Counting the Number of Three-Player Partizan Cold Games

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Abstract. We give upper and lower bounds on $S_3[n]$ equal to the number of three-player partizan cold games born by day n. In particular, we give an upper bound of $O(S_2[n]^3)$ and a lower bound of $\Omega(S_2[n])$ where $S_2[n]$ is the number of surreal numbers born by day n.

1 Introduction

Games represent a conflict of interests between two or more parties and, as a consequence, they are a good framework to study complex problem-solving strategies. Typically, a real-world economical, social or political conflict involves more than two parties and a winning strategy is often the result of coalitions. For this reason, it is important to determine the winning strategy of a player in the worst scenario, i.e., assuming that all his/her opponents are allied against him/her.

It is therefore, a challenging and fascinating problem to extend the field of combinatorial game theory [1,3] so as to allow more than two players. Past effort to classify impartial three-player combinatorial games (the theories of Li [5] and Straffin [8]) have made various restrictive assumptions about the rationality of one's opponents and the formation and behavior of coalitions. Loeb [6] introduces the notion of a stable winning coalition in a multi-player game as a new system of classification of games. Differently, Propp [7] adopts in his work an agnostic attitude toward such issues, and seeks only to understand in what circumstances one player has a winning strategy against the combined forces of the other two.

Cincotti [2] presents a theory to classify three-player partizan games adopting the same attitude. Such a theory represents a possible extension of Conway's theory of partizan games [3,4] and it is therefore both a theory of games and a theory of numbers.

In order to understand the mathematical structure of three-player partizan games, counting the number of cold games born by day n is a crucial point. We recall that the number of surreal numbers born by day n is $S_2[n] = 2^{n+1} - 1$. Moreover, a lower and upper bound of two-player games is given by Wolfe and Fraser in [9].

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2 Three-Player Partizan Games

For the sake of self-containment we recall in this section the main results concerning three-player partizan games obtained in the previous work [2].

2.1 Basic Definitions

Definition 1. *If L, C, R are any three sets of numbers previously defined and*

1. *no element of L is $\geq_L$ any element of $C \cup R$, and*
2. *no element of C is $\geq_C$ any element of $L \cup R$, and*
3. *no element of R is $\geq_R$ any element of $L \cup C$,*

then $\{L|C|R\}$ is a number. All numbers are constructed in this way.

This definition for numbers is based on the definition and comparison operators for games given in the following two definitions.

Definition 2. *If L, C, R are any three sets of games previously defined then $\{L|C|R\}$ is a game. All games are constructed in this way.*

We introduce three different relations ($\geq_L$,$\geq_C$,$\geq_R$) that represent the subjective point of view of every player which is independent from the point of view of the other players.

Definition 3. *We say that*

1. *$x \geq_L y$ iff ($y \geq_L$ no x^C and $y \geq_L$ no x^R and no $y^L \geq_L x$),*
2. *$x \geq_C y$ iff ($y \geq_C$ no x^L and $y \geq_C$ no x^R and no $y^C \geq_C x$),*
3. *$x \geq_R y$ iff ($y \geq_R$ no x^L and $y \geq_R$ no x^C and no $y^R \geq_R x$).*

Numbers are totally ordered with respect to $\geq_L$, $\geq_C$, and $\geq_R$ but games are just partially-ordered, e.g., there exist games x and y for which we have neither $x \geq_L y$ nor $y \geq_L x$.

Definition 4. *We say that*

1. *$x =_L y$ if and only if ($x \geq_L y$ and $x \leq_L y$),*
2. *$x =_C y$ if and only if ($x \geq_C y$ and $x \leq_C y$),*
3. *$x =_R y$ if and only if ($x \geq_R y$ and $x \leq_R y$),*
4. *$x = y$ if and only if ($x =_L y$, $x =_C y$, and $x =_R y$),*
5. *$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\}$.*

Moreover, it is possible to classify numbers in 11 classes as shown in Table 1. The entries '?' are unrestricted and indicate that we can have different outcomes. For further details, see [2].

