Complexity and Approximation Results for the Connected Vertex Cover Problem

Bruno Escoffier[1], Laurent Gourvès[1], and Jérôme Monnot[1]

CNRS-LAMSADE, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, F-75775 Paris Cedex 16, France
{escoffier,laurent.gourves,monnot}@lamsade.dauphine.fr

Abstract. We study a variation of the vertex cover problem where it is required that the graph induced by the vertex cover is connected. We prove that this problem is polynomial in chordal graphs, has a PTAS in planar graphs, is APX-hard in bipartite graphs and is 5/3-approximable in any class of graphs where the vertex cover problem is polynomial (in particular in bipartite graphs).

Keywords: Connected vertex cover, chordal graphs, bipartite graphs, planar graphs, **APX**-complete, approximation algorithm.

1 Introduction

In this paper, we study a variation of the vertex cover problem where the subgraph induced by any feasible solution must be connected. Formally, a vertex cover of a simple graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ which covers all edges, *i.e.* which satisfies: $\forall e = \{x, y\} \in E$, $x \in S$ or $y \in S$. The vertex cover problem (MinVC in short) consists in finding a vertex cover of minimum size. MinVC is known to be **APX**-complete in cubic graphs [1] and **NP**-hard in planar graphs, [13]. MinVC is 2-approximable in general graphs, [3] and admits a polynomial approximation scheme in planar graphs, [5]. On the other hand, MinVC is polynomial for several classes of graphs such as bipartite graphs, chordal graphs, graphs with bounded treewidth, etc. [7].

The connected vertex cover problem, denoted by MinCVC, is the variation of the vertex cover problem where, given a connected graph $G = (V, E)$, we seek a vertex cover S^* of minimum size such that the subgraph induced by S^* is connected. This problem has been introduced by Garey and Johnson, [12] where it is proved to be **NP**-hard in planar graphs of maximum degree 4. As indicated in [19], this problem has some applications in the domain of wireless network design. In such a model, the vertices of the network are connected by transmission links. We want to place a minimum number of relay stations on vertices such that any pair of relay stations are connected (by a path which uses only relay stations) and every transmission link is incident to a relay station. This is exactly the connected vertex cover problem.

A. Brandstädt, D. Kratsch, and H. Müller (Eds.): WG 2007, LNCS 4769, pp. 202–213, 2007.

1.1 Previous Related Works

The main complexity and approximability results known on this problem are the following: in [21], it is shown that MinCVC is polynomially solvable when the maximum degree of the input graph is at most 3. However, it is **NP**-hard in planar bipartite graphs of maximum degree 4, [10], as well as in 3-connected graphs, [22]. Concerning the positive and negative results of the approximability of this problem, MinCVC is 2-approximable in general graphs, [20,2] but it is **NP**-hard to approximate within ratio $10\sqrt{5}-21$, [10]. Finally, recently the fixed-parameter tractability of MinCVC with respect to the vertex cover size or to the treewidth of the input graph has been studied in [10,14,17,18,19]. More precisely, in [10] a parameterized algorithm for MinCVC with complexity $O^*(2.9316^k)$ is presented improving the previous algorithm with complexity $O^*(6^k)$ given in [14] where k is the size of an optimal connected vertex cover. Independently, the authors of [17,18] have also obtained FPT algorithms for MinCVC and they obtain in [18] an algorithm with complexity $O^*(2.7606^k)$. In [19], the author gives a parameterized algorithm for MinCVC with complexity $O^*(2^t \cdot t^{3t+2}n)$ where t is the treewidth of the graph and n the number of vertices.

