5
Methods of knowledge representation using type-2 fuzzy sets

5.1 Introduction

The fuzzy sets, discussed in the previous chapter, are called type-1 fuzzy sets. They are characterized by the membership function, while the value of this function for a given element x is called the grade of membership of this element to a fuzzy set. In case of type-1 fuzzy sets, the membership grade is a real number taking values in the interval $[0, 1]$. This chapter will present another concept of a fuzzy description of uncertainty. According to this concept, the membership grade is not a number any more, but it has a fuzzy character. Figure 5.1 shows a graphic illustration of type-1 fuzzy sets $A_1, ..., A_5$ and corresponding type-2 fuzzy sets $\widetilde{A}_1, ..., \widetilde{A}_5$. It should be noted that in case of type-2 fuzzy sets, for any given element x, we cannot speak of an unambiguously specified value of the membership function. In other words, the membership grade is not a number, as in case of type-1 fuzzy sets.

In subsequent points of this chapter, basic definitions concerning type-2 fuzzy sets will be presented and operations on these sets will be discussed. Then type-2 fuzzy relations and methods of transformation of type-2 fuzzy sets into type-1 fuzzy sets will be introduced.

In the last part of this chapter, the theory of type-2 fuzzy sets will serve for the construction of the fuzzy inference system. Particular blocks of such system will be discussed in details, including type-2 fuzzification, type-2 rules base, type-2 inference mechanisms and the two-stage defuzzification consisting of type-reduction and defuzzification.

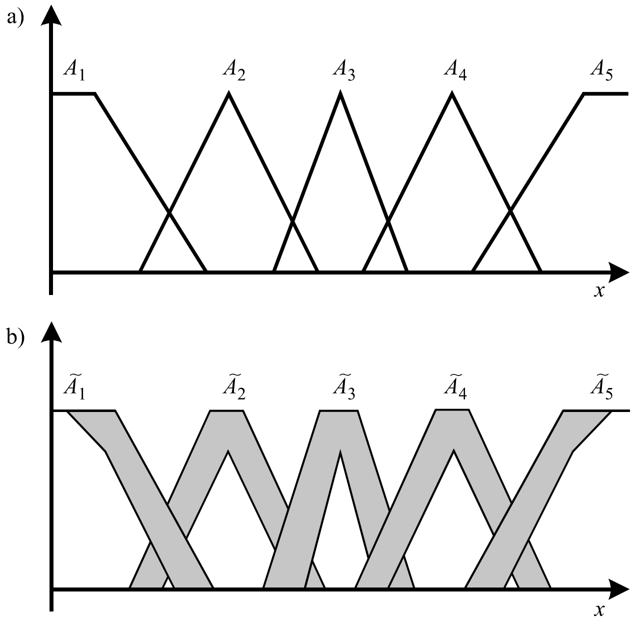

FIGURE 5.1. Graphic illustrations of type-1 fuzzy sets and corresponding type-2 fuzzy sets

5.2 Basic definitions

Definition 5.1
Type-2 fuzzy set $\widetilde{A}$ defined on a universe of discourse X, which is denoted as $\widetilde{A} \subseteq X$, is a set of pairs

$$\{x, \mu_{\widetilde{A}}(x)\}, \tag{5.1}$$

where x is an element of a fuzzy set, and its grade of membership $\mu_{\widetilde{A}}(x)$ in the fuzzy set $\widetilde{A}$ is a type-1 fuzzy set defined in the interval $J_x \subset [0,1]$, i.e.

$$\mu_{\widetilde{A}}(x) = \int_{u \in J_x} f_x(u)/u. \tag{5.2}$$

Function $f_x : [0,1] \to [0,1]$ will be called *the secondary membership function*, and its value $f_x(u)$ will be called *the secondary grade* or *secondary membership*. Of course, u is an argument of the secondary membership function. The interval J_x, being a domain of the secondary membership function f_x, is called *the primary membership of* element x. The fuzzy set $\widetilde{A}$ may be notated, in the notation of fuzzy sets, as follows:

$$\widetilde{A} = \int_{x \in X} \mu_{\widetilde{A}}(x)/x \tag{5.3}$$

or

$$\widetilde{A} = \int \mu_{\widetilde{A}}(x)/x = \int_{x \in X} \left[\int_{u \in J_x} f_x(u)/u \right] /x, \quad J_x \subseteq [0,1]. \tag{5.4}$$

Example 5.1

Fig. 5.2a depicts the method of construction of type-2 fuzzy sets. For a given element x_1 we get the interval $J_{x_1} = [0.4, 0.7]$ being a domain of the secondary membership function f_{x_1}. Figures 5.2b, 5.2c and 5.2d show exemplary secondary membership functions of triangular, interval and Gaussian types with a finite support [171].

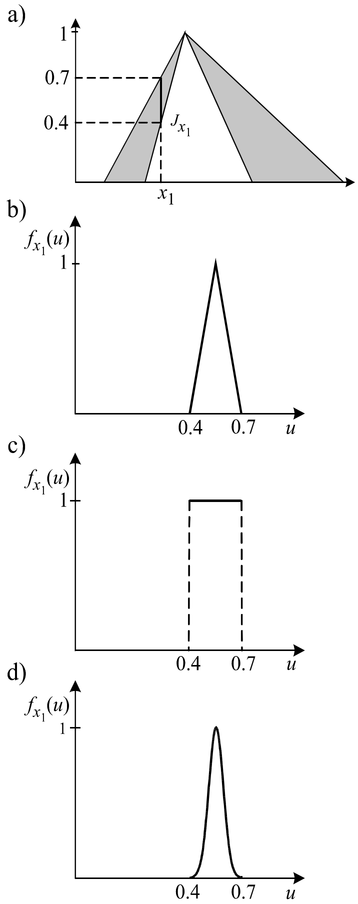

FIGURE 5.2. Illustration of type-2 fuzzy set and secondary membership functions for $J_{x_1} = [0.4; 0.7]$

Figure 5.3 depicts the same type-2 fuzzy set, but another element x_2 is chosen, $x_2 \in X$, as well as a corresponding membership grade being a type-1 fuzzy set (of a triangular, interval or Gaussian type with a finite support) defined on the interval $J_{x_2} = [0.1, 0.6]$.

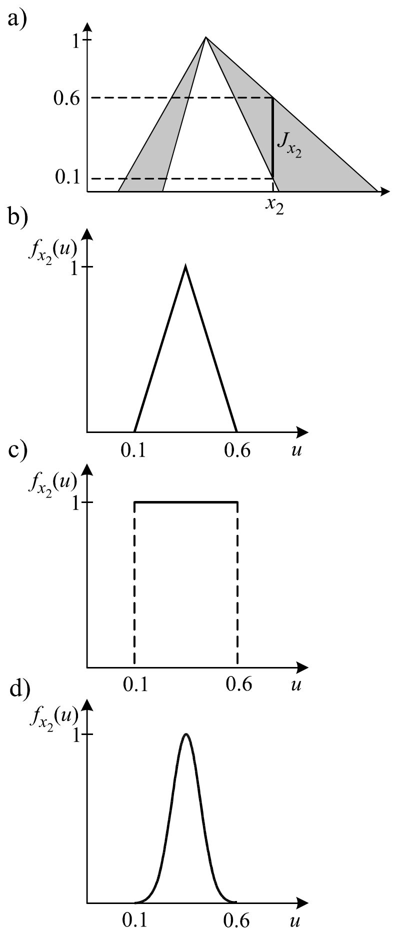

FIGURE 5.3. Illustration of type-2 fuzzy set and secondary membership functions for $J_{x_2} = [0.1; 0.6]$

In a discrete case, the type-2 fuzzy set will be defined in a similar way, i.e.

$$\widetilde{A} = \sum_{x \in X} \mu_{\tilde{A}}(x)/x \tag{5.5}$$

and

$$\mu_{\tilde{A}}(x) = \sum_{u \in J_x} f_x(u)/u. \tag{5.6}$$

Let us assume that the set X has been discretized and takes R values $x_1, ..., x_R$. Moreover, the intervals J_x corresponding to these values, have been discretized and each of them takes M_i values, $i = 1, ..., R$. We can then note

$$\widetilde{A} = \sum_{x \in X} \left[\sum_{u \in J_x} f_x(u)/u \right] /x = \sum_{i=1}^{R} \left[\sum_{u \in J_{x_i}} f_{x_i}(u)/u \right] /x_i \tag{5.7}$$
$$= \left[\sum_{k=1}^{M_1} f_{x_i}(u_{1k})/u_{1k} \right] /x_1 + \ldots + \left[\sum_{k=1}^{M_R} f_{x_R}(u_{Rk})/u_{Rk} \right] /x_R.$$

Remark 5.1
The fuzzy membership grade can take two characteristic and extreme forms of the type-1 fuzzy set:
$\mu_{\widetilde{A}}(x) = 1/1$ meaning a full membership of element x to the fuzzy set $\widetilde{A}$,
$\mu_{\widetilde{A}}(x) = 1/0$ meaning the lack of membership of element x to the fuzzy set $\widetilde{A}$,

Example 5.2
Let us assume that $X = \{1, 2, 3\}$ and $J_{x_1} = \{0.2, 0.5, 0.7\}$, $J_{x_2} = \{0.5, 1\}$, $J_{x_3} = \{0.1, 0.3, 0.5\}$. If we assign appropriate grades of secondary membership to particular elements of sets $J_{x_1}, J_{x_2}, J_{x_3}$ we may define the following type-2 fuzzy set:

$$\widetilde{A} = (0.5/0.2 + 1/0.5 + 0.5/0.7)/1 + (0.5/0.5 + 1/1)/2 \tag{5.8}$$
$$+ (0.5/0.1 + 1/0.3 + 0.5/0.5)/3.$$

Definition 5.2
Let us assume that each secondary membership function f_x of a type-2 fuzzy set takes value 1 only for one element $u \in J_x$. Then the union of elements u forms a so-called *principal membership function*, i.e.

$$\mu_{A_g}(x) = \int_{x \in X} u/x, \quad \text{where} \quad f_x(u) = 1. \tag{5.9}$$

The principal membership function defines the appropriate type-1 fuzzy set denoted as A_g.

Remark 5.2
In case where the secondary membership function f_x is an interval function, then the principal membership function will be determined as a union of all the elements u being mid-points of the primary membership J_x,$x \in X$.