Table 1. Classification of numbers

Short notation	Class	Left starts	Center starts	Right starts
$=0$	$=_L 0,=_C 0,=_R 0$	Right wins	Left wins	Center wins
$>_L 0$	$>_L 0,<_C 0,<_R 0$	Left wins	Left wins	Left wins
$>_C 0$	$<_L 0,>_C 0,<_R 0$	Center wins	Center wins	Center wins
$>_R 0$	$<_L 0,<_C 0,>_R 0$	Right wins	Right wins	Right wins
$=_{LC} 0$	$=_L 0,=_C 0,<_R 0$	Center wins	Left wins	Center wins
$=_{LR} 0$	$=_L 0,<_C 0,=_R 0$	Right wins	Left wins	Left wins
$=_{CR} 0$	$<_L 0,=_C 0,=_R 0$	Right wins	Right wins	Center wins
$<_{CR} 0$	$=_L 0,<_C 0,<_R 0$	?	Left wins	Left wins
$<_{LR} 0$	$<_L 0,=_C 0,<_R 0$	Center wins	?	Center wins
$<_{LC} 0$	$<_L 0,<_C 0,=_R 0$	Right wins	Right wins	?
<0	$<_L 0,<_C 0,<_R 0$	?	?	?

2.2 Examples of Numbers

According to the construction procedure, every number has the form $\{L|C|R\}$, where L, C, and R are three sets of earlier constructed numbers. At day zero, we have only the empty set $\emptyset$ therefore the earliest constructed number could only be $\{L|C|R\}$ with $L = C = R = \emptyset$, or in the simplified notation $\{\ |\ |\ \}$. We denote it by 0.

The first day we have only three new numbers which we call $1_L = \{0|\ |\ \}$, $1_C = \{\ |0|\ \}$, and $1_R = \{\ |\ |0\}$. We observe that $\{0|0|\ \}$, $\{0|\ |0\}$, $\{\ |0|0\}$, and $\{0|0|0\}$ are not numbers. Table 2 shows the numbers created the second day.

Note 1. In [2] the list of numbers created the second day was incomplete because we can create 24 (not 18) different numbers.

Table 2. Numbers created the second day

$\{1_L|\ |\ \}$	$\{\ |1_C|\ \}$	$\{\ |\ |1_R\}$	$\{1_C|1_L|\ \}$
$\{0,1_C,1_R|\ |\ \}$	$\{\ |0,1_L,1_R|\ \}$	$\{\ |\ |0,1_L,1_C\}$	$\{1_R|\ |1_L\}$
$\{0,1_C|\ |\ \}$	$\{\ |0,1_L|\ \}$	$\{\ |\ |0,1_L\}$	$\{\ |1_R|1_C\}$
$\{0,1_R|\ |\ \}$	$\{\ |0,1_R|\ \}$	$\{\ |\ |0,1_C\}$	$\{1_C,1_R|\ |\ \}$
$\{0|1_L|\ \}$	$\{1_C|0|\ \}$	$\{1_R|\ |0\}$	$\{\ |1_L,1_R|\ \}$
$\{0|\ |1_L\}$	$\{\ |0|1_C\}$	$\{\ |1_R|0\}$	$\{\ |\ |1_L,1_C\}$

3 Counting the Numbers

How many numbers will be created by day n?

Definition 5. *Let S_2 and S_3 be respectively Conway's surreal numbers and the new set of numbers previously defined. We define three different maps $\pi : S_3 \to S_2$ as follows:*

1. $\pi_L(\{x^L|x^C|x^R\}) = \{\pi_L(x^L)|\pi_L(x^C), \pi_L(x^R)\}$
2. $\pi_C(\{x^L|x^C|x^R\}) = \{\pi_C(x^C)|\pi_C(x^L), \pi_C(x^R)\}$
3. $\pi_R(\{x^L|x^C|x^R\}) = \{\pi_R(x^R)|\pi_R(x^L), \pi_R(x^C)\}$

