On the Support Size of Stable Strategies in Random Games*

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Abstract. In this paper we study the support sizes of evolutionary stable strategies (ESS) in random evolutionary games. We prove that, when the elements of the payoff matrix behave either as uniform, or normally distributed independent random variables, almost all ESS have support sizes $o(n)$, where n is the number of possible types for a player. Our arguments are based exclusively on the severity of a *stability* property that the payoff submatrix indicated by the support of an ESS must satisfy. We then combine our normal–random result with a recent result of McLennan and Berg (2005), concerning the expected number of Nash Equilibria in normal–random bimatrix games, to show that the expected number of ESS is significantly smaller than the expected number of symmetric Nash equilibria of the underlying symmetric bimatrix game.

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1 Introduction

In this work we study the distribution of the support sizes of evolutionary stable strategies (ESS) in random evolutionary games, whose payoff matrices have elements that behave as independent, identically distributed random variables. Arguing about the existence of a property in random games may actually reveal information about the (in)validity of the property in the vast majority of payoff matrices. In particular, a vanishing probability of ESS existence would prove that this notion of stability is rather rare among payoff matrices, dictating the need for a new, more widely applicable notion of stability. Etessami and Lochbihler [2] recently proved both the **NP**−hardness and **coNP**−hardness of even detecting the existence of an ESS for an arbitrary evolutionary game.

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The concept of ESS was formally introduced by Maynard–Smith and Price [14]. Haigh [4] provided an alternative characterization, via a set of necessary and sufficient conditions (called *feasibility*, *superiority* and *stability* conditions), for a strategy $\mathbf{x}$ being an ESS. As will be clear later, the first two conditions imply that the profile $(\mathbf{x}, \mathbf{x})$ is a symmetric Nash equilibrium (SNE) of the underlying symmetric bimatrix game $\langle A, A^T \rangle$.

A series of works about thirty years ago (eg, [3], [9], [6]) have investigated the probability that an evolutionary game with an $n \times n$ payoff A whose elements behave as uniform random variables in $[0, 1]$, possesses a *completely mixed strategy* (ie, assigning positive probability to all possible types) which is an ESS. Karlin [6] had already reported experimental evidence that the stability condition is far more restrictive than the feasibility condition in this case wrt[1] the existence of ESS (the superiority condition becomes vague in this case since we refer to completely mixed strategies).

Kingman [7] also worked on the severity of the stability condition, in a work on the size of polymorphisms which, interpreted in random evolutionary games, corresponds to random payoff matrices that are *symmetric*. For the case of uniform distribution, he proved that almost all ESS in a random evolutionary game with symmetric payoff matrix, have support size *less than* $2.49\sqrt{n}$. Consequently, Haigh [5] extended this result to the case of asymmetric random payoff matrices. Namely, for a particular probability measure with density $\phi(x) = \exp(-x)/\sqrt{\pi x}$, he proved that almost all ESS have support size *at most* $1.636 n^{2/3}$. He also conjectured that similar results should also hold for a wide range of probability measures with continuous density functions.

Another (more recent) line of research concerns the expected number of Nash equilibria in random bimatrix games. Initially McLennan [10] studied this quantity for arbitrary normal form games and provided a formula for this number. Consequently, McLennan and Berg [11] computed asymptotically tight bounds for this formula, for the special case of bimatrix games. They proved that the expected number of NE in normal–random bimatrix games is asymptotically equal to $\exp(0.2816n + \mathcal{O}(\log n))$, while almost all NE have support sizes that concentrate around $0.316n$. Recently Roberts [13] calculated this number in the case zero sum games $\langle A, -A \rangle$ and coordination games $\langle A, A \rangle$, when the Cauchy probability measure is used for the entries of the payoff matrix.

In a previous work of ours [8] we had attempted to study the support sizes of ESS in random games, under the uniform probability measure. In that work we had calculated an exponentially small upper bound on the probability of any given support of size r being the support of an ESS (actually, being the support of a submatrix that satisfies the stability condition). Nevertheless, this bound proved to be insufficient for answering Haigh's conjecture for the uniform case, due to the extremely large number of supports.

In this work we resolve affirmatively the conjecture of Haigh for the cases of both the uniform distribution in $[0, 1]$ and the standard normal distribution (actually, shifted by a positive number). In both cases we prove the crucial probability

[1] With respect to.

of stability for a given support to be significantly smaller than exponential. This is enough to prove that almost all ESS have *sublinear* support size. We then proceed to combine our result on satisfaction of the stability condition in random evolutionary games wrt the (standard, shifted) normal distribution, with the result of McLennan and Berg [11] on the expected number of NE in normal–random bimatrix games. Our observation is that ESS in random evolutionary games are significantly less than SNE in the underlying symmetric bimatrix games.

The structure of the rest of the paper is the following: In Section 2 we provide some notation and some elementary background on (symmetric) bimatrix and evolutionary games. In Section 3 we calculate the probability of the stability condition holding (unconditionally) for given support sizes, in the case of the uniform distribution (cf. Subsection 3.2) and in the case of the normal distribution (cf. Subsection 3.3). We then use these bounds to give concentration results on the support sizes of ESS for these two random models of evolutionary games (cf. Subsection 3.4). In Section 4 we prove that the stability condition is more severe than the (symmetric) Nash property in symmetric games, by showing that the expected number of ESS in a evolutionary game with a normal–random payoff matrix A is significantly less than the expected number of Symmetric Nash Equilibria in the underlying symmetric bimatrix game $\langle A, A^T \rangle$.

2 Preliminaries

Notation. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_{\geqslant 0}$ is the set of nonnegative reals, and $\mathbb{N}$ is the set of nonnegative integer numbers. For any $k \in \mathbb{N} \setminus \{0\}$, we denote the set $\{1, 2, \ldots, k\}$ by $[k]$. $\mathbf{e_i} \in \mathbb{R}^n$ is the vector with all its elements equal to zero, except for its $i-$th element, which is equal to one. $\mathbf{1} = \sum_{i \in [n]} \mathbf{e_i}$ is the all–one vector, while $\mathbf{0}$ is the all–zero vector in $\mathbb{R}^n$.

