Algebraic Structures for Bipolar Constraint-Based Reasoning

Hélène Fargier[1] and Nic Wilson[2]

[1] IRIT, 118 route de Narbonne 31062 Toulouse Cedex, France
[2] Cork Constraint Computation Centre, University College Cork, Ireland
fargier@irit.fr, n.wilson@4c.ucc.ie

Abstract. The representation of both scales of cost and scales of benefit is very natural in a decision-making problem: scales of evaluation of decisions are often bipolar. The aim of this paper is to provide algebraic structures for the representation of bipolar rules, in the spirit of the algebraic approaches of constraint satisfaction. The structures presented here are general enough to encompass a large variety of rules from the bipolar literature, as well as having appropriate algebraic properties to allow the use of CSP algorithms such as forward-checking and algorithms based on variable elimination.

1 The Introduction

Soft constraints frameworks usually consider that preferences are expressed in a negative way. For instance, Valued Constraint Satisfaction Problems (VCSPs, [16]) aim at minimising an increasing combination of the violation costs provided by the constraints. This also is the case for all instances of semiring-based CSPs [3], where the combination of the successive valuations provided by the constraints decreases (worsens) the global evaluation. But problems often also contain positive preference constraints which increase the global satisfaction degree, so it is desirable to extend constraints approaches to such situations. For example, if one is choosing a holiday apartment, one has to balance the (positive) benefits of a decision, such as having a sea view, against the (negative) monetary cost. This bipolar characteristic of the preferences in CSPs has recently been advocated by Bistarelli, Pini, Rossi and Venable [4,14,15]. Bipolarity is also an important focus of research in several domains, e.g. psychology [17,18], multicriteria decision making [8,9], and more recently in AI: argumentation [1] and qualitative reasoning [10,2,7].

There are basically two ways of representing a bipolar notion on a scale. The first one is the so called *univariate model* proposed by Osgood et al [13]. It consists of a scale with a central neutral element ranging from negative values (below the neutral element) to positive values (higher than the neutral element). This kind of model has recently been introduced into constraint programming in [4,14,15]. Unlike this first model the *bivariate model* introduced by Cacioppo and al. [5] (see for instance [9,8,7]) does not use one but two scales; this can

K. Mellouli (Ed.): ECSQARU 2007, LNAI 4724, pp. 623–634, 2007.

be pictured with a horizontal axis encoding the intensity of positive values, and the vertical axis the intensity of the negative ones. Thus the evaluation is not necessarily totally positive nor totally negative, but can have both positive and negative components. The original motivation for such a model comes from the fact that a subject may feel at the same time a positive response and a negative one for the same characteristic of an object. For a house, being close to a bus station is both good (time is saved) and bad (it is noisy).

The aim of the present paper is to provide algebraic structures for the representation of bivariate bipolar rules, in the spirit of the algebraic approaches of constraint satisfaction. This is to enable combinatorial optimisation over expressive languages of constraints, where both costs and benefits can be expressed. The structure should be rich enough to encompass a large variety of rules from the bipolar literature; but it should have appropriate algebraic properties to allow the use of soft CSP algorithms. The next section discusses classes of bipolar decision rules. Section 3 describes our basic algebraic structure and shows how to represent some decision rules from the literature using this; special subclasses are also examined. Section 4 describes bipolar systems of constraints and a forward checking algorithm for optimisation. In Section 5 we define a richer algebraic structure, bipolar semirings, which allows more complex propagation algorithms.

2 Bipolar Decision Rules

The purpose of a bipolar decision making procedure is to provide a comparison relation $\succeq$ between alternatives, given, for each alternative d, a multiset $P(d)$ of positive evaluations and a multiset set $N(d)$ of negative ones. In the context of preference-based CSPs, $N(d)$ corresponds to preference valuations provided by some negative constraints, as in fuzzy CSPs and more generally, semiring-based CSPs, and $P(d)$ corresponds to the positive valuations provided by reward-based constraints. The basic property of bipolar decision processes is that the bigger $P(d)$ (respectively, $N(d)$) is, the better (resp., worse) d is:

$$P(d') \subseteq P(d) \text{ and } N(d') \supseteq N(d) \implies d \succeq d'$$

Cumulative prospect theory [18] adds to this “bimonotonicity” axiom a second principle: $P(d)$ and $N(d)$ must be separately evaluated by means of two functions that provide an overall positive degree $p(d)$ and an overall negative degree $n(d)$. According to bimonotonicity, p should be maximised and n minimised.