MinCVC is related to the unweighted version of tree cover. The tree cover problem has been introduced in [2] and consists, given a connected graph $G = (V, E)$ with non-negative weights w on the edges, in finding a tree $T = (S, E')$ of G with $S \subseteq V$ and $E' \subseteq E$ which spans all edges of G and such that $w(T) = \sum_{e \in E'} w(e)$ is minimum. In [2], the authors prove that the tree cover problem is approximable within factor 3.55 and the unweighted version is 2-approximable. Recently, (weighted) tree cover has been shown to be approximable within a factor of 3 in [16], and a 2-approximation algorithm is proposed in [11]. Clearly, the unweighted version of tree cover is (asymptotically) equivalent to the connected version since S is a connected vertex cover of G iff there exists a tree cover $T' = (S, E')$ for some subset E' of edges. Since in this latter case, the weight of T' is $|S| - 1$, the result follows.

1.2 Our Contribution

In this article, we mainly deal with complexity and approximability issues for MinCVC in particular classes of graphs. More precisely, we first present some structural properties on connected vertex covers (Section 2). Using these properties, we show that MinCVC is polynomial in chordal graphs (Section 3). Then, in Section 4, we prove that MinCVC is **APX**-complete in bipartite graphs of maximum degree 4, even if each vertex of one block of the bipartition has a degree at most 3. On the other hand, if each vertex of block part of the bipartition has a degree at most 2 and the vertices of the other block have an arbitrary degree, then MinCVC is polynomial. Section 5 deals with the approximability of MinCVC. We first show that MinCVC is 5/3-approximable in any class of graphs where MinVC is polynomial (in particular in bipartite graphs, or more generally in perfect graphs). Then, we present a polynomial approximation scheme for MinCVC in planar graphs.

Notation. All graphs considered are undirected, simple and without loops. Unless otherwise stated, n and m will denote the number of vertices and edges, respectively, of the graph $G = (V, E)$ considered. $N_G(v)$ denotes the *neighborhood* of v in G, ie., $N_G(v) = \{u \in V : \{u, v\} \in E\}$ and $d_G(v)$ its *degree* that is $d_G(v) = |N_G(v)|$. Finally, $G[S]$ denotes the subgraph of G induced by S.

2 Structural Properties

We present in this subsection some properties on vertex covers or connected vertex covers. These properties will be useful in the rest of the article to devise polynomial algorithms that solve MinCVC either optimally (chordal graphs) or approximately (bipartite graphs,...).

2.1 Vertex Cover and Graph Contraction

For a subset $A \subseteq V$ of a graph $G = (V, E)$, the *contraction* of G with respect to A is the simple graph $G_A = (V', E')$ where we replace A in V by a new vertex v_A (so, $V' = (V \setminus A) \cup \{v_A\}$) and $\{x, y\} \in E'$ iff either $x, y \notin A$ and $\{x, y\} \in E$ or $x = v_A$, $y \neq v_A$ and there exists $v \in A$ such that $\{v, y\} \in E$. The *connected contraction* of G following $V' \subseteq V$ is the graph $G^c_{V'}$ corresponding to the iterated contractions of G with respect to the connected components of V' (note that contraction is associative and commutative). Formally, $G^c_{V'}$ is constructed in the following way: let $A_1, \cdots, A_q$ be the connected components of the subgraph induced by V'. Then, we inductively apply the contraction with respect to A_i for $i = 1, \cdots, q$. Thus, $G^c_{V'} = G_{A_1 \circ \cdots \circ A_q}$. Finally, let $New(G^c_{V'}) = \{v_{A_1}, \cdots, v_{A_q}\}$ be the new vertices of $G^c_{V'}$ (those resulting from the contraction). The following Lemma concerns contraction properties that will, in particular, be the basis of the approximation algorithm presented in Subsection 5.1.