We consider any $n \times 1$ matrix as a column vector and any $1 \times n$ matrix as a row vector of $\mathbb{R}^n$. A vector is denoted by small boldface letters (eg, $\mathbf{x}, \mathbf{p}, \ldots$) and is typically considered as a column vector. For any $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$, its $i-$th row (as a row vector) is denoted by A^i and its $j-$th column (as a column vector) is denoted by A_j. The $(i, j)-$th element of A is denoted by $A_{i,j}$ (or, A_{ij}). A^T is the transpose matrix of A. For any positive integer $k \in \mathbb{N}$, $\Delta_k \equiv \{\mathbf{z} \in \mathbb{R}^k_{\geqslant 0} : \mathbf{1}^T \mathbf{z} = 1\}$ is the $(k-1)-$simplex, ie, the set of probability vectors over $k-$element sets. For any $\mathbf{z} \in \Delta_k$, its **support** is the subset of $[k]$ of actions that are assigned positive probability mass: $supp(\mathbf{x}) \equiv \{i \in [k] : z_i > 0\}$.

For any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and any event $\mathcal{E} \in \mathcal{F}$, $\mathbb{P}\{\mathcal{E}\}$ is the probability of this event occurring, while $\mathbb{I}_{\{\mathcal{E}\}}$ is the indicator variable of $\mathcal{E}$ holding. For a random variable X, $\mathbb{E}\{X\}$ is its expectation and $\mathbb{V}\mathrm{ar}\{X\}$ its variance. In order to denote that a random variable X gets its value according to a probability distribution F, we use the following notation: $X \in_{\mathrm{R}} F$. For example, for a uniform random variable in $[0, 1]$ we write $X \in_{\mathrm{R}} \mathcal{U}(0, 1)$, while for a random variable drawn from the standard normal distribution we write $X \in_{\mathrm{R}} \mathcal{N}(0, 1)$.

Bimatrix Games. The subclass of symmetric bimatrix games provides the basic setting for much of Evolutionary Game Theory. Indeed, every evolutionary game implies an underlying symmetric bimatrix game, that is repeatedly played between randomly chosen opponents from the population. Therefore we provide the main game theoretic definitions with respect to symmetric bimatrix games.

Definition 1. *For arbitrary $m \times n$ real matrices $A, B \in \mathbb{R}^{m\times n}$, the* **bimatrix game** $\Gamma = \langle A, B\rangle$ *is a game in strategic form between two players, in which the first (row) player has m possible actions and the second (column) player has n possible actions. A* **mixed strategy** *for the row (column) player is a probability distribution $\mathbf{x} \in \Delta_m$ ($\mathbf{y} \in \Delta_n$), according to which she chooses her own action,* independently *of the other player's choice. A strategy $\mathbf{x} \in \Delta_m$ is* **completely mixed** *if and only if $supp(\mathbf{x}) = [m]$. The* **payoffs** *of the row and the column player, when the row and column players adopt strategies $\mathbf{e_i}$ and $\mathbf{e_j}$, are A_{ij} and B_{ij} respectively. If the two players adopt the strategies $\mathbf{p} \in \Delta_m$ and $\mathbf{q} \in \Delta_n$, then the* **(expected) payoffs** *of the row and column player are $\mathbf{p}^T A\mathbf{q}$ and $\mathbf{p}^T B\mathbf{q}$ respectively. Some special cases of bimatrix games are the* **zero sum** *($B = -A$), the* **coordination** *($B = A$), and the* **symmetric** *($B = A^T$) games.*

Note that in case of a symmetric bimatrix game, the two players have exactly the same set of possible actions (say, $[n]$). The standard notion of equilibrium in strategic games are the Nash Equilibria [12]:

Definition 2. *For any bimatrix game $\langle A, B\rangle$, a strategy profile $(\mathbf{x}, \mathbf{y}) \in \Delta_m \times \Delta_n$ is called a* **Nash Equilibrium** *(NE in short), if and only if $\mathbf{x}^T A\mathbf{y} \geqslant \mathbf{z}^T A\mathbf{y}$, $\forall z \in \Delta_m$ and $\mathbf{x}^T B\mathbf{y} \geqslant \mathbf{x}^T B\mathbf{z}$, $\forall z \in \Delta_n$. If additionally $supp(\mathbf{x}) = [m]$ and $supp(\mathbf{y}) = [n]$, then $(\mathbf{x}, \mathbf{y})$ is called a* **completely mixed Nash Equilibrium** *(CMNE in short). A profile $(\mathbf{x}, \mathbf{x})$ that is NE for $\langle A, B\rangle$ is called a* **symmetric Nash Equilibrium** *(SNE in short).*

Observe that the payoff matrices in a symmetric bimatrix game need not be symmetric. Note also that not all NE of a symmetric bimatrix game need be symmetric. However it is known that there is at least one such equilibrium:

Theorem 1 ([12]). *Each finite symmetric bimatrix game has at least one SNE.*

When we wish to argue about the vast majority of symmetric bimatrix games, one way is to assume that the real numbers in the set $\{A_{i,j} : (i, j) \in [n]\}$ are independently drawn from a probability distribution F. For example, it can be the uniform distribution in an interval $[a, b] \in \mathbb{R}$, denoted by $\mathcal{U}(a, b)$. Then, a random symmetric bimatrix game Γ is just an instance of the implied random experiment that is described in the following definition:

Definition 3. *A symmetric bimatrix game $\Gamma = \langle A, A^T\rangle$ is an instance of a (symmetric 2-player)* random game *wrt the probability distribution F, if and only if $\forall i, j \in [n]$, the real number $A_{i,j}$ is an independently and identically distributed random variable drawn from F.*

Evolutionary Stable Strategies. For some $A \in \mathbb{R}^{n\times n}$, fix a symmetric game $\Gamma = \langle A, A^T\rangle$. Suppose that all the individuals of an infinite population are programmed to play the same (either pure or mixed) *incumbent* strategy $\mathbf{x} \in \Delta_n$, whenever they are involved in Γ. Suppose also that at some time a small group of *invaders* appears in the population. Let $\varepsilon \in (0,1)$ be the share of invaders in the post–entry population. Assume that all the invaders are programmed to play the (pure or mixed) strategy $\mathbf{y} \in \Delta_n$ whenever they are involved in Γ.