2.1 Univariate Models

Such models represent a situation where p and n are on the same scale and the decision strategy can be modelled by computing a *net predisposition*: $\mathrm{NP}(d) = f(p(d), n(d))$, where f is increasing in its first argument and decreasing in its second. Alternatives d are then ranked increasingly with respect to NP. The most famous example is based on an aggregation by a sum:

$$\mathrm{NP}^{+}(d) = p(d) - n(d) = \sum_{v \in P(d)} v - \sum_{v \in N(d)} v.$$

Another example is provided by qualitative reasoning [10]:

$$\mathrm{NP}^{qual}(d) = min(p(d), 1 - n(d))$$

where $p(d) = \max_{v \in P(d)} v$ and $n(d) = \max_{v \in N(d)} v$. More generally, we can consider that $p(d)$ and $n(d)$ are obtained by monotonic and associative combinations of the valuations they contain, namely by a pair of t-conorms[1] $(\otimes^{+}, \otimes^{-})$: $p(d) = \bigotimes^{+}_{v \in P(d)} v$ and $n(d) = \bigotimes^{-}_{v \in N(d)} v$. It should be noticed that $\otimes^{+}$ and $\otimes^{-}$ can be different from each other—for some subjects, their combination of positive effects is more or less isomorphic to a sum, while for the negative scale, the worst value is taken, i.e. $\otimes^{-} = max$. The NP model thus encompasses more than just the simple additive rule. In [4], net predisposition is generalised to semiring-valued constraints through use of (i) two semirings, one, L^{+}, for representing positive degrees of preference, and the other, L^{-}, for representing negative degrees of preference, equipped with their respective multiplications $\otimes^{+}$ and $\otimes^{-}$ and (ii) an operator $\otimes$ defined within $L^{+} \cup L^{-}$ for combining positive and negative elements. The framework then aims at maximising $(\bigotimes^{+}_{v \in P(d)} v) \otimes (\bigotimes^{-}_{v \in N(d)} v)$.

2.2 Bivariate Models

Since they are fundamentally single-scaled, univariate models are not well suited to the representation of all decision making situations. For instance, a conflicting set whose strongest positive argument is equally strong as its strongest negative argument is often difficult to rank (see e.g. [17]). Since a univariate model aggregates a positive and a negative value into either a positive or a negative value, and since such scales are totally ordered, it cannot account for situations of incomparability. Hence the necessity of bivariate models as first proposed by [5] (see also [8]). As discussed in the introduction, a second reason is the necessity of taking into account arguments that have both a positive and a negative aspect. Classical examples of such rules are provided by Pareto rules. In contrast to net predisposition, these do not make any aggregation of p and n, but rather consider that each of the two dimensions is a criterion and that the scales of the criteria are not commensurate. Decision is then made on the basis of a Pareto comparison:

$$\text{Pareto: } d \succeq d' \iff p(d) \geq p(d') \text{ and } n(d) \leq n(d')$$

Letting $\otimes^{+} = \otimes^{-} = max$, one recovers the qualitative rule proposed in [7], but once again, $\otimes^{+}$ and $\otimes^{-}$ can be two different t-conorms.

The Pareto ordering is obviously rather weak, and it is natural to strengthen it by adding extra orderings to represent tradeoffs. For example, in a Pareto

[1] A t-conorm is an increasing associative and commutative operation on some ordered scale $L = [0_L, 1_L]$ with 0_L as unit element and 1_L as absorbing element. We formulate here the rules in the way they apply in a constraint-based setting. Some of them admit a more general definition accounting for non-independent arguments.

system with both scales being $\{0, 1, 2, \dots\} \cup \{\infty\}$, and both combinations being addition, we might add extra orderings such as $(1, 3) \succeq (0, 0)$. The new ordering $\succeq$ is then defined to be the smallest transitive relation which (i) extends both the Pareto ordering and the extra orderings and (ii) satisfies the property that $\otimes$ is monotone over $\succeq$ (see Definition 1 below). In this example we could deduce using the monotonicity property also that $(2, 4) \succeq (1, 1)$. Other instances of the bivariate model in [5] are provided by qualitative reasoning, namely the order of magnitudes formalism in [19] and the “bilexi” qualitative rule in [7].

3 Bipolar Valuation Structures

The constituent elements of a bipolar framework should be a set of valuations A containing a subset of positive valuations (say, A^+), and a set of negative valuations (say, A^-), a combination operator $\otimes$ and a comparison relation[2] $\succeq$ on A. $\succeq$ is a partial order (i.e., $\succeq$ is reflexive, antisymmetric and transitive, but need not be complete).

A^- contains a worst element, say $\bot$ (which could be received upon the violation of some hard constraint), and A^+ contains a best element $\top$ (which could be received upon the ideal satisfaction of the goal(s)). Both share the *neutral* or “indifferent” valuation, that should not modify the evaluation of a decision.

We also will need algorithms for optimisation in the combinatorial case, e.g. branch and bound algorithms. This further restricts the algebraic framework we are looking for: $\otimes$ should not be sensitive to the order in which the constraints are considered, so is assumed to be commutative and associative; it also should be monotonic w.r.t. $\succeq$.