Lemma 1. *Let $G = (V, E)$ be a connected graph and let $S \subseteq V$ be a vertex cover of G. Let $G_0 = (V_0, E_0) = G^c_S$ be the connected contraction of G following S where $A_1, \cdots, A_q$ are the connected components of the subgraph induced by S. The following assertions hold:*

- *(i) G_0 is connected and bipartite.*
- *(ii) If $S = S^*$ is an optimal vertex cover of G, then $New(G_0)$ is an optimal vertex cover of G_0.*
- *(iii) If $S = S^*$ is an optimal vertex cover of G and $v \in V \setminus S^*$ with $d_{G^c_{S^*}}(v) \geq 2$, then $New(G_0)$ is an optimal vertex cover of $G_0 = G^c_{S^* \cup \{v\}}$.*

2.2 Connected Vertex Covers and Biconnectivity

Now, we deal with connected vertex covers. It is easy to see that if the removal of a vertex v disconnects the input graph (v is called a *cut-vertex*, or an *articulation point*), then v has to be in any connected vertex cover. In this section we show that, informally, solving MinCVC in a graph is equivalent to solve it on the biconnected components of the graph, under the constraint of including all cut vertices.

Formally, a connected graph $G = (V, E)$ with $|V| \geq 3$ is *biconnected* if for any two vertices x, y there exists a simple cycle in G containing both x and y. A *biconnected component* (also called *block*) $G_i = (V_i, E_i)$ is a maximal connected subgraph of G that is biconnected. For a connected graph $G = (V, E)$, V_c denotes the set of cut-vertices of G and $V_{i,c}$ its restriction to V_i.

Lemma 2. *Let $G = (V, E)$ be a connected graph. $S \subseteq V$ is a connected vertex cover of G iff for each biconnected component $G_i = (V_i, E_i)$, $i = 1, \cdots, p$, $S_i = S \cap V_i$ is a connected vertex cover of G_i containing $V_{i,c}$.*

Lemma 2 allows us to characterize the optimal connected vertex covers of G.

Corollary 1. *Let $G = (V, E)$ be a connected graph. $S^* \subseteq V$ is an optimal connected vertex cover of G iff for each biconnected component $G_i = (V_i, E_i)$, $i = 1, \cdots, p$, $S^*_i = S^* \cap V_i$ is an optimal connected vertex cover of G_i among the connected vertex covers of G_i containing $V_{i,c}$.*

For instance, using Corollary 1, we deduce that for the class of *trees* or *split graphs* MinCVC is polynomial. More generally, we will see in Section 3 that this result holds in chordal graphs. If we denote by MinPrextCVC (by analogy with the well known PreExtension Coloring problem) the variation of MinCVC where given $G = (V, E)$ and $V_0 \subseteq V$, we seek a connected vertex cover S of G containing V_0 and of minimal size, we obtain the following result:

Lemma 3. *Let $\mathcal{G}$ be a class of connected graphs defined by a hereditary property. Solving* MinCVC *in $\mathcal{G}$ polynomially reduces to solve* MinPrextCVC *in the biconnected graphs of $\mathcal{G}$. Moreover, if $\mathcal{G}$ is closed by pendant addition (ie., is closed under addition of a new vertex v and a new edge $\{u, v\}$ where $u \in V$), then they are polynomially equivalent.*

3 Chordal Graphs

The class of *chordal graphs* is a very well known class of graphs which arises in many practical situations. A graph G is chordal if any cycle of G with a size at least 4 has a chord (*i.e.*, an edge linking two non-consecutive vertices of the cycle). There are many characterizations of chordal graphs, see for instance [7].

In this section, we devise a polynomial time algorithm to compute an optimal CVC in chordal graphs. To achieve this, we need the following lemma.

Lemma 4. *Let $G = (V, E)$ be a connected chordal graph and let S be a vertex cover of G. The following properties hold:*

(*i*) *The connected contraction $G_0 = (V_0, E_0) = G^c_S$ of G following S is a tree.*
(*ii*) *If G is biconnected, then S is a connected vertex cover of G.*

Proof. Let S be a vertex cover of G.