Example 5.3
We are going to discuss a type-2 fuzzy set given by formula (5.8). Upon the basis of Definition 5.2 we may determine the following fuzzy set A_g:

$$A_g = \frac{0.5}{1} + \frac{1}{2} + \frac{0.3}{3}. \tag{5.10}$$

5.3 Footprint of uncertainty

The type-2 fuzzy set may be described using the notion of the footprint of uncertainty.

Definition 5.3
Let us assume that $J_x \subset [0,1]$ means the primary membership of element x. *The footprint of uncertainty (FOU)* of a type-2 fuzzy set $\widetilde{A} \subseteq X$ will be a bounded region consisting of all the points of primary membership of elements x, i.e.

$$FOU\left(\widetilde{A}\right) = \bigcup_{x \in X} J_x. \tag{5.11}$$

Example 5.4
Let us discuss the family of membership functions of the type-1, fuzzy set which is described by the Gaussian function with the assumption that a standard deviation σ changes in the interval $[\sigma_1, \sigma_2]$, i.e.:

$$\mu_A(x) = N(m, \sigma; x) = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right], \quad \sigma \in [\sigma_1, \sigma_2]. \tag{5.12}$$

The family of membership functions (5.12) forms a type-2 fuzzy set. A full description of this set would require to define the secondary membership function for each point x and the corresponding interval J_x. Figure 5.4 shows the footprint of uncertainty of the discussed type-2 fuzzy set.

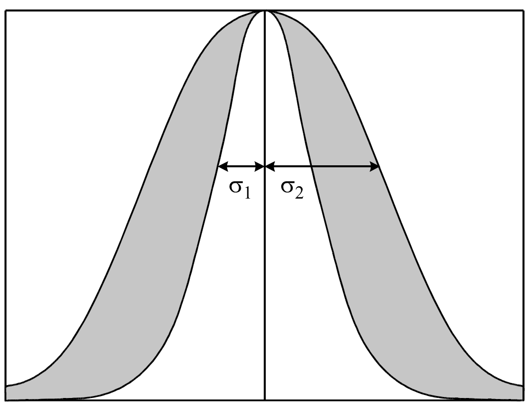

FIGURE 5.4. Footprint of uncertainty of a type-2 fuzzy set: $\sigma \in [\sigma_1, \sigma_2]$

Example 5.5
Let us discuss the family of membership functions of the type-1 fuzzy set which is described by the Gaussian function with the assumption that the average value m changes in interval $[m_1, m_2]$, i.e.

$$\mu_A(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right], \quad m \in [m_1, m_2]. \tag{5.13}$$

The family of membership functions (5.13) forms a type-2 fuzzy set. As in the previous example, a full description of this set would require to define the secondary membership function for each point x and the corresponding interval J_x. Figure 5.5 shows the footprint of uncertainty of the discussed type-2 fuzzy set.

Let us assume that $J_x = \left[\underline{J}_x, \overline{J}_x\right], \quad x \in X$.

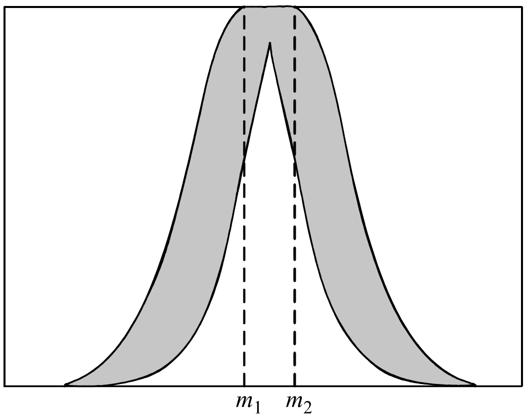

FIGURE 5.5. Footprint of uncertainty of a type-2 fuzzy set: $m \in [m_1, m_2]$

Definition 5.4
The upper membership function (UMF) is the membership function of the type-1 fuzzy set defined by:

$$\overline{\mu}_{\widetilde{A}}(x) = UMF(\widetilde{A}) = \bigcup_{x \in X} \overline{J_x} \quad \forall x \in X. \tag{5.14}$$

Definition 5.5
The lower membership function (LMF) is the membership function of the type-1 fuzzy set defined by:

$$\underline{\mu}_{\widetilde{A}}(x) = LMF(\widetilde{A}) = \bigcup_{x \in X} \underline{J_x} \quad \forall x \in X. \tag{5.15}$$

Example 5.6
We are going to determine the footprint of uncertainty for the type-2 fuzzy set given in Example 5.4. It is easy to notice that the upper membership function takes the form

$$\overline{\mu}_{\widetilde{A}}(x) = N(m, \sigma_2; x), \tag{5.16}$$

and the lower membership function is given by

$$\underline{\mu}_{\widetilde{A}}(x) = N(m, \sigma_1; x). \tag{5.17}$$

Example 5.7
For the type-2 fuzzy set given in Example 5.5 the upper membership function takes the form

$$\overline{\mu}_{\widetilde{A}}(x) = \begin{cases} N(m_1, \sigma; x) & \text{for} \quad x < m_1, \\ 1 & \text{for} \quad m_1 \le x \le m_2, \\ N(m_2, \sigma; x) & \text{for} \quad x > m_2, \end{cases} \tag{5.18}$$

and the lower membership function is given by

$$\underline{\mu}_{\widetilde{A}}(x) = \begin{cases} N(m_2, \sigma; x) & \text{for} \quad x \le \dfrac{m_1 + m_2}{2}, \\ N(m_1, \sigma; x) & \text{for} \quad x > \dfrac{m_1 + m_2}{2}. \end{cases} \tag{5.19}$$

5.4 Embedded fuzzy sets

In type-2 fuzzy sets we can distinguish between so-called embedded type-1 and embedded type-2 fuzzy sets.

Definition 5.6
From each interval J_x, $x \in X$, we will select only one element $\theta \in J_x$.
The embedded type-2 set in set $\widetilde{A}$ is set $\widetilde{A_o}$

$$\widetilde{A}_o = \int_{x \in X} \left[f_x(\theta) / \theta \right] / x \quad \theta \in J_x \subseteq U = [0, 1]. \tag{5.20}$$

Of course, there is an uncountable number of embedded sets $\widetilde{A}_\mathrm{o}$ in set $\widetilde{A}$. In a discrete case, the embedded set $\widetilde{A}_0$ is defined as follows:

$$\widetilde{A}_\mathrm{o} = \sum_{i=1}^{R} \left[f_{x_i} \left(\theta_i \right) / \theta_i \right] / x_i \quad \theta_i \in J_{x_i} \subseteq U = [0,1]. \tag{5.21}$$

It is easy to notice that there are $\prod_{i=1}^{R} M_i$ embedded fuzzy sets $\widetilde{A}_\mathrm{o}$ in set $\widetilde{A}$.

Example 5.8
Let us assume that

$$\begin{aligned} \widetilde{A} = & \left(0.5/0.2 + 1/0.5 + 0.5/0.7\right)/2 + \left(0.3/0.5 + 1/1\right)/3 \\ & + \left(0.5/0.1 + 1/0.3 + 0.5/0.5\right)/4. \end{aligned} \tag{5.22}$$

Then one of the 18 embedded fuzzy sets $\widetilde{A}_\mathrm{o}$ takes the form

$$\widetilde{A_\mathrm{o}} = \left(0.5/0.7\right)/2 + \left(0.3/0.5\right)/3 + \left(1/0.3\right)/4. \tag{5.23}$$

Each embedded type-2, fuzzy set $\widetilde{A_\mathrm{o}}$ is connected with an embedded type-1 fuzzy set denoted as A_o.

Definition 5.7
The embedded type-1 set is defined as follows:

$$A_\mathrm{o} = \int_{x \in X} \theta / x \quad \theta \in J_x \subseteq U = [0,1]. \tag{5.24}$$

There is an uncountable number of embedded fuzzy sets A_o. In a discrete case, formula (5.24) becomes

$$A_\mathrm{o} = \sum_{i=1}^{R} \theta_i / x_i \quad \theta_i \in J_{x_i} \subseteq U = [0,1]. \tag{5.25}$$

The number of all sets A_o is $\prod_{i=1}^{R} M_i$.

A particular case of an embedded type-1 set is fuzzy set A_g defined by the principal membership function given by formula (5.9). Furthermore, it should be noted that embedded fuzzy set A_o looses all the information about secondary grades. Thus, upon the basis of the family of embedded sets A_o, it is not possible to reconstruct the type-2 fuzzy set, but only its footprint of uncertainty. However, the notion of embedded set will turn to be especially useful when discussing the fast algorithm of type-reduction presented further in this chapter.

Example 5.9
Figure 5.6 shows three different embedded type-1 sets.

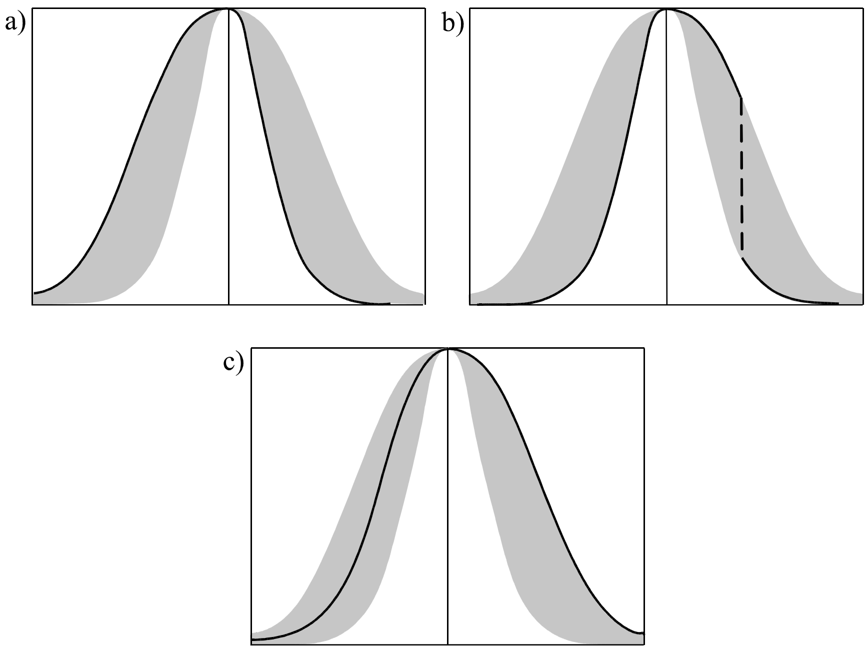

FIGURE 5.6. Embedded type-1 fuzzy sets

Example 5.10
Let us discuss a type-2 fuzzy set given by

$$\widetilde{A} = (0.6/0.3 + 1/0.7)\,/3 + (0.4/0.4 + 1/1)\,/5. \tag{5.26}$$

We can distinguish four embedded fuzzy sets A_{o} in set $\widetilde{A}$:

$$\begin{aligned} A_{\text{o}} &= 0.3/3 + 0.4/5, \\ A_{\text{o}} &= 0.7/3 + 0.4/5, \\ A_{\text{o}} &= 0.3/3 + 1/5, \\ A_{\text{o}} &= 0.7/3 + 1/5. \end{aligned} \tag{5.27}$$

5.5 Basic operations on type-2 fuzzy sets

The extension principle (Subchapter 4.4) allows to extend operations on type-1 fuzzy sets to operations on type-2 sets.