Theorem 1. *For any $x, y \in S_3$*

1. $x \leq_L y \iff \pi_L(x) \leq \pi_L(y)$
2. $x \leq_C y \iff \pi_C(x) \leq \pi_C(y)$
3. $x \leq_R y \iff \pi_R(x) \leq \pi_R(y)$

Proof. *1.* If $x \leq_L y$ then $\nexists x^L \geq_L y$ and, by the inductive hypothesis

$$\nexists \pi_L(x^L) \geq \pi_L(y) \Rightarrow \nexists \pi_L(x)^L \geq \pi_L(y). \tag{1}$$

Moreover, $\nexists y^C \leq_L x$, $\nexists y^R \leq_L x$, and by the inductive hypothesis

$$\left.\begin{array}{l}\nexists \pi_L(y^C) \leq \pi_L(x)\\ \nexists \pi_L(y^R) \leq \pi_L(x)\end{array}\right\} \Rightarrow \nexists \pi_L(y)^R \leq \pi_L(x). \tag{2}$$

Conversely, if $\pi_L(x) \leq \pi_L(y)$ then

$$\nexists \pi_L(x)^L \geq \pi_L(y) \Rightarrow \nexists \pi_L(x^L) \geq \pi_L(y) \tag{3}$$

and by the inductive hypothesis $\nexists x^L \geq_L y$.
Also,

$$\nexists \pi_L(y)^R \leq \pi_L(x) \Rightarrow \left\{\begin{array}{l}\nexists \pi_L(y^C) \leq \pi_L(x)\\ \nexists \pi_L(y^R) \leq \pi_L(x)\end{array}\right. \tag{4}$$

and by the inductive hypothesis $\nexists y^C \leq_L x$ and $\nexists y^R \leq_L x$.

2. Analogous to *1.*

3. Analogous to *1.* □

We have two corollaries of the above theorem.

Corollary 1. *If $x \in S_3$ is a number then $\pi_L(x)$, $\pi_C(x)$, and $\pi_R(x)$ are numbers.*

Corollary 2. *Let $x, y \in S_3$ be two numbers. Then $x = y$ if and only if $\pi_L(x) = \pi_L(y)$, $\pi_C(x) = \pi_C(y)$, and $\pi_R(x) = \pi_R(y)$.*

It follows that to every number $x \in S_3$ there corresponds a unique triple $(\pi_L(x), \pi_C(x), \pi_R(x))$ of surreal numbers. Table 3 shows all numbers born by day 2 and their corresponding triples of surreal numbers.

Theorem 2. *Let $x = \{x^L|x^C|x^R\} \in S_3[n]$ be a number born by day n. Then $\pi_L(x)$, $\pi_C(x)$, and $\pi_R(x) \in S_2[n]$.*

Proof. By the hypothesis x^L, x^C, and $x^R \in S_3[n-1]$ and by the inductive hypothesis $\pi_L(x^L)$, $\pi_L(x^C)$, and $\pi_L(x^R) \in S_2[n-1]$ therefore $\pi_L(x) \in S_2[n]$. Analogously, $\pi_C(x)$ and $\pi_R(x)$ are numbers born by day n. □

Unfortunately, the above theorem is not reversible.

Table 3. Numbers born by day 2 and their corresponding triples of surreal numbers