3.1 Definition and Basic Properties

Definition 1. *A bipolar valuation structure is a tuple $\mathcal{A} = \langle A, \otimes, \succeq \rangle$ where:*

- *$\succeq$ is a (possibly partial) order on A with a unique maximum element $\top$ and a unique minimum element $\bot$ (so for all $a \in A$, $\top \succeq a \succeq \bot$);*
- *$\otimes$ is a commutative and associative binary operation on A with neutral element $\mathbf{1}$ (for all $a \in A$, $a \otimes \mathbf{1} = a$); furthermore $\otimes$ is monotone over $\succeq$: if $a \succeq b$ then for all $c \in A$, $a \otimes c \succeq b \otimes c$.*

Notice that the assumption of the existence of elements $\top$ and $\bot$ is not restrictive. If A does not contain them then we can add them whilst maintaining the properties of $\otimes$, $\succeq$, $\mathbf{1}$.

An element a is said to be *positive* if $a \succeq \mathbf{1}$, and it is said to be *negative* if $a \preceq \mathbf{1}$. We write the set of positive elements of A as A^+, and the set of negative elements as A^-. The following proposition gives some basic properties.

Proposition 1. *Let $\mathcal{A} = \langle A, \otimes, \succeq \rangle$ be a bipolar valuation structure. Then*

(i) $\otimes$ is increasing (resp. decreasing) with respect to positive (resp. negative) elements: if $a \in A$ and $p \succeq \mathbf{1} \succeq n$ then $a \otimes p \succeq a \succeq a \otimes n$;

[2] Given a relation $\succeq$, we use $\succ$ to mean the strict part of $\succ$, so that $a \succ b$ if and only if $a \succeq b$ and $b \not\succeq a$ (i.e., $b \succeq a$ does not hold).

(ii) $\perp$ *(resp.* $\top$*) is an absorbing element in* A^- *(resp.* A^+*);*
(iii) for all $p \in A^+$ *and* $n \in A^-$*,* $p \otimes n \succeq n$ *and* $p \succeq p \otimes n$*, so that* $p \otimes n$ *is between* n *and* p*.*

(i) is a key property for bipolar systems, related to bimonotonicity mentioned above. The third property follows from (i) and means that $p \otimes n$ is somewhere between p and n—but it does not imply that $p \otimes n$ is either positive or negative. It may happen that neither $p \otimes n \succeq \mathbf{1}$ nor $\mathbf{1} \succeq p \otimes n$: the set A can contain more elements than purely positive and purely negative ones, which gives it the ability to represent conflicting values that have both a positive and a negative component.

Define A^{PN} to be the set of all those elements of A which can be written as a combination of a positive and a negative element, i.e.,

$$A^{PN} = \{a \in A \,:\, a = p \otimes n, \quad n \preceq \mathbf{1} \preceq p\}.$$

A^{PN} contains A^+, A^- and all the valuations that are obtained by combining positive and negative values: it is the core of the bipolar representation.

Proposition 2. *Let* $\mathcal{A} = \langle A, \otimes, \succeq \rangle$ *be a bipolar valuation structure, then* A^-*,* A^+ *and* A^{PN} *are each closed under* $\otimes$*. Moreover,* A^{PN} *contains* $A^+ \cup A^-$*, including* $\mathbf{1}, \perp$ *and* $\top$*. Hence* $\langle A^{PN}, \otimes, \succeq \rangle$ *is also a bipolar valuation structure.*

Definition 2
A bipolar structure is bivariate *iff* $A = A^{PN}$*. It is* univariate *iff* $A = A^+ \cup A^-$*.*

In particular, in a univariate system, the combination of a positive element and a negative element is always comparable to the neutral element.

The framework of bipolar valuation structures is general enough to allow valuations outside A^{PN}, but they do not have such a simple interpretation in terms of positive and negative values. Since we are interested in the representation of bipolarity, we focus the paper on bivariate systems (which includes univariate systems).

3.2 Examples

Additive net predisposition For representing NP^+ we will use $\mathcal{A} = \mathbb{R} \cup \{-\infty, +\infty\}$ with $\otimes = +$ and $\succeq = \geq$. So, the neutral element $\mathbf{1}$ equals 0, $A^+ = \mathbb{R}^+ \cup \{+\infty\}$, $A^- = \mathbb{R}^- \cup \{-\infty\}$. We define $-\infty \otimes +\infty = -\infty$, since getting a conflict is very uncomfortable and should be avoided. However, in practice, $+\infty$ is never allocated by any constraint.

