The Algebras of Łukasiewicz Many-Valued Logic: A Historical Overview*

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To Daniele Mundici on his 60th birthday

Abstract. An outline of the history of the algebras corresponding to Łukasiewicz many-valued logic from the pioneering work by G. Moisil in 1940 until D. Mundici's work in 1986.

1 Łukasiewicz and Post Many-Valued Logics

The three-valued system of propositional calculus was constructed by Jan Łukasiewicz in the year 1920 and described in a lecture given at the Polish Philosophical Society in Lwów. A short paper in Polish, based on his lecture, was published the same year [40].

The n-valued systems, discovered by Łukasiewicz in 1922, were briefly described in a textbook on Mathematical Logic published in 1929 [41]. A joint paper with Alfred Tarski, published in German in 1930 [44], contains, among other things, an account of results obtained by several Polish logicians on n-valued systems of propositional calculi, where n is either an integer ≥ 2 or $n = \aleph_0$. In [42], published in the same year and also in German, Łukasiewicz explains the philosophical ideas about determinism and modalities that leaded him to the construction of the n-valued calculi.

An idea of Łukasiewicz's philosophical motivation for the introduction of many-valued logic can be grasped from the following paragraphs of his Farewell Lecture as Rector of Warsaw University, on March 7, 1918 [10]:

> I have declared a spiritual war upon all coercion that restricts man's creative activity. There are two kinds of coercion. One of them is physical [...].
> The other kind of coercion is logical. We must accept self evident principles and the theorems resulting therefrom. This coercion is much stronger than the physical; there is no hope for liberation. No physical or intellectual force can overcome the principles of logic and mathematics.

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> That coercion originated with the rise of Aristotelian logic and Euclidean geometry.
> The concept was born of science as a system of principles and theorems connected by logical relationship. [...]
> In the universe conceived in this way there is no place for a creative act resulting not from a law but from a spontaneous impulse [...].
> The creative mind revolts against this concept of science, the universe and life. A brave individual, conscious of his value, does not want to be just a link in the chain of cause, but wants himself affect the course of events. This was always been the background of the opposition between science and art. [...]
> He has two paths to choose from: either to submerge himself in scepticism and abandon research, or to come to grips with the concept of science based on Aristotelian logic. I have chose that second path.[...]
> In striving to transform the concept of science based on Aristotelian logic I had to forge weapons stronger than that logic. It was symbolic logic that became such a weapon for me.

As the Referee pointed out, it is worthwhile to stress the fact that Łukasiewicz's idea of a third truth-value as a way to 'liberation' from 'the coercion originated with the rise of Aristotelian logic and Euclidean geometry' grew up in the discussions on determinism which took place immediately before World War I among polish scholars (see [70,92]).

In his 1920 thesis at Columbia University [69], Emil Leon Post[1] developed systems of n-valued propositional calculi, for n an integer ≥ 2, as natural generalizations of the truth-table approach to classical propositional calculus.

In contrast with Łukasiewicz, Post had no philosophical motivations.

The following paragraph is taken from the Introduction of the published paper:

> Whether these "non-Aristotelian" logics and the general development which includes them will have a direct application we do not know; but we believe that, inasmuch as the theory of elementary propositions is at the base of the complete system of *Principia*, this broadened outlook upon the theory will serve to prepare us for a similar analysis of that complete system, and so ultimately of mathematics.

Other systems of many-valued logic were considered by different authors, even before the publication of [40][2]. Only Post's work is explicitly mentioned because we will see that for each finite $n \geq 2$, Łukasiewicz and Post n-valued logics are strongly related from the algebraic point of view.

[1] Although Post was born in Poland, he arrived in the United States when he was seven years old, so had no influence of the Polish philosophical school.

[2] A system of three-valued logic, different from the one of Łukasiewicz was considered by Charles S. Peirce in 1912, see [24]. More than two truth values were used by Paul Bernays in his *Habilitationsschrift* at the University of Göttingen (1918), to give independence proof for postulates of classical propositional calculus. Parts of this work were published eight yeas latter [5] (see [94]).

