative) and $x_1(1) = 7/9 = x_2(1)$. The solution $\overline{X}_c$ exists, with α-cuts $[x_1(\alpha), x_2(\alpha)]$, shown in Figure 10.1.

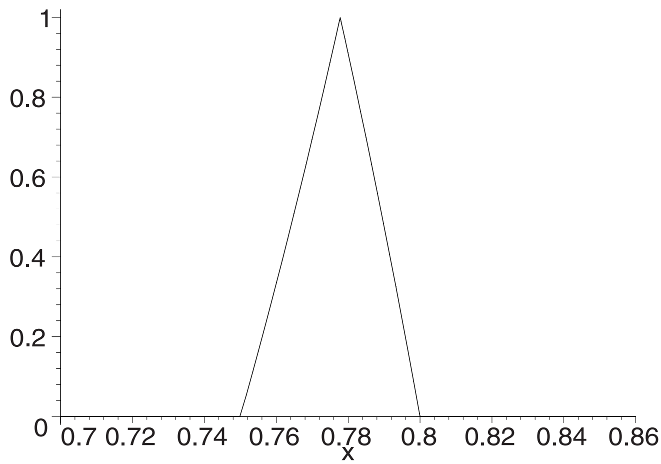

Fig. 10.1. Solution to Example 10.2.2, $\overline{X}_c \approx (0.75/0.77\overline{7}/0.8)$

Working more examples, like Examples 10.2.1 and 10.2.2 above, we conclude that too often fuzzy equations have no solution ($\overline{X}_c$). This motivated the authors in ([5]-[15]) to propose other solutions for fuzzy equations. These new solutions will be introduced in the next section. The classical solution, plus the new solutions, will be used throughout this chapter.

10.2.1 Other Solutions

We continue working with the fuzzy equation $\overline{A}\,\overline{X} + \overline{B} = \overline{C}$. The other solutions simply fuzzify the crisp solution $(c-b)/a$, $a \neq 0$. The fuzzified crisp solution is

$$(\overline{C} - \overline{B})/\overline{A}, \tag{10.10}$$

where we assume zero does not belong to the support of $\overline{A}$. There are two ways to evaluate equation (10.10). The first method is the extension principle. If $\overline{X}_e$ is the value of equation (10.10) using the extension principle, then

$$\overline{X}_e(x) = \max\{\pi(a,b,c) \mid (c-b)/a = x\}, \tag{10.11}$$

where

$$\pi(a,b,c) = \min\left\{\overline{A}(a), \overline{B}(b), \overline{C}(c)\right\}. \tag{10.12}$$

Since the expression $(c-b)/a$, $a \neq 0$, is continuous in a, b, c we know how to find α-cuts of $\overline{X}_e$ (Section 2.4.1)

$$x_{e1}(\alpha) = \min\left\{(c-b)/a \mid a \in \overline{A}[\alpha], b \in \overline{B}[\alpha], c \in \overline{C}[\alpha]\right\}, \tag{10.13}$$

$$x_{e2}(\alpha) = \max\left\{(c-b)/a \mid a \in \overline{A}[\alpha], b \in \overline{B}[\alpha], c \in \overline{C}[\alpha]\right\}, \tag{10.14}$$

where

$$\overline{X}_e[\alpha] = [x_{e1}(\alpha), x_{e2}(\alpha)], \tag{10.15}$$

$0 \leq \alpha \leq 1$. $\overline{X}_e$ will be a triangular shaped fuzzy number when $\overline{A}$, $\overline{B}$, $\overline{C}$ are triangular fuzzy numbers. It is not difficult to show that if $\overline{X}_c$ exists, $\overline{X}_c \leq \overline{X}_e$. In this chapter $\overline{X}_c \leq \overline{X}_e$ means that $\overline{X}_c$ is a fuzzy subset of $\overline{X}_e$ (Section 2.2.3).

An important fact about $\overline{X}_c$ is that it will satisfy the fuzzy equation. That is $\overline{A} \cdot \overline{X}_c + \overline{B} = \overline{C}$ holds using α-cuts and interval arithmetic. However, $\overline{X}_e$ may, or may not, satisfy the fuzzy equation. However, $\overline{X}_e$ will always exist but $\overline{X}_c$ may fail to exist.

The second way to evaluate equation (10.10) is to use α-cuts and interval arithmetic. If the result is $\overline{X}_I$, we have

$$\overline{X}_I[\alpha] = \frac{\overline{C}[\alpha] - \overline{B}[\alpha]}{\overline{A}[\alpha]}, \tag{10.16}$$

to be simplified by interval arithmetic, $0 \leq \alpha \leq 1$. It is also not too difficult to argue that $\overline{X}_e \leq \overline{X}_I$.

$\overline{X}_I$ may or may not satisfy the fuzzy equation. $\overline{X}_I$ will be a triangular shaped fuzzy number when $\overline{A}$, $\overline{B}$, $\overline{C}$ are all triangular fuzzy numbers. We summarize these results as:

1. If $\overline{X}_c$ exists, then $\overline{X}_c \leq \overline{X}_e \leq \overline{X}_I$;
2. $\overline{X}_c$ always satisfies the fuzzy equation;
3. $\overline{X}_e \leq \overline{X}_I$.

Up to now our general strategy for solving fuzzy equations will be:

1. the solution is $\overline{X}_c$ when it exists;
2. if $\overline{X}_c$ fails to exist, the solution is $\overline{X}_e$; and
3. if $\overline{X}_c$ fails to exist and $\overline{X}_e$ is difficult to construct, use $\overline{X}_I$ as the (approximate) solution.

For more complicated fuzzy equations $\overline{X}_e$ will be difficult to compute. However, $\overline{X}_I$ is usually easily constructed, since it uses only max, min and the arithmetic of real numbers. For this reason we suggest approximating $\overline{X}_e$ by $\overline{X}_I$ when we do not have $\overline{X}_e$.

Example 10.2.1.1

This continues Example 10.2.1 where $\overline{X}_c$ does not exist. To calculate $\overline{X}_e$ we need to evaluate equations (10.13) and (10.14). But this is easily done since $(c-b)/a$ is increasing in c and decreasing in both b and a. So

$$x_{e1}(\alpha) = \frac{c_1(\alpha) - b_2(\alpha)}{a_2(\alpha)} = \frac{4+2\alpha}{3-\alpha}, \tag{10.17}$$

$$x_{e2}(\alpha) = \frac{c_2(\alpha) - b_1(\alpha)}{a_1(\alpha)} = \frac{8-2\alpha}{1+\alpha}. \tag{10.18}$$

$\overline{X}_e$ is shown in Figure 10.2.

In calculating $\overline{X}_I[\alpha]$ we get

$$\overline{X}_I[\alpha] = [c_1(\alpha) - b_2(\alpha), c_2(\alpha) - b_1(\alpha)] \left[\frac{1}{a_2(\alpha)}, \frac{1}{a_1(\alpha)}\right], \tag{10.19}$$

which is the same as $\overline{X}_e[\alpha]$ because intervals in equation (10.19) are positive. In this example, we get $\overline{X}_e = \overline{X}_I$, whose support $\overline{X}_e[0] = [\frac{4}{3}, 8]$.

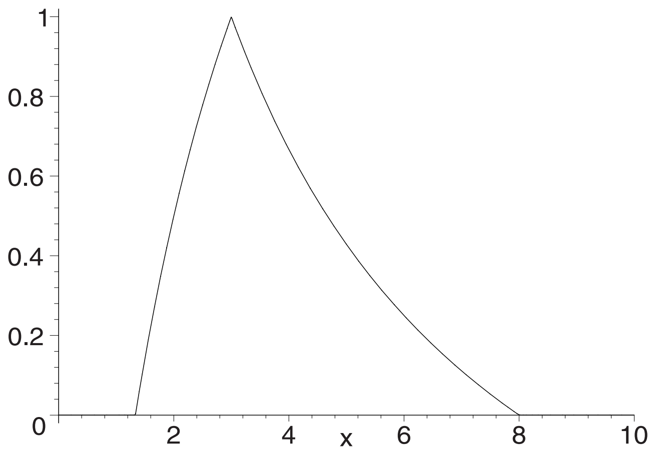

Fig. 10.2. Solution to Example 10.2.1.1, $\overline{X}_e = \overline{X}_I \approx (\frac{4}{3}/3/8)$

Example 10.2.1.2

This continues Example 10.2.2. We notice that, since $\overline{A} > 0$ and $\overline{C} - \overline{B} > 0$, $\frac{\partial}{\partial c}[\frac{c-b}{a}] = \frac{1}{a} > 0$, $\frac{\partial}{\partial b}[\frac{c-b}{a}] = -\frac{1}{a} < 0$ and $\frac{\partial}{\partial a}[\frac{c-b}{a}] = \frac{b-c}{a^2} < 0$. This means that the expression $\frac{c-b}{a}$ is increasing in c but decreasing in a and b. Then equations (10.13) and (10.14) become

$$x_{e1}(\alpha) = \frac{c_1(\alpha) - b_2(\alpha)}{a_2(\alpha)} = \frac{4+3\alpha}{10-\alpha}, \tag{10.20}$$

$$x_{e2}(\alpha) = \frac{c_2(\alpha) - b_1(\alpha)}{a_1(\alpha)} = \frac{10 - 3\alpha}{8 + \alpha}. \tag{10.21}$$

As in Example 10.2.1.1 we obtain $\overline{X}_I = \overline{X}_e$. $\overline{X}_c$ and $\overline{X}_e$ are shown in Figure 10.3. The support of $\overline{X}_e = \overline{X}_I$ is $\overline{X}_e[0] = [0.4, 1.25]$.