We are going to discuss two type-2 fuzzy sets, $\widetilde{A}$ and $\widetilde{B}$, defined as follows:

$$\widetilde{A} = \int_{x\in X} \left(\int_{u\in J_x^u} f_x(u)/u \right) /x \tag{5.28}$$

and

$$\widetilde{B} = \int_{x\in X} \left(\int_{v\in J_x^v} g_x(v)/v \right) /x, \tag{5.29}$$

where $J_x^u, J_x^v \subset [0,1]$. The sum of sets $\widetilde{A}$ and $\widetilde{B}$ is a type-2 fuzzy set notated as $\widetilde{A} \cup \widetilde{B}$ and defined as follows:

$$\begin{aligned} \mu_{\widetilde{A}\cup\widetilde{B}} &= \int_{w\in J_x^w} h_x(w)/w = \phi\left(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\right) \\ &= \phi\left(\int_{u\in J_x^u} f_x(u)/u, \int_{v\in J_x^v} g_x(v)/v \right). \end{aligned} \tag{5.30}$$

In this case, the extended ϕ function is any t-conorm, but its arguments are not common numbers, but type-1 fuzzy sets $\mu_{\widetilde{A}}(x)$ and $\mu_{\widetilde{B}}(x)$ for the given $x \in X$. In accordance with the extension principle

$$\phi\left(\int_{u\in J_x^u} f_x(u)/u, \int_{v\in J_x^v} g_x(v)/v \right) = \int_{u\in J_x^u} \int_{v\in J_x^v} f_x(u) \overset{T}{*} g_x(v)/\phi(u,v). \tag{5.31}$$

After substituting t-conorm in place of function ϕ, the sum of type-2 fuzzy sets is given by the fuzzy membership function

$$\mu_{\widetilde{A}\cup\widetilde{B}}(x) = \int_{u\in J_x^u} \int_{v\in J_x^v} f_x(u) \overset{T}{*} g_x(v)/u \overset{S}{*} v. \tag{5.32}$$

The formula above allows to determine the sum of type-2 fuzzy sets for each value of x. The membership function of the resulting set is the highest value of expression $f_x(u) \overset{T}{*} g_x(v)$ for all the pairs (u,v), which give the same element $w = u \overset{S}{*} v$ as a result.

Example 5.11
This example will explain in details the manner of determining the sum of type-2 fuzzy sets. We assume the minimum operation as the t-norm, and the maximum operation as the t-conorm. We are going to discuss two type-2 fuzzy sets, $\widetilde{A}$ and $\widetilde{B}$, defined as follows:

$$\begin{aligned} \widetilde{A} &= (0.5/0.2 + 1/0.5 + 0.5/0.7)/1 + (0.5/0.5 + 1/1)/2 \\ &\quad + (0.5/0.1 + 1/0.3 + 0.5/0.5)/3 \end{aligned} \tag{5.33}$$

and

$$\widetilde{B} = (1/0)/1 + (0.5/0.5 + 1/0.8)/2 + (1/0.6 + 0.5/1)/3. \tag{5.34}$$

In accordance to formula (5.32) for $x = 1$ we have

$$\begin{aligned} \mu_{\widetilde{A}\cup\widetilde{B}}(1) &= \frac{0.5 \wedge 1}{0.2 \vee 0} + \frac{1 \wedge 1}{0.5 \vee 0} + \frac{0.5 \wedge 1}{0.7 \vee 0} \\ &= \frac{0.5}{0.2} + \frac{1}{0.5} + \frac{0.5}{0.7}. \end{aligned} \tag{5.35}$$

For $x = 2$ we obtain

$$\begin{aligned} \mu_{\widetilde{A}\cup\widetilde{B}}(2) &= \frac{0.5 \wedge 0.5}{0.5 \vee 0.5} + \frac{0.5 \wedge 1}{0.5 \vee 0.8} + \frac{\max(1 \wedge 0.5, 1 \wedge 1)}{1} \\ &= \frac{0.5}{0.5} + \frac{0.5}{0.8} + \frac{1}{1}. \end{aligned} \tag{5.36}$$

For $x = 3$ we have

$$\begin{aligned} \mu_{\widetilde{A}\cup\widetilde{B}}(3) &= \frac{\max(0.5\wedge 1, 1\wedge 1, 0.5\wedge 1)}{0.6} + \frac{\max(0.5\wedge 0.5, 1\wedge 0.5, 0.5\wedge 0.5)}{1} \\ &= \frac{1}{0.6} + \frac{0.5}{1}. \end{aligned} \tag{5.37}$$

Thus, the sum of sets $\widetilde{A}$ and $\widetilde{B}$ is

$$\begin{aligned} \widetilde{A} \cup \widetilde{B} = {} & (0.5/0.2+1/0.5+0.5/0.7)/1 + (0.5/0.5+0.5/0.8+1/1)/2 \\ & + (1/0.6+0.5/1)/3. \end{aligned} \tag{5.38}$$

The intersection of sets $\widetilde{A}$ and $\widetilde{B}$ is a type-2 fuzzy set with the fuzzy membership function given by the following formula

$$\begin{aligned} \mu_{\widetilde{A}\cap\widetilde{B}}(x) &= \int_{w\in J_x^w} h_x(w)/w = \phi\left(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\right) \\ &= \phi\left(\int_{u\in J_x^u} f_x(u)/u, \int_{v\in J_x^v} g_x(v)/v\right), \end{aligned} \tag{5.39}$$

where the extended function ϕ is any t-norm this time. The arguments of the function ϕ are type-1 fuzzy sets, i.e, $\mu_{\widetilde{A}}(x)$ and $\mu_{\widetilde{B}}(x)$. Thus, the intersection of type-2 fuzzy sets is specified as follows:

$$\mu_{\widetilde{A}\cap\widetilde{B}}(x) = \int_{u\in J_x^u} \int_{v\in J_x^v} f_x(u) \overset{T^*}{*} g_x(v) / u \overset{T}{*} v. \tag{5.40}$$

In formula (5.40), the t-norm aggregating secondary memberships has been denoted by T^*, and its form can be selected irrespectively of the selection of the extended t-norm T. Also in this case the membership function of the resulting set is the highest value of the expression $f_x(u) \overset{T^*}{*} g_x(v)$ for all the pairs (u, v), which bring the same element $w = u \overset{T}{*} v$ as a result.

Example 5.12
We are going to determine the intersection of the type-2 fuzzy sets discussed in Example 5.11. We assume the minimum operation as t-norm T^* and T. In accordance with formula (5.40), for $x = 1$ and $x = 2$, we obtain

$$\mu_{\widetilde{A} \cap \widetilde{B}}(1) = \frac{\max(0.5 \wedge 1, 1 \wedge 1, 0.5 \wedge 1)}{0} = \frac{1}{0} \tag{5.41}$$

and

$$\begin{aligned} \mu_{\widetilde{A} \cap \widetilde{B}}(2) &= \frac{\max(0.5 \wedge 0.5, 0.5 \wedge 1, 1 \wedge 0.5)}{0.5} + \frac{1 \wedge 1}{1 \wedge 0.8} \\ &= \frac{0.5}{0.5} + \frac{1}{0.8}. \end{aligned} \tag{5.42}$$

A complement of the type-2 fuzzy set is a type-2 fuzzy set with the fuzzy membership function given by the formula

$$\begin{aligned} \mu_{\widehat{\widetilde{A}}}(x) &= \phi\left(\mu_{\widetilde{A}}(x)\right) \\ &= \int_{u \in J_x^u} f_x(u) / (1 - u). \end{aligned} \tag{5.43}$$

Example 5.13
Let us discuss a type-2 fuzzy set defined by the following formula:

$$\mu_{\widetilde{A}}(x) = (0.4/0.6 + 1/0.7)/9. \tag{5.44}$$

In accordance with formula (5.43), we have

$$\mu_{\widehat{\widetilde{A}}}(x) = (0.4/0.4 + 1/0.3)/9. \tag{5.45}$$

Remark 5.3
Sum (5.32) and intersection (5.40) of type-2 fuzzy sets may be treated as a result of applying the operator of the extended t-norm $\widetilde{T}$ and extended t-conorm $\widetilde{S}$. These operators may also be discussed in the context of type-1 fuzzy sets defined in the interval $[0, 1]$. We are going to discuss two such sets

$$F = \int_{u \in J^u} f(u)/u \quad \text{and} \quad G = \int_{v \in J^v} g(v)/v. \tag{5.46}$$

The operator of the extended t-norm, whose arguments and resulting value are type-1 fuzzy sets defined within the universe of discourse $[0, 1]$ is given by

$$F \overset{\widetilde{T}}{*} G = \int_{u \in J^u} \int_{v \in J^v} g(u) \overset{T^*}{*} f(v) / u \overset{T}{*} v. \tag{5.47}$$

An analogic result may be obtained in a discrete case. We are going to discuss two type-1 sets

$$F = \sum_{u \in J^u} f(u)/u \quad \text{and} \quad G = \sum_{v \in J^v} g(v)/v. \tag{5.48}$$

The operator of the extended t-norm is given by the formula

$$\mu_{\widetilde{A}} \overset{\widetilde{T}}{*} \mu_{\widetilde{B}} = \sum_{u \in J^u} \sum_{v \in J^v} \left(f(u) \overset{T^*}{*} g(v) \right) / u \overset{T}{*} v, \tag{5.49}$$

and the operator of the extended t-conorm takes the form

$$\mu_{\widetilde{A}} \overset{\widetilde{S}}{*} \mu_{\widetilde{B}} = \sum_{u \in J^u} \sum_{v \in J^v} \left(f(u) \overset{T}{*} g(v) \right) / u \overset{S}{*} v. \tag{5.50}$$

The introduction of extended triangular norms operating on type-1 sets allows to simplify considerably the notation of complicate operations on type-2 fuzzy sets.