Pareto: Pareto$^{\otimes^-, \otimes^+}$ denotes any Pareto rule built from two t-conorms $\otimes^-$ and $\otimes^+$, respectively in $L^- = [0^-, 1^-]$ and $L^+ = [0^+, 1^+]$. The combination is performed pointwise (using the two conorms) and pair (n, p) is preferred to (n', p') if and only if it is better on each co-ordinate. The encoding of such a rule is done using the a product structure: $\mathcal{A} = \langle L^- \times L^+, (\otimes^-, \otimes^+), \succeq^{par} \rangle$ with $\mathbf{1} = (0^-, 0^+), \perp = (1^-, 0^+), \top = (0^-, 1^+)$ where $\succeq^{par}$ is simply defined by the Pareto principle: $(n,p) \succeq^{par} (n',p') \iff n \leq n'$ and $p \geq p'$. As a particular case, the qualitative $Pareto^{max}$ rule corresponds to the structure $\mathcal{A} = \langle [0,1] \times [0,1], (\max, \max), \succeq^{par} \rangle$ with $\mathbf{1} = (0,0)$, $\bot = (1,0)$, $\top = (0,1)$.

Additive net prediposition is obviously univariate. The rules of the form $Pareto^{\otimes^+,\otimes^-}$ are not univariate but bivariate. So also is the following rule.

Order of magnitude calculus (OOM): In the system of order of magnitude reasoning described in [19], the elements are pairs $\langle \sigma, r \rangle$ where $\sigma \in \{+,-,\pm\}$, and $r \in \mathbb{Z} \cup \{\infty\}$. The system is interpreted in terms of "order of magnitude" values of utility, so, for example, $\langle -, r \rangle$ represents something which is negative and has order of magnitude K^r (for a large number K). Element $\langle \pm, r \rangle$ arises from the sum of $\langle +, r \rangle$ and $\langle -, r \rangle$. $\langle \pm, r \rangle$ can be thought of as the interval between $\langle -, r \rangle$ and $\langle +, r \rangle$, since the sum of a positive quantity of order K^r and a negative quantity of order K^r can be either positive or negative and of any order less than or equal to r. Let $A_{oom} = \{\langle \pm, -\infty \rangle\} \cup \{\langle \sigma, r \rangle : \sigma \in \{+,-,\pm\}, r \in \mathbb{Z} \cup \{+\infty\}\}$. We write also $\langle -, +\infty \rangle$ as $\bot$, and $\langle +, +\infty \rangle$ as $\top$.

The interpretation leads to defining $\otimes$ by: $\langle \sigma, r \rangle \otimes \langle \sigma', r' \rangle = \langle \sigma, r \rangle$ if $r > r'$; it's equal to $\langle \sigma', r' \rangle$ if $r < r'$; and is equal to $\langle \sigma \oplus \sigma', r \rangle$ if $r = r'$, where $\oplus$ is given by: $+ \oplus + = +$ and $- \oplus - = -$, and otherwise, $\sigma \oplus \sigma' = \pm$. Operation $\otimes$ is commutative and associative with neutral element $\langle \pm, -\infty \rangle$. $\succeq$ is defined by the following instances:[3] (i) for all r and s, $\langle +, r \rangle \succeq \langle -, s \rangle$; (ii) for all $\sigma \in \{+,-,\pm\}$, and all r, r' with $r \geq r'$: $\langle +, r \rangle \succeq \langle \sigma, r' \rangle \succeq \langle -, r \rangle$. The relation $\succeq$ is a partial order with unique minimum element $\bot$ and unique maximum element $\top$. The positive elements and the negative elements are both totally ordered, and $A_{oom} = A^{PN}$. However, there are incomparable elements, e.g. $\langle \pm, r \rangle$ and $\langle \pm, s \rangle$ when $r \neq s$.

3.3 Important Subclasses of Bipolar Valuation Structures

Below we discuss some properties and special kinds of bipolar structures.

Unipolar scales: First of all, let us say that $\mathcal{A}$ is purely positive (resp., purely negative) iff $A = A^+$ (resp. $A = A^-$). In such a structure, $\bot = \mathbf{1}$ (resp. $\top = \mathbf{1}$). The most classical example is provided by semiring-based CSPs where $A = A^-$, while purely positive preference structures are considered in [4].

Totally ordered scales: In most of the bipolar rules encountered in the literature, $\succeq$ is complete on $A^+ \cup A^-$, e.g. NP^+, $Pareto^{max,max}$ and A_{oom}. Unless the structure is univariate, this does not imply that $\succeq$ is complete, but that the restriction of $\otimes$ on A^+ (resp. A^-) is a t-conorm (resp. a t-norm).

Strict monotonicity: $\mathcal{A} = \langle A, \otimes, \succeq \rangle$ is said to be *strictly monotonic* if for all $a, b \in A$ and for all $c \neq \top, \bot$, we have $a \succ b \Rightarrow a \otimes c \succ b \otimes c$. Qualitative rules based on max and min operations are not strictly monotonic, while addition-based frameworks often are. Failure of strict monotonicity corresponds to the

[3] This definition is slightly stronger than the one in [19], which doesn't allow $\langle +, r \rangle \succeq \langle \pm, r \rangle \succeq \langle -, r \rangle$; either order can be justified, but our choice has better computational properties.

well known "drowning effect": without strict monotonicity, it may happen that a decision d is not necessarily strictly preferred to d' even though it is strictly preferred to d' by all constraints apart from one that judges them equally.