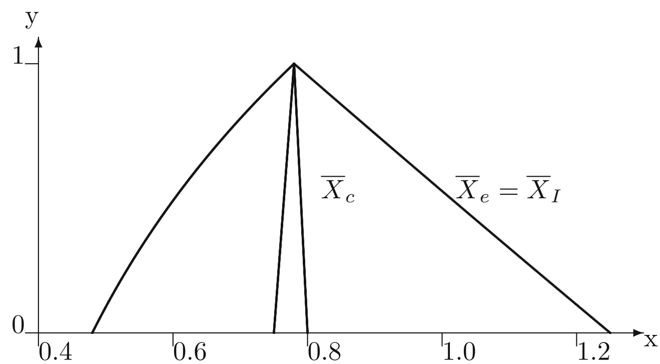

Fig. 10.3. Solutions to Example 10.2.1.2, $\overline{X}_c \approx (0.75/0.77\overline{7}/0.8)$ and $\overline{X}_e \approx (0.4/0.77\overline{7}/1.25)$

10.2.2 Fuzzy Monte Carlo Method

Let **Q** be the set of triangular fuzzy numbers and the set of quadratic fuzzy numbers (Section 4.3.2 called QBGFNs). Let D be some metric on **Q**. Then D has the following properties: for $\overline{M}$, $\overline{N}$ and $\overline{P}$ in **Q**

1. $D(\overline{M}, \overline{N}) \geq 0$;
2. $D(\overline{M}, \overline{N}) = 0$ implies that $\overline{M} = \overline{N}$;
3. $D(\overline{M}, \overline{N}) = D(\overline{N}, \overline{M})$; and
4. $D(\overline{M}, \overline{N}) \leq D(\overline{M}, \overline{P}) + D(\overline{P}, \overline{N})$.

Then our new solution will be $\overline{X}^*$ that solves the minimization problem

$$min\{D(\overline{A} \cdot \overline{X} + \overline{B}, \overline{C}) | \overline{X} \in \mathbf{Q}\}. \tag{10.22}$$

That is, $\overline{X}^*$ is a fuzzy number from **Q** that makes $\overline{A} \cdot \overline{X}^* + \overline{B}$ as close as possible to $\overline{C}$, where the distance is measured by the metric D.

Our fuzzy Monte Carlo method may be applied to approximate $\overline{X}^*$. Randomly generate a sequence $\overline{X}_1$, $\overline{X}_2$,... from **Q**, compute the distance D between $\overline{A} \cdot \overline{X}_i + \overline{B}$ and $\overline{C}$, and keep the $\overline{X}_i$ that makes this distance the smallest. In this way we may compute better and better, for longer and longer sequences, approximations to $\overline{X}^*$.

Now we will rework the examples in the previous section using our fuzzy Monte Carlo method and compare the results. But first we need to select a metric D. Of course, the answers can vary depending on the choice of D. Metrics on fuzzy numbers were discussed in [8]. We first give an example of a "horizontal" metric and then an example of a "vertical" metric.

Let $\overline{M}[\alpha] = [m_1(\alpha), m_2(\alpha)]$, $\overline{N}[\alpha] = [n_1(\alpha), n_2(\alpha)]$, $L(\alpha) = |m_1(\alpha) - n_1(\alpha)|$ and $R(\alpha) = |m_2(\alpha) - n_2(\alpha)|$. Then

$$D(\overline{M}, \overline{N}) = max\{max(L(\alpha), R(\alpha))|0 \leq \alpha \leq 1\}, \tag{10.23}$$

is a metric. In [8] the authors show that $D(\overline{M}, \overline{N}) = 1$ for $\overline{M} = (1/2/4)$ and $\overline{N} = (1/3/4)$. For triangular/trapezoidal fuzzy numbers it is easy to compute the distance between them using equation (10.23). It is

$$D(\overline{M}, \overline{N}) = max\{|m_1(0) - n_1(0)|, |m_1(1) - n_1(1)|, |m_2(0) - n_2(0)|\}, \tag{10.24}$$

Also, the Hamming distance measure in equation (2.45) in Chapter 2 is a metric. Also in [8] the authors argue that the Hamming distance between $\overline{M} = (1/2/4)$ and $\overline{N} = (1/3/4)$ is 0.75. Let us use the first metric, the "horizontal metric" in our fuzzy Monte Carlo studies.

We will use the metric defined by equation (10.23) in equation (10.22). Let $D(\overline{X}) = D(\overline{A} \cdot \overline{X} + \overline{B}, \overline{C})$ and let $\epsilon \in (0, 1]$ be the "threshold". If the fuzzy Monte Carlo method produces a $\overline{X}$ so that $D(\overline{X}) < \epsilon$ we will say that we have found an acceptable approximate solution $\overline{X}_a^* = \overline{X}$ with $\overline{X}_a^* \approx \overline{X}^*$. We begin with $D(\overline{X})$ very large (100) and will accept a solution only if it minimizes to $D(\overline{X}) < \epsilon = 0.5$.

Example 10.2.2.1

This continues Examples 10.2.1 and 10.2.1.1. We wish to use our fuzzy Monte Carlo method to compute $\overline{X}_a^* \approx \overline{X}^*$ and then compare $\overline{X}_a^*$ to the other solution $\overline{X}_e = \overline{X}_I$. Recall that in this example the classical solution $\overline{X}_c$ does not exist and $\overline{X}_a^*$ is in $\mathbf{Q}$ and must satisfy $D(\overline{X}_a^*)$ small; e.g., $D(\overline{X}_a^*) < \epsilon = 0.5$.

We have already generated and studied 100,000 crisp random vectors $v = (x_1, ..., x_5)$ in $[0, 1]^5$ using a Sobol quasi-random number generator (Chapter 3). In Section 4.3.2 we relate how we create our vectors v. Next we determine an interval $[a, b]$, which will depend on the application, for the random quadratic fuzzy numbers. Then we map v into a QBGFN.

To map v into a QBGFN (Figure 4.4), first we sort, translate and transform $\{x_1, x_2, x_3\}$ to $\{z_1, z_2, z_3\}$ using $z_i = (b-a)x_i+a$, $i = 1, 2, 3$. Additionally we map x_4, x_5 into parameters for the left and right membership functions, respectively, using $z_4 = (2x_4 - 1)(z_2 - z_1 + 1)$ and $z_5 = (2x_5 - 1)(z_3 - z_2 + 1)$. In Section 5.3 we explain why we know these QBGFNs will cover our search space.

We modified the fuzzy Monte Carlo program used for Chapters 6-9 to optimize the minimization problem of equation (10.22). Since for this problem the classical solution does not exist, we choose $[a, b]$ as the support of $\overline{X}_e = \overline{X}_I$, which is $[\frac{4}{3}, 8]$. To compute $D(\overline{A} \cdot \overline{X} + \overline{B}, \overline{C})$, we compute 100 α-cuts of $\overline{A} \cdot \overline{X} + \overline{B}$ and $\overline{C}$ for given $\overline{A}$, $\overline{B}$ and $\overline{C}$, where $\overline{X}$ is one of 30,000 generated random quadratic fuzzy numbers.

Table 10.1. $(1/2/3) \cdot \overline{X} + (-3/-2/-1) = (3/4/5)$, Example 10.2.1, First Interval

Solution	$\overline{X}$	$\overline{A} \cdot \overline{X} + \overline{B}$
$\overline{X}_c$	does not exist	$(3/4/5) = \overline{C}$
$\overline{X}_I = \overline{X}_e$	$\approx (1.33\overline{3}/3/8)$	$\approx (-1.66\overline{6}/4/23)$
$\overline{X}^*_a$,	$(2.9655/2.9980/3.0176/0.1286/0.6457)$, $D(\overline{X}) = 3.0527$	$\approx (-0.03/4.00/8.05)$

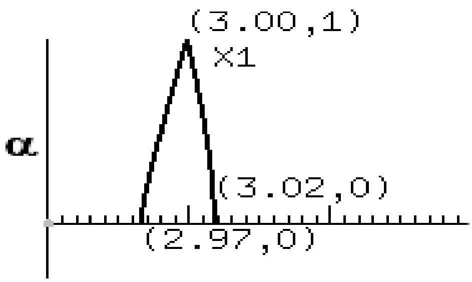

Fig. 10.4. $\overline{X}^*_a$ of Example 10.2.2.1

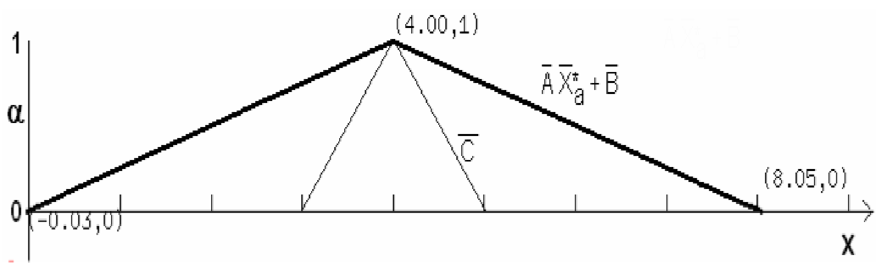

Fig. 10.5. $\overline{A} \cdot \overline{X}^*_a + \overline{B}$ of Example 10.2.2.1

We allow our program to execute for $30,000$ iterations and obtain the results shown in Table 10.1.

Searching for $\overline{X}$ in the interval $[1.33, 8.00]$, the smallest "horizontal" distance for equation (10.23) that we got was 3.0527. Increasing the numbers of fuzzy numbers to $50,000$ does not produce a change. In other words we obtained a quadratic fuzzy number $\overline{X}^*_a \approx (2.9655/2.9980/3.0176)$ (Figure 10.4) such that $D(\overline{A} \cdot \overline{X}^*_a + \overline{B}, \overline{C}) \approx 3.0527$. The graph of $\overline{X}^*_a$ is nearly crisp. $\overline{A} \cdot \overline{X}^*_a + \overline{B} \approx (-0.03/4.00/8.05)$. But this error is too large.

We changed the interval $[a, b]$ and saw that there is a good reduction in the distance between $\overline{A} \cdot \overline{X} + \overline{B}$ and $\overline{C}$. From an optimization over $[0, 5]$ we minimized at $D(\overline{A} \cdot \overline{X} + \overline{B}, \overline{C}) = 2.00$ for $\overline{X} = \overline{X}^*_a = (2.5000, 2.5000, 2.5000, -0.0000, 0.5000)$. This error is still too large with $\epsilon = 2.00$ and $\overline{A} \cdot \overline{X}^*_a + \overline{B} \approx (1.00/3.00/5.00)$ (Figure 10.5).

Our fuzzy Monte Carlo method was unable to get an acceptable approximate solution to this fuzzy equation. We believe that no quadratic fuzzy number $\overline{X}$ can make $D(\overline{X}) < \epsilon = 0.5$.