Remark 5.4
The extended function ϕ may also be a function of many variables. Then the operations of the extended t-norm and t-conorm take the following forms:

$$\overset{n}{\underset{i=1}{\widetilde{T}}} F_i = \int_{u_1 \in J_1} \cdots \int_{u_n \in J_n} \overset{n}{\underset{i=1}{T}}{}^* f_i(u_i) \Big/ \overset{n}{\underset{i=1}{T}} u_i, \tag{5.51}$$

$$\overset{n}{\underset{i=1}{\widetilde{S}}} F_i = \int_{u_1 \in J_1} \cdots \int_{u_n \in J_n} \overset{n}{\underset{i=1}{T}} f_i(u_i) \Big/ \overset{n}{\underset{i=1}{S}} u_i, \tag{5.52}$$

where $F_i = \int_{u_i \in J_i} f_i(u_i)/u_i, i = 1, \ldots, n$.

Remark 5.5
The operations of extended t-norm and t-conorm are easier to be made with specified assumptions concerning the membership function of particular fuzzy sets. We are going to discuss n convex, normal type-1 fuzzy sets $F_1, ..., F_n$ with membership functions $f_1, ..., f_n$. Let us assume that $f_1(v_1) = f_2(v_2) = \cdots = f_n(v_n) = 1$, where $v_1, v_2, ..., v_n$ are real numbers such that $v_1 \leq v_2 \leq \cdots \leq v_n$ Then the extended minimum type t-norm, known as the meet operation, is specified as follows ([97, 134]):

$$\mu_{\cap_{i=1}^n F_i}(\theta) = \begin{cases} \vee_{i=1}^n f_i(\theta), & \theta < v_1, \\ \wedge_{i=1}^k f_i(\theta), & v_k \leq \theta < v_{k+1}, 1 \leq k \leq n-1, \\ \wedge_{i=1}^n f_i(\theta), & \theta \geq v_n, \end{cases} \tag{5.53}$$

whereas the extended maximum type t-conorm takes the form

$$\mu_{\cup_{i=1}^n F_i}(\theta) = \begin{cases} \wedge_{i=1}^n f_i(\theta), & \theta < v_1, \\ \wedge_{i=k+1}^n f_i(\theta), & v_k \leq \theta < v_{k+1}, 1 \leq k \leq n-1, \\ \vee_{i=1}^n f_i(\theta), & \theta \geq v_n. \end{cases} \tag{5.54}$$

Remark 5.6
Let us discuss n Gaussian fuzzy sets $F_1, F_2, ..., F_n$ with means $m_1, m_2, ..., m_n$ and with standard deviations $\sigma_1, \sigma_2, ..., \sigma_n$. Then, as a result of an approximate extended operation of the algebraic t-norm we have [97]

$$\mu_{F_1 \cap F_2 \cap \cdots \cap F_n}(\theta) \approx \mathrm{e}^{(-1/2)((\theta - m_1 m_2 \cdots m_n)/\overline{\sigma})^2}, \tag{5.55}$$

while

$$\overline{\sigma} = \sqrt{\sigma_1^2 \prod_{i;i\neq 1} m_i^2 + \cdots + \sigma_j^2 \prod_{i;i\neq j} m_i^2 + \cdots + \sigma_n^2 \prod_{i;i\neq n} m_i^2}, \tag{5.56}$$

where $i = 1, ..., n$.

5.6 Type-2 fuzzy relations

At first, we are going to define the Cartesian product of type-2 fuzzy sets.

Definition 5.8
The Cartesian product of n type-2 fuzzy sets $\widetilde{A}_1 \subseteq X_1, \widetilde{A}_2 \subseteq X_2, \ldots, \widetilde{A}_n \subseteq X_n$ is the fuzzy set $\widetilde{A} = \widetilde{A}_1 \times \widetilde{A}_2 \times \ldots \times \widetilde{A}_n$ defined on set $X_1 \times X_2 \times \ldots \times X_n$, while the membership function of set $\widetilde{A}$ is given by the formula

$$\mu_{\widetilde{A}}(\mathbf{x}) = \mu_{\widetilde{A}_1 \times \widetilde{A}_2 \times \ldots \times \widetilde{A}_n}(x_1, x_2, \ldots, x_n) = \mathop{\widetilde{T}}_{i=1}^{n} \mu_{\widetilde{A}_i}(x_n), \tag{5.57}$$

where $x_1 \in X_1, ..., x_n \in X_i$, and the operation of the extended t-norm is described by dependency (5.51).

Definition 5.9
The binary type-2 fuzzy relation $\widetilde{R}$ between two non-empty non-fuzzy sets X and Y is the type-2 fuzzy set determined on the Cartesian product $X \times Y$, i.e.

$$\widetilde{R}(X, Y) = \int_{X \times Y} \mu_{\widetilde{R}}(x, y) / (x, y), \tag{5.58}$$

while $x \in X, y \in Y$, and the membership grade of the pair (x, y) to the fuzzy set $\widetilde{R}$ is a type-1 fuzzy set defined in the interval $J_{x,y}^v \subset [0, 1]$, i.e.

$$\mu_{\widetilde{R}}(x, y) = \int_{v \in J_{x,y}^v} r_{x,y}(v) / v, \tag{5.59}$$

where $r_{x,y}(v)$ is the secondary grade.

Example 5.13
Let $X = \{3, 4\}$ and $Y = \{4, 5\}$. We are going to formalize an imprecise statement "y is more or less equal to x". At first, we are going to determine the type-1 relation R in the following way:

$$R = \frac{0.8}{(3.4)} + \frac{0.6}{(3.5)} + \frac{1}{(4.4)} + \frac{0.8}{(4.5)}. \tag{5.60}$$

An analogic type-2 fuzzy relation may take on the form

$$\begin{aligned}\widetilde{R} = &(0.6/0.7 + 1/0.8 + 0.5/0.6) \,/\, (3, 4) \\ &+ (0.3/0.5 + 1/0.6 + 0.4/0.3) \,/\, (3, 5) \\ &+ (1/1 + 1/1 + 1/1) \,/\, (4, 4) \\ &+ (0.6/0.7 + 1/0.8 + 0.5/0.6) \,/\, (4.5).\end{aligned} \tag{5.61}$$

Example 5.14
We are going to formalize an imprecise statement "number x slightly differs from number y". This problem may be solved with the type-1 fuzzy relation described by the membership function

$$\mu_R(x, y) = \max\{(4 - |x - y|)/4.0\}. \tag{5.62}$$

An analogic type-2 fuzzy relation may take on the form

$$\mu_{\widetilde{R}}(x, y) = \int_{v \in [0,1]} \exp\left[-\left(\frac{v - m(x, y)}{\sigma}\right)^2\right] / v\,, \tag{5.63}$$

where $\sigma > 0$ and

$$m(x, y) = \max\{(4 - |x - y|)/4.0\}. \tag{5.64}$$

Alternatively, the secondary membership function of Gaussian type may be substituted by a fuzzy triangular number. Figure 5.7a depicts the illustration of the type-1 fuzzy relation given by formula (5.62). Figure 5.7b depicts the possibility of uncertainty in the specification of the statement "number x slightly differs from number y". The figure depicts the footprint of uncertainty while the level of shading corresponds to the value of the secondary grade. Figure 5.7c presents the triangular secondary membership function defined in the interval $J_x = [0.2, 0.4]$.

It is worth mentioning that fuzzy relations may be made with the use of extended norms. We are going to discuss the membership function of the type-2 fuzzy set defined on set X, $\widetilde{A} \subseteq X$, i.e.

$$\mu_{\widetilde{A}}(x) = \int_{u \in J_x^u} f_x(u)/u \tag{5.65}$$

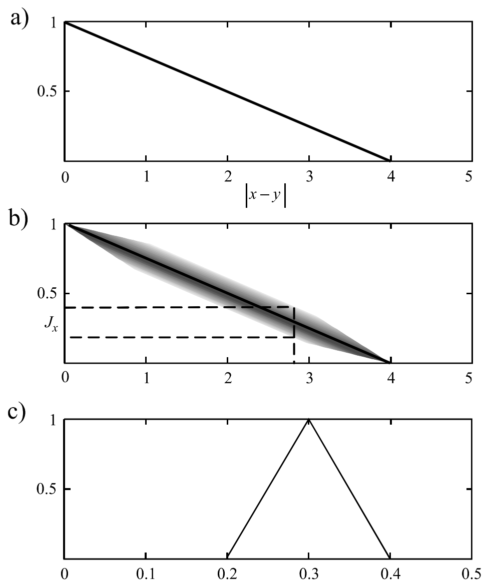

FIGURE 5.7. Illustration of type-1 and type-2 fuzzy relations

and the membership function of fuzzy set $\widetilde{B}$ defined on another set Y, $\widetilde{B} \subset Y$, i.e.

$$\mu_{\widetilde{B}}(y) = \int_{v \in J_y^v} g_y(v) / v, \tag{5.66}$$

where $J_x^u, J_y^v \subset [0,1]$. The extended t-conorm of the type-2 fuzzy sets defined on different spaces forms a certain fuzzy relation $\widetilde{R}$, determined as follows:

$$\mu_{\widetilde{R}}(x,y) = \mu_{\widetilde{A}}(x) \overset{\widetilde{S}}{*} \mu_{\widetilde{B}}(y) = \int_{u \in J_x^u} \int_{v \in J_y^v} f_x(u) \overset{T}{*} g_y(v) / u \overset{S}{*} v \tag{5.67}$$
$$= \int_{w \in J_{x,y}^w} r_{x,y}(w) / w.$$