	Day 0	Day 1	Day 2	
$=$	$\{ \mid \mid \}$ (0,0,0)			
$>_L$		$\{0\mid \mid \}$ (1,-1,-1)	$\{1_L\mid \mid \}$	(2,-2,-2)
$>_L$			$\{0, 1_C, 1_R\mid \mid \}$	(1,-2,-2)
$>_L$			$\{0, 1_C\mid \mid \}$	(1,-1,-2)
$>_L$			$\{0, 1_R\mid \mid \}$	(1,-2,-1)
$>_L$			$\{0\mid 1_L\mid \}$	(1/2,-1/2,-2)
$>_L$			$\{0\mid \mid 1_L\}$	(1/2,-2,-1/2)
$>_C$		$\{ \mid 0\mid \}$ (-1,1,-1)	$\{ \mid 1_C\mid \}$	(-2,2,-2)
$>_C$			$\{ \mid 0, 1_L, 1_R\mid \}$	(-2,1,-2)
$>_C$			$\{ \mid 0, 1_L\mid \}$	(-1,1,-2)
$>_C$			$\{ \mid 0, 1_R\mid \}$	(-2,1,-1)
$>_C$			$\{1_C\mid 0\mid \}$	(-1/2,1/2,-2)
$>_C$			$\{ \mid 0\mid 1_C\}$	(-2,1/2,-1/2)
$>_R$		$\{ \mid \mid 0\}$ (-1,-1,1)	$\{ \mid \mid 1_R\}$	(-2,-2,2)
$>_R$			$\{ \mid \mid 0, 1_L, 1_C\}$	(-2,-2,1)
$>_R$			$\{ \mid \mid 0, 1_L\}$	(-1,-2,1)
$>_R$			$\{ \mid \mid 0, 1_C\}$	(-2,-1,1)
$>_R$			$\{1_R\mid \mid 0\}$	(-1/2,-2,1/2)
$>_R$			$\{ \mid 1_R\mid 0\}$	(-2,-1/2,1/2)
$=_{LC}$			$\{1_C\mid 1_L\mid \}$	(0,0,-2)
$=_{LR}$			$\{1_R\mid \mid 1_L\}$	(0,-2,0)
$=_{CR}$			$\{ \mid 1_R\mid 1_C\}$	(-2,0,0)
$<_{CR}$			$\{1_C, 1_R\mid \mid \}$	(0,-2,-2)
$<_{LR}$			$\{ \mid 1_L, 1_R\mid \}$	(-2,0,-2)
$<_{LC}$			$\{ \mid \mid 1_L, 1_C\}$	(-2,-2,0)

Example 1. Let's consider $1_L + 1_C + 1_R = \{\{ \mid 1_R\mid 1_C\}\mid\{1_R\mid \mid 1_L\}\mid\{1_C\mid 1_L\mid \}\}$. We observe that $\pi_L(x) = \pi_C(x) = \pi_R(x) = -1$ therefore they all belong to $S_2[1]$ but $1_L + 1_C + 1_R \notin S_3[1]$ because it will be created only the third day.

It follows that a rough upper bound on $S_3[n]$ is given by the number of distinct triples of surreal numbers born by day n, i.e., $(S_2[n])^3$. Moreover, a simple lower bound is given by $S_2[n]$.

Theorem 3. *For any $x, y \in S_3$*

1. $\pi_L(x + y) = \pi_L(x) + \pi_L(y)$
2. $\pi_C(x + y) = \pi_C(x) + \pi_C(y)$
3. $\pi_R(x + y) = \pi_R(x) + \pi_R(y)$

Proof. *1.*

$$\begin{aligned}\pi_L(x + y) &= \pi_L(\{x^L + y, x + y^L\mid x^C + y, x + y^C\mid x^R + y, x + y^R\}) \qquad (5)\\ &= \{\pi_L(x^L + y), \pi_L(x + y^L)\mid \\ &\quad \pi_L(x^C + y), \pi_L(x + y^C), \pi_L(x^R + y), \pi_L(x + y^R)\}\end{aligned}$$

$$= \{\pi_L(x^L) + \pi_L(y), \pi_L(x) + \pi_L(y^L)|$$
$$\pi_L(x^C) + \pi_L(y), \pi_L(x) + \pi_L(y^C),$$
$$\pi_L(x^R) + \pi_L(y), \pi_L(x) + \pi_L(y^R)\}$$
$$= \pi_L(x) + \pi_L(y)$$

2. Analogous to *1.*
3. Analogous to *1.* □

Theorem 4. *Let $x \in S_3$ be a number. Then*

1. $\pi_L(x) + \pi_C(x) \leq 0$,
2. $\pi_L(x) + \pi_R(x) \leq 0$,
3. $\pi_C(x) + \pi_R(x) \leq 0$.