Pairs of individuals in this *dimorphic* post–entry population are now repeatedly drawn at random to play always the same symmetric game Γ against each other. Recall that, due to symmetry, it is exactly the same for each player to be either the row or the column player. If an individual is chosen to participate, the probability that her (random) opponent will play strategy $\mathbf{x}$ is $1-\varepsilon$, while that of playing strategy $\mathbf{y}$ is ε. This is equivalent to saying that the opponent is an individual who plays the *mixed* strategy $\mathbf{z} = (1-\varepsilon)\mathbf{x}+\varepsilon\mathbf{y}$. The post–entry payoff to the incumbent strategy $\mathbf{x}$ is then $\mathbf{x}^T A\mathbf{z}$ and that of the invading strategy $\mathbf{y}$ is just $\mathbf{y}^T A\mathbf{z}$. Intuitively, evolutionary forces will select *against* the invader if $\mathbf{x}^T A\mathbf{z} > \mathbf{y}^T A\mathbf{z}$. The most popular notion of stability in evolutionary games is the Evolutionary Stable Strategy (ESS):

Definition 4. *A strategy* $\mathbf{x}$ *is* **evolutionary stable** *(ESS in short) if for any strategy* $\mathbf{y} \neq \mathbf{x}$ *there exists a barrier* $\bar{\varepsilon} = \bar{\varepsilon}(\mathbf{y}) \in (0,1)$ *such that* $\forall 0 < \varepsilon \leqslant \bar{\varepsilon},\ \mathbf{x}^T A\mathbf{z} > \mathbf{y}^T A\mathbf{z}$ *where* $\mathbf{z} = (1-\varepsilon)\mathbf{x} + \varepsilon\mathbf{y}$.

The following lemma states that the "hard cases" of evolutionary games are *not* the ones in which there exists a completely mixed ESS:

Lemma 1 (Haigh 1975 [4]). *If a completely mixed strategy* $\mathbf{x} \in \Delta$ *is an ESS, then it is the* unique *ESS of the evolutionary game.*

Indeed, it is true that, if for an evolutionary game with payoff matrix $A \in \mathbb{R}^{n\times n}$ it holds that some strategy $\mathbf{x} \in \Delta_n$ is an ESS, then no strategy $\mathbf{y} \in \Delta_n$ such that $supp(\mathbf{y}) \subseteq supp(\mathbf{x})$ may be an ESS as well.

Haigh [4] also provided an alternative characterization of ESS in evolutionary games, which is the *conjunction* of the following sentences, and will prove to be very useful for our discussion:

Theorem 2 (Haigh [4]). *A strategy* $\mathbf{p} \in \Delta_n$ *in an evolutionary game with payoff matrix* $A \in \mathbb{R}^{n\times n}$ *is an ESS if and only if the following* necessary and sufficient *conditions simultaneously hold:*

[H1]: <u>Nash Property</u> *There is a constant* $c \in \mathbb{R}$ *such that:*

- **[H1.1]:** <u>Feasibility</u> $\sum_{j\in supp(\mathbf{p})} A_{ij}p_j = A^i\mathbf{p} = c,\ \forall i \in supp(\mathbf{p})$.
- **[H1.2]:** <u>Superiority</u> $\sum_{j\in supp(\mathbf{p})} A_{ij}p_j = A^i\mathbf{p} \leqslant c,\ \forall i \notin supp(\mathbf{p})$.

[H2]: <u>Stability</u> $\forall \mathbf{x} \in \mathbb{R}^n$:
IF $(\mathbf{x} \neq \mathbf{0} \wedge supp(\mathbf{x}) \subseteq supp(\mathbf{p}) \wedge \mathbf{1}^T\mathbf{x} = 0)$ **THEN** $\mathbf{x}^T A\mathbf{x} < 0$

Observe that [H1] assures that $(\mathbf{p},\mathbf{p})$ is a *symmetric Nash Equilibrium* (SNE) of the underlying symmetric bimatrix game $\langle A, A^T\rangle$. This is because $\forall i,j \in [n], i \in supp(\mathbf{p}) \Rightarrow A^i\mathbf{p} \geqslant A^j\mathbf{p}$ and $\forall i,j \in [n], i \in supp(\mathbf{p}) \Rightarrow \mathbf{p}^T(A^T)_i = \mathbf{p}^T(A^i)^T \geqslant \mathbf{p}^T(A^j)^T = \mathbf{p}^T(A^T)_j$. Since in this work we deal with evolutionary games with *random* payoff matrices (in particular, whose entries behave as independent, identically distributed continuous random variables), we can safely assume that almost surely [H1.2] holds with strict inequality. As for [H2], this is the one that guarantees the stability of the strategy against (sufficiently small) invasions.

3 Probability of Stability

In this section we study the probability of a strategy with support size $r \in [n]$ also being an ESS. In the next section we shall use this to calculate an upper bound on the support sizes of almost all ESS in a random game.

Assume a probability distribution F, whose density function $\phi : \mathbb{R} \mapsto [0,1]$ exists, according to which the random variables $\{A_{ij}\}_{(i,j)\in[n]\times[n]}$ determine their values. We focus on the cases of: (i) the uniform distribution $\mathcal{U}(0,1)$, with density function $\phi_u(x) = \mathbb{I}_{\{x\in[0,1]\}}$ and distribution function $\Phi_u(x) = x \cdot \mathbb{I}_{\{x\in[0,1]\}} + \mathbb{I}_{\{x>1\}}$, and (ii) the standard normal distribution $\mathcal{N}(0,1)$, with density function $\phi_g(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}}$ and distribution function $\Phi_g(x) = \int_{-\infty}^{x} n(t)dt$. Our goal is to study the severity of [H2] for a strategy being an ESS. We follow Haigh's generalization of the interesting approach of Kingman (for random *symmetric* payoff matrices) to the case of asymmetric matrices. Our findings are analogous to those of Haigh [5], who gave the general methodology and then focused on a particular distribution. Here we resolve the cases of uniform distribution and standard normal distributions, which were left open in [5].