Example 10.2.2.2

This continues Examples 10.2.2 and 10.2.1.2. We apply our fuzzy Monte Carlo method to compute $\overline{X}_a^* \approx \overline{X}^*$ and then compare $\overline{X}_a^*$ to the other solutions $\overline{X}_c$ and $\overline{X}_e = \overline{X}_I$. Recall that in this example the classical solution exists and is shown in Figure 10.1. The constraints are still $\overline{X}_a^* \in \mathbf{Q}$ and it must satisfy $D(\overline{X}_a^*)$ small; e.g., $D(\overline{X}_a^*) < \epsilon = 0.5$. As in Example 10.2.2.1 we generate a solution using a Sobol quasi-random number generator to produce random quadratic

Table 10.2. $(8/9/10) \cdot \overline{X} + (-3/-2/-1) = (3/5/7)$, Example 10.2.2.2

Solution	$\overline{X}$	$\overline{A} \cdot \overline{X} + \overline{B}$
$\overline{X}_c$	$\approx (0.75/0.77\overline{7}/0.80)$	$\approx (3/5/7) = \overline{C}$
$\overline{X}_I = \overline{X}_e$	$\approx (0.4/0.77\overline{7}/1.25)$	$\approx (0.2/5/11.5)$
$\overline{X}_a^*$,	$(0.7523, 0.7786, 0.7999, 0.7788, 0.7816)$, $D(\overline{X}) = 0.083976$	$\approx (3.02/5.01/7.00)$

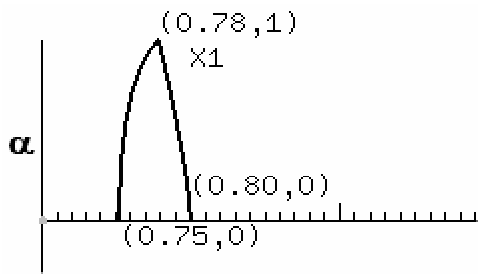

Fig. 10.6. $\overline{X}_a^*$ of Example 10.2.2.2

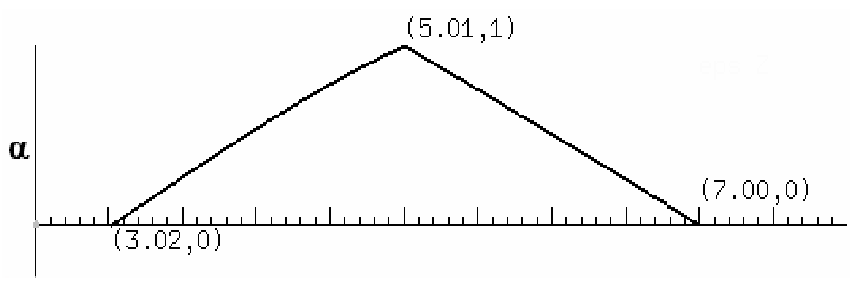

Fig. 10.7. $\overline{A} \cdot \overline{X}_a^* + \overline{B}$ of Example 10.2.2.2

fuzzy numbers determined by vectors of length 5. We consider fuzzy numbers in $[0.4, 1.25]$ which is the support of $\overline{X}_e = \overline{X}_I$.

We allow our program to execute for $50,000$ iterations and obtain the results shown in Table 10.2. $\overline{X}_a^*$ (Figure 10.6) is very close to $\overline{X}_c$ (Figure 10.1). Note too how $\overline{A} \cdot \overline{X}_a^* + \overline{B}$ (Figure 10.7) matches $\approx (3/5/7)$. Our fuzzy Monte Carlo method found an acceptable solution.

10.3 Fuzzy Quadratic Equation

In this section we wish to discuss solutions to

$$\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C} = \overline{D}, \tag{10.25}$$

for triangular fuzzy numbers $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ and $\overline{X}$ a triangular shaped fuzzy number. We know a crisp quadratic equation can have real number solutions and complex number solutions. The same is true of the fuzzy quadratic. However, we will not consider fuzzy complex numbers in this book so we will only work with fuzzy quadratics that have no solution or the solutions are real triangular shaped fuzzy numbers. In Section 10.2 we looked at three possible solutions to the fuzzy linear equation: classical $(\overline{X}_c)$, extension principle $(\overline{X}_e)$ and the interval arithmetic $(\overline{X}_I)$. In this section we only consider the classical solution.

Let $\overline{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$, $\overline{B}[\alpha] = [b_1(\alpha), b_2(\alpha)]$, $\overline{C}[\alpha] = [c_1(\alpha), c_2(\alpha)]$, $\overline{D}[\alpha] = [d_1(\alpha), d_2(\alpha)]$, and $\overline{X}[\alpha] = [x_1(\alpha), x_2(\alpha)]$. We use $\alpha-$cuts and interval arithmetic to solve for $x_1(\alpha)$ and $x_2(\alpha)$. Equation (10.25) becomes

$$[a_1(\alpha), a_2(\alpha)][x_1(\alpha), x_2(\alpha)]^2 + [b_1(\alpha), b_2(\alpha)][x_1(\alpha), x_2(\alpha)] + \\ +[c_1(\alpha), c_2(\alpha)] = [d_1(\alpha), d_2(\alpha)], \tag{10.26}$$

for all α. We do the interval arithmetic (Section 2.3.2), which depends on $\overline{A}, \overline{B}$ and $\overline{X}$ being positive or negative, and solve for $x_1(\alpha)$ and $x_2(\alpha)$. We have a solution if $x_1(0) < x_1(1) \leq x_2(1) < x_3(0)$ and $dx_1(\alpha)/d\alpha > 0$, $dx_2(\alpha)/d\alpha < 0$.

Now we look at two examples where the first has a solution and the second does not have a solution. More details on the fuzzy quadratic can be found in ([6]-[9],[11],[13]). Then we turn to our fuzzy Monte Carlo method to see what approximate answers it can give.

Example 10.3.1

Let $\overline{A} = (3/4/5)$, $\overline{B} = (1/2/3)$, $\overline{C} = (0/1/2)$ and $\overline{D} = (1/3/5)$. We will look for a solution where $\overline{X} \geq 0$. Then equation (10.26) becomes

$$[3 + \alpha, 5 - \alpha][x_1^2(\alpha), x_2^2(\alpha)] + [1 + \alpha, 3 - \alpha][x_1(\alpha), x_2(\alpha)] \\ +[\alpha, 2 - \alpha)] = [1 + 2\alpha, 5 - 2\alpha)], \tag{10.27}$$

for $\alpha \in [0, 1]$. If $\overline{X} \approx (x_1/x_2/x_3)$ we first solve for the x_i getting $x_1 = 0.4343 < x_2 = 0.5000 < x_3 = 0.5307$. Looks like we will get a solution. Next we look at $x_1(\alpha)$ which is

$$x_1(\alpha) = \frac{-(1+\alpha) + \sqrt{5\alpha^2 + 18\alpha + 13}}{6 + 2\alpha}, \tag{10.28}$$

and then $x_2(\alpha)$

$$x_2(\alpha) = \frac{-(3-\alpha) + \sqrt{5\alpha^2 - 38\alpha + 69}}{10 - 2\alpha}, \tag{10.29}$$

for $0 \le \alpha \le 1$. We find that $dx_1(\alpha)/d\alpha > 0$ and $dx_2(\alpha)/d\alpha < 0$ and $\overline{X}$ is a solution. The graph of this solution is in Figure 10.8.

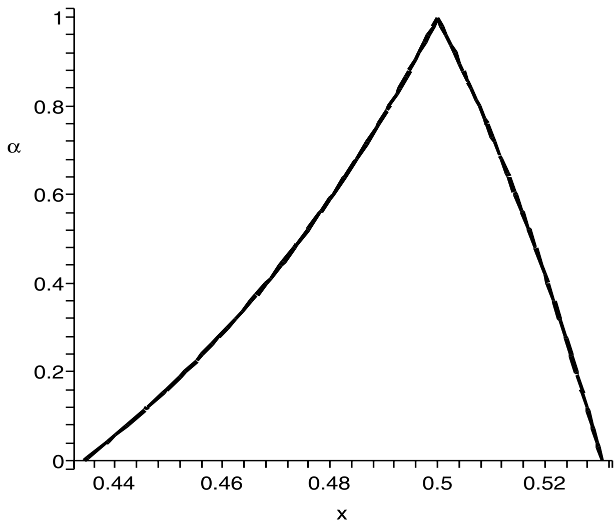

Fig. 10.8. Solution to Example 10.3.1, $\overline{X}_c \approx (0.4343/0.5/0.5307)$

Example 10.3.2

This example will have no (classical) solution for $\overline{X}$. Let $\overline{A} = (2/4/6)$, $\overline{B} = (0/2/4)$, $\overline{C} = 0$, $\overline{D} = (0.5/1/1.5)$ and $\overline{X}$ a non-negative triangular shaped fuzzy number. Let $\overline{X} \approx (x_1/x_2/x_3)$. So $x_1(0) = x_1$ and $x_2(0) = x_3$. Now we set up an equation, like equation (10.27), for $\alpha = 0$ and obtain two equations to solve

$$2x_1^2(0) = 0.5, \tag{10.30}$$

and

$$6x_2^2(0) + 4x_2(0) = 1.5. \tag{10.31}$$

If $\overline{X}$were a solution, its support would be $[x_1(0), x_2(0)]$. However $[x_1(0), x_2(0)] = [0.5, 0.2676]$; i.e., $x_2(0) < x_2(0)$. Thus there is no classical solution.

10.3.1 Fuzzy Monte Carlo Method

We follow the same strategy as outlined in Section 10.2.2 for generating approximate solutions to fuzzy linear equations using fuzzy Monte Carlo methods. **Q** will be the same set of fuzzy numbers and we use the same metric between fuzzy numbers given in equation (10.23). We call our new solution $\overline{X}^*$, the fuzzy number in **Q** that minimizes the distance $D(\overline{X})$ between $\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$ and $\overline{D}$. Using our fuzzy Monte Carlo method we obtain $\overline{X}_a^*$ an approximation to $\overline{X}^*$. We use the same "threshold" ϵ discussed in Section 10.2.2. If the fuzzy Monte Carlo method produces a $\overline{X}$ so that $D(\overline{X}) < \epsilon$ we will say that we have found an acceptable approximate solution $\overline{X}_a^* = \overline{X}$ with $\overline{X}_a^* \approx \overline{X}^*$. We begin with $D(\overline{X})$ very large (100) and will accept a solution only if it minimizes to $D(\overline{X}) < \epsilon = 0.5$.

Now we will rework Examples 10.3.1 and 10.3.2 using our fuzzy Monte Carlo method.