For (i): from Lemma 1, we know that $G_0 = (V_0, E_0) = G_S^c$ is bipartite and connected. Assume that G_0 is not a tree, and let Γ be a cycle of G_0 with a minimal size. By construction, Γ is chordless, has a size at least 4 and alternates vertices of $New(G_0)$ and vertices of $V \setminus S$. From Γ, we can build a cycle Γ' of G using the following rule: if $\{x, v_{A_i}\} \in \Gamma$ and $\{v_{A_i}, y\} \in \Gamma$ where $x, y \notin S$ and $v_{A_i} \in New(G_0)$ (where we recall that A_i is some connected component of $G[S]$), then we replace these two edges by a shortest path $\mu_{x,y}$ from x to y in G among the paths from x to y in G which only use vertices of A_i (such a path exists since A_i is connected and is linked to x and y); by repeating this operation, we obtain a cycle Γ' of G with $|\Gamma'| \geq |\Gamma| \geq 4$. Let us prove that Γ' is chordless which will lead to a contradiction since G is assumed to be chordal. Let v_1, v_2 be two non consecutive vertices of Γ'. If $v_1 \notin S$ and $v_2 \notin S$, then $\{v_1, v_2\} \notin E$ since otherwise Γ would have a chord in G_0. So, we can assume that $v_1 \in (\mu_{x,y} \setminus \{x, y\})$ and $v_2 \in \mu_{x,y}$ (since there is no edge linking two vertices of disjoint paths $\mu_{x,y}$ and $\mu_{x',y'}$); in this case, using edge $\{v_1, v_2\}$, we could obtain a path which uses strictly less edges than $\mu_{x,y}$.

For (ii): Suppose that S is not connected. Then, from (i) we deduce that G_0 is not a star and thus, there are two edges $\{v_{A_i}, x\}$ and $\{x, v_{A_j}\}$ in G_0 where A_i and A_j are two connected components of S. We deduce that x would be a cut-vertex of G, contradiction since G is assumed to be biconnected.

In particular, using (ii) of Lemma 4, we deduce that any optimal vertex cover S^* of a biconnected chordal graph G is also an optimal connected vertex cover.

Now, we give a simple linear algorithm for computing an optimal connected vertex cover of a chordal graph.

Theorem 1. MinCVC *is polynomial in chordal graphs. Moreover, an optimal solution can be found in linear time.*

Proof. Following Lemma 3, solving MinCVC in a chordal graph $G = (V, E)$ can be done by solving MinPrextCVC in each of the biconnected components $G_i = (V_i, E_i)$ of G. Since G_i is both biconnected and chordal, by Lemma 4, MinPrextCVC is the same problem as MinPrextVC (in G_i). But, by adding a pendant edge to vertices required to be taken in the vertex cover, we can easily reduce MinPrextVC to MinVC (note that the graph remains chordal). Since computing the biconnected components and solving MinVC in a chordal graph can be done in linear time (see [7]), the result follows.

4 Bipartite Graphs

A bipartite graph $G = (V, E)$ is a graph where the vertex set is partitioned into two independent sets L and R. Using the result of [10], we already know that MinCVC is **NP**-hard in planar bipartite graphs of maximum 4. Using Lemma 3, we can strengthen this result:

Lemma 5. MinCVC *is **NP**-hard in biconnected planar bipartite graphs of maximum degree 4.*

Now, one can show that MinCVC has no PTAS in bipartite graphs of maximum degree 4.

Theorem 2. MinCVC *is not* 1.001031*-approximable in connected bipartite graphs* $G = (L, R; E)$ *where* $\forall l \in L$, $d_G(l) \leq 4$ *and* $\forall r \in R$, $d_G(r) \leq 3$, *unless **P**=**NP**.*

In Theorem 2, we proved in particular that MinCVC is **NP**-hard when all the vertices of one part of the bipartition have a degree at most 3. It turns out that if all the vertices of one part of this bipartition have a degree at most 2, the problem becomes easy. This property will be very useful to devise our approximation algorithm in Subsection 5.1.