Similarly, the extended t-norm creates a fuzzy relation in the form of

$$\mu_{\widetilde{R}}(x,y) = \mu_{\widetilde{A}}(x) \overset{\widetilde{T}}{*} \mu_{\widetilde{B}}(y) = \int_{u \in J_x^u} \int_{v \in J_y^v} f_x(u) \overset{T^*}{*} g_y(v) / u \overset{T}{*} v \tag{5.68}$$
$$= \int_{w \in J_{x,y}^w} r_{x,y}(w) / w.$$

In the application of the theory of fuzzy sets to the construction of inference systems it is necessary to use the concept of the composition of fuzzy relations, which, in the context of type-2 fuzzy sets, are defined as follows:

Definition 5.10
The sup-T type (sup-star) *extended composition of* type-2 fuzzy relations $\widetilde{R} \subseteq X \times Y$ and $\widetilde{S} \subseteq Y \times Z$ is the fuzzy relation $\widetilde{R} \circ \widetilde{S} \subseteq X \times Z$ with the membership function

$$\mu_{\widetilde{R} \circ \widetilde{S}}(\mathbf{x}, \mathbf{z}) = \underset{y \in Y}{\widetilde{S}} \left(\mu_{\widetilde{R}}(\mathbf{x}, \mathbf{y}) \overset{\widetilde{T}}{*} \mu_{\widetilde{S}}(\mathbf{x}, \mathbf{z}) \right). \tag{5.69}$$

Definition 5.11
The extended composition of the type-2 fuzzy set $\widetilde{A}$, $\widetilde{A} \subset X$, and of the type-2 fuzzy relation $\widetilde{R} \subseteq X \times Y$ is denoted as $\widetilde{A} \circ \widetilde{R}$ and determined as follows:

$$\mu_{\widetilde{B}}(y) = \underset{\mathbf{x} \in X}{\widetilde{S}} \left(\mu_{\widetilde{A}}(\mathbf{x}) \overset{\widetilde{T}}{*} \mu_{\widetilde{R}}(\mathbf{x}, y) \right). \tag{5.70}$$

5.7 Type reduction

The defuzzification of the type-2 fuzzy sets consists of two stages: At first, a so-called *type reduction* should be made, which is the transformation of the type-2 fuzzy set into the type-1 fuzzy set. This way we are going to obtain the type-1 fuzzy set called a *centroid*, which may be defuzzified to a non-fuzzy value. We are going to show the method for the determination of the centroid of the type-2 fuzzy set.

Let us discuss a fuzzy set A (type-1) defined on set X. Let us assume that set X has been discretized and takes R values $x_1, ..., x_R$. The centroid of fuzzy set A is determined as follows:

$$C_A = \frac{\sum_{k=1}^{R} x_k \mu_A(x_k)}{\sum_{k=1}^{R} \mu_A(x_k)}. \tag{5.71}$$

We are going to determine the centroid of the type-2 fuzzy set, $\widetilde{A} = \left\{ \left(x, \mu_{\widetilde{A}}(x) \right) | x \in X \right\}$, which, as a result of an analogic discretizaton is notated as follows:

$$\widetilde{A} = \sum_{k=1}^{R} \left[\int_{u \in J_{x_k}} f_{x_k}(u) / u \right] / x_k. \tag{5.72}$$

Applying the extension principle to formula (5.71) we get

$$C_{\widetilde{A}} = \int_{\theta_1 \in J_{x_1}} \cdots \int_{\theta_R \in J_{x_R}} \left[f_{x_1}(\theta_1) * \cdots * f_{x_R}(\theta_R) \right] / \frac{\sum_{k=1}^{R} x_k \theta_k}{\sum_{k=1}^{R} \theta_k}. \tag{5.73}$$

Of course, the centroid $C_{\widetilde{A}}$ is a type-1 fuzzy set. Let us note that any selection of elements $\theta_1 \in J_{x_1}, ..., \theta_R \in J_{x_R}$ along with corresponding secondary grades $f_{x_1}(\theta_1), ..., f_{x_R}(\theta_R)$, creates an embedded fuzzy set $\widetilde{A}_o$ (type-2).

Example 5.15
Let $\mathbf{X} = \{2, 5\}$. We are going to perform the type reduction of the following type-2 fuzzy set:

$$\widetilde{A} = (0.6/0.4 + 1/0.8)/2 + (0.3/0.7 + 1/0.6)/5. \tag{5.74}$$

The centroid of the type-2 fuzzy set given by formula (5.74) is a type-1 fuzzy set taking the form

$$\begin{aligned} C_{\widetilde{A}} &= \frac{0.6 \times 0.3}{a_1} + \frac{0.6 \times 1}{a_2} + \frac{1 \times 0.3}{a_3} + \frac{1 \times 1}{a_4} \\ &= \frac{0.18}{a_1} + \frac{0.6}{a_2} + \frac{0.3}{a_3} + \frac{1}{a_4}, \end{aligned} \tag{5.75}$$

while

$$\begin{aligned} a_1 &= \frac{2 \times 0.4 + 5 \times 0.7}{0.4 + 0.7} = \frac{43}{11}, \\ a_2 &= \frac{2 \times 0.4 + 5 \times 0.6}{0.4 + 0.6} = 3.8, \\ a_3 &= \frac{2 \times 0.8 + 5 \times 0.7}{0.8 + 0.7} = 3.4, \\ a_4 &= \frac{2 \times 0.8 + 5 \times 0.6}{0.8 + 0.6} = \frac{23}{7}. \end{aligned}$$

In a continuous case, the determination of the centroid of the type-2 fuzzy set is a much more complicated task from the computational point of view. The problem becomes easier to solve, if the secondary membership functions are interval ones. Then formula (5.73) takes the form

$$C_{\widetilde{A}} = \int_{\theta_1 \in J_{x_1}} \cdots \int_{\theta_R \in J_{x_R}} 1 \Big/ \frac{\sum_{k=1}^{R} x_k \theta_k}{\sum_{k=1}^{R} \theta_k}. \tag{5.76}$$

We are going to show the method of the determination of the centroid of the type-2 fuzzy set having an interval secondary membership function. With reference to formula (5.76), let us define

$$s(\theta_1, ..., \theta_R) = \frac{\sum_{k=1}^{R} x_k \theta_k}{\sum_{k=1}^{R} \theta_k}. \tag{5.77}$$

It is obvious that centroid (5.76) will be an interval type-1 fuzzy set, i.e.

$$C_{\tilde{A}} = \int_{x \in [x_l, x_p]} 1/x \equiv [x_l, x_p]. \tag{5.78}$$

From the observations shown above, it may be concluded that the determination of centroid (5.76) comes down to the optimization (maximization and minimization) with respect to θ_k of function given by formula (5.77), taking account of constraints

$$\theta_k \in \left[\underline{\theta}^k, \overline{\theta}^k\right], \tag{5.79}$$

where $k = 1, ..., R$ and

$$\underline{\theta}^k = \underline{J_x}, \overline{\theta}^k = \overline{J}_x. \tag{5.80}$$

Differentiating expression (5.77) with respect θ_j, we get

$$\begin{aligned}
\frac{\partial}{\partial \theta_j} s\left(\theta_1, ..., \theta_R\right) &= \frac{\partial}{\partial \theta_j}\left[\frac{\sum\limits_{k=1}^{R} x_k \theta_k}{\sum\limits_{k=1}^{R} \theta_k}\right] = \frac{\partial}{\partial \theta_j}\left[\frac{x_j \theta_j + \sum\limits_{k \neq j} x_k \theta_k}{\theta_j + \sum\limits_{k \neq j} \theta_k}\right] \\
&= \left[\frac{1}{\theta_j + \sum\limits_{k \neq j} \theta_k}\right] (x_j) \left(x_j \theta_j + \sum_{k \neq j} x_k \theta_k\right)\left[\frac{-1}{\left(\theta_j + \sum\limits_{k \neq j} \theta_k\right)^2}\right] \\
&= \frac{x_j}{\sum\limits_{k=1}^{R} \theta_k} - \frac{\sum\limits_{k=1}^{R} x_k \theta_k}{\left(\sum\limits_{k=1}^{R} \theta_k\right)^2} = \frac{x_j}{\sum\limits_{k=1}^{R} \theta_k} - \left[\frac{\sum\limits_{k=1}^{R} x_k \theta_k}{\sum\limits_{k=1}^{R} \theta_k}\right] \frac{1}{\sum\limits_{k=1}^{R} \theta_k} \\
&= \frac{x_j - s\left(\theta_1, ..., \theta_R\right)}{\sum\limits_{k=1}^{R} \theta_k}.
\end{aligned} \tag{5.81}$$

Of course $\sum_{k=1}^{R} \theta_k > 0$. Hence, from the last equality we have

$$\frac{\partial}{\partial \theta_j} s\left(\theta_1, ..., \theta_R\right) \geq 0, \quad \text{if} \quad x_j \geq s\left(\theta_1, ..., \theta_R\right) \tag{5.82}$$

and

$$\frac{\partial}{\partial \theta_j} s\left(\theta_1, ..., \theta_R\right) \leq 0, \quad \text{if} \quad x_j \leq s\left(\theta_1, ..., \theta_R\right). \tag{5.83}$$

When equating the right side of expression (5.81) to zero we get

$$\frac{\sum_{k=1}^{R} x_k\theta_k}{\sum_{k=1}^{R}\theta_k} = x_j. \tag{5.84}$$

Therefore

$$\sum_{k=1}^{R} x_k\theta_k = x_j\sum_{k=1}^{R}\theta_k \tag{5.85}$$

and

$$x_j\theta_j + \sum_{\substack{k=1\\k\neq j}}^{R} x_k\theta_k = x_j\theta_j + x_j\sum_{\substack{k=1\\k\neq j}}^{R}\theta_k. \tag{5.86}$$

In consequence

$$\frac{\sum_{k\neq j} x_k\theta_k}{\sum_{k\neq j}\theta_k} = x_j. \tag{5.87}$$

We find out that the necessary condition for the extremum s to exist does not depend in any way on parameter θ_k with respect to which the derivative was calculated. However, inequalities (5.82) and (5.83) show in which direction we should go in order to increase or decrease the value of expression $s(\theta_1, ..., \theta_R)$. Upon the basis of these inequalities we conclude that

i) if $x_j > s(\theta_1, ..., \theta_R)$, then$s(\theta_1, ..., \theta_R)$ is increasing along with the decrease of parameter θ_j,

ii) if $x_j < s(\theta_1, ..., \theta_R)$, then$s(\theta_1, ..., \theta_R)$ is increasing along with the increase of parameter θ_j.