Example 10.3.1.1

This continues Example 10.3.1. We wish to use our fuzzy Monte Carlo method to compute $\overline{X}_a^* \approx \overline{X}^*$ and then compare $\overline{X}_a^*$ to the classical solution $\overline{X}_c$ in Figure 10.8. Recall that in this example the classical solution $\overline{X}_c$ exists and $\overline{X}_a^*$ is in **Q** and must satisfy $D(\overline{X}_a^*) < \epsilon = 0.5$. We follow the same procedure discussed in Example 10.2.2.1.

We have already generated and studied 100,000 crisp random vectors $v = (x_1, ..., x_5)$ in $[0,1]^5$ using a Sobol quasi-random number generator (Chapter 3). In Section 4.3.2 we relate how we create our vectors v. Next we determine an interval $[a,b]$, which will depend on the application, for the random quadratic fuzzy numbers. Then we map v into a QBGFN.

Since for this problem the classical solution does exist, we choose $[a,b]$ as the support of $\overline{X}_c$, which is $[0.4343, 0.5307]$. To compute $D(\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}, \overline{D})$, we compute 100 α-cuts of $\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$ and $\overline{D}$ for given $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$, where $\overline{X}$ is one of 50,000 generated random quadratic fuzzy numbers.

Searching for $\overline{X}$ in the interval $[0.4343, 0.5307]$, the smallest "horizontal" distance for equation (10.23) that we got in $50,000$ iterations was 0.014840 (Table 10.3), in the 19931^{th} iteration. However, we found $D(\overline{X}) < 0.20$ on the 7^{th} iteration. Figure 10.9 shows $\overline{X}_a^*$; Figure 10.10 shows $\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$ for $\overline{X} = \overline{X}_a^*$.

Wanting to investigate how a change in the interval $[a,b]$ might affect our result, we executed a test using $[a,b] = [0,1]$. and also saw a very good reduction in the distance between $\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$ and $\overline{D}$. The graphs of these two solutions are so similar that we do not show graphs for this follow-up experiment. As early as the 637^{th} iteration, an acceptable solution with $D(\overline{X}) = 0.193726$ was found. Although we performed $50,000$ iterations, our acceptable solution was found on the $20,726^{th}$ iteration. From an optimization over $[0,1]$ we minimized at $D(\overline{X}) = 0.036417$ (Table 10.3).

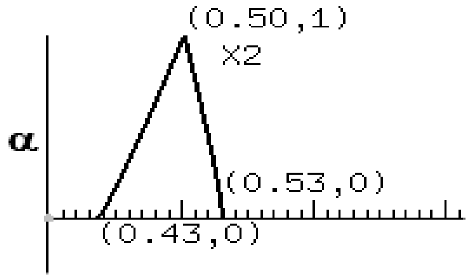

Fig. 10.9. $\overline{X}_a^*$ of Example 10.3.1.1

Table 10.3. $(3/4/5) \cdot \overline{X}^2 + (1/2/3) \cdot \overline{X} + (0/1/2) = (1/3/5)$, Example 10.3.1.1

Solution	$\overline{X}$	$\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$
$\overline{X}_c$	$\approx (0.4343/0.5/0.5307)$	$\approx (1/3/5) = \overline{D}$
$\overline{X}_a^*$,	$[a,b] = [0.4343, 0.5307]$ $(0.4348, 0.5016, 0.5306, -0.0074, 0.8749)$, $D(\overline{X}) = 0.014840$	$\approx (1.00/3.01/5.00)$
$\overline{X}_a^*$,	$[a,b] = [0.0, 1.0]$ $(0.4419, 0.5021, 0.5341, 0.0980, 0.9120)$, $D(\overline{X}) = 0.036417$	$\approx (1.03/3.01/5.03)$

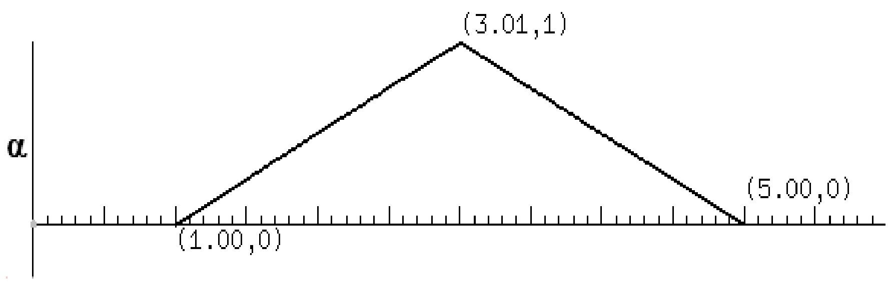

Fig. 10.10. $\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$ for $\overline{X} = \overline{X}_a^*$ of Example 10.3.1.1

Example 10.3.1.2

This continues Examples 10.3.2. We apply our fuzzy Monte Carlo method to compute $\overline{X}_a^* \approx \overline{X}^*$. Recall that in this example the classical solution does not exist. The constraints are still $\overline{X}_a^* \in \mathbf{Q}$ and it must satisfy $D(\overline{X}_a^*) < \epsilon = 0.5$. We follow the same procedure discussed in Example 10.3.1.1.

Table 10.4. $(2/4/6) \cdot \overline{X}^2 + (0/2/4) \cdot \overline{X} + (0/0/0) = (0.5/1/1.5)$, Example 10.3.2

Solution	$\overline{X}$	$\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$
$\overline{X}_c$	does not exist	$(0.5/1/1.5) = \overline{D}$
$\overline{X}_a^*$,	$(0.3058, 0.3088, 0.3107, -0.3767, -0.8683)$, $D(\overline{X}) = 0.321659$	$\approx (0.19/1.00/1.82)$

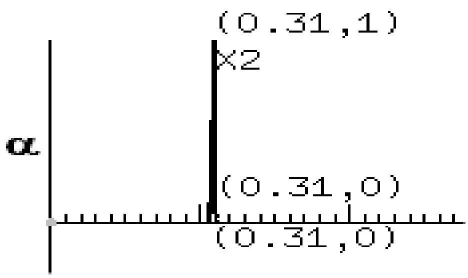

Fig. 10.11. $\overline{X}_a^*$ of Example 10.3.1.2

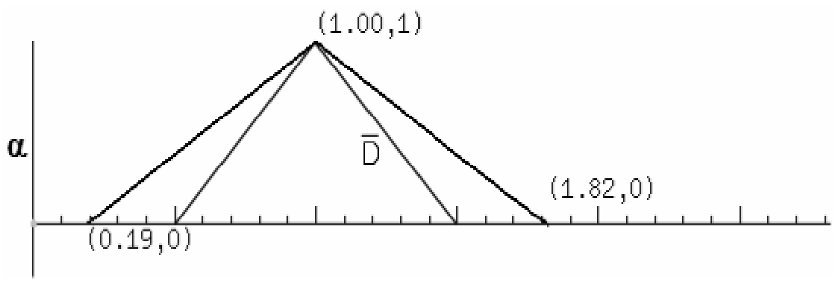

Fig. 10.12. $\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$ for $\overline{X} = \overline{X}_a^*$ of Example 10.3.1.2

Now we need to find an interval $[a, b]$ for our random quadratic fuzzy numbers. The classical solution does not exist. Choose the vertex values for the $\overline{A}$,...,$\overline{D}$ and consider $4x^2 + 2x + 0 = 1$ which has positive solution $x = 0.3090$. We begin with intervals centered at this value so we begin with the interval $I = [0, 0.6180]$.

To compute $D(\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}, \overline{D})$, we compute 100 α-cuts of $\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C}$ and $\overline{D}$ for given $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$, where $\overline{X}$ is one of 50,000 generated random quadratic fuzzy numbers.

From an optimization over $[0, 0.6180]$ we minimized at $D(\overline{X}) = 0.3217$ (Table 10.4 and Figure 10.12). Our fuzzy Monte Carlo result generated an "acceptable" solution. Perhaps we could reduce the error measure with more iterations.

10.4 Fuzzy Matrix Equation

This section is based on ([6],[8],[12],[13]). Let $\overline{A} = [\overline{a}_{ij}]$ be a $n \times n$ matrix of triangular fuzzy numbers $\overline{a}_{ij}$, $\overline{B}^t = (\overline{b}_1, \ldots, \overline{b}_n)$ a $n \times 1$ vector of triangular fuzzy numbers $\overline{b}_i$ and $\overline{X}^t = (\overline{x}_1, \ldots, \overline{x}_n)$ a $n \times 1$ vector of unknown triangular shaped fuzzy numbers $\overline{x}_j$. Set $\overline{a}_{ij} = (a_{ij1}/a_{ij2}/a_{ij3})$, $\overline{b}_i = (b_{i1}/b_{i2}/b_{i3})$, and $\overline{x}_j \approx (x_{j1}/x_{j2}/x_{j3})$. We wish to solve

$$\overline{A}\,\overline{X} = \overline{B}, \tag{10.32}$$

for $\overline{X}$.

We need to introduce some more notation. Define

$$a[\alpha] = \prod_{i,j=1}^{n} a_{ij}[\alpha], \tag{10.33}$$

$$b[\alpha] = \prod_{i=1}^{n} b_i[\alpha], \tag{10.34}$$

for $0 \le \alpha \le 1$. Let $v = (a_{11}, a_{12}, \ldots, a_{nn}) \in \mathbb{R}^k$, $k = n^2$, be a vector in $a[0]$. Each $v \in a[0]$ determines a crisp $n \times n$ matrix $A = [a_{ij}]$. Also, $b^t = (b_1, \ldots, b_n) \in \mathbb{R}^n$ is a vector in $b[0]$. As in our previous research on this topic we assume A^{-1} exists for all v in $a[0]$. The existence of A^{-1} over $a[0]$ simplifies the discussion of the joint solution to be introduced below.

The joint solution $\overline{X}_J$, a fuzzy subset of $\mathbb{R}^n$, is based on the extension principle

$$\overline{X}_J(x) = \max\left\{ \pi(v, b) \mid x = A^{-1}b \right\}, \tag{10.35}$$

where

$$\pi(v, b) = \min\left\{ \overline{a}_{ij}(a_{ij}), \overline{b}_i(b_i) \mid \text{all } i, j \right\}. \tag{10.36}$$

The vertex of $\overline{X}_J(x)$, where the membership value is equal to one, is at $x = A^{-1}b$ for $v = (a_{112}, a_{122}, \ldots, a_{nn2})$, $b^t = (b_{12}, \ldots, b_{n2})$. In the crisp case the solution to $Ax = b$ is a vector $x = A^{-1}b$ in $\mathbb{R}^n$, so for the fuzzy case $\overline{A}\,\overline{X} = \overline{B}$, the (joint) solution is a fuzzy vector about the crisp solution $A^{-1}b$, for v and b at the vertex values of all the triangular fuzzy numbers.