3.1 Kingman's Approach

Consider an arbitrary strategy $\mathbf{p} \in \Delta_n$, for which we assume (without loss of generality) that its support is $supp(\mathbf{p}) = [r]$. Since condition [H2] has to hold for any non-zero real vector $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\} : \mathbf{1}^T\mathbf{x} = 0 \wedge supp(\mathbf{x}) \subseteq [r]$, we can also apply it for all vectors $\mathbf{x(i,j)} = \mathbf{e_i} - \mathbf{e_j} : 1 \leqslant i < j \leqslant r$, as was observed in [7]. This immediately implies the following *necessary condition* for $\mathbf{p}$ being an ESS:

$$\forall 1 \leqslant i < j \leqslant r,\ \ A_{ij} + A_{ji} > A_{ii} + A_{jj} \tag{1}$$

Mimicking Kingman and Haigh's notation [7,5], we denote by D_I the event that our random matrix A has the property described by inequality (1), if $r = |I|$ and we rearrange the rows and columns of A so that $I = [r]$. As was demonstrated in [5], the probability of this event is expressed by the following form:

$$\mathbb{P}\{D_I\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\leqslant i<j\leqslant r} [1 - G(a_{ii} + a_{jj})] \cdot \prod_{i\in[r]} [\phi(a_{ii})]\ da_{11} \cdots da_{rr} \tag{2}$$

where $G(x) = \int_{-\infty}^{x} g(t)dt$ is the distribution function of any random variable $X_{ij} = A_{ij} + A_{ji} :\ 1 \leqslant i < j \leqslant r$ (the sum of two **iid** random variables with density function ϕ). Note that the density function g is the *convolution* of f with itself. This formula was studied in [5] for the special case $\phi(x) = \exp(-x)/\sqrt{\pi x}$.

In the next two subsections we do the same for the uniform and (shifted) standard normal distribution. Then we bound the support sizes of almost all ESS in uniformly–random and normal–random evolutionary games.

3.2 The Case of $\mathcal{U}(0,1)$

If we adopt $\mathcal{U}(0,1)$ as our basic probability distribution, then of course $f(x) = \phi_u(x) = \mathbb{I}_{\{x\in(0,1)\}}$ and the distribution function G can be easily computed: $\forall 0 \leqslant x \leqslant 1,\quad G(x) = \int_0^x f(a_{ii}) \left(\int_0^{x-a_{ii}} f(a_{jj})da_{jj}\right) da_{ii} = \frac{x^2}{2}$ and $\forall 1 \leqslant x \leqslant 2,\ \ G(x) = \int_0^1 f(a_{ii}) \left(\int_0^{\min\{1,x-a_{ii}\}} f(a_{jj})da_{jj}\right) da_{ii} = 2x - 1 - \frac{x^2}{2}$. Therefore we conclude that the following holds (also mentioned in [5]): $\forall x \in \mathbb{R}$,

$$1 - G(x) = \left(1 - \frac{x^2}{2}\right) \cdot \mathbb{I}_{\{0\leqslant x\leqslant 1\}} + \frac{1}{2}(2-x)^2 \cdot \mathbb{I}_{\{1<x\leqslant 2\}} \tag{3}$$

Observe now that each $1-G(a_{ii}+a_{jj})$ factor in equation (2) expresses the probability that the random variable $X_{ij} \equiv A_{ij} + A_{ji}$ is strictly larger than a certain value $a_{ii} + a_{jj}$. On the other hand, all the $f(a_{ii}) = \phi_u(a_{ii})$ factors in equality (2) assure that each of the diagonal elements in A (ie, the random variables A_{ii}) get the assumed values (ie, $A_{ii} = a_{ii}$), which have to be *nonnegative*. We use the following trivial upper bound on each of the $1 - G(a_{ii} + a_{jj})$ factors, which exploits only the fact of *non negative* values of the elementary random variables $A_{ij} \in_{\mathrm{R}} \mathcal{U}(0,1)$ that we consider: $\forall 1 \leqslant i < j \leqslant r,\ 1 - G(a_{ii} + a_{jj}) = \mathbb{P}\{X_{ij} > a_{ii} + a_{jj}\} \leqslant \mathbb{P}\{X_{ij} > a_{ii}\} = 1 - G(a_{ii})$, to get the following from (2):

$$\begin{aligned}
\mathbb{P}\{D_I\} &\leqslant \int_0^1 \cdots \int_0^1 \prod_{1\leqslant i<j\leqslant r} [1 - G(a_{ii})] \ \ da_{11}\cdots da_{rr} \\
&= \prod_{i\in[r-1]} \left(\int_0^1 [1 - G(a_{ii})]^{r-i} \ \ da_{ii}\right)
\end{aligned} \tag{4}$$

using the facts that $f(x) = \mathbb{I}_{\{x\in(0,1)\}}$ and $\int_0^1 f(a_{rr})da_{rr} = 1$. Plugging in the form of $1-G(x)$ in case of the uniform distribution (eq. (3)), we get the following:

$$\mathbb{P}\{D_I\} \leqslant \prod_{i\in[r-1]} \left(\int_0^1 \left[1 - \frac{1}{2}a_{ii}^2\right]^{r-i} \ \ da_{ii}\right) \tag{5}$$