2 Moisil's Łukasiewicz Algebras

In 1940 Gregorie Moisil introduced in [49] three-valued and four-valued Łukasiewicz algebras. To my knowledge, this was *the first attempt to give algebras corresponding to Łukasiewicz many-valued logic.*

In 1942 Paul C. Rosenbloom [75] introduced Post algebras, the algebras of Post many-valued logics.

Moisil's motivations and aims are clearly established in the following paragraphs, extracted from the Introduction of [49]:

> La logique formelle, en tant que science symbolique indépendante, est en possession de deux méthodes différentes. La première, appelée d'habitude méthode axiomatique est celle qui a été presque unanimement utilisée, celle qu'on trouve, par exemple, dans les traités de MM. Hilbert et Ackermann, de MM. Hilbert et Bernays, dans *Principia Mathematica.* On considère la logique comme un ensemble de *thèses*, en appelant thèse une expression qui est toujours vraie. Les thèses sont déduites d'un certain nombre d'axiomes à l'aide de certains schémas déductifs (tels que le modus ponens ou la règle de substitution). Cette méthode sera appelée *calcul des thèses.*
>
> Une seconde méthode est celle introduite par MM. Gentzen et Jaskowski [...].
>
> A chacune de ces méthodes purement logiques (c'est-à-dire ne supposant pas les Mathématiques constituées) on peut faire correspondre une branche des Mathématiques [...].
>
> Au calcul des thèses correspond ce qu'il convient d'appeler d'Algèbre de la Logique, en donnant à ce terme la signification générale d'étude algébrique des systèmes suggérés par le calcul des thèses. Les systèmes les plus intéressantes sont ceux qui ont été appelés structures (Oystein Ore), lattice (G. Birkhoff), Verbände (F. Klein) ou logiques. Ce sont des systèmes à deux lois de composition. Parmi les structures on a étudié les structures modulaires, distributives, avec éléments complémentaires el les algèbres de Boole.
>
> La relation entre le calcul des thèses et l'Algèbre de la Logique est établie par la méthode des matrices. Un premier problème consiste à définir une matrice telle que le calcul des thèses considéré soit celui qui est remplie ou satisfait par cette matrice (Tarski [84]) . Le calcul des propositions classiques a été définie a l'aide de la matrice L_2 à deux éléments ("le vrai" et "le faux"), celui de M. Heyting pour une matrice infinie (Jaskowski [32]), les logiques de M. Łukasiewicz à l'aide des matrices qui sont des structures simplement ordonnées.
>
> Un second problème qui se pose est celui caractériser algébriquement toutes les matrices qui correspondent à un calcul des thèses donné. Ce problème est résolu pour le calcul des thèses classique auquel correspond l'étude des algèbres de Boole.
>
> MM. Birkhoff [6] et Stone [83] ont montré que toute algèbre de Boole finie est le produit de structures L_2 et que toute algèbre de Boole infinie peut

> être représentée comme une algèbre de classes, c'est-à-dire comme une sous-structure de L_2^E, où E est un certain ensemble. C'est là un troisième problème fondamental, celui de la représentation des différentes algèbres suggérés par la logique.
> C'est à l'étude de ceux deus derniers problèmes pour les logiques trivalentes et tétravalentes de M. Łukasiewicz qu'est dédié ce Mémoire.
> Nous avons tout d'abord caractérisé algébriquement ces logiques, en créant un calcul qui les rend très maniables pour l'algorithmiste.
> En second lieu nous avons démontré que, dans le cas finie, ce calcul est adéquat à ces logiques, toute algèbre qui satisfait ses axiomes étant un produit cartésien de structures L_2, L_3 respectivement L_2, L_3, L_4.