Idempotent structures: An element $a \in A$ is said to be *idempotent* if $a \otimes a = a$, and $\otimes$ is said to be idempotent if every element of $\mathcal{A}$ is idempotent. The idempotence of $\otimes$ is very useful for having simple and efficient constraint propagation algorithms. Idempotence, which is at work e.g. in $Pareto^{max,\max}$, A_{oom} and in many unipolar structures (e.g. fuzzy CSPs), induces the drowning effect. Naturally, idempotence and strict monotonicity are highly incompatible properties. The range of compatibility of idempotence with a univariate scale is also very narrow—it reduces the structure to a very special form:

Proposition 3. *If bipolar valuation structure $\mathcal{A}$ is idempotent and univariate, then for all $p \in A^+$ and $n \in A^-$, either $p \otimes n = p$ or $p \otimes n = n$.*

Invertibility: The notion of cancellation is captured by the property of invertibility. Element a is said to be *invertible* if there exists element $b \in A$ with $a \otimes b = \mathbf{1}$. A structure is said to be *invertible* if every element in $A - \{\top, \bot\}$ is invertible. $A - \{\top, \bot\}$ then forms a commutative group under $\otimes$. This property is important for the framework in [4,14,15] and fits well with univariate scales. For instance, it is easy to show that when $\succeq$ is complete on A^+, invertibility is a sufficient condition for making a bivariate system univariate.

On the other hand, associativity implies that $\mathbf{1}$ is the only element a which is both idempotent and invertible, since if $a \otimes b = \mathbf{1}$ then $a = a \otimes \mathbf{1} = a \otimes (a \otimes b) = (a \otimes a) \otimes b = a \otimes b = \mathbf{1}$. This means that when $\otimes$ is idempotent a positive argument can never be exactly cancelled by a negative argument: invertibility is strongly related to strict monotonicity.

Proposition 4. *If bipolar valuation structure $\mathcal{A}$ is invertible then it is strictly monotonic.*

This problem is avoided in [4,14,15] by not assuming associativity on their univariate scale. But invertibility should not be considered as a norm, and bivariate systems are generally not invertible.

4 Bipolar Constraints and Optimisation

4.1 Bipolar Systems of Constraints

Let X be a set of variables, where variable $x \in X$ has domain $D(x)$. For $U \subseteq X$, we define $D(U)$ to be the set of all possible assignments to U, i.e., $\prod_{x \in U} D(x)$.

Let $\mathcal{A} = \langle A, \otimes, \succeq \rangle$ be a bipolar valuation structure. An $\mathcal{A}$-*constraint* φ [over X] is defined to be a function from $D(s_\varphi)$ to A, where s_φ, the *scope* of φ, is a subset of variables associated with φ. We shall also refer to φ as a *bipolar constraint.*

Definition 3. *A bipolar system of constraints, over a bipolar valuation structure $\mathcal{A}$, is a triple (X, D, C) where X is a set of variables, D the associated domains and C a multiset of $\mathcal{A}$-constraints over X.*

Bipolar constraint φ allocates a valuation $\varphi(d)$ to any assignment d to its scope. More generally, if d is an assignment to a superset of s_φ, and e is the projection of d to s_φ, then we define $\varphi(d)$ to be $\varphi(e)$. For any assignment d of X, the *bipolar evaluation of d* is $val(d) = \bigotimes_{\varphi \in C} \varphi(d)$.

Many requests can be addressed to a bipolar system of constraints. The most classical one, the optimisation request, searches for one undominated solution: d is undominated if and only if there does not exist any d' such that $val(d') \succ val(d)$. Variants include the search for several (or all the) undominated solutions. The associated decision problem, for a given bipolar valuation structure $\mathcal{A} = \langle A, \otimes, \succeq \rangle$ can be written as:

[BCSP$_\mathcal{A}$]: *Given a bipolar system of constraints over $\mathcal{A}$ and $a \in A$, does there exist an assignment d such that $val(d) \succ a$.*

Proposition 5. *Let $\mathcal{A} = \langle A, \otimes, \succeq \rangle$ be a bipolar valuation structure. Suppose that testing $b \succ a$ is polynomial, and computing the combination of a multiset of elements in A is polynomial. Suppose also that A contains at least two elements. Then BCSP$_\mathcal{A}$ is* NP*-complete.*

Indeed, given these assumptions, for any $\mathcal{A}$, the problem BCSP$_\mathcal{A}$ is in NP, since we can guess assignment d, and test $val(d) \succ a$ in polynomial time. It is NP-hard if A has more than one element since then $\top$ and $\bot$ must be different and so either $\mathbf{1} \neq \bot$ or $\mathbf{1} \neq \top$ (or both); in either case we can use a reduction from 3SAT, by considering bipolar constraints which only take two different values: $\mathbf{1}$ and either $\bot$ or $\top$.