In the crisp case the marginals, the x_i, are just the components of the vector $x = A^{-1}b$. In the fuzzy case we obtain the marginals $\overline{X}_{Ji}$ by projecting $\overline{X}_J$ onto the coordinate axes. Then

$$\overline{X}_{Ji}(w) = \max\left\{\overline{X}_J(x) \mid x \in \mathbb{R}^n, x_i = w \right\}, \tag{10.37}$$

for $1 \le i \le n$. Obviously, it will be difficult to compute $\overline{X}_J$ and $\overline{X}_{Ji}$, $1 \le i \le n$, for $n \ge 4$. We will determine the joint solution, and its marginals, in two examples at the end of this section for $n = 2$.

Since $\overline{X}_J$ is difficult to determine we now turn to methods of finding the marginals directly without first computing the joint solution. As in the

Section 10.2 there will be three solutions: (1) the classical solution $\overline{X}_{ci}$; (2) the extension principle solution $\overline{X}_{ei}$; and (3) the interval arithmetic solution $\overline{X}_{Ii}$.

The classical solution is determined by substituting the intervals $\overline{a}_{ij}[\alpha]$, $\overline{b}_i[\alpha]$ and $\overline{X}_i[\alpha] = [x_{i1}(\alpha), x_{i2}(\alpha)]$ into $\overline{A}\ \overline{X} = \overline{B}$ and solving for the $x_{i1}(\alpha)$, $x_{i2}(\alpha)$, $1 \le i \le n$. The resulting equations are evaluated using interval arithmetic. If the intervals $[x_{i1}(\alpha), x_{i2}(\alpha)]$ define a triangular shaped fuzzy number $\overline{X}_i$ for $0 \le \alpha \le 1$, $1 \le i \le n$, then this solution is called the classical solution and we write $\overline{X}_{ci} = \overline{X}_i$, $1 \le i \le n$. The conditions for $[x_{i1}(\alpha), x_{i2}(\alpha)]$ to define $\overline{X}_{ci}$ were discussed in Section 10.2. The equations to solve for $x_{i1}(\alpha)$ and $x_{i2}(\alpha)$ are

$$\sum_{j=1}^{n} [a_{ij1}(\alpha), a_{ij2}(\alpha)][x_{j1}(\alpha), x_{j2}(\alpha)] = [b_{i1}(\alpha), b_{i2}(\alpha)], \tag{10.38}$$

for $1 \le i \le n$, where $\overline{a}_{ij}[\alpha] = [a_{ij1}(\alpha), a_{ij2}(\alpha)]$, $\overline{b}_i[\alpha] = [b_{i1}(\alpha), b_{i2}(\alpha)]$. After using interval arithmetic we obtain a $(2n) \times (2n)$ system to solve for $x_{i1}(\alpha)$, $x_{i2}(\alpha)$, $0 \le \alpha \le 1$.

Too often the $\overline{X}_{ci}$ fail to exist. We only need $\overline{X}_{ci}$, for one value of i, to fail to exist for the classical solution to not exist. When the classical solution does not exist we turn to $\overline{X}_{ei}$, $1 \le i \le n$.

We will use Cramer's rule on $Ax = b$ to solve for each x_i. A comes from $v \in a[0]$ and let $b \in b[0]$. Let A_j be A with its j–th column replaced by b. Then

$$x_j = \frac{|A_j|}{|A|}, \tag{10.39}$$

$1 \le j \le n$, where $|\cdot|$ denotes the determinant. We fuzzify equation (10.39), using the extension principle, to get

$$\overline{X}_{ej}(x_j) = \max\left\{\pi(v, b) \mid x_j = |A_j|/|A|\ \right\}, \tag{10.40}$$

$1 \le j \le n$. If $\overline{X}_{ej}[\alpha] = [x_{ej1}(\alpha), x_{ej2}(\alpha)]$, we may find the α-cuts of $\overline{X}_{ej}$ as (Section 2.4.1)

$$x_{ej1}[\alpha] = \min\left\{\frac{|A_j|}{|A|} \mid v \in a[\alpha], b \in b[\alpha]\ \right\}, \tag{10.41}$$

$$x_{ej2}[\alpha] = \max\left\{\frac{|A_j|}{|A|} \mid v \in a[\alpha], b \in b[\alpha]\ \right\}, \tag{10.42}$$

To get the $\overline{X}_{Ij}$ we evaluate equation (10.39) using α-cuts and interval arithmetic. Substitute intervals $\overline{a}_{ij}[\alpha]$ and $\overline{b}_i[\alpha]$ for a_{ij} and b_i in $|A_j|/|A|$, evaluate using interval arithmetic, and the result is $\overline{X}_{Ij}[\alpha]$, $0 \le \alpha \le 1$, $1 \le j \le n$.

We have the following result: If the $\overline{X}_{ci}$ exist, $1 \le i \le n$, then $\overline{X}_{ci} \le \overline{X}_{Ji} \le \overline{X}_{ei} \le \overline{X}_{Ii}$, $1 \le i \le n$.

Our solution strategy is: (1) use $\overline{X}_{ci}$, $1 \le i \le n$, if it exists; (2) if the classical solution does not exist use $\overline{X}_{Ji}$, $1 \le i \le n$. However, if the joint solution is too difficult to compute use $\overline{X}_{ei}$, $1 \le i \le n$. Equations (10.41) and (10.42) may be hard to evaluate to get the $\overline{X}_{ei}$. One can always use the $\overline{X}_{Ii}$ because they are the easiest to calculate. Notice how the fuzziness grows (the supports do not decrease) as we go from $\overline{X}_{ci}$ to $\overline{X}_{Ii}$. The only solution guaranteed to satisfy the fuzzy equations is the classical solution.

In the following two examples we only consider 2×2 fuzzy matrices since then we can easily see pictures of α-cuts of the joint solution. It is known that, in general, α-cuts of the joint solution need not be convex ([19],[20]). For example, in two dimensions $\overline{X}_J[\alpha]$ need not be a rectangle.

Example 10.4.1

Let

$$\overline{A} = \begin{pmatrix} \overline{a}_{11} & 0 \\ 0 & \overline{a}_{22} \end{pmatrix}, \tag{10.43}$$

and $\overline{B}^t = (\overline{b}_1, \overline{b}_2)$, where $\overline{a}_{11} = (4/5/7)$, $\overline{a}_{22} = (6/8/12)$, $\overline{b}_1 = (1/2/3)$ and $\overline{b}_2 = (2/5/8)$.

Then $\overline{a}_{11}[\alpha] = [4+\alpha, 7-2\alpha]$, $\overline{a}_{22}[\alpha] = [6+2\alpha, 12-4\alpha]$, $\overline{b}_1[\alpha] = [1+\alpha, 3-\alpha]$ and $\overline{b}_2[\alpha] = [2+3\alpha, 8-3\alpha]$. Since all the fuzzy numbers are positive we will solve for $\overline{X}_{ci} > 0$, $i = 1, 2$. The equations are

$$[4+\alpha, 7-2\alpha] \cdot [x_{c11}(\alpha), x_{c12}(\alpha)] = [1+\alpha, 3-\alpha]\,, \tag{10.44}$$

$$[6+2\alpha, 12-4\alpha] \cdot [x_{c21}(\alpha), x_{c22}(\alpha)] = [2+3\alpha, 8-3\alpha]\,, \tag{10.45}$$

which define triangular shaped fuzzy numbers,

$$\overline{X}_{c1}[\alpha] = \left[\frac{1+\alpha}{4+\alpha}, \frac{3-\alpha}{7-2\alpha}\right], \tag{10.46}$$

$$\overline{X}_{c2}[\alpha] = \left[\frac{2+3\alpha}{6+2\alpha}, \frac{8-3\alpha}{12-4\alpha}\right], \tag{10.47}$$

$0 \leq \alpha \leq 1$.

We have shown before that a way to find α-cuts of $\overline{X}_J$ is

$$\overline{X}_J[\alpha] = \{A^{-1}b \mid v \in a[\alpha], b \in b[\alpha]\ \}. \tag{10.48}$$

Now $A^{-1}b = (b_1/a_{11}, b_2/a_{22})^t$ so that

$$\overline{X}_J[\alpha] = \frac{\overline{b}_1[\alpha]}{\overline{a}_{11}[\alpha]} \times \frac{\overline{b}_2[\alpha]}{\overline{a}_{22}[\alpha]}, \tag{10.49}$$

which is a rectangle in $\mathbb{R}^2$ for all $0 \leq \alpha \leq 1$.

$\overline{X}_J[\alpha]$ is expressed in equation (10.49) as the product of two factors. The first factor of $\overline{X}_J[\alpha]$ is $\overline{X}_{J1}[\alpha]$ and $\overline{X}_{J2}[\alpha]$ is the second. Then

$$\overline{X}_{J1}[\alpha] = \left[\frac{1+\alpha}{7-2\alpha}, \frac{3-\alpha}{4+\alpha}\right], \tag{10.50}$$

$$\overline{X}_{J2}[\alpha] = \left[\frac{2+3\alpha}{12-4\alpha}, \frac{8-3\alpha}{6+2\alpha}\right], \tag{10.51}$$

$0 \leq \alpha \leq 1$.

Next we find that $|A_1|/|A| = b_1/a_{11}$ and $|A_2|/|A| = b_2/a_{22}$. From equations (10.41) and (10.42) we obtain $\overline{X}_{ei} = \overline{X}_{Ji}$, $i = 1, 2$.

Finally, we substitute the intervals $\overline{a}_{11}[\alpha]$, $\overline{a}_{22}[\alpha]$, $\overline{b}_1[\alpha]$ and $\overline{b}_2[\alpha]$ into $x_1 = b_1/a_{11}$ and $x_2 = b_2/a_{22}$ and we see that $\overline{X}_{Ii} = \overline{X}_{ei}$, $i = 1, 2$.