Using the trivial bound $(1-x)^a \leqslant \exp(-ax), \forall x > 0, \forall a \geqslant 1$, we have:

$$\begin{aligned}
\mathbb{P}\{D_I\} &\leqslant \prod_{i\in[r-1]} \left[\int_0^1 \exp\left(-\frac{r-i}{2}a_{ii}^2\right) \ \ da_{ii}\right] \\
&\leqslant \prod_{i\in[r-1]} \left[\frac{1}{\sqrt{r-i}} \cdot \int_0^1 \exp\left(-\frac{\left(a_{ii}\sqrt{r-i}\right)^2}{2}\right) \ \ \sqrt{r-i}\, da_{ii}\right]
\end{aligned}$$

$$= \prod_{i\in[r-1]} \left[\frac{1}{\sqrt{r-i}} \cdot \int_0^{\sqrt{r-i}} \exp\left(-\frac{\beta_i^2}{2}\right) \quad d\beta_i \right]$$

$$< \prod_{i\in[r-1]} \left[\frac{1}{2\sqrt{r-i}} \right] = \exp\left(-(r-1)\ln 2 - \frac{1}{2}\sum_{j=1}^{r-1} \ln j \right)$$

$$< \exp\left(-(r-1)\ln 2 - \frac{1}{2}\left[(r-1) + \sum_{j=1}^{r-1} H_j \right] \right)$$

$$= \exp\left(-(r-1)\ln 2 - \frac{1}{2}\left[-2(r-1) + rH_{r-1} \right] \right)$$

$$= \exp\Big((r-1)(1-\ln 2) - \frac{r}{2} - \frac{r}{2}\ln(r-1) \Big) = \exp\left(-\frac{r\ln r}{2} + \mathcal{O}(r) \right) \quad (6)$$

since, $\int_0^{\sqrt{r-i}} \exp\left(-\frac{\beta_i^2}{2}\right) d\beta_i < \int_0^{\infty} \exp\left(-\frac{\beta_i^2}{2}\right) d\beta_i = \frac{1}{2}$. We used the following properties of harmonic numbers: If $H_{r-1} = \sum_{i=1}^{r-1} \frac{1}{i}$ is the $(r-1)-$th harmonic number, then $\sum_{i=1}^{r-1} H_i = rH_{r-1} - (r-1)$ and $\ln(r-1) < H_{r-1} < \ln(r-1)+1$.

3.3 The Case of $\mathcal{N}(\xi, 1)$

Assume now, for some $\xi > 0$ that will be fixed later, that each element of the payoff matrix behaves as a normally distributed independent random variable with mean ξ and variance 1: $\forall (i,j) \in [n]\times[n], A_{ij} \in_{\mathrm{R}} \mathcal{N}(\xi, 1)$. Then it also holds that all the X_{ij} variables (for $1 \leqslant i < j \leqslant r$) behave also as normally distributed random variables, with mean 2ξ and variance 2. That is: $\forall (i,j) \in [r]\times[r] : i \neq j, X_{ij} \in_{\mathrm{R}} \mathcal{N}(2\xi, 2)$. Then the following hold: $\forall t \in \mathbb{R}, f(t) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(t-\xi)^2}{2}\right)$ and $g(t) = \frac{1}{2\sqrt{\pi}}\exp\left(-\frac{(t-2\xi)^2}{4}\right)$. Moreover, $\forall x \in \mathbb{R}, 1-F(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty}\exp\left(-\frac{(t-\xi)^2}{2}\right) dt \Rightarrow 1-F(x) = \frac{1}{\sqrt{2\pi}}\int_{x-\xi}^{\infty}\exp\left(-\frac{z^2}{2}\right) dz$ (by the change in variable $z = t-\xi$) and $1-G(x) = \frac{1}{2\sqrt{\pi}}\int_x^{\infty}\exp\left(-\frac{(t-2\xi)^2}{4}\right) dt \Rightarrow 1-G(x) = \frac{1}{\sqrt{2\pi}}\int_{\frac{x-2\xi}{\sqrt{2}}}^{\infty}\exp\left(-\frac{z^2}{2}\right) dz$ (by setting $z = \frac{t-2\xi}{\sqrt{2}}$). The following property is useful for bounding the distribution function of a normal random variable (cf. Theorem 1.4 of [1]): $\forall x > 0,\ \left(1-x^{-2}\right)\frac{\exp(x^2/2)}{x} \leqslant \int_x^{\infty}\exp(-z^2/2)dz \leqslant \frac{\exp(x^2/2)}{x}$. A simple corollary of this property is the following:

Corollary 1. *Assume that $F(x), G(x)$ are the distribution functions of $\mathcal{N}(\xi,1)$ and $\mathcal{N}(2\xi,2)$ respectively. Then: $\forall x > \xi,\ 1-F(x) \in \left[\left(1-\frac{1}{(x-\xi)^2}\right), 1\right]\cdot\frac{1}{\sqrt{2\pi}}\cdot\frac{\exp(-(x-\xi)^2/2)}{x-\xi}$ and $\forall x > 2\xi,\ 1-G(x) \in \left[\left(1-\frac{2}{(x-2\xi)^2}\right), 1\right]\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\exp(-(x-2\xi)^2/4)}{x-2\xi}$.*

Recall now that

$$\mathbb{P}\{D_I\} = \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{r\text{ times}} \prod_{1\leqslant i<j\leqslant r}\left[1-G(a_{ii}+a_{jj})\right]\cdot\prod_{i\in[r]}\left(f(a_{ii})da_{ii}\right)$$

$$\leqslant \sum_{k=0}^{r} \binom{r}{k} \prod_{i=1}^{k} \left(\int_{-\infty}^{0} f(a_{ii}) da_{ii} \right) \cdot \prod_{i=k+1}^{r} \left(\int_{0}^{\infty} [1 - G(a_{ii})]^{r-i} f(a_{ii}) da_{ii} \right)$$
$$= \sum_{k=0}^{r} \binom{r}{k} (F(0))^k \cdot \prod_{i=0}^{r-k-1} \mu_i$$