Lemma 6. MinCVC *is polynomial in bipartite graphs* $G = (L, R; E)$ *such that* $\forall r \in R$, $d_G(r) \leq 2$. *Moreover, if* $L_2 = \{l \in L : d_G(l) \geq 2\}$, *then* $opt(G) = |L| + |L_2| - 1$.

5 Approximation Results

MinCVC is trivially **APX**-complete in k-connected graphs for any $k \geq 2$ since starting from graph $G = (V, E)$, instance of MinVC, we can add a clique K_k of size k and link each vertex of G to each vertex of K_k. This new graph G' is obviously k-connected and S is a vertex cover of G iff S union the k vertices of K_k (we can always assume that $S \neq V$) is a connected vertex cover of G'. Thus, using the negative result of [15] it is quite improbable that one can improve the approximation ratio of 2 for MinCVC, even in k-connected graphs. Thus, in this subsection we deal with the approximability of MinCVC in particular classes of graphs.

In Subsection 5.1, we devise a 5/3-approximation algorithm for any class of graphs where the classical vertex cover problem is polynomial. In Subsection 5.2, we show that MinCVC admits a PTAS in planar graphs.

5.1 When MinVC Is Polynomial

Let $\mathcal{G}$ be a class of connected graphs where MinVC is polynomial (for instance, the connected bipartite graphs). The underlying idea of the algorithm is simple: we first compute an optimal vertex cover, and then try to connect it by adding vertices (either using high degree vertices or Lemma 6). The analysis leading to the ratio 5/3 is based on Lemma 1 which deals with graph contraction.

Now, let us formally describe the algorithm. Recall that given a vertex set V', $G^c_{V'}$ denotes the connected contraction of V following V', and $New(G^c_{V'})$ denotes the set of new vertices (one for each connected component of $G[V']$).

$\texttt{algo}_{CVC}$ input: A graph $G = (V, E)$ of $\mathcal{G}$ with at least 3 vertices.

1 Find an optimal vertex cover S^* of G such that in $G^c_{S^*}$, $\forall v \in New(G^c_{S^*})$, $d_{G^c_{S^*}}(v) \geq 2$;
2 Set $G_1 = G^c_{S^*}$, $N_1 = New(G^c_{S^*})$, $S = S^*$ and $i = 1$;
3 While $|N_i| \geq 2$ and there exists $v \notin N_i$ such that v is linked in G_i to at least 3 vertices of N_i do
  3.1 Set $S := S \cup \{v\}$ and $i := i + 1$;
  3.2 Set $G_i := G^c_S$ and $N_i = New(G^c_S)$;
4 If $|N_i| \geq 2$, apply the polynomial algorithm of Lemma 6 on G_i (let S' be the produced solution) and set $S := S \cup (V \cap S')$;
5 Output S;

Now, we show that $\texttt{algo}_{CVC}$ outputs a connected vertex cover of G in polynomial time. First of all, given an optimal vertex cover S^* of a graph G (assumed here to be computable in polynomial time), we can always transform it in such a way that $\forall v \in New(G^c_{S^*})$, $d_{G^c_{S^*}}(v) \geq 2$. Indeed, if a vertex of $G^c_{S^*}$ corresponding to a connected component of S^* has only one neighbor in $G^c_{S^*}$, then we can take this neighbor in S^* and remove one vertex on this connected component (and the number of such 'leaf' connected components decreases, as soon as $G^c_{S^*}$ has at least 3 vertices). Now, using (ii) of Lemma 1, we know that $New(G^c_{S^*})$ is an optimal vertex cover of $G^c_{S^*}$. Then, from $New(G^c_{S^*})$, we can find such a solution within polynomial time.