For this 2×2 fuzzy diagonal matrix $\overline{A}$ we get

$$\overline{X}_{ci} \subsetneq \overline{X}_{Ji} = \overline{X}_{ei} = \overline{X}_{Ii}, \tag{10.52}$$

$i = 1, 2$. The graphs of $\overline{X}_{ci}$ and $\overline{X}_{Ji}$, $i = 1, 2$ are in Figures 10.13 and 10.14.

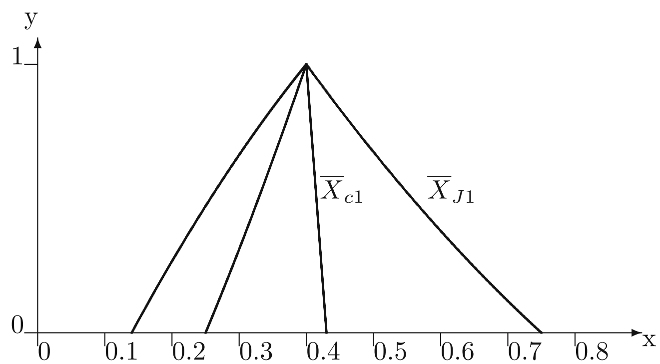

Fig. 10.13. $\overline{X}_{c1} \approx (\frac{1}{4}/\frac{2}{5}/\frac{3}{7})$ and $\overline{X}_{J1} \approx (\frac{1}{7}/\frac{2}{5}/\frac{3}{4})$ in Example 10.4.1

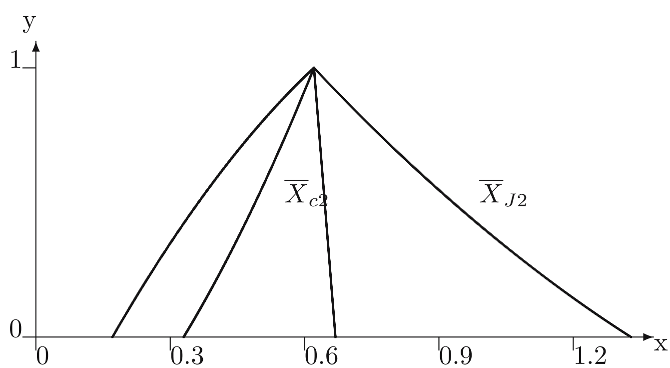

Fig. 10.14. $\overline{X}_{c2} \approx (\frac{1}{3}/\frac{5}{8}/\frac{3}{4})$ and $\overline{X}_{J2} \approx (\frac{1}{6}/\frac{5}{8}/\frac{4}{3})$ in Example 10.4.1

Example 10.4.2

Let

$$\overline{A} = \begin{pmatrix} \overline{a}_{11} & 0 \\ 1 & \overline{a}_{22} \end{pmatrix}, \tag{10.53}$$

and $\overline{B}^t = (\overline{b}_1, \overline{b}_2)$, where $\overline{a}_{11} = (1/2/3)$, $\overline{a}_{22} = (2/5/8)$, $\overline{b}_1 = (4/5/7)$ and $\overline{b}_2 = (6/8/12)$. Then $\overline{a}_{11}[\alpha] = [1+\alpha, 3-\alpha]$, $\overline{a}_{22}[\alpha] = [2+3\alpha, 8-3\alpha]$, $\overline{b}_1[\alpha] = [4+\alpha, 7-2\alpha]$ and $\overline{b}_2[\alpha] = [6+2\alpha, 12-4\alpha]$.

As in Example 10.4.1 we solve for $\overline{X}_{c1} > 0$ and obtain

$$\left[\frac{4+\alpha}{1+\alpha}, \frac{7-2\alpha}{3-\alpha}\right], \tag{10.54}$$

which does not define a fuzzy number since $\partial/\partial\alpha[(4+\alpha)/(1+\alpha)] < 0$, or $(4+\alpha)/(1+\alpha)$ is a decreasing function of α in $[0,1]$. The classical solution does not exist.

We find α-cuts of $\overline{X}_J$ using equation (10.48). We only go through the details for $\alpha = 0$ and $\alpha = 1$. An equivalent expression to equation (10.48) is

$$\overline{X}_J[\alpha] = \{\, x \in \mathbb{R}^n \mid Ax = b, v \in a[\alpha], b \in b[\alpha]\}\,. \tag{10.55}$$

For $\alpha = 1$ we get $x = (2.5, 1.1)$. For $\alpha = 0$ first assume $x_1 \geq 0$, $x_2 \geq 0$. Then we want all solutions for x_1 and x_2 so that

$$([1,3]x_1 + [0,0]x_2) \cap [4,7] \neq \emptyset, \tag{10.56}$$

$$([1,1]x_1 + [2,8]x_2) \cap [6,12] \neq \emptyset. \tag{10.57}$$

We have used the $\alpha = 0$ cuts of $\overline{a}_{11}$, $\overline{a}_{22}$, $\overline{b}_1$ and $\overline{b}_2$. This means

$$x_1 \leq 7, \tag{10.58}$$

$$3x_1 \geq 4, \tag{10.59}$$

$$x_1 + 2x_2 \leq 12, \tag{10.60}$$

$$x_1 + 8x_2 \geq 6, \tag{10.61}$$

for $x_1 \geq 0$, $x_2 \geq 0$ in the first quadrant. Now x_1 must be non–negative so we can now only consider the fourth quadrant.

Assume $x_1 \geq 0$ and $x_2 \leq 0$. Then the equations become

$$x_1 \leq 7, \tag{10.62}$$

$$3x_1 \geq 4, \tag{10.63}$$

$$x_1 + 8x_2 \leq 12, \tag{10.64}$$

$$x_1 + 2x_2 \geq 6, \tag{10.65}$$

for $x_1 \geq 0$, $x_2 \leq 0$. The solution $\overline{X}_J[0]$ is shown in Figure 10.15. It is not convex since the line joining $(4/3, 7/12)$ and $(7, -1/2)$ is not entirely in $\overline{X}_J[0]$.

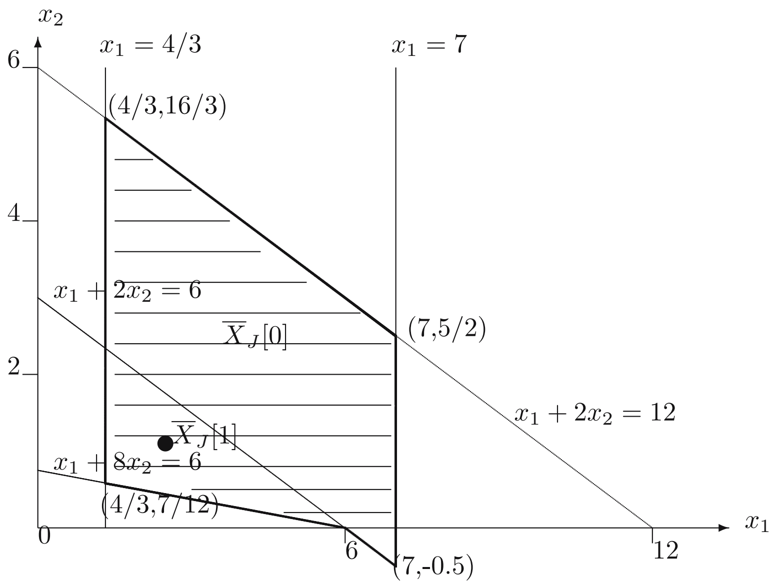

Fig. 10.15. Support of the Joint Solution in Example 10.4.2

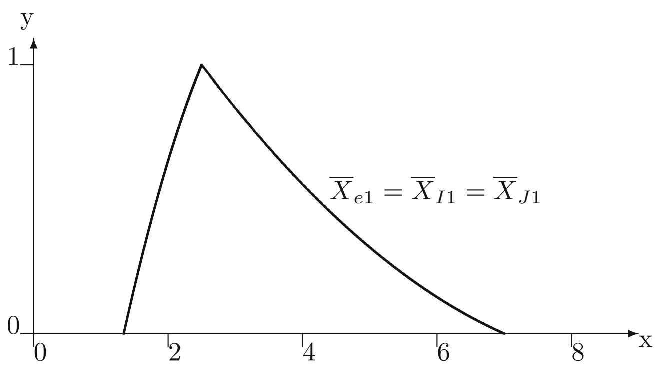

Fig. 10.16. $\overline{X}_{e1} = \overline{X}_{I1} = \overline{X}_{J1} \approx (\frac{4}{3}/\frac{5}{2}/7)$ in Example 10.4.2

Projecting $\overline{X}_J$ onto the x_i–axes produces $\overline{X}_{Ji}$, $i = 1, 2$. These marginals are in Figures 10.16 and 10.17.

We find the α-cuts of the $\overline{X}_{Ji}$ as follows: (1) we first construct a diagram like Figure 10.15 for $\overline{X}_J[\alpha]$ for each $0 \leq \alpha \leq 1$; (2) project the diagram onto the x_1-axis to get $\overline{X}_{J1}[\alpha]$; and (3) project the picture onto the x_2-axis to obtain $\overline{X}_{J2}[\alpha]$. It turns out, for this example, that $\overline{X}_{J1} = \overline{X}_{e1}$ and $\overline{X}_{J2} = \overline{X}_{e2}$.

Using equations (10.41) and (10.42) we find α-cuts of $\overline{X}_{ej}$, $j = 1, 2$. It is easy to see that

$$\overline{X}_{e1}[\alpha] = \left[\frac{4+\alpha}{3-\alpha}, \frac{7-2\alpha}{1+\alpha}\right], \tag{10.66}$$

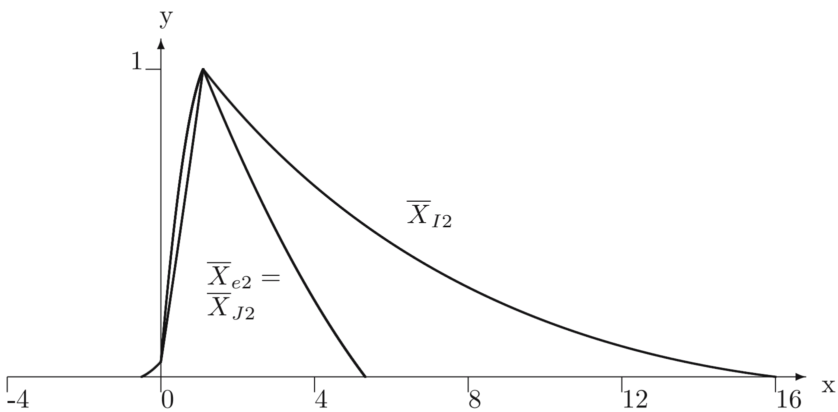

Fig. 10.17. $\overline{X}_{e2} = \overline{X}_{J2}, \overline{X}_{I2} \approx (-\frac{1}{2}/\frac{11}{10}/16)$ in Example 10.4.2

$0 \leq \alpha \leq 1$. However, $\overline{X}_{e2}[\alpha]$ is a little more difficult since we need to find the max and min of

$$\frac{a_{11}b_2 - b_1}{a_{11}a_{22}}, \tag{10.67}$$

for $a_{11} \in \overline{a}_{11}[\alpha]$, $a_{22} \in \overline{a}_{22}[\alpha]$, $b_1 \in \overline{b}_1[\alpha]$ and $b_2 \in \overline{b}_2[\alpha]$. We did this and $\overline{X}_{e2}$ is shown in Figure 10.17.