Proof. *1.* We observe that

$$\pi_L(x^L) + \pi_C(x) < \pi_L(x^L) + \pi_C(x^L) \tag{6}$$

and by the inductive hypothesis

$$\pi_L(x^L) + \pi_C(x^L) \leq 0. \tag{7}$$

Analogously,

$$\pi_L(x) + \pi_C(x^C) < \pi_L(x^C) + \pi_C(x^C) \leq 0 \tag{8}$$

therefore no left option of $\pi_L(x) + \pi_C(x)$ is ≥ 0.
2. Analogous to *1.*
3. Analogous to *1.* □

Theorem 5. *Let $x \in S_3[n]$ and $y \in S_3[m]$ be two numbers. Then $x + y \in S_3[n+m]$.*

Proof. We recall that $x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\}$. By the hypothesis, x^L, x^C, and x^R belong to $S_3[n-1]$ and y^L, y^C, y^R belong to $S_3[m-1]$. By the inductive hypothesis, $x^L + y$, $x + y^L$, $x^C + y$, $x + y^C$, $x^R + y$, $x + y^R$ belong to $S_3[n+m-1]$ therefore $x + y \in S_3[n+m]$. □

3.1 Lower and Upper Bound

Below we give a more accurate upper and lower bound for each class. We start recalling four statements.

1. The number of surreal numbers born by day n is

$$S_2[n] = 2^{n+1} - 1 \tag{9}$$

2. The number of positive (negative) surreal numbers born by day n is

$$\frac{1}{2}(S_2[n] - 1) \tag{10}$$

3. The number of positive (negative) dyadic fraction born by day n is

$$\frac{1}{2}(S_2[n] - 2n - 1) \tag{11}$$

4. The following equality holds

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{12}$$

Definition 6. *We define*

$$(i+1)_L = \{i_L| \mid \} \tag{13}$$
$$(j+1)_C = \{ \mid j_C| \} \tag{14}$$
$$(k+1)_R = \{ \mid \mid k_R\} \tag{15}$$

where $i, j, k \in \mathbf{N}$ *and* $0_L = 0_C = 0_R = 0$.

Definition 7. *We define*

$$\left(\frac{2p+1}{2^{q+1}}\right)_{LC} = \left\{\left(\frac{p}{2^q}\right)_{LC}\middle|\left(\frac{p+1}{2^q}\right)_{LC}\middle|\right\} \tag{16}$$
$$\left(\frac{2p+1}{2^{q+1}}\right)_{LR} = \left\{\left(\frac{p}{2^q}\right)_{LR}\middle|\middle|\left(\frac{p+1}{2^q}\right)_{LR}\right\} \tag{17}$$

where $p,\ q \in \mathbf{N}$.

Note 2. If $\left(\frac{p}{2^q}\right) \in \mathbf{N}$ then $\left(\frac{p}{2^q}\right)_{LC} = \left(\frac{p}{2^q}\right)_{LR} = \left(\frac{p}{2^q}\right)_{L}$. Analogously, if $\left(\frac{p+1}{2^q}\right) \in \mathbf{N}$ then $\left(\frac{p+1}{2^q}\right)_{LC} = \left(\frac{p+1}{2^q}\right)_{LR} = \left(\frac{p+1}{2^q}\right)_{L}$.

Theorem 6. *If* $x = \left(\frac{2p+1}{2^{q+1}}\right)_{LC}$ *is born the* b^{th} *day then* $\pi_R(x) = -b$.