4.2 Forward Checking Algorithm

This section describes a generalization of the Forward Checking algorithm for finding an undominated complete assignment in bipolar systems of constraints. For the sake of brevity, we assume that all the constraints in C are either unary or binary; however, it is not hard to modify the algorithm to be able to deal with constraints of higher arity.

We assume that we have implemented a function $\mathtt{UB}(S)$ that, given a finite subset S of A, returns some upper bound of them (with respect to $\succeq$). $\mathtt{UB}(S)$ might be implemented in terms of repeated use of a function $\vee(\cdot, \cdot)$ where $\vee(a, b)$ is an upper bound of both a and b (i.e., $\vee(a, b) \succeq a, b$). For example, if least upper bounds exist, we can set $\vee(a, b)$ to be some least upper bound of a and b. In particular, if $\succeq$ is a total order, we can use max. However, very often $\succeq$ is not a total order, e.g., for the Pareto rule. For each constraint φ we write also $\mathtt{UB}(\varphi)$ for $\mathtt{UB}(\{\varphi(d) : d \in D(s_\varphi)\})$, i.e. an upper bound over the values of φ. $\mathtt{UB}(\varphi)$ is an important parameter: in a bipolar system of constraints, future constraints cannot be neglected, since they can increase the current evaluation. In other terms, setting $\mathtt{UB}(\varphi) = \mathbf{1}$ is not sound—while $\mathtt{UB}(\varphi) = \top$ is sound but generally inefficient, both because $\top$ is generally not provided by any constraint (nothing is perfect) and because it is not far from being absorbing (and is so on A^+). In practice, for each φ, $\mathtt{UB}(\varphi)$ can be pre-computed.

The handling of $\mathtt{UB}(\varphi)$ is the main difference between classical Forward Checking and bipolar Forward Checking. The structure of the algorithm is very classical: the top level procedure, `BestSol`, returns global parameter d^*, which will then be an undominated solution, and global parameter b^* which equals $val(d^*)$. The algorithm performs a tree search over assignments, pruning only when there can be no complete assignment below this point with better *val* than the current best valuation b^* (which is initialised as $\bot$).

Without loss of generality, we assume that for each $x \in X$ there exists exactly one unary constraint φ_x on x (if there exists more, we can combine them; if there exists none, we can set $\varphi_x(v) = \mathbf{1}$ for all $v \in D(x)$). The algorithm involves, for each variable x, a unary constraint μ_x, which is initially set to being equal to φ_x. The backtracking is managed with the help of two procedures: `StoreDomainsUnary`(i) takes a backup copy of the variable domains and the values of the unary constraints μ_x at tree depth i; `RestoreDomainsUnary`(i) restores them as they were at point `StoreDomainsUnary`(i).

We write an assignment d to a set of n variables as a set of assignments $x := v$. In particular, $\{\}$ designates the assignment to the empty set of variables.

procedure `BestSol`
$b^* := \bot$
`for all variables` x`, for all` $v \in D(x)$`, set` $\mu_x(v) := \varphi_x(v)$
`FC(`0`,` $\{\}$`,` $\mathbf{1}$`)`
`Return` d^* `and` b^*

procedure `FC`$(i, d, \mathit{CurrentVal})$
`If` $i = n$ `then`
  `if` $\mathit{CurrentVal} \succ b^*$ `then` $b^* := \mathit{CurrentVal}$; $d^* := d$
`Else`
  `Choose an unassigned variable` x
  `StoreDomainsUnary(`i`)`
  `For all` v `in` $D(x)$
    `If PropagateFC`$(x, v, \mathit{CurrentVal})$ `then FC`$(i+1, d \cup \{x := v\}, \mathit{CurrentVal} \otimes \mu_x(v))$
    `RestoreDomainsUnary(`i`)`

boolean function `PropagateFC`$(x, v, \mathit{CurrentVal})$
$\mathit{PastVal} := \mathit{CurrentVal} \otimes \mu_x(v)$
$\mathit{futureConstr} = \{\varphi$ `linking two unassigned variables`$\}$
$\mathit{FutureVal} := \bigotimes_{\varphi \in \mathit{futureConstr}} \mathtt{UB}(\varphi)$
`// Propagate forward on the future variables:`
`For all` φ `linking` x `to an unassigned variable` y
  `For all values` v' `in` $D(y)$
    `set` $\mu_y(v') := \mu_y(v') \otimes \varphi(x = v, y = v')$
  $\mathit{Upper}_y := \mathtt{UB}(\{\mu_y(v') : v' \in D(y)\})$

```
// Pruning the domains
For all unassigned variables y and all v' ∈ D(y)
   VarsVal_y := ⊗_{y' unassigned, y'≠y} Upper_{y'}
   UppBd_y(v') := μ_y(v') ⊗ VarsVal_y ⊗ PastVal ⊗ FutureVal
   If not(UppBd_y(v') ≻ b*) remove v' from D(y)
   If D(y) = ∅ then return FALSE (and exit PropagateFC)
Return TRUE
```