Lastly, we see that $\overline{X}_{I1} = \overline{X}_{e1}$ and

$$\overline{X}_{I2}[\alpha] = \left[x_{I21}(\alpha), x_{I22}(\alpha)\right], \tag{10.68}$$

with $x_{I21}(\alpha) = N_1(\alpha)/D_2(\alpha)$, $0 \leq \alpha \leq 0.0981 = (\sqrt{108} - 10)/4$, $x_{I21}(\alpha) = N_1(\alpha)/D_1(\alpha)$ for $0.0981 \leq \alpha \leq 1$ and $x_{I22}(\alpha) = N_2(\alpha)/D_2(\alpha)$ for all α and

$$N_1(\alpha) = (1 + \alpha)(6 + 2\alpha) - (7 - 2\alpha), \tag{10.69}$$

$$N_2(\alpha) = (3 - \alpha)(12 - 4\alpha) - (4 + \alpha), \tag{10.70}$$

$$D_1(\alpha) = (3 - \alpha)(8 - 3\alpha), \tag{10.71}$$

$$D_2(\alpha) = (1 + \alpha)(2 + 3\alpha). \tag{10.72}$$

The reason for the change in the denominator for $x_{I21}(\alpha)$ is that $N_1(\alpha)$ is negative for $0 \leq \alpha \leq 0.0981$. We used the fact that $[a, b][c, d] = [ad, bd]$ if $a < 0 < b$ and $0 < c < d$ but $[a, b][c, d] = [ac, bd]$ when $0 < a$ and $0 < c$. $\overline{X}_{ei}$ and $\overline{X}_{Ii}$ are in Figures 10.16 and 10.17 for $i = 1, 2$.

10.4.1 Fuzzy Monte Carlo Method

We follow the same strategy as outlined in Section 10.2.2 for generating approximate solutions to fuzzy linear equations using fuzzy Monte Carlo methods. **Q** will be the same set of fuzzy numbers and we use the same metric between fuzzy numbers given in equation (10.23). But now we need to extend that metric to a distance measure between fuzzy vectors. Let $\overline{X}^t = (\overline{x}_1, ..., \overline{x}_n)$ and

$$\overline{W}_i = \overline{a}_{i1}\overline{x}_1 + ... + \overline{a}_{in}\overline{x}_n, \tag{10.73}$$

for $i = 1, 2, ..., n$. Then $\overline{A}\ \overline{X} = \overline{B}$ is the same as $\overline{W} = \overline{B}$ where $\overline{W}^t = (\overline{W}_1, ..., \overline{W}_n)$. Given a $\overline{X}$ the distance between $\overline{W} = \overline{A}\ \overline{X}$ and $\overline{B}$ will be

$$D(\overline{W}, \overline{B}) = max\{D(\overline{W}_i, \overline{b}_i) | i = 1, 2, ..., n\}, \tag{10.74}$$

where $D(\overline{W}_i, \overline{b}_i)$ is from equation (10.23). Our new solution $\overline{X}^*$, whose elements $\overline{x}_i^*$ are fuzzy numbers in **Q**, solves

$$min\{D(\overline{W}, \overline{B}) | \overline{x}_i \in \mathbf{Q}\ all\ i\}. \tag{10.75}$$

Let $(\overline{X}^*)^t = (\overline{x}_1^*, ..., \overline{x}_n^*)$. Let $D(\overline{X}) = D(\overline{W}, \overline{B})$. Using our fuzzy Monte Carlo method we obtain $\overline{X}_a^*$ an approximation to $\overline{X}^*$. Let $\overline{X}_a^* = (\overline{x}_{a1}^*, ..., \overline{x}_{an}^*)$. We use the same "threshold" ϵ discussed in Section 10.2.2. If the fuzzy Monte Carlo method produces a $\overline{X}$ so that $D(\overline{X}) < \epsilon$ we will say that we have found an acceptable approximate solution $\overline{X}_a^* = \overline{X}$ with $\overline{X}_a^* \approx \overline{X}^*$. We begin with $D(\overline{X})$ very large (100) and will accept a solution only if it minimizes to $D(\overline{X}) < \epsilon = 0.5$.

Now we will rework Examples 10.4.1 and 10.4.2 using our fuzzy Monte Carlo method.

Example 10.4.1.1

This continues Example 10.4.1. We wish to use our fuzzy Monte Carlo method to compute $\overline{X}_a^* \approx \overline{X}^*$ and then compare $\overline{X}_a^*$ to $\overline{X}_c = (\overline{X}_{c1}, \overline{X}_{c2})$ shown in Figures 10.16 and 10.17. Recall that in this example the classical solution $\overline{X}_c$ exists and the components of $\overline{X}_a^* = (\overline{x}_{a1}^*, \overline{x}_{a2}^*)$ are in **Q** and we must have $D(\overline{X}_a^*) < \epsilon = 0.5$.

The fuzzy matrix equation in this example may be written $\overline{a}_{11}\ \overline{x}_1 = \overline{b}_1$ and $\overline{a}_{22}\ \overline{x}_2 = \overline{b}_2$; hence, we can solve for the $\overline{x}_i$, $i = 1, 2$, separately. However, we choose to solve for them simultaneously. Although we may determine individually lower $D(\overline{x}_{ai}^*)$ (having only to satisfy them one at a time), it is more algorithmically convenient to solve them simultaneously as we will do for this Example and for Example 10.4.1.2 discussed below. The difference between the methods will be whether we take crisp quasi-random numbers as 5-tuples for each $\overline{x}_i$, or as 10-tuples for $(\overline{x}_1, \overline{x}_2)$.

We solve for the "best" $\overline{x}_1$ and $\overline{x}_2$ simultaneously. We generate a Sobol quasi-random number 10-tuple with which we generate $\overline{x}_1$ with the first five and $\overline{x}_2$ with the last five. Using $\overline{X} = (\overline{x}_1, \overline{x}_2)$ we compute $\overline{W}_1 = \overline{a}_{11}\overline{x}_1 + \overline{a}_{12}\overline{x}_2$, and

$\overline{W}_2 = \overline{a}_{21}\overline{x}_1 + \overline{a}_{22}\overline{x}_2$ (even though for Example 10.4.1 $\overline{a}_{12} = 0$ and $\overline{a}_{21} = 0$). Then $D(\overline{W}, \overline{B}) = max\{D(\overline{W}_1, \overline{b}_1), D(\overline{W}_2, \overline{b}_2)\}$ is determined for this $\overline{X}$. As we evaluate these $\overline{X}$'s we find the least $D(\overline{W}, \overline{B})$, capturing its $\overline{X}$ as $\overline{X}^*_a$.

Next we determine intervals, which will depend on the application, for the random quadratic fuzzy numbers (QBGFNs). We modified the fuzzy Monte Carlo program to optimize the minimization problem of equations (10.74) and (10.75) for fuzzy matrix equations. For this problem the classical solution does exist. For $\overline{x}_1$ we choose $[a, b] = [0.1429, 0.7500]$ which is approximately the support of $\overline{X}_{J1}$. For $\overline{x}_2$ we choose $[a, b] = [0.1666, 1.3333]$ which is approximately the support of $\overline{X}_{J2}$.

To compute $D(\overline{A} \cdot \overline{X}, \overline{B})$, we used 100 α-cuts of $\overline{A} \cdot \overline{X}$ and $\overline{B}$ for given $\overline{A}$ and $\overline{B}$, where $\overline{X}$ is one of 50,000 pairs of random quadratic fuzzy numbers.

The smallest $D(\overline{W}, \overline{B})$ we obtained was 0.302329. The $\overline{X}$ that produced this value we had saved as $\overline{X}^*_a = (\overline{x}^*_{a2}, \overline{x}^*_{a2})$. $\overline{x}^*_{a1}$ and $\overline{x}^*_{a2}$ are displayed in Figure 10.18. As one can see in Table 10.5, solutions of $\left[\begin{smallmatrix} (4/5/7) & 0 \\ 0 & (6/8/12) \end{smallmatrix}\right] \cdot \overline{X} = \left[\begin{smallmatrix} (1/2/3) \\ (2/5/8) \end{smallmatrix}\right]$, we have acceptable solutions for $\overline{x}^*_{a1}$ and $\overline{x}^*_{a2}$. Figure 10.19 and Figure 10.20 are graphs for $\overline{A} \cdot \overline{X}$.