Let us remind that $\underline{\theta}_k \leq \theta_k \leq \overline{\theta}_k$. Hence, function s reaches the maximum if

a) $\theta_k = \overline{\theta}_k$ for these values k, for which $x_k > s$,

b) $\theta_k = \underline{\theta}_k$ for these values k, for which $x_k < s$,

Upon this basis we are going to present an iterative algorithm (known as Karnik - Mendel type reduction algorithm) for the search of the maximum of function s:

1) Determine $\theta_k = \frac{\underline{\theta}_k + \overline{\theta}_k}{2}$, $k = 1, ..., R$, calculate $s' = s(\theta_1, ..., \theta_R)$.

2) Find $j\,(1 \leq j \leq R-1)$ so that $x_j \leq s' < x_{j+1}$.

3) Substitute $\theta_k = \underline{\theta}_k$ for $k \leq j$ and $\theta_k = \overline{\theta}_k$ for $k > j$.
Calculate $s'' = s\left(\underline{\theta}_1, ..., \underline{\theta}_j, \overline{\theta}_{j+1}, ..., \overline{\theta}_R\right)$.

4) If $s'' = s'$ then s'' is the maximum value of function s.
If $s'' \neq s'$ then pass on to step 5.

5) Substitute $s' = s''$ and pass on to step 2.

In an analogic way, we may determine the minimum of function s. This function reaches the minimum if

a) $\theta_k = \overline{\theta}_k$ for these values k, for which $x_k < s$,

b) $\theta_k = \underline{\theta}_k$ for these values k, for which $x_k > s$, The iterative algorithm for the search of the minimum of function s is given as follows:

1) Determine $\theta_k = \frac{\underline{\theta}_k + \overline{\theta}_k}{2}, k = 1, ..., R$, calculate $s' = s(\theta_1, ..., \theta_R)$.

2) Find j $(1 \leq j \leq R - 1)$ so that $x_j < s' \leq x_{j+1}$.

3) Substitute $\theta_k = \overline{\theta}_k$ for $k < j$ and $\theta_k = \underline{\theta}_k$ for $k \geq j$, calculate $s'' = s\left(\overline{\theta}_1, ..., \overline{\theta}_j, \underline{\theta}_{j+1}, ..., \underline{\theta}_R\right)$.

4) If $s'' = s'$ then, s'' is the minimum value of function s.
If $s'' \neq s'$ then pass on to step 5.

5) Substitute $s' = s''$ and pass on to step 2.

Example 5.16
Figures 5.8 – 5.10 depict the method of working of the iterative algorithm for the search of the centroid of the type-2 fuzzy set with the interval secondary membership function. In Fig. 5.8, the footprint of uncertainty of the type-2 fuzzy set, which will be subject to type reduction, is marked. The thick line in this picture corresponds with point 1 of the iterative algorithm, which starts with the determination of the centre of particular intervals $J_x, x \in X$ and the value of expression (5.77).

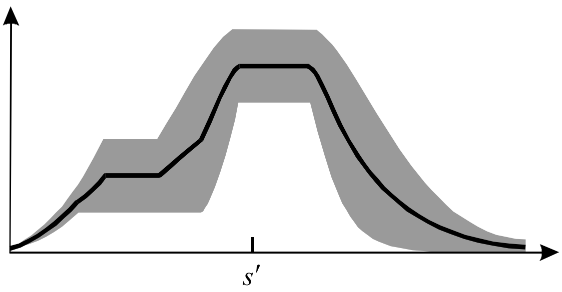

FIGURE 5.8. The footprint of uncertainty of type-2 fuzzy set; the thick line corresponds with point 1 of the K-M iterative type-reduction algorithm

The centroid is a type-1 fuzzy set given by formula (5.78). By iteration, we search for point x_p, determining the centroid of an embedded fuzzy set (Fig. 5.9) consisting first of a piece of the lower membership function, and then of a piece of the upper membership function. Similarly, we search for point x_l, determining the centroid of an embedded fuzzy set (Fig. 5.10) consisting first of a piece of the upper membership function, and then of a piece of the lower membership function.

The obtained fuzzy set $C_{\widetilde{A}} = [x_l, x_p]$ may be deffuzified (Fig. 5.11) in the following way:

$$\widehat{x}_w = \frac{x_l + x_p}{2}. \tag{5.88}$$

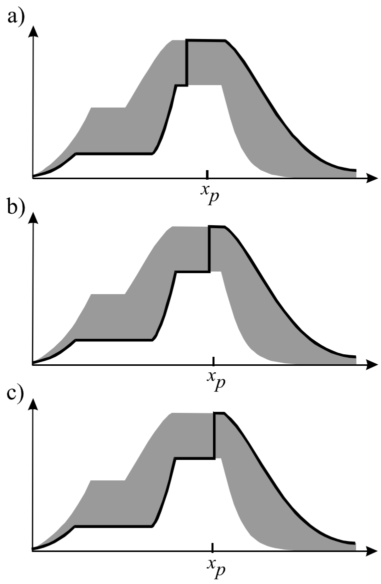

FIGURE 5.9. Iterative search for point x_p determining the centroid of an embedded fuzzy set

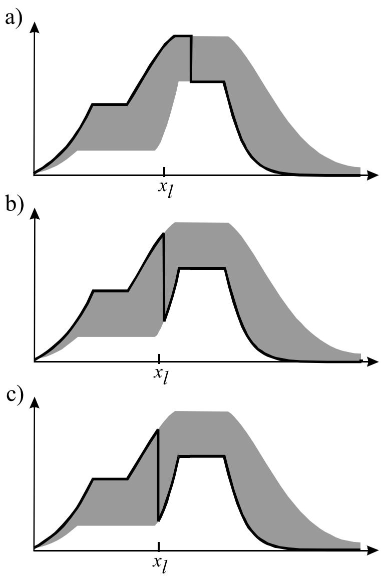

FIGURE 5.10. Iterative search for point x_l determining the centroid of an embedded fuzzy set

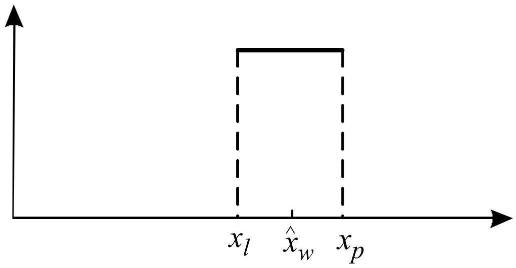

FIGURE 5.11. Fuzzy set $C_{\widetilde{A}}$

5.8 Type-2 fuzzy inference systems

We are going to discuss the type-2 fuzzy system having n input variables $x_i \in X_i \subset R, i = 1, \ldots, n$, and a scalar output $y \in Y$. Figure 5.12 depicts the block diagram of such a system. It consists of the following elements: the type-2 fuzzification block, rule base described by type-2 fuzzy relations, type-2 inference mechanism, and the deffuzification block.

The deffuzification has two stages: at first, the type reduction is performed (Subchapter 5.7) and then the classic defuzzification is applied (Subchapter 4.9).

5.8.1 Fuzzification block

Let $\overline{x} = (x_1, ..., x_n)^T \in \mathbf{X} = X_1 \times X_2 \times \cdots \times X_n$ be the input signal of the fuzzy inference system. In type-1 fuzzy systems, the singleton type fuzzification is applied. Its equivalence in type-2 fuzzy systems is the singleton-singleton type fuzzification defined as follows:

$$\widetilde{\mu}_{A'}(\mathbf{x}) = \begin{cases} 1/1, & \text{if} \quad \mathbf{x} = \overline{\mathbf{x}}, \\ 1/0, & \text{if} \quad \mathbf{x} \neq \overline{\mathbf{x}}. \end{cases} \tag{5.89}$$

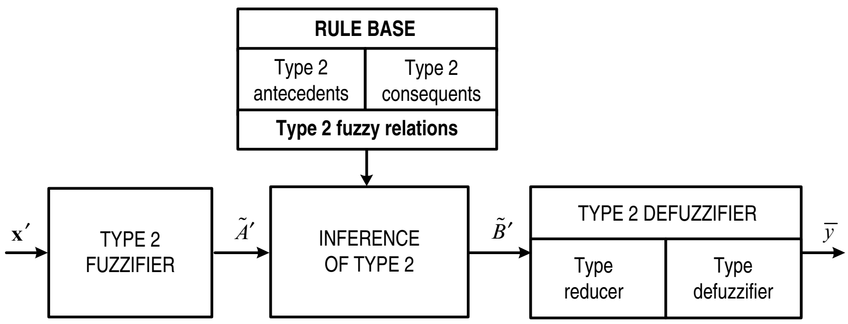

FIGURE 5.12. Block diagram of a type-2 fuzzy inference system

In case of the independence of particular input variables, the above-mentioned operation takes the form of:

$$\widetilde{\mu}_{A'}(x_i) = \begin{cases} 1/1\,, & \text{if} \quad x_i = \overline{x}_i, \\ 1/0\,, & \text{if} \quad x_i \neq \overline{x}_i. \end{cases} \tag{5.90}$$

for $i = 1, ..., n$. As a result of the fuzzification, we obtain n input type-2 fuzzy sets described by:

$$\widetilde{A}'_i = (1/1)\,/\overline{x}_i, \quad i = 1, \ldots, n, \tag{5.91}$$

where $\overline{x}_i$ is a specific value of i-th input variable.

It is worth mentioning that other methods for the fuzzification of the input signal are also possible. Figure 5.13 depicts a graphic illustration of these methods. For instance, the fuzzification of singleton-interval type (Fig. 5.13b) means that the secondary membership function is an interval fuzzy set. The non-singleton-singleton fuzzification (Fig. 5.13c) means that the secondary membership function is a singleton type fuzzy set, and in this case the fuzzification is identical to the non-singleton type fuzzification for type-1 fuzzy sets. The non-singleton-triangular fuzzification (Fig. 5.13e) means that the secondary membership function is triangular fuzzy set, while the level of shading on Fig. 5.13e reflects the value of the secondary membership function (triangular) for given element $u \in J_x$.