The soundness of the pruning condition is ensured by the monotonicity of $\otimes$ and the transitivity of $\succeq$. But it can also be sound in some structures that do not fulfill these conditions. In particular, even if $\otimes$ is not monotone over $\succeq$ then the algorithm will still be correct if operator $\vee$ ensures that $\forall c \in A$, $\vee(a, b) \otimes c$ is an upper bound of $a \otimes c$ and $b \otimes c$.

The family of Forward Checking algorithms includes more complex versions than the one extended here, e.g. Reversible Directional Arc Consistency(RDAC) and other improvements [12]. The algebraic structure presented in Section 3 is rich enough to allow them to work soundly. But more sophisticated algorithms for constraint optimisation, which use more complex constraint propagation (e.g. using variable elimination), require more than a simple upper bound operator. This is the topic of the next section.

5 Bipolar Semirings

An important computational technique for multivariate problems (such as CSPs) is sequential variable elimination (bucket elimination). This calls for the structure to be rich enough to allow the definition of an internal operator $\vee$ that not only provides an upper bound of its operands (and thus admits $\top$ as absorbing element and $\bot$ as a neutral element) but is also assumed to be associative, commutative and idempotent. Unsurprisingly, the kind of structure needed is a semiring, but of a more general form than the semirings usually used in constraint programming. A (commutative) semiring is a set A endowed with two operations $\vee$ and $\otimes$ which are both commutative and associative and such that $\otimes$ distributes over $\vee$.

Definition 4. *A bipolar semiring is a tuple $\langle A, \otimes, \vee, \succeq \rangle$ where: $\langle A, \otimes, \succeq \rangle$ is a bipolar monotonic valuation structure; $\vee$ is an associative, commutative and idempotent operation on A with neutral element $\bot$ and absorbing element $\top$, satisfying:*

Distributivity: for all $a, b, c \in A$, $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$;
Monotonicity of $\vee$ over $\succeq$: i.e., $a \succeq b \implies a \vee c \succeq b \vee c$.

Notice that, since $a \succeq \bot$, and $\vee$ is monotone over $\succeq$, we have $a \vee b \succeq \bot \vee b = b$. We therefore have the following:

Proposition 6. *Let $\langle A, \otimes, \vee, \succeq \rangle$ be a bipolar semiring. Then for any $a, b \in A$, $a \vee b \succeq a, b$.*

Hence Definition 4 implicitly requires $a \vee b$ to be an $\succeq$-upper bound for a and b, which is an important property for branch-and-bound and variable elimination algorithms. When $\succeq$ is a total order, finding a suitable $\vee$ is immediate: choose $\vee = \max$. When $\succeq$ is an upper semi-lattice, $a \vee b$ will be the least upper bound of a and b. For instance, when A is the product of a totally ordered positive scale and a totally ordered negative scale, as in the Pareto case, we can use pointwise application of maximum. In the *OOM* framework $\langle \sigma, r \rangle \vee \langle \sigma', r' \rangle$ is the better of the two elements if they are comparable; otherwise, their least upper bound is equal to $\langle +, \max(r, r') \rangle$. It can be shown that $\langle A_{oom}, +, \vee, \succeq, \rangle$ is a bipolar semiring.

Importantly, semiring properties are sufficient for variable elimination to be correct (see e.g., [11]). Hence Definition 4 enables the use of such methods within a branch and bound tree search as a way of generating global upper bounds of a set of bipolar constraints (in particular, as a way to compute a stronger value of *FutureVal* in the above algorithm). However, sequential variable elimination is only practical in certain situations, in particular, if the scopes of the constraints are such that the treewidth is small. Otherwise one can use a mini-buckets [6] approach for generating an upper bound of the least upper bound, since it has been shown that sufficient conditions for this technique to be applicable to general soft constraints, are that $\mathcal{A}$ forms a semiring, the two operators are monotone over the ordering, and $a \vee b \succeq a, b$ for all $a, b \in A$.

Notice that $\vee$ itself defines a comparison relation $\succeq_\vee$ on A, as in semiring-based CSPs: for all $a, b \in A$, $a \succeq_\vee b \iff a \vee b = a$. It follows that for any $a \in A$, we have $\bot \preceq_\vee a \preceq_\vee \top$ and that $\vee$ and $\otimes$ are monotone with respect to $\succeq_\vee$. Hence if $\langle A, \otimes, \vee, \succeq \rangle$ is a bipolar semiring then $\langle A, \otimes, \vee, \succeq_\vee \rangle$ is as well. It is also easy to show that $\succeq_\vee$ is a partial order (it is antisymmetric) but is not necessarily complete. Moreover, by Proposition 6, $a \succeq_\vee b \Rightarrow a \succeq b$. Hence if is optimal (i.e. non dominated) with respect to $\succeq$ then it is optimal w.r.t. $\succeq_\vee$.