where we have exploited the facts that $\forall x \in \mathbb{R}, 1 - G(x) = \mathbb{P}\{X > x\} \leqslant 1$ and $\forall y, z \geqslant 0, 1 - G(y+z) = \mathbb{P}\{X > y+z\} \leqslant \mathbb{P}\{X > z\} = 1 - G(z)$ and we set $\mu_i \equiv \int_0^\infty [1 - G(x)]^i f(x) dx,\ \forall i \in \mathbb{N}$. Exploiting Corollary 1 and the symmetry of the normal distribution, we have: $F(0)^k = (1 - F(2\xi))^k \leqslant \left(\frac{\exp(-\xi^2/2)}{\xi \cdot \sqrt{2\pi}} \right)^k = \exp\left(-\frac{k \ln(2\pi)}{2} - \frac{k\xi^2}{2} - k \ln \xi \right)$. As for the product of the μ_i's, since $\forall i \geqslant 0,\ \mu_i = \int_0^\infty [1 - G(x)]^i f(x) dx \leqslant [1 - G(0)]^i \cdot (1 - F(0))$, we conclude that: $\prod_{i=0}^{r-k-1} \mu_i \leqslant (1 - F(0))^{r-k} (1 - G(0))^{(r-k)(r-k-1)/2} < \exp\left(-\frac{(r-k)(r-k-1)G(0)}{2} \right)$. Therefore we get the following bound:

$$\mathbb{P}\{D_I\} \leqslant \exp\left(-\frac{r(r-1)}{2} \cdot G(0) \right) + \sum_{k=1}^{r} \exp\left(k \ln\left(\frac{r}{k}\right) - \frac{(r-k)(r-k-1)}{2} \cdot G(0) - \frac{k\xi^2}{2} - \mathcal{O}(k \ln \xi) \right) \quad (7)$$

Assume now that, for some sufficiently small $\delta > 0$, it holds that $\xi = \sqrt{(1-\delta)\ln r}$. Observe that for some constant $\varepsilon > 0$ and all $0 \leqslant k \leqslant \varepsilon r$, $\prod_{i=0}^{r-k-1} \mu_i < \exp\left(-\frac{(1-\varepsilon)^2}{2} r^2 \cdot G(0) \right) = \exp\left(-\frac{(1-\varepsilon)^2}{2} r \ln r \cdot e^{\delta \ln r - \mathcal{O}(\ln \ln r)} \right) < \exp\left(-\frac{(1-\varepsilon)^2}{2} r \ln r \right)$ for $\delta = \Omega\left(\frac{\ln \ln r}{\ln r}\right)$, exploiting the fact that $G(0) = \exp\left(-\xi^2 - \ln \xi - \mathcal{O}(1) \right)$ (cf. Corollary 1). On the other hand, for all $\varepsilon r < k \leqslant r$, observe that $F(0)^k \leqslant \exp\left(-\frac{k\xi^2}{2} - k \ln \xi - \mathcal{O}(k) \right) < \exp\left(-\frac{1-\delta}{2} \varepsilon r \ln r - \mathcal{O}(k \ln \ln r) \right) < \exp\left(-\frac{(1-\delta)\varepsilon}{2} r \ln r \right)$. Since for $\varepsilon = \frac{3-\sqrt{5}}{2}$ it holds that $\frac{(1-\varepsilon)^2}{2} \geqslant \frac{(1-\delta)\varepsilon}{2}$, we conclude that each term in the right hand side of inequality (7) is upper bounded by $\exp(-\varepsilon(1-\delta)/2 \cdot r \ln r + \mathcal{O}(r))$ and so we get the following: $\mathbb{P}\{D_I\} \leqslant \exp\left(-\frac{(1-\delta)\varepsilon}{2} \cdot r \ln r + \mathcal{O}(r) \right)$.

3.4 Support Sizes of Almost All ESS

In the previous subsections we calculated upper bounds on the probability $\mathbb{P}\{D_I\}$ of a size–r subset $I \subset [n]$ (say, $I = [r]$) satisfying [H2] (and thus being a candidate support for an ESS), for the cases of $\mathcal{U}(0,1)$ and $\mathcal{N}\left(\sqrt{(1-\delta)\ln r}, 1\right)$. We now apply the following counting argument introduced by Kingman and used also by Haigh: Let d_r be the event that there exists a submatrix of the random matrix A, of size *at least* $r \times r$, such that D_I is satisfied. Then the probability of this event occurring is upper by the following formula (cf. [5][eq. 10]): $\forall 1 \leqslant s \leqslant r \leqslant n,\ \mathbb{P}\{d_r\} = \mathbb{P}\{\exists \text{ submatrix with } |I| \geqslant r \text{ s.t. } D_I \text{ holds}\} \leqslant \binom{n}{s} \cdot \mathbb{P}\{D_s\} \cdot \binom{r}{s}^{-1}$. Using Stirling's formula, $k! = \sqrt{2\pi k} \cdot (k/e)^k \cdot (1 + \Theta(1/k))$, where $e = \exp(1)$, we write: $\forall 1 \leqslant s \leqslant r \leqslant n$, $\binom{n}{s} \cdot \binom{r}{s}^{-1} = \left(\frac{n}{r}\right)^{s+1/2} \cdot \left(\frac{n}{n-s}\right)^{n-s} \cdot \left(\frac{r-s}{r}\right)^{r-s} \cdot \left(\frac{r-s}{n-s}\right)^{1/2} \cdot (1 + \mathrm{o}(1))$. Assume now that $r = An^a > s = Bn\beta$, for some $1 > a > \beta > 0$ and $A \geqslant B$. Then: $\binom{n}{s} \cdot \binom{r}{s}^{-1} = (1 + \mathrm{o}(1)) \cdot \left(\frac{n^{1-a}}{A}\right)^{Bn^\beta + 1/2} \cdot \left(1 - \frac{B}{n^{1-\beta}}\right)^{-n^\beta(n^{1-\beta} - B)} \cdot \left(1 - \frac{B}{An^{a-\beta}}\right)^{n^\beta(An^{a-b} - B)} \cdot \left(\frac{A}{n^{1-a}} \cdot \frac{1 - B/(An^{a-\beta})}{1 - B/n^{1-\beta}}\right)^{1/2} = \exp\left((1-a)Bn^\beta \ln n + \mathcal{O}(n^\beta)\right)$. We proved for the uniform distribution $\mathcal{U}(0,1)$ that for any subset $I \subseteq [n]$ such that $|I| = An^a$, $\mathbb{P}\{D_I\} = \exp\left(-\frac{Aa}{2} n^a \ln n + \mathcal{O}(n^a)\right)$. Therefore, in this case, $\mathbb{P}\{d_{An^a}\} \leqslant \exp\left[-\left(\frac{a}{2} - 1 + a\right) Bn^\beta \ln n + \mathcal{O}(n^\beta)\right]$, which tends to zero for all $a > 2/3$.