Łukasiewicz built up his logic from the connectives of implication$\to$and negation$\neg$, whose "truth-tables" are defined, for $x, y \in [0,1]$ as

$$\neg x := 1 - x, \tag{2.1}$$

$$x \to y := \min(1 - x + y, 1). \tag{2.2}$$

When n is an integer ≥ 2, the n-valued calculus is obtained by restricting the values of x, y to

$$L_n := \{0, \frac{1}{n-1}, \frac{2}{n-1} \cdots \frac{n-2}{n-1}, 1\} \subseteq [0,1], \tag{2.3}$$

and for $n = \aleph_0$, x, y are allowed to take any rational value in $[0,1]$.

Notice that for $x, y \in [0,1]$,

$$\max(x, y) = (x \to y) \to y, \tag{2.4}$$

and

$$\min(x, y) = \neg \max(\neg x, \neg y). \tag{2.5}$$

Thus the order structure of $[0,1]$ can be recovered from $\neg$ and $\to$.

The unary operator ∇, defined by the truth-table

$$\nabla x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{2.6}$$

can be interpreted as a modal operator of possibility.

Tarski, then a collaborator of Łukasiewicz, observed that ∇ can be defined on L_3 by

$$\nabla x = \neg x \to x = \min(x + x, 1).$$

Moisil defined three-valued Łukasiewicz algebras as systems

$$\langle A, \vee, \wedge, \neg, \nabla, 0, 1 \rangle$$

such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice with smallest element 0 and greatest element 1, $\neg$ and ∇ are unary operations that correspond to negation and to possibility, respectively.

L_3, equipped with the natural lattice operations and $\neg$ and ∇ as given respectively by (2.1) and (2.6) is an example of a three-valued Łukasiewicz-algebra, which has L_2, the two-element Boolean algebra, as a subalgebra.

Moisil showed that Łukasiewicz's implication (2.2) is definable in L_3, and that each *finite* three-valued Łukasiewicz algebra is a direct product of algebras L_3 and L_2.

In the subsequent paper [50], Moisil introduced the following example of a three-valued Łukasiewicz algebra: Let B be a Boolean algebra, and let $B^{[2]} := \{(x, y) \in B \times B : x \leq y\}$. Then $B^{[2]}$ with the lattice operations defined pointwise, and $\neg(x, y) := (\neg y, \neg x)$, and $\nabla(x, y) = (y, y)$ is a three-valued Łukasiewicz-algebra.

Then he proved that for each three-valued Łukasiewicz algebra A there is a Boolean algebra B such that A is embedded in $B^{[2]}$, improving a result already obtained in [49]. In this way, and taking into account the results of Stone [83], he obtained a representation of three-valued Łukasiewicz algebras as pairs of sets.

Inspired by the relations discovered by Stone between Boolean algebras and rings [82], Moisil investigated the relations between three-valued Łukasiewicz algebras and rings that are a product of a ring of characteristic 2 and a ring of characteristic 3 [51]. An equational characterization of three-valued Łukasiewicz algebra was given in [52].

On each L_n Moisil considered $n - 1$ unary operations $\nabla_1^n, \ldots, \nabla_{n-1}^n$ defined as follows:

$$\nabla_i^n\left(\frac{j}{n-1}\right) = \begin{cases} 1 & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n. \end{cases} \tag{2.7}$$

Notice that $\nabla_2^3 = \nabla$ and $\nabla_1^3 = \neg\nabla\neg$.

Moisil considered these operations as generalized modal operators, ∇_1^n, which assigns the value 0 to each $x \neq 1$, correspond to necessity, and ∇_{n-1}^n corresponds to possibility.

In his paper [50] he also introduced n-valued Łukasiewicz algebras for $2 \leq n < \aleph_0$ as bounded distributive lattices equipped with an involutive negation satisfying the De Morgan laws, and $n - 1$ unary operations corresponding to the modal operators (2.7).

Moisil showed that each n-valued Łukasiewicz algebra can be embedded in a product of algebras L_n, and also in $B^{[n]}$, for some Boolean algebra B.