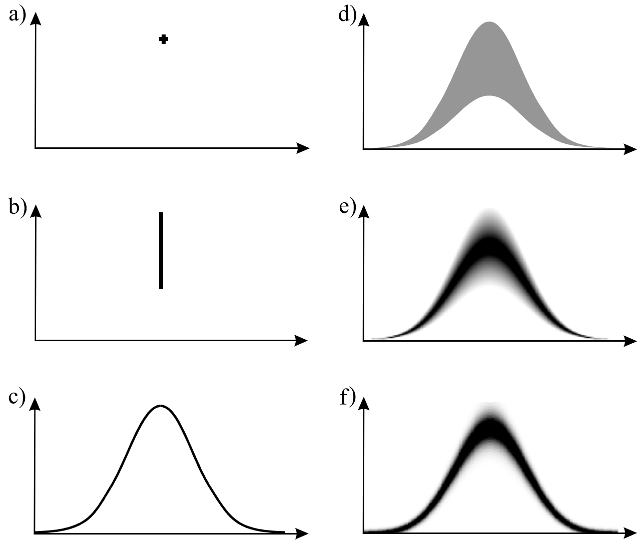

FIGURE 5.13. Illustration of different fuzzification methods: a) singleton-singleton, b) singleton-interval, c) nonsingleton-singleton, d) nonsigleton-interval, e) nonsingleton-triangular, f) nonsingleton-gaussoidal

5.8.2 Rules base

The linguistic model consists of N rules in the form of:

$$\widetilde{R}^k : \textbf{IF } x_1 \text{ is } \widetilde{A}_1^k \textbf{ AND } x_2 \text{ is } \widetilde{A}_2^k \textbf{ AND} \ldots \textbf{AND } x_n \text{ is } \widetilde{A}_n^k$$
$$\textbf{THEN } y \text{ is } \widetilde{B}^k, k = 1, \ldots, N. \tag{5.92}$$

Denote

$$\widetilde{A} = \widetilde{A}^k = \widetilde{A}_1^k \times \widetilde{A}_2^k \times \ldots \times \widetilde{A}_n^k. \tag{5.93}$$

Of course

$$\mu_{\widetilde{A}^k}(\mathbf{x}) = \mathop{\widetilde{\mathbf{T}}}_{i=1}^{n} \mu_{\widetilde{A}_i^k}(\overline{x}_i). \tag{5.94}$$

It is easily seen that rule (5.92) may be presented in the form of implication

$$\widetilde{A}^k \to \widetilde{B}^k, \quad k = 1, ..., n. \tag{5.95}$$

5.8.3 Inference block

At first, we are going to determine membership function $\mu_{\widetilde{A}^k \to \widetilde{B}^k}(\mathbf{x}, y)$. Each k-th rule is represented in a fuzzy system by a certain type-2 fuzzy relation.

$$\widetilde{R}^k(\mathbf{x}, y) = \int_{\mathbf{X} \times Y} \mu_{\widetilde{R}^k}(\mathbf{x}, y) / (\mathbf{x}, y), \tag{5.96}$$

where

$$\mu_{\widetilde{R}^k}(\mathbf{x}, y) = \int_{v \in V_{\mathbf{x},y}} r_{\mathbf{x},y}^k(v) / v. \tag{5.97}$$

Therefore

$$\mu_{\widetilde{A}^k \to \widetilde{B}^k}(\mathbf{x}, y) = \mu_{\widetilde{R}^{(k)}}(\mathbf{x}, y). \tag{5.98}$$

Membership function $\mu_{\widetilde{A}^k \to \widetilde{B}^k}(\mathbf{x}, y)$ will be determined, analogically as in case of type-1 systems, upon the basis of the knowledge of membership function $\mu_{\widetilde{A}^k}(\mathbf{x})$ and $\mu_{\widetilde{B}^k}(y)$. Using the operator of the extended t-norm we have

$$\mu_{\widetilde{A}^k \to \widetilde{B}^k}(\mathbf{x}, y) = \mu_{\widetilde{A}^k}(\mathbf{x}) \overset{\widetilde{T}}{*} \mu_{\widetilde{B}^k}(y). \tag{5.99}$$

The Mamdani and Larsen rules used in type-1 systems now take the form of

- extended min rule (Mamdami)

$$\mu_{\widetilde{A}^k \to \widetilde{B}^k}(\mathbf{x}, y) = \int_{u \in J_{\mathbf{x}}^u} \int_{v \in J_y^\nu} \left(f_{\mathbf{x}}(u) \overset{T}{*} g_y(v) \right) / \min(u, v), \tag{5.100}$$

- extended product rule (Larsen)

$$\mu_{\widetilde{A}^k \to \widetilde{B}^k}(\mathbf{x}, y) = \int_{u \in J_{\mathbf{x}}^u} \int_{v \in J_y^\nu} \left(f_{\mathbf{x}}(u) \overset{T}{*} g_y(v) \right) / uv. \tag{5.101}$$

At the output of the inference block, we obtain a type-2 fuzzy set $\widetilde{B}'^k$. This set is determined by the composition of the input fuzzy set $\widetilde{A}'$ and the fuzzy relation $\widetilde{R}^k$, i.e.

$$\widetilde{B}'^k = \widetilde{A}' \circ \widetilde{R}^k = \widetilde{A}' \circ \left(\widetilde{A}^k \rightarrow \widetilde{B}^k\right). \tag{5.102}$$

Using Definition 5.11, we determine the membership function of the fuzzy set $\widetilde{B}'^k$

$$\begin{aligned} \mu_{\widetilde{B}'^k}(y) = \mu_{\widetilde{A}' \circ \widetilde{R}^k}(y) &= \underset{\mathbf{x} \in \mathbf{X}}{\widetilde{S}} \left(\mu_{\widetilde{A}'}(\mathbf{x}) \overset{\widetilde{T}}{*} \mu_{\widetilde{R}^k}(\mathbf{x}, y) \right) \\ &= \underset{\mathbf{x} \in \mathbf{X}}{\widetilde{S}} \left(\mu_{\widetilde{A}'}(\mathbf{x}) \overset{\widetilde{T}}{*} \mu_{\widetilde{A}' \rightarrow \widetilde{B}^k}(\mathbf{x}, y) \right). \end{aligned} \tag{5.103}$$

In case of singleton-singleton type fuzzification (5.84) the formula above takes the form

$$\mu_{\widetilde{B}'^k}(y) = \mu_{\widetilde{A}^k \rightarrow \widetilde{B}^k}(\overline{\mathbf{x}}, y). \tag{5.104}$$

Using formulae (5.99) and (5.94), we obtain

$$\mu_{\widetilde{B}'^k}(y) = \mu_{\widetilde{A}_1^k \times \ldots \times \widetilde{A}_n^k}(\overline{\mathbf{x}}) \overset{\widetilde{T}}{*} \mu_{\widetilde{B}^k}(y) = \left(\overset{n}{\underset{i=1}{\widetilde{T}}} \mu_{\widetilde{A}_i^k}(\overline{x}_i) \right) \overset{\widetilde{T}}{*} \mu_{\widetilde{B}^k}(y). \tag{5.105}$$

Let us denote the firing strength of k-th rule in the following way:

$$\tau_k = \overset{n}{\underset{i=1}{\widetilde{T}}} \mu_{\widetilde{A}_i^k}(\overline{x}_i). \tag{5.106}$$

Then dependency (5.105) takes the form

$$\mu_{\widetilde{B}'^k}(y) = \tau_k \overset{\widetilde{T}}{*} \mu_{\widetilde{B}^k}(y). \tag{5.107}$$

Remark 5.7
In case of type-1 fuzzy sets the firing strength of τ_k rule is a real number while $\tau_k \in [0, 1]$. In case of type-2 fuzzy sets the firing strength of τ_k rule is a type-1 fuzzy set defined in $[0, 1]$.

Having inference results $\widetilde{B}'^k$ for all N rules, we make an aggregation using the operator of the extended t-conorm

$$\mu_{\widetilde{B}'}(y) = \overset{N}{\underset{k=1}{\widetilde{S}}} \mu_{\widetilde{B}'^k}(y). \tag{5.108}$$

We are going to show the inference process in interval systems. In such systems the secondary membership functions of fuzzy sets $\widetilde{A}_i^k$ and $\widetilde{B}^k$, $i = 1, ..., n$, $k = 1, ..., N$, are constant functions taking value 1 in all intervals $J_x, x \in X$. Within further discussion we are going to apply two properties ([97, 134]) of interval type-1 fuzzy sets $F_1, ..., F_n$, defined on intervals $[l_1, p_1], ..., [l_n, p_n]$, where $l_i \geq 0$ and $p_i \geq 0$, $i = 1, ..., n$.

1) Extended t-norm $\widetilde{T}_{i=1}^n F_i$ is an interval type-1 fuzzy set defined on interval $\left[\left(l_1 \overset{T}{*} l_2 \overset{T}{*} \ldots \overset{T}{*} l_n\right), \left(p_1 \overset{T}{*} p_2 \overset{T}{*} \ldots \overset{T}{*} p_n\right)\right]$, where $\overset{T}{*}$ denotes t-norm of minimum type or product.

2) Extended t-conorm $\widetilde{S}_{i=1}^n F_i$ is an interval type-1 fuzzy set defined in the interval $[(l_1 \vee l_2 \vee \ldots \vee l_n), (p_1 \vee p_2 \vee \ldots \vee p_n)]$, where $\vee$ means a maximum operation.