Similarly, we proved for the normal distribution $\mathcal{N}\left(\sqrt{(1-\delta)\ln(Bn^\beta)}, 1\right)$ that for any $I \subseteq [n] : |I| = Bn^\beta$, $\mathbb{P}\{D_I\} = \exp\left(-\frac{\varepsilon(1-\delta)B\beta}{2} \cdot n^\beta \ln n + \mathcal{O}(n^\beta)\right)$, where $\varepsilon = \frac{3-\sqrt{5}}{2}$. Therefore we conclude that: $\mathbb{P}\{d_{An^a}\} \leqslant \exp\left[-\left(\frac{\varepsilon(1-\delta)\beta}{2} - 1 + a\right) Bn^\beta \ln n + \mathcal{O}(n^\beta)\right]$, which tends to zero for all $a > \frac{4}{7 - \sqrt{5} - (3-\sqrt{5})\delta} \cong 0.8396$, since $\delta = \Theta\left(\frac{\ln \ln n}{\ln n}\right) = \mathrm{o}(1)$ (for $n \to \infty$). Thus we conclude with the following theorem concerning the support sizes of ESS in a random evolutionary game:

Theorem 3. *Consider an evolutionary game with a random $n \times n$ payoff matrix A. Fix arbitrary positive constant $\zeta > 0$.*

1. *If $A_{ij} \in_{\mathrm{R}} \mathcal{U}(0,1), \forall (i,j) \in [n] \times [n]$, then, as $n \to \infty$, it holds that: $\mathbb{P}\left\{\exists \text{ ESS with support size at least } n^{(2+2\zeta)/3}\right\} \leqslant \exp\left(-\frac{5\zeta}{6} \cdot n^{(2+\zeta)/3} \cdot \ln n + \mathcal{O}\left(n^{(2+\zeta)/3}\right)\right) \to 0$.*
2. *If $A_{ij} \in_{\mathrm{R}} \mathcal{N}(\xi, 1), \forall (i,j) \in [n] \times [n]$, where $\xi = \Theta\left(\sqrt{\ln n}\right)$, then, as $n \to \infty$, it holds that: $\mathbb{P}\left\{\exists \text{ ESS with support size at least } n^{0.8397+\zeta}\right\} \leqslant \exp\left(-1.19\zeta \cdot n^{0.8397+\zeta/2} \cdot \ln n + \mathcal{O}\left(n^{0.8397+\zeta/2}\right)\right) \to 0$.*

Remark: Indeed the above theorem upper bounds the *unconditional* probability of [H2] being satisfied by any submatrix of A that is determined by an index set $I \subseteq [n] : |I| > n^{2/3}$ (for the uniform case) or $|I| > n^{0.8397}$ (for the case of the normal distribution). We adopt the particular presentation for purposes of comparison with the corresponding results of Haigh [5] and Kingman [7].

4 An Upper Bound on the Expected Number of ESS

We now combine our result on the probability of [H2] being satisfied in random evolutionary games wrt $\mathcal{N}(\xi, 1)$, with a result of McLennan and Berg [11] on the expected number of NE in random bimatrix games wrt $\mathcal{N}(0, 1)$. The goal is to show that ESS in random evolutionary games are significantly less than SNE in the underlying symmetric bimatrix games.

We start with some additional notation, that will assist the clearer presentation of the argument. Let A, B be normal–random $n \times n$ (payoff) matrices: $\forall (i,j) \in [n] \times [n], A_{ij}, B_{ij} \in_{\mathrm{R}} \mathcal{N}(\xi, 1)$. $E^{nash}_{n,r}$ is the expected number of NE with support sizes equal to r for both strategies, in $\langle A, B\rangle$. $E^{sym}_{n,r}$ is the expected number of SNE with support sizes equal to r for both strategies, in $\langle A, A^T\rangle$. $E^{ess}_{n,r}$ is the expected number of ESS of support size r, in the random evolutionary game, with payoff matrix A. Finally, $E^{stable}_{n,r}$ is the expected number of strategies with support size r that satisfy property [H2], in the random evolutionary game, with payoff matrix A. We shall prove now the following theorem:

Theorem 4. *If the $n \times n$ payoff matrix A of an evolutionary game is randomly chosen so that each of its elements behaves as an independent $\mathcal{N}(\xi, 1)$ random variable, then it holds that $E^{ess}_n = \mathrm{o}(E^{sym}_n)$, as $n \to \infty$.*

Proof: First of all we should mention that the concept of Nash Equilibrium is invariant under *affine transformations* of the payoff matrices. Therefore, we may safely assume that the results of [11] on the expected number of NE in $n \times n$ bimatrix games, in which the values of both the payoff matrices are treated as standard normal random variables, are also valid if we shift both the payoff matrices by any positive number ξ (or equivalently, if we consider the normal distribution $\mathcal{N}(\xi, 1)$ for the elements of the payoff matrices). The main theorem of the work of McLennan and Berg concerns $E^{nash}_{n,r}$ in $\langle A, B\rangle$ [2].