It follows that the modal operations ∇_i^n can be defined on L_n from Łukasiewicz implication and negation, but as was observed by Alan Rose while he was visiting the University of Bahía Blanca in 1965, it is not possible to define Łukasiewicz implication from the lattice operations, the negation and the modal operators when $n \geq 5$. Hence, *for $n \geq 5$ n-valued Łukasiewicz algebras do not correspond to n-valued Łukasiewicz logic.*

A correct algebrization of the n-valued calculus, for $n \geq 5$, can be obtained by adding to the operations of n-valued Łukasiewicz algebras a set of $\frac{n(n-5)+2}{2}$ binary operations satisfying some simple equations. In this way n-valued Łukasiewicz propositional calculus can be considered as an expansion of the intuitionistic calculus. The algebras so expanded are called *proper n-valued Łukasiewicz algebras* [16,17].

It is worthwhile to remark that Dana Scott [81], without reference to Moisil, considered the operators ∇_i^n on L_n as two-valued valuations, and he showed that they are related with n-valued Łukasiewicz implication as follows:

$\nabla_i^n(x \to y) = 1$ iff whenever $i + j \leq k + 1$ and $\nabla_j^n x = 1$, then $\nabla_k^n y = 1$.

Moisil also defined infinite-valued Łukasiewicz algebras, where the modal operators are indexed by a totally ordered set of arbitrary cardinality. But they are not related with Łukasiewicz infinite-valued calculus. Moisil considered these algebras in relation with fuzzy logic.[3]

Three-valued Łukasiewicz algebras were intensively investigated by Antonio Monteiro during the early sixties of the last century. Monteiro's work during that period was mostly shown in his lectures at the University of Bahía Blanca (Argentina), and it is partially summarized in his posthumous paper [56, Chapitre VII].

Besides given a simple equational characterization of three-valued Łukasiewicz algebras [54], he introduced the weak implication $\Rightarrow$ by the formula:

$$x \Rightarrow y := \nabla \neg x \vee y \tag{2.8}$$

and he showed that the Łukasiewicz implication (2.2) and the weak implication are related as follows:

$$x \to y = (x \Rightarrow y) \wedge (\neg y \Rightarrow \neg x),$$

$$x \Rightarrow y = x \to (x \to y).$$

Hence a subset of a three-valued Łukasiewicz algebra containing the greatest element is closed under modus ponens with respect to Łukasiewicz implication if and only if it is closed under modus ponens with respect to weak implication. Hence both implications are equivalent from the point of view of deduction in three-valued Łukasiewicz logic (see [56]).

Let me mention the following important results obtained by Monteiro:

1. Three-valued Łukasiewicz algebras coincide with the semisimple Nelson algebras, i. e., the algebras of the constructive logic with strong negation considered by D. Nelson and A. A. Markov. Consequently, *three-valued Łukasiewicz logic is an axiomatic extension of the constructive logic with strong negation* (see [56,57]).

[3] The monograph [9] is the standard reference for Moisil's Łukasiewicz algebras. They are also considered in [1, Chapter XI]. For historical remarks and updated references see [26].

2. It is possible to define from each monadic Boolean algebra A (as defined by Halmos [29]) a three-valued Łukasiewicz algebra $L(A)$, and each three-valued Łukasiewicz algebra is isomorphic to $L(A)$ for a suitable monadic Boolean algebra A (see [55,56]).

As a matter of fact, it turns out that the relation between three-valued Łukasiewicz algebras and monadic Boolean algebras is functorial (see [45]).

Since it was shown by Halmos that monadic Boolean algebras are the algebraic counterpart of classical first order monadic calculus, Monteiro considered that the representation of three-valued Łukasiewicz algebras into monadic Boolean algebras gives a proof of the consistency of Łukasiewicz three-valued logic relative to classical logic.

It is fair to say that Monteiro's results on three-valued Łukasiewicz algebras inspired most of the research done in the theory of Łukasiewicz n-valued Łukasiewicz algebras (see [9] and the references given there).

3 Chang's MV-Algebras

A deep result on Łukasiewicz infinite valued-logic was proved by Robert MacNaughton in 1951 [48], characterizing the propositional formulas of n variables, modulo logical equivalence, by means of $[0,1]$-valued piecewise linear continuous functions on the hypercube $[0,1]^n$ equipped with the usual product topology.