Moreover, using (i) of Lemma 1 with S^*, we deduce that the graph G_i is bipartite, for each possible value of i. Assume that $G_i = (N_i; R_i, E_i)$ for iteration i where N_i is the left set corresponding to the contracted vertices and R_i is the right set corresponding to the remaining vertices and let p be the last iteration. Clearly, if $|N_p| = 1$, the the output solution S is connected. Otherwise, the algorithm uses step 4; we know that G_p is bipartite and by construction $\forall r \in R_p$, $d_{G_p}(r) \leq 2$. Thus, we can apply Lemma 6 on G_p. Moreover, a simple proof also gives that $\forall l \in N_p$, $d_{G_p}(l) \geq 2$. Indeed, otherwise there exists $l \in N_p$ such that l has a unique neighbor $r_0 \in R_p$. Let $\{x_1, \cdots, x_j\} \subseteq N_{p-1}$ with $j \geq 3$ and r_1 be the vertices contracted in G_{p-1} in order to obtain G_p. We conclude that the neighborhood of $\{x_1, \cdots, x_j\}$ is $\{r_0, r_1\}$ in G_{p-1} which is impossible since on the one hand, N_{p-1} is an optimal vertex cover of G_{p-1} (using (iii) of Lemma 1), and on the other hand, by flipping $\{x_1, \cdots, x_j\}$ with $\{r_0, r_1\}$, we obtain another vertex cover of G_{p-1} with smaller size than N_{p-1}! Finally, using Lemma 6, an optimal connected vertex cover of G_p consists of taking N_p and $|N_p| - 1$ of R_p. In conclusion, S is a connected vertex cover of G.

We now prove that this algorithm improves the ratio 2.

Theorem 3. *Let $\mathcal{G}$ be a class of connected graphs where* MinVC *is polynomial. Then, $\texttt{algo}_{CVC}$ is a 5/3-approximation for* MinCVC *in $\mathcal{G}$.*

Proof. Let $G = (V, E) \in \mathcal{G}$. Let S be the approximate solution produced by $\texttt{algo}_{CVC}$ on G. Using the previous notations and Lemma 6, the solution S has a value $apx(G)$ satisfying:

$$apx(G) = |S^*| + p - 1 + |N_p| - 1 \tag{1}$$

where p is the number of iterations of step 3. Obviously, we have:

$$opt(G) \geq |S^*| \tag{2}$$

Now let us prove that for any $i = 1, \cdots, p-1$, we also have $opt(G_i) \geq opt(G_{i+1}) + 1$. Let S_i^* be an optimal connected vertex cover of G_i. Let $r_i \in R_i$ be the vertex added to S during iteration i and let $N_{G_i}(r_i)$ be the neighbors of r_i in G_i. The graph G_{i+1} is obtained from the contraction of G_i with respect to the subset $S_i = \{r_i\} \cup N_{G_i}(r_i)$. Thus, if v_{S_i} denotes the new vertex resulting from the contraction of S_i, then $(S_i^* \setminus S_i) \cup \{v_{S_i}\}$ is a connected vertex cover of G_{i+1}. Moreover, $|S_i^* \cap S_i| \geq 2$ since either $r_i \in S_i^*$ and at least one of these neighbors must belong to S_i^* (S_i^* is connected and $i < p$) or $N_{G_i}(r_i) \subseteq S_i^*$ since S_i^* is a vertex cover. Thus $opt(G_{i+1}) \leq |S_i^* \setminus S_i| + 1 = opt(G_i) - |S_i^* \cap S_i| + 1 \leq opt(G_i) - 1$. Summing up these inequalities for $i = 1$ to $p - 1$, and using that $opt(G) \geq opt(G_1)$, we obtain:

$$opt(G) \geq opt(G_p) + p - 1 \tag{3}$$

Moreover, thanks to Lemma 6, we know that $opt(G_p) = 2|N_p| - 1$. Together with equation (3), we get:

$$opt(G) \geq 2|N_p| - 1 + p - 1 \tag{4}$$

Finally, since each vertex chosen in step 3 has degree at least 3, we get $|N_{i+1}| \leq |N_i| - 2$. This immediately leads to $|N_1| \geq |N_p| + 2(p-1)$. Since $|S^*| \geq |N_1|$, we get:

$$|S^*| \geq |N_p| + 2(p-1) \tag{5}$$

Combination of equations (2), (4) and (5) with coefficients 4, 1 and 1 (respectively) gives:

$$5opt(G) \geq 3|S^*| + 3|N_p| - 1 + 3(p-1) \tag{6}$$

Then, equation (1) allows to conclude.