We are going to introduce a symbolic notation, according to which the interval fuzzy set A will be denoted as

$$A = \int_{x \in [a,b]} 1/x \equiv [a, b]. \tag{5.109}$$

Using property 1, we are going to express the firing strength of rule τ_k, being now an interval type-1 fuzzy set, through the values of the lower and upper membership functions of fuzzy sets $\widetilde{A}_i^k$. Based on property 1, we may denote

$$\tau_k = \left[\underline{\tau}_k, \overline{\tau}_k\right], \tag{5.110}$$

where

$$\underline{\tau}_k (\overline{\mathbf{x}}) = \underline{\mu}_{\widetilde{A}_1^k} (\overline{x}_1) * \ldots * \underline{\mu}_{\widetilde{A}_n^k} (\overline{x}_n) \tag{5.111}$$

and

$$\overline{\tau}_k (\overline{\mathbf{x}}) = \overline{\mu}_{\widetilde{A}_1^k} (\overline{x}_1) * \cdots * \overline{\mu}_{\widetilde{A}_n^k} (\overline{x}_n). \tag{5.112}$$

Using formulas (5.107), (5.110), and property 1, we obtain

$$\mu_{\widetilde{B}'^k} (y) = \mu_{\widetilde{B}^k} (y) \overset{\widetilde{T}}{*} \left[\underline{\tau}^k, \overline{\tau}^k\right] \equiv \left[\underline{b}^k (y), \overline{b}^k (y)\right], \quad y \in Y, \tag{5.113}$$

where

$$\underline{b}^k (y) = \underline{\tau}^k \overset{T}{*} \underline{\mu}_{\widetilde{B}^k} (y) \tag{5.114}$$

and

$$\overline{b}^k (y) = \overline{\tau}^k \overset{T}{*} \overline{\mu}_{\widetilde{B}^k} (y). \tag{5.115}$$

Using formulas (5.113), (5.108), and property 2, we may determine

$$\mu_{\widetilde{B}'} (y) = \underset{k=1}{\overset{N}{\widetilde{S}}} \mu_{\widetilde{B}'^k} (y) = \underset{k=1}{\overset{N}{\widetilde{S}}} \left[\underline{b}^k (y), \overline{b}^k (y)\right] = \left[\underline{b} (y), \overline{b} (y)\right], \tag{5.116}$$

where

$$\underline{b} (y) = \underline{b}^1 (y) \vee \underline{b}^2 (y) \vee \ldots \vee \underline{b}^N (y) \tag{5.117}$$

and

$$\overline{b}(y) = \overline{b}^1(y) \vee \overline{b}^2(y) \vee \ldots \vee \overline{b}^N(y). \sum_{k \neq j} . \tag{5.118}$$

Example 5.17
Figure 5.14 shows the method of determining the firing strength of a type-2 system with two rules. As the t-norm the minimum operation was chosen. Therefore,

$$\underline{\tau}_k = \min\left[\underline{\mu}_{\widetilde{A}_1^k}(\overline{x}_1), \underline{\mu}_{\widetilde{A}_2^k}(\overline{x}_2)\right] \tag{5.119}$$

and

$$\overline{\tau}_k = \min\left[\overline{\mu}_{\widetilde{A}_1^k}(\overline{x}_1), \overline{\mu}_{\widetilde{A}_2^k}(\overline{x}_2)\right] \tag{5.120}$$

for $k = 1, 2$. As we have emphasized earlier, the firing strengths are interval type-1 fuzzy sets.

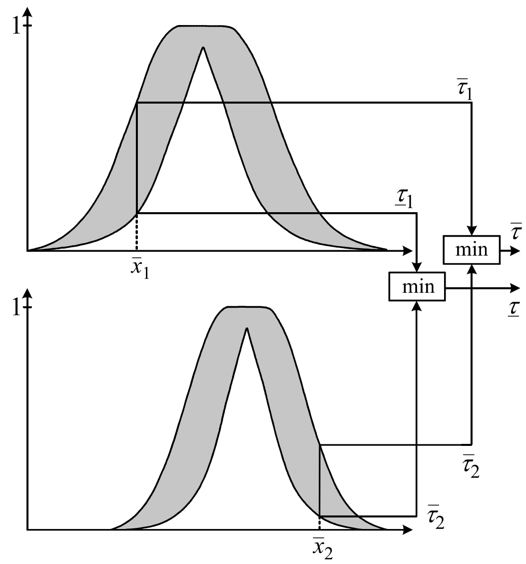

FIGURE 5.14. The method of determining the firing strength of a type-2 fuzzy system with singleton-singleton fuzzification

Example 5.18
Figures 5.15 and 5.16 show output type-2 fuzzy sets $\widetilde{B}^1$ and $\widetilde{B}^2$, as well as fuzzy sets (shaded ones) $\widetilde{B}'^1$ and $\widetilde{B}'^2$ resulting from inference given by formula (5.113).

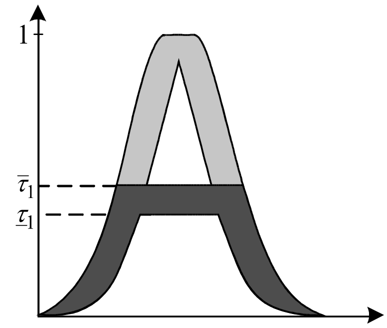

FIGURE 5.15. Output type-2 fuzzy set $\widetilde{B}^1$ and corresponding inferred type-2 fuzzy set $\widetilde{B}'^1$

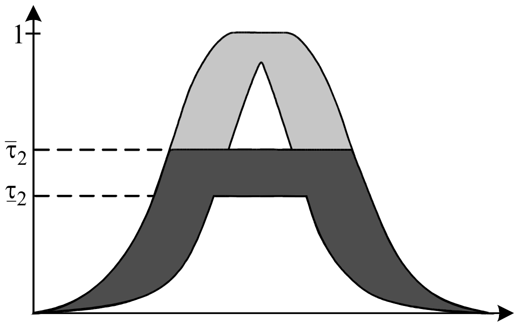

FIGURE 5.16. Output type-2 fuzzy set $\widetilde{B}^2$ and corresponding inferred type-2 fuzzy set $\widetilde{B}'^2$

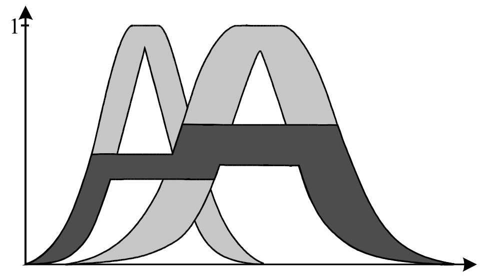

FIGURE 5.17. Type-2 fuzzy set $\widetilde{B}'$ resulting from the aggregation of fuzzy sets $\widetilde{B}'^1$ and $\widetilde{B}'^2$

Figure 5.17 presents fuzzy set (shaded one) $\widetilde{B}'$ given by formula (5.116) and resulting from the aggregation of fuzzy sets $\widetilde{B}'^1$ and $\widetilde{B}'^2$. In order to determine this set we have used the operation

$$\max\left(\min \overline{\tau}_1, \overline{\mu}_{\widetilde{B}^1}(y), \min \overline{\tau}_2, \overline{\mu}_{\widetilde{B}^2}(y)\right) \tag{5.121}$$

and

$$\max\left(\min \underline{\tau}_1, \underline{\mu}_{\widetilde{B}^1}(y), \min \underline{\tau}_2, \underline{\mu}_{\widetilde{B}^2}(y)\right). \tag{5.122}$$

Example 5.19

Examples 5.17 and 5.18 present the results obtained for interval type-2 fuzzy systems with singleton fuzzification given by formula (5.89). These results can be generalized for the case where the input signal is a type-1 fuzzy set (nonsingleton-singleton fuzzification) or an interval type-2 fuzzy

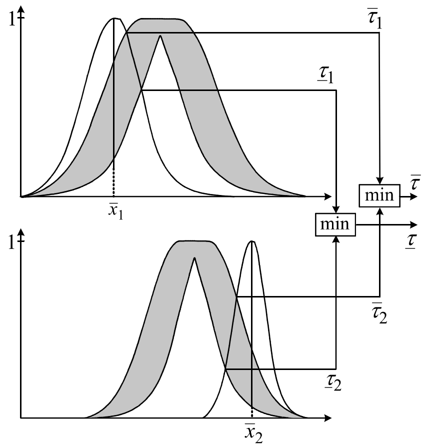

FIGURE 5.18. The method of determining the firing strength of a type-2 fuzzy system with nonsigleton-singleton fuzzification

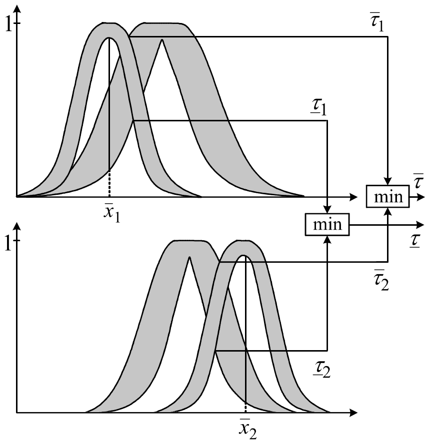

FIGURE 5.19. The method of determining the firing strength of a type-2 fuzzy system with nonsigleton (type-2) – interval fuzzification

set (type-2 nonsingleton fuzzification – interval). Figures 5.18 and 5.19 show the method of determining the firing strength illustrating both cases.

5.9 Notes

The notion of the type-2 fuzzy set has been introduced by Lotfi Zadeh [266]. In his article the author also defines the amount and the intersection of the type-2 fuzzy sets using the extension principle for that purpose. The basic notions characterizing type-2 fuzzy sets, i.e. the secondary membership functions and grades, the upper and lower membership functions, as well as the notions of embedded fuzzy sets and the footprint of uncertainty, have been successively introduced to the global literature by Mendel, and their review is contained in his monography [134]. The method of inference with the use of interval type-2 fuzzy sets was first described by Gorzałczany [64]. Basic operations on type-2 fuzzy sets have been provided by Dubois and Prade [42], and Karnik and Mendel [97,100]. The interval fuzzy sets of higher levels have been examined by Hisdal [80]. The iterative algorithm of type reduction for the interval type-2 fuzzy sets has been introduced by Karnik and Mendel [97, 101]. This has allowed to construct the interval type-2 fuzzy logic systems. The first such constructions have been presented by Karnik, Mendel and Liang [99]. The analysis of the differences between the interval inference systems and type-1 systems is presented in an article by Starczewski [240]. An interesting method of type reduction has been presented by Wu and Mendel in article [261]. The interval type-2 systems have been used for the prediction of chaotic series [98]. A novelty is the construction of the type-2 fuzzy inference system with the triangular secondary membership function, presented by Starczewski [238]. On the webpage http://ieee-cis.org/standards/ Mendel, Hagras and John have presented basic information on the type-2 fuzzy sets. This subject is also discussed on http://www.type2fuzzylogic.org/.