MacNaughton also characterized the functions from L_n^k into L_n that represent the formulas of Łukasiewicz n-valued propositional calculus.

Łukasiewicz had conjectured that a propositional formula φ is a tautology of the $\aleph_0$-valued calculus if and only if φ can be derived by the rules of detachment and substitution from five formulas that he proposed as axioms (see [44]). Mordechaj Wajsberg, who in 1931 had given an axiomatization of Łukasiewicz three-valued logic, claimed in [91] that he had proved the conjecture, but his proof was never published. Wajsberg was killed during the Second World War.

The first printed proof of Łukasiewicz conjecture, due A. Rose and B. J. Rosser, appeared in 1958 [74]. They use in their proof MacNaughton's theorem.

The same year C. C. Chang [13] introduced MV-algebras, with the intention of proving Łukasiewicz conjecture by algebraic means.

Notice that in the real segment $[0,1]$ we have that

$$x \oplus y := \min(1, x+y) = \neg x \to y, \tag{3.9}$$

and

$$x \to y = \neg x \oplus y. \tag{3.10}$$

Hence Chang defined MV-algebras essentially in terms of a binary operation $\oplus$ that corresponds to the truncated addition in $[0,1]$ and the negation $\neg$ that have to satisfy certain equations. Thus MV-algebras form a variety or equational class. (The operation $\odot$ can be defined as $x \odot y = \neg(\neg x \oplus \neg y)$.

If we add to the axioms given by Chang to define MV-algebras the requirement that the operation $\oplus$ be idempotent, $x \oplus x = x$, then we obtain the characterization of Boolean algebras as complemented distributive lattices.

Of course, the segment $[0, 1]$ with truncated addition and Łukasiewicz negation $\neg$ given by (2.1) is an MV-algebra, known as the *standard MV-algebra*.

Moreover, Chang proved that the Lindenbaum-Tarski algebra of Łukasiewicz $\aleph_0$-valued calculus is an MV-algebra, and that a formula φ is provable from Łukasiewicz axioms by detachment and substitution if and only if its equivalence class is the unit of this algebra.

Hence to prove Łukasiewicz conjecture turns out to be equivalent to prove that the standard MV-algebra generates the variety of MV-algebras.

Chang proved that (in the current universal algebra language) the simple MV-algebras are the standard MV-algebra and its subalgebras.

Then one way to prove that the standard MV-algebra generates the whole variety is to prove that all algebras in the variety are semisimple, i. e., subdirect products of subalgebras of the standard MV-algebra $[0, 1]$. Notice that it is the method used by Rasiowa and Sikorski [71] to prove the completeness of some axiomatizations of the classical propositional calculus with respect to two-valued tautologies.

But (fortunately) this is not the case, because Chang constructed an example of a non-semisimple MV-algebra.

As a matter of fact, the paper develops a very sophisticated mathematical theory that ends with some weak applications to Łukasiewicz logic (see [15]).

The next year, Chang published in the same journal another paper [14], where he observed that if u is a positive element of a totally ordered abelian group G, then the segment $[0, u] = \{x \in G : 0 \leq x \leq u\}$ becomes an MV-algebra if we define the operations $\oplus$ and $\neg$ as

$$x \oplus y = \min(u, (x + y)),$$

and

$$\neg x = u - x.$$

Then, given a totally ordered MV-algebra A, he was able to construct a totally ordered abelian group $G(A)$ and $u > 0$ in $G(A)$ such that A is isomorphic to the MV-algebra $[0, u]$.

He also proved that every MV-algebra is a subdirect product of totally ordered MV-algebras. From these results he could prove Łukasiewicz's conjecture by translating it into a problem in the first order theory of totally ordered abelian groups.

In the early sixties Chang and his student Belluce published some papers concerning with the predicate calculus based on Łukasiewicz infinite-valued logic. In particular monadic MV-algebras were considered [4,2].