6 Conclusion

The representation of both scales of cost and scales of benefit is very natural in a decision-making problem. We have abstracted the kind of properties assumed in such bipolar reasoning to produce algebraic valuation structures which, firstly, allow the representation of many natural forms of bipolar reasoning, and secondly, have sufficient structure to allow optimisation algorithms. As well as bipolar univariate models, our framework can also represent bivariate models for bipolar reasoning, which allow the kind of incomparability found in many natural systems for reasoning with positive and negative degrees of preference.

This paper has proposed a generalization of the forward checking algorithm for handling the optimization in bipolar structures. This algorithm actually applies to rather general algebraic structures, even to structures similar to bipolar valuation structures but which are not fully monotonic.

Acknowledgements. This material is based partly on works supported by the Science Foundation Ireland under Grant No. 00/PI.1/C075 and also Grant No. 05/IN/I886, and benefitted from the Enterprise Ireland Ulysses travel grant No. FR/2006/36.

References

1. Amgoud, L., Bonnefon, J.F., Prade, H.: An argumentation-based approach to multiple criteria decision. In: Godo, L. (ed.) ECSQARU 2005. LNCS (LNAI), vol. 3571, pp. 10–17. Springer, Heidelberg (2005)
2. Benferhat, S., Dubois, D., Kaci, S., Prade, H.: Bipolar representation and fusion of preferences on the possibilistic logic framework. In: Proc. KR2002, pp. 421–448 (2002)
3. Bistarelli, S., Montanari, U., Rossi, F.: Semiring-based Constraint Solving and Optimization. JACM 44(2), 201–236 (1997)
4. Bistarelli, S., Pini, M.S., Rossi, F., Venable, K.B.: Positive and negative preferences. In: Proc. CP 2005 workshop on preferences and soft constraints (Soft 2005) (2005)
5. Cacioppo, J.T., Gardner, W.L., Berntson, G.G.: Beyond bipolar conceptualizations and measures: The case of attitudes and evaluative space. Personality and Social Psychology Review 1, 3–25 (1997)
6. Dechter, R., Rish, I.: Mini-buckets: A general scheme for bounded inference. J. ACM 50(2), 107–153 (2003)
7. Dubois, D., Fargier, H.: Qualitative decision making with bipolar information. In: KR'06, pp. 286–297 (2006)
8. Grabisch, M., Labreuche, Ch.: Bi-capacities — parts I and II. Fuzzy Sets and Systems 151(2), 211–260 (2005)
9. Greco, S., Matarazzo, B., Slowinski, R.: Bipolar Sugeno and Choquet integrals. In: Proc. EUROFUSE 2002 Workshop on Information Systems (2002)
10. Kaci, S., Prade, H.: Bipolar goals. a possibilisitc logic characterization of preferred choices. In: Proc. ECAI'04 Workshop on local computation for logics and uncertainty, pp. 13–18 (2004)
11. Kohlas, J.: Valuation algebras induced by semirings. Technical Report 04-03, Department of Informatics, University of Fribourg (2004)
12. Larrosa, J., Meseguer, P., Schiex, T., Verfaillie, G.: Reversible DAC and other improvements for solving max-CSP. In: Proc. AAAI'96, pp. 347–352 (1998)
13. Osgood, C.E., Suci, G.J., Tannenbaum, P.H.: The Measurement of Meaning. University of Illinois Press, Chicago (1957)
14. Pini, M.S., Rossi, F., Venable, K.B., Bistarelli, S.: Bipolar preference problems. In: Proc. ECAI 2006 (poster paper) (2006)
15. Pini, M.S., Rossi, F., Venable, K.B., Bistarelli, S.: Modelling and solving bipolar preference problems. In: Proc. 2006 ERCIM workshop on constraints (2006)
16. Schiex, T., Fargier, H., Verfaillie, G.: Valued constraint satisfaction problems: Hard and easy problems. In: Proc. IJCAI'95, pp. 631–637 (1995)
17. Slovic, P., Finucane, M., Peters, E., MacGregor, D.G.: Rational actors or rational fools? Implications of the affect heuristic for behavioral economics. The Journal of Socio-Economics 31, 329–342 (2002)
18. Tversky, A., Kahneman, D.: Advances in prospect theory: Cumulative representation of uncertainty. Journal of Risk and Uncertainty 5, 297–323 (1992)
19. Wilson, N.: An order of magnitude calculus. In: Proc. UAI'95, pp. 548–555 (1995)