5.2 Planar Graphs

Given a planar embedding of a planar graph $G = (V, E)$, the level of a vertex is defined as follows (see for instance [4]): the vertices on the exterior face are at level 1. Given vertices at level i, let f be an interior face of the subgraph induced by vertices at level i. If G_f denotes the subgraph induced by vertices included

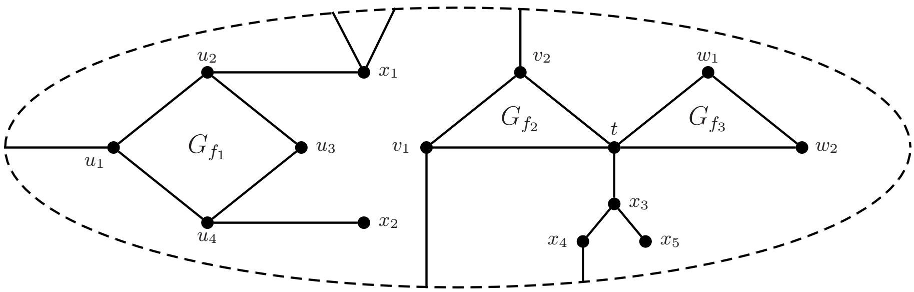

Fig. 1. Level of a planar graph

in f, then the vertices on the exterior face of G_f are at level $i+1$. The set of vertices at level i is called the layer L_i.

This is illustrated on Figure 1. The dashed ellipse represents an interior face on level $i-1$. Depicted vertices are at level i. There are 3 interior faces (constituted respectively by the u_i's, by $\{v_1, v_2, t\}$ and $\{t, w_1, w_2\}$).

Baker gave in [4] a polynomial time approximation scheme for several problems including vertex cover in planar graphs. The underlying idea is to consider k-outerplanar subgraphs of G constituted by k consecutive layers. The polynomiality of vertex cover in k-outerplanar graphs (for a fixed k) allows to achieve a $(k+1)/k$ approximation ratio.

We adapt this technique in order to achieve an approximation scheme for MinCVC (MinCVC is **NP**-hard in planar graphs, see [12]). First of all, note that k-outerplanar graphs have treewidth bounded above by $3k-1$, [6]. Since MinCVC is polynomially solvable for graphs with bounded treewidth, [19], MinCVC is polynomial for k-outerplanar graphs.

Theorem 4. MinCVC *admits an approximation scheme in planar graphs.*

Proof. Given an embedding of a planar (connected) graph G, we define, as previously, the layers $L_1, \cdots, L_q$ of G. For each layer L_i, we define F_i as the set of vertices of L_i that are in an interior face of L_i. For instance, in Figure 1, all vertices but the x_i's are in F_i.

Following the principle of the approximation scheme for vertex cover, we define an algorithm for any integer $k > 0$. Let $V_i = F_i \cup L_{i+1} \cup L_{i+2} \cup \ldots \cup L_{i+k}$, and G_i be the graph induced by V_i. Note that G_i is not necessarily connected since for example there can be several disjoint faces in F_i (there are two connected components in Figure 1).

Let S^* be an optimum connected vertex cover on G with value $opt(G)$, and $S_i^* = S^* \cap V_i$. Then of course S_i^* is a vertex cover of G_i. However, even restricted to a connected component of G_i, it is not necessarily connected. Indeed, S^* is connected but the path(s) connecting two vertices of S^* in a connected component of G_i may use vertices out of this connected component. To overcome this problem, notice that only vertices in F_i or in F_{i+k} connect V_i to $