In our work we are concerned about $E^{ess}_{n,r}$, the expected number of ESS with support size r, in a random evolutionary game with payoff matrix A. Our purpose is to demonstrate the severity of [H2] (compared to the Nash Property [H1] that must also hold for an ESS), therefore we shall compare the expected number of SNE in $\langle A, A^T\rangle$ with the expected number of ESS in the random evolutionary game with payoff matrix A. Although the main result of [11] concerns arbitrary (probably asymmetric) normal–random bimatrix games, if one adapts their calculations for SNE in symmetric bimatrix games, then one can easily observe that similar concentration results hold for this case as well. The key formula of [11] is the following: $\forall 1 \leqslant r \leqslant n, E^{nash}_{n,r} = \binom{n}{r}^2 \cdot 2^{2-2r} \cdot (R(r-1, n-r))^2$, where, $R(a,b) = \int_{-\infty}^{\infty} \phi_g(x) \cdot \left(\Phi_g\left(\frac{x}{\sqrt{a+1}}\right)\right)^b dx$ is the probability of e_0 getting a value greater than $\sqrt{a+1}$ times the maximum value among $e_1, \ldots, e_b$, where

[2] In such a random game, strategy profiles in which the two player don't have the *same* support sizes, are *not* NE with probability asymptotically equal to one. This is why we only focus on profiles in which both players have the same support size r.

$e_0, e_1, \ldots, e_b \in_R \mathcal{N}(0,1)$. For the case of a random symmetric bimatrix game $\langle A, A^T \rangle$, the proper shape of the formula for SNE in $\langle A, A^T \rangle$ is the following: $\forall 1 \leqslant r \leqslant n, E^{sym}_{n,r} = \binom{n}{r} \cdot 2^{1-r} \cdot (R(r-1, n-r))$.

As for the asymptotic result that the support sizes r of NE are sharply concentrated around $0.316n$, this is also valid for SNE in symmetric games. The only difference is that as one increases n by one, the expected number of NE in the symmetric game goes up, *not* by an asymptotic factor of $\exp(0.2816) \approx 1.3252$, but rather by its square root $\exp(0.1408) \approx 1.1512$. So, we can state this extension of the McLennan-Berg result as follows: There exists a constant $\beta \approx 0.316$, such that for any $\varepsilon > 0$, it holds (as $n \to \infty$) that $\sum_{r=\lfloor(1-\varepsilon)\beta n\rfloor}^{\lceil(1+\varepsilon)\beta n\rceil} E^{sym}_{n,r} \geqslant \varepsilon E^{sym}_n$. From this we can easily deduce that $\sum_{r=1}^{\lfloor(1-\varepsilon)\beta n\rfloor -1} E^{sym}_{n,r} \leqslant (1-\varepsilon) E^{sym}_n$. It is now rather simple to observe that for any $1 \leqslant Z \leqslant n$, $E^{ess}_n \equiv \sum_{r=1}^{n} E^{ess}_{n,r} = \sum_{r=1}^{Z} E^{ess}_{n,r} + \sum_{r=Z+1}^{n} E^{ess}_{n,r} \leqslant \sum_{r=1}^{Z} E^{sym}_{n,r} + \sum_{r=Z+1}^{n} E^{stable}_{n,r}$, since $ess = stable \wedge sym$. If we set $Z = n^{0.8397+\zeta}$ for some $\zeta > 0$, then: $\sum_{r=1}^{Z} E^{sym}_{n,r} \leqslant \frac{n^{-0.1603+\zeta}}{\beta} E^{sym}_n$ and $\sum_{r=Z+1}^{n} E^{stable}_{n,r} \leqslant \sum_{r=Z+1}^{n} E^{ess}_n \cdot \mathbb{P}\{\exists \text{ ESS with support } \geqslant r\} < \sum_{Z=r+1}^{n} E^{sym}_n \cdot \mathbb{P}\{\exists \text{ ESS with support } \geqslant r\} < E^{sym}_n \cdot \exp\left(\log n - 1.19\zeta \cdot n^{0.8397+\zeta/2} \cdot \ln n + \mathcal{O}\left(n^{0.8397+\zeta/2}\right)\right)$. Therefore, we conclude that $E^{ess}_n = \mathcal{O}\left(n^{-0.16} \cdot E^{sym}_n\right) = \mathrm{o}(E^{sym}_n)$. ■

References

1. Durrett, R.: Probability: Theory and Examples, 2nd edn. Duxbury Press (1996)
2. Etessami, K., Lochbihler, A.: The computational complexity of evolutionary stable strategies. Int. J. of Game Theory (to appear, 2007)
3. Gillespie, J.H.: A general model to account for enzyme variation in natural populations. III: Multiple alleles. Evolution 31, 85–90 (1977)
4. Haigh, J.: Game theory and evolution. Adv. in Appl. Prob. 7, 8–11 (1975)
5. Haigh, J.: How large is the support of an ESS? J. of Appl. Prob. 26, 164–170 (1989)
6. Karlin, S.: Some natural viability systems for a multiallelic locus: A theoretical study. Genetics 97, 457–473 (1981)
7. Kingman, J.F.C.: Typical polymorphisms maintained by selection at a single locus (special issue). J. of Appl. Prob. 25, 113–125 (1988)
8. Kontogiannis, S., Spirakis, P.: Counting stable strategies in random evolutionary games. In: Deng, X., Du, D.-Z. (eds.) ISAAC 2005. LNCS, vol. 3827, pp. 839–848. Springer, Heidelberg (2005)
9. Lewontin, R.C., Ginzburg, L.R., Tuljapurkar, S.D.: Heterosis as an explanation of large amounts of genic polymorphism. Genetics 88, 149–169 (1978)
10. McLennan, A.: The expected numer of nash equilibria of a normal form game. Econometrica 73(1), 141–174 (2005)
11. McLennan, A., Berg, J.: The asymptotic expected number of nash equilibria of two player normal form games. Games & Econ. Behavior 51, 264–295 (2005)
12. Nash, J.: Noncooperative games. Annals of Mathematics 54, 289–295 (1951)
13. Roberts, D.P.: Nash equilibria of cauchy-random zero-sum and coordination matrix games. Int. J. of Game Theory 34, 167–184 (2006)
14. Smith, J.M., Price, G.: The logic of animal conflict. Nature 246, 15–18 (1973)