Proof. By definition

$$\pi_R\left(\left(\frac{2p+1}{2^{q+1}}\right)_{LC}\right) = \left\{ \middle|\pi_R\left(\left(\frac{p}{2^q}\right)_{LC}\right), \pi_R\left(\left(\frac{p+1}{2^q}\right)_{LC}\right)\right\} \tag{18}$$

We observe that either $\left(\frac{p}{2^q}\right)_{LC}$ or $\left(\frac{p+1}{2^q}\right)_{LC}$ must be born the $(b-1)^{th}$ day therefore by the inductive hypothesis we have $\pi_R(x) = \{ \mid -(b-1)\} = -b$. □

Below we make eight observations

1. The first class contains only the number 0.
2. In the class $>_L 0$, $\pi_L(x)$ is positive, $\pi_C(x)$ and $\pi_R(x)$ are negative therefore we have an upper bound of $\frac{1}{8}(S_2[n]-1)^3$. Using Theorem 4 and the equality 12 we can refine this value obtaining $\frac{1}{24}(S_2[n]^2 - 1)S_2[n]$.

In contrast, we can express every triple $(i,-j,-k)$ by the number $\{(i-1)_L,(k-1)_C,(j-1)_R|\,|\,\}$ where $i-j \le 0, i-k \le 0$ and $i,j,k \in \mathbf{Z}^+$. Moreover, for every positive dyadic fraction we can create two different numbers $\left(\frac{2p+1}{2^{q+1}}\right)_{LC}$ and $\left(\frac{2p+1}{2^{q+1}}\right)_{LR}$ corresponding respectively to $\left(\frac{2p+1}{2^{q+1}}, -\frac{2p+1}{2^{q+1}}, -b\right)$ and $\left(\frac{2p+1}{2^{q+1}}, -b, -\frac{2p+1}{2^{q+1}},\right)$ where b is the day $\left(\frac{2p+1}{2^{q+1}}\right)$ was born. Summing up, we have a lower bound of $\frac{1}{6}n(n+1)(2n+1)+S_2[n]-2n-1$.

3. The classes $>_C 0$ and $>_R 0$ are analogous to the class $>_L 0$.
4. If $x =_{LC} 0$ then $x^R = \emptyset$ therefore $\pi_R(x) = \{\ |\pi_R(x^L), \pi_R(x^C)\} \in \mathbf{Z}^-$. If $x \in S_3[n]$ then by Theorem 2, $-n \le \pi_R(x)$ therefore n is an upper bound. Moreover, we observe that the number $\{\{\ |0|1_C\}|\{0|\ |1_L\}|\ \}$ corresponding to $(0,0,-1)$ belongs to $S_3[3]$ therefore the lower bound is n with $n > 2$.
5. The classes $=_{LR} 0$ and $=_{CR} 0$ are analogous to the class $=_{LC} 0$.
6. In the class $<_{CR} 0$, $\pi_L(x) = 0$, $\pi_C(x)$ and $\pi_R(x)$ are negative therefore we have an upper bound of $\frac{1}{4}(S_2[n]-1)^2$.
   In contrast, we can express every triple $(0,-j,-k)$ by the number $\{(k-1)_C,(j-1)_R|\ |\ \}$ where $j,k \in \mathbf{Z}^+$ with $j,\,k \ge 2$. Moreover, for every positive dyadic fraction we can create the number $\left(\frac{2p+1}{2^{q+1}}\right)_{LC} + \left(\frac{2p+1}{2^{q+1}}\right)_{RL}$ corresponding to $\left(0, -b-\frac{2p+1}{2^{q+1}}, -b+\frac{2p+1}{2^{q+1}}\right)$ where b is the day $\left(\frac{2p+1}{2^{q+1}}\right)$ was born. Summing up, we have a lower bound of $(n-1)^2+\frac{1}{2}(S_2[\lfloor n/2\rfloor]-2\lfloor n/2\rfloor-1)$.
7. The classes $<_{LR} 0$ and $<_{LC} 0$ are analogous to the class $<_{LR} 0$.
8. In the last class, we have an upper bound of $\frac{1}{8}(S_2[n]-1)^3$ because $\pi_L(x)$, $\pi_C(x)$, and $\pi_R(x)$ are all negative.
   We recall that
   (a) If $x <_{CR} 0 \in S_3[n-2]$ and $y = \{\ |1_R|1_C\} \in S_3[2]$ then $x+y < 0 \in S_3[n]$.
   (b) If $x <_{LR} 0 \in S_3[n-2]$ and $y = \{1_R|\ |1_L\} \in S_3[2]$ then $x+y < 0 \in S_3[n]$.
   (c) If $x <_{LC} 0 \in S_3[n-2]$ and $y = \{1_C|1_L|\ \} \in S_3[2]$ then $x+y < 0 \in S_3[n]$.
   To be sure that the sets of numbers given by (a), (b), and (c) are disjoint, we do not consider the numbers $x <_{CR} 0 \in S_3[n-2]$ corresponding to $(0,-j,-k)$ where either $j=2$ or $k=2$. Analogously, we do not consider the numbers $x <_{LR} 0 \in S_3[n-2]$ corresponding to $(-j,0,-k)$ where either $j=2$ or $k=2$ and the numbers $x <_{LC} 0 \in S_3[n-2]$ corresponding to $(-j,-k,0)$ where either $j=2$ or $k=2$. Therefore, we have a lower bound of $\frac{3}{2}S_2[\lfloor n/2\rfloor-1]+3n^2-24n-3\lfloor n/2\rfloor+\frac{99}{2}$.