Table 10.5. Solutions for Example 10.4.1, $\overline{A} \cdot \overline{X} = \overline{B}$

Solution	$\overline{X}$	$\overline{A} \cdot \overline{X}$
$\overline{X}_{c1}$	$\approx (0.25/0.4/0.4286)$	$\approx (1/2/3) = \overline{b}_1$
$\overline{X}_{c2}$	$\approx (0.333\overline{3}/0.625/0.75)$	$\approx (2/5/8) = \overline{b}_2$
$\overline{X}_{J1}$	$\approx (0.1429/0.4/0.75)$	$\approx (0.5716/2/5.25)$
$\overline{X}_{J2}$	$\approx (0.166\overline{6}/0.625/1.333\overline{3})$	$\approx (1/5/20)$
$\overline{x}^*_{a1}$,	$(0.2473, 0.3973, 0.4436, -0.7088, 0.4686)$, $D(\overline{W}_1, \overline{b}_1) = 0.119918$	$\approx (0.99/1.99/3.11)$
$\overline{x}^*_{a2}$,	$(0.3646, 0.6203, 0.6908, -0.0618, 0.0208)$, $D(\overline{W}_2, \overline{b}_2) = 0.302329$ (min)	$\approx (2.19/4.96/8.29)$

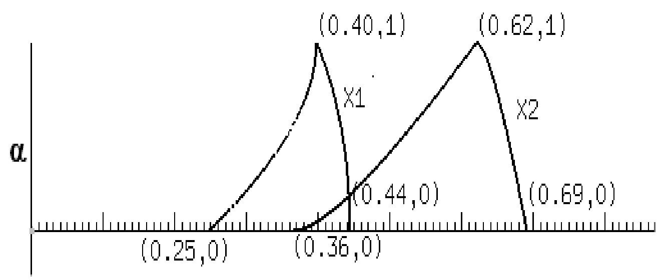

Fig. 10.18. $\overline{x}^*_{a1}$ and $\overline{x}^*_{a2}$ of Example 10.4.1.1

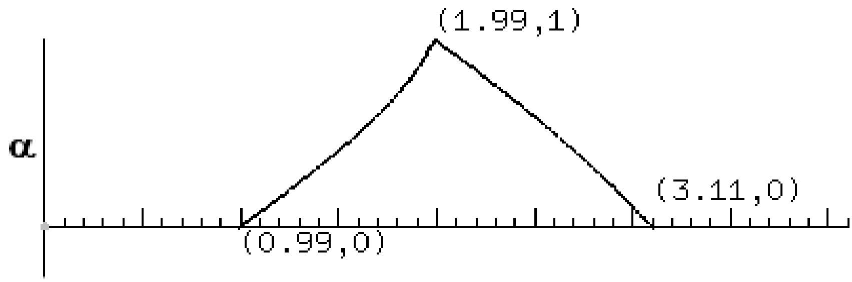

Fig. 10.19. $(4/5/7) \cdot \overline{x}_{a1}^*$ of Example 10.4.1.1

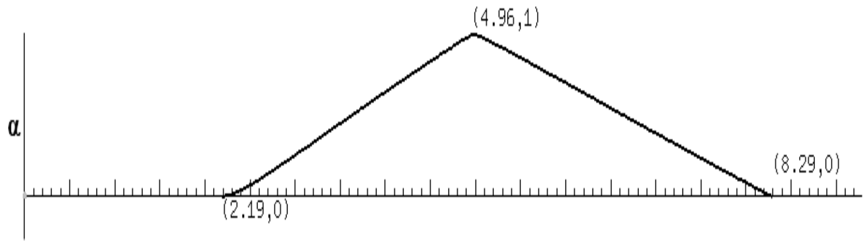

Fig. 10.20. $(6/8/12) \cdot \overline{x}_{a2}^*$ of Example 10.4.1.1

Example 10.4.1.2

This continues Examples 10.4.2. We apply our fuzzy Monte Carlo method to compute $\overline{X}_a^* \approx \overline{X}^*$. Recall that in this example the classical solution does not exist but we do have $\overline{X}_e$, $\overline{X}_J$ and $\overline{X}_I$ shown in Figures 10.16 and 10.17. The constraints are still that the components of $\overline{X}_a^*$ are in **Q** and it must satisfy $D(\overline{X}_a^*) < \epsilon = 0.5$.

Table 10.6. Solutions for Example 10.4.2, $\overline{A} \cdot \overline{X} = \overline{B}$

	$\overline{X}$	$\overline{A} \cdot \overline{X}$
$\overline{X}_c$	does not exist	$\approx [(4/5/7)\ (6/8/12)]^t$
$\overline{X}_{I1}$	$\approx (1.333\overline{3}/2.5/7)$	$\approx (1.333\overline{3}/5/21)$
$\overline{X}_{I2}$	$\approx (-0.5/1.1/16)$	$\approx (1.333\overline{3}/8/135)$
$\overline{x}_{a1}^*$,	$(1.5847, 1.6205, 3.3639, 0.6678, -2.6900)$,	$\approx (1.58/3.24/10.09)$
	$D(\overline{W}_1, \overline{b}_1) = 3.091642$	
$\overline{x}_{a2}^*$,	$(0.4937, 0.7202, 1.3369, -0.8838, -1.1810)$,	$\approx (2.57/5.22/14/06)$
	$D(\overline{W}_1, \overline{b}_1) = 3.427969$ (min)	

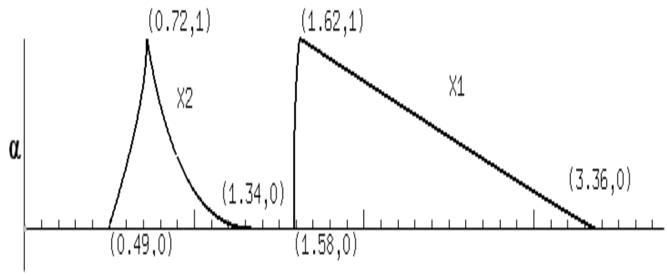

Fig. 10.21. $\overline{x}^*_{a1}$ and $\overline{x}^*_{a2}$ of Example 10.4.1.2

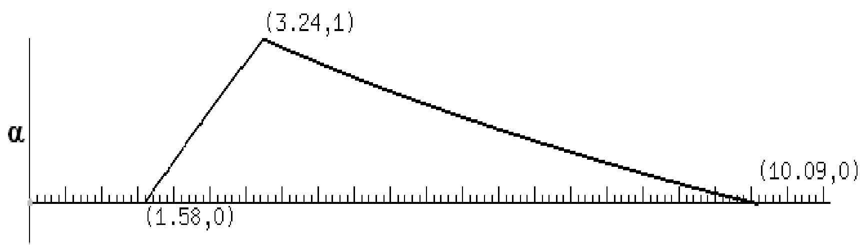

Fig. 10.22. $(1/2/3) \cdot \overline{x}^*_{a1}$ of Example 10.4.1.2

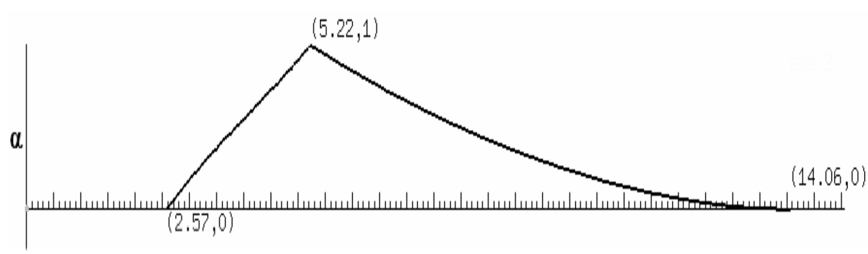

Fig. 10.23. $\overline{x}^*_{a1} + (2/5/8) \cdot \overline{x}^*_{a2}$ of Example 10.4.1.2

For this problem the classical solution does not exist. For $\overline{x}_1$ we choose $[a, b] = [1.3333, 7]$ which is approximately the support of $\overline{X}_{I1}$. For $\overline{x}_2$ we choose $[a, b] = [0, 16]$ which covers the positive portion of the support of $\overline{X}_{I2}$.

To compute $D(\overline{A} \cdot \overline{X}, \overline{B})$, we used 100 α-cuts of $\overline{A} \cdot \overline{X}$ and $\overline{B}$ for given $\overline{A}$ and $\overline{B}$, where $\overline{X}$ is one of 50,000 pairs of random quadratic fuzzy numbers.

The distance measure D is given in equations (10.74) and (10.75). The smallest $D(\overline{W}, \overline{B})$ we obtained was 3.427969. The $\overline{X}$ that produced this value we had saved as $\overline{X}^*_a = (\overline{x}^*_{a1}, \overline{x}^*_{a2})$. $\overline{X}^*_a$ is shown in Figure 10.21. $(1/2/3) \cdot \overline{x}^*_{a1}$ is shown in Figure 10.22. The corresponding $\overline{x}^*_{a1} + (2/5/8) \cdot \overline{x}^*_{a2}$ is shown in Figure 10.23.

Our fuzzy Monte Carlo method was unable to get an acceptable approximate solution to this fuzzy matrix equation. Table 10.6 lists solutions of $[\begin{smallmatrix}(1/2/3) & 0 \\ 1 & (2/5/8)\end{smallmatrix}] \cdot \overline{X} = [\begin{smallmatrix}(4/5/7) \\ (6/8/12)\end{smallmatrix}]$ for Example 10.4.2. As our value of $D(\overline{X})$ indicates, we do not have good correspondence with the right had side of our fuzzy matrix equation. We believe that no fuzzy quadratic fuzzy vector $\overline{X}$ can make this $D(\overline{X}) < \epsilon = 0.5$.

10.5 Summary and Conclusions

Through several examples of three types of fuzzy equations (fuzzy linear equations $\overline{A} \cdot \overline{X} + \overline{B} = \overline{C}$, fuzzy quadratic equations $\overline{A} \cdot \overline{X}^2 + \overline{B} \cdot \overline{X} + \overline{C} = \overline{D}$, and fuzzy matrix equations $\overline{A} \cdot \overline{X} = \overline{B}$) we have demonstrated a use of fuzzy Monte Carlo optimization to obtain solutions. Where possible we demonstrated computation of the "classical" solution. We showed how one may compute an "extension principle" solution; and we showed how one may determine a "interval arithmetic" solution.

Also, for every example we used fuzzy Monte Carlo optimization to produce some solution. In those examples where a "classical" solution could be computed, fuzzy Monte Carlo found an acceptable solution. In one case, Example 10.3.1.2, were no classical solution existed, fuzzy Monte Carlo determined an acceptable approximate solution.

This study supports the viability of this method for solving fuzzy equations. Our choice of a value ϵ, and performance of $D(\overline{X})$ during simulations, indicates we may in the future choose a smaller $\epsilon = 0.2$ to differentiate "tight" solutions from "loose" ones.

These optimizations were performed on various Windows XP machines running in the 2Ghz range with over 1GB RAM.

$\overline{A} \cdot \overline{X} + \overline{B} = \overline{C}$ completed $50,000$ iterations in about 1.5 hours. Fuzzy quadratic exercises and fuzzy matrix equations completed $50,000$ iterations in $2-3$ hours. Convergence to an acceptable $D(\overline{X})$ was evident within a few hundred iterations.

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