In 1973 appeared Piero Mangani's paper [46], in which the author derived from a few axioms many important properties of MV-algebras. For instance, he showed that the algebras L_n are quasi-primal.

Mangani's paper was followed by papers by Saeli and Lacava [77,38,35,36,37], all published in Italian, were some interesting results on MV-algebras are obtained. For instance, Lacava [35] observed that if u is a positive element of a lattice ordered abelian group G, then the segment $[0, u] := \{x \in G : 0 \leq x \leq u\}$ becomes an MV-algebra by defining

$$x \oplus y = u \wedge (x + y) \text{ and } \neg x = u - x.$$

Moreover, using the fact that each MV-algebra is a subdirect product of totally ordered MV-algebras together with Chang's results, Lacava embedded each MV-algebra in a segment of a lattice-ordered abelian group. Lacava also characterized the subdirectly irreducible MV-algebras and showed that (lattice) complete MV-algebras are semisimple [36].

In 1977, Revaz Grigolia [28] gave an equational characterization of the subvarieties of the variety of MV-algebras generated by the finite chains L_n, considered as subalgebras of the standard MV-algebra $[0, 1]$. The algebras in such subvarieties were called MV$_n$-algebras.

Grigolia used MV$_n$-algebras to give an axiomatization for each n-valued Łukasiewicz calculus.

MV$_n$-algebras and proper n-valued Łukasiewicz algebras are term-wise equivalent. Besides being defined with just two operations, MV$_n$-algebras have the advantage that all belong to the same variety, independently of n. This is not the case with proper n-valued Łukasiewicz algebras, because the first order language used to define them depends on n.

On each n-valued Łukasiewicz algebra $\mathbf{A}$ define the operators J_i, for $i = 1, \ldots, n-1$:

$$J_i(x) = \sigma^n_{n-i}(x) \wedge \neg\sigma^n_{n-i-1}(x),$$

where $\sigma^n_0(x) = 0$ and $\sigma^n_n(x) = 1$. Notice that in L_n we have:

$$J_i\left(\frac{j}{(n-1)}\right) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus for $j = 1, \ldots, n-1$, the sentence

"The proposition p has truth-value $\frac{j}{n-1}$"

can be expressed in Łukasiewicz n-valued logic.

But it follows from the mentioned results of MacNaughton that such kind of operations cannot be defined in the infinite-valued logic. In some cases this kind of operations can be added, as it is the case of the so called Baaz operation.

As I already mentioned, Post algebras of order n, the algebras of n-valued Post logic, have been introduced by Rosenbloom in 1941. They were further investigated during the sixties by G. Epstein [23], T. Traczyk [87,88], G. Rousseau [76] , Ph. Dwinger [22] (see also [1,9] and the references given there). It turned out that they can be characterized as n-valued Łukasiewicz algebras with $n-2$ constants, satisfying some simple equations, added [1,9]. Post algebras were also considered by Dietrich Schwartz [78,79] under the name of *MV-algebras of finite order*. An MV-algebra of order n is an MV-algebra A satisfying the equation $x^{n-1} \oplus x = x$ and endowed with a constant c that satisfies equations that guarantee that the map $\frac{k}{n-1} \mapsto k.c$ is an MV-homomorphism from L_n into A.

The (lattice) complete Post algebras of order n can be characterized as the injective objects in the category of MV_n-algebras.

H. W. Buff [12] considered decidability problems of MV-algebras, and L. P. Belluce [3] gave a functional representation of semi-simple MV-algebras, initiate the study of the prime spectra of MV-algebras, and consider some problems on (lattice) complete MV-algebras.

4 Other Approaches to MV-Algebras

In 1966 Y. Imai and K. Iseki [30] introduced BCK-algebras as a common abstraction of the algebras corresponding to the implicative fragments of several logics existing in the literature, including classical and intuitionistic logic. Since then a lot of papers concerned with these algebras were published.