Summarizing, we have an upper bound of

$$\frac{1}{4}S_2[n]^3+\frac{3}{8}S_2[n]^2-\frac{5}{4}S_2[n]+3n+\frac{13}{8}=O(S_2[n]^3) \tag{19}$$

and a lower bound of

$$3S_2[n]+\frac{3}{2}S_2[\lfloor n/2\rfloor]+\frac{3}{2}S_2[\lfloor n/2\rfloor-1]+ \\ n^3+\frac{15}{2}n^2-\frac{65}{2}n-6\lfloor n/2\rfloor+49=\Omega(S_2[n]) \tag{20}$$

Table 4 shows the results so far obtained but to establish the exact value of $S_3[n]$ as well as the canonical form of a three-player game is still an open problem.

Table 4. Results obtained so far

x	Lower bound	Upper bound
$= 0$	1	1
$>_L 0$	$S_2[n] + \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{11}{6}n - 1$	$\frac{1}{24}(S_2[n]^2 - 1)S_2[n]$
$>_C 0$	$S_2[n] + \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{11}{6}n - 1$	$\frac{1}{24}(S_2[n]^2 - 1)S_2[n]$
$>_R 0$	$S_2[n] + \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{11}{6}n - 1$	$\frac{1}{24}(S_2[n]^2 - 1)S_2[n]$
$=_{LC} 0$	n, $n > 2$	n
$=_{LR} 0$	n, $n > 2$	n
$=_{CR} 0$	n, $n > 2$	n
$<_{CR} 0$	$\frac{1}{2}S_2[\lfloor n/2 \rfloor] + n^2 - 2n - \lfloor n/2 \rfloor + \frac{1}{2}, n > 1$	$\frac{1}{4}(S_2[n] - 1)^2$
$<_{LR} 0$	$\frac{1}{2}S_2[\lfloor n/2 \rfloor] + n^2 - 2n - \lfloor n/2 \rfloor + \frac{1}{2}, n > 1$	$\frac{1}{4}(S_2[n] - 1)^2$
$<_{LC} 0$	$\frac{1}{2}S_2[\lfloor n/2 \rfloor] + n^2 - 2n - \lfloor n/2 \rfloor + \frac{1}{2}, n > 1$	$\frac{1}{4}(S_2[n] - 1)^2$
< 0	$\frac{3}{2}S_2[\lfloor n/2 \rfloor - 1] + 3n^2 - 24n - 3\lfloor n/2 \rfloor + \frac{99}{2}$, $n > 3$	$\frac{1}{8}(S_2[n] - 1)^3$

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