The bounded commutative BCK-algebras , a class of BCK-algebras defined by K. Iseki and S. Tanaka [31], was intensively investigated by W. H. Cornish [18,19], A. Romanowska and T. Traczyk [72,73] at the end of the seventies and beginning of the eighties. It was proved by Font, Rodríguez and Torrens [25], and independently, by Daniele Mundici [59], that these algebras coincide with MV-algebras. As a consequence, some results on MV-algebras were rediscovered in terms of BCK-algebras.

In particular, relations between a class of bounded commutative BCK-algebras and lattice ordered abelian groups were obtained by Cornish [18], corresponding to the relation between perfect MV-algebras and lattice-ordered abelian groups established by Di Nola and Lettieri [21].

Bruno Bosbach [11] introduced MV-algebras under the name of *symmetric bricks*. He was lead to bricks by his investigations on the algebraic structure of positive cones of (non necessarily abelian) lattice ordered groups. He developed the theory of bricks in an independent way. The paper contains, among other things, representations theorems that generalize the characterization of Boolean algebras as Boolean rings, and results on the structure of complete bricks.

In 1981, Yuichi Komori [34] investigated the axiomatic extensions of infinite-valued Łukasiewicz propositional calculus using algebraic tools. Although Komori was acquainted with the papers [13,14] and in a few places refers to them, his work was rather independent from Chang's. He introduced *CN algebras*, that were presented in the original language of Łukasiewicz, i. e., implication and negation, and with axioms that were straightforward adaptations of the axioms conjectured by Łukasiewicz. Komori made explicit use of the completeness of the first order theory of a special class of totally ordered abelian groups, previously introduced by him in [33]. Komori's CN-algebras are term-wise equivalent to Chang's MV-algebras, hence Komori determined *the lattice of subvarieties of the variety of MV-algebras.*

In his Doctoral Dissertation of 1981 at the University of Barcelona, Antonio Jesús Rodríguez also presented MV-algebras in the original language of Łukasiewicz, and he called the algebras so defined *Wajsberg algebras.* The main parts of the dissertation was published in [25]. Trough this paper Willem Blok became acquainted with the algebras of Łukasiewicz infinite-valued logic, and with his collaborators connected them with the theory of *hoops* [8,7].

A. Torrens [86] started the classification of Wajsberg algebras in terms of Boolean products.

5 Mundici's Work

In the papers described before, the algebras related to Łukasiewicz many-valued logic were considered as interesting algebraic structures that could eventually be applied to obtain some logical results.

A turning point of the theory was the paper [58] by Daniele Mundici *Interpretation of AF C*-Algebras in Łukasiewicz Sentential Calculus* published in the Journal of Functional Analysis in 1986.

It was certainly surprising to see the words "Łukasiewicz sentential calculus" in the title of an article of about fifty pages published in the Journal of Functional Analysis, and even more surprising to see that the paper was communicated by the 1982 Fields Medal Alan Connes.

In that paper it is proved that Chang's MV-algebras are categorically equivalent to lattice-ordered abelian groups with a strong unit. This result allowed Daniele to relate, via dimension groups, countable MV-algebras with A(pproximately) F(inite-dimensional) C*-algebras, an important class of algebras considered in Functional Analysis (see, for instance, [27]). Since MV-algebras are the Lindenbaum algebras of Łukasiewicz propositional calculus modulo a theory, a theory of the calculus is associated with the corresponding AFC*-algebra, and Daniele showed, among other things, that to simple AFC*-algebras correspond finitely axiomatizable theories. Daniele continued these investigations in several papers (see, for instance, [60,61,62,65,66,67]).

In subsequent papers Daniele gave a semantics for MV_n-algebras in terms of Ulam games, paving the way to apply MV-algebras to coding theory [63,64,67]. Moreover, he discovered that deduction in Łukasiewicz logic are related to desingularization of toric varieties [68].

These results stimulated further researches by Daniele and many other people in the theory of MV-algebras and their connections with other mathematical structures. But this is not history, but present. The evolution of these ideas should be consider in the future.

Perhaps another old professor will explain them during the celebration of the 60^{th} birthday of some of the young organizers of this meeting.

Thanks for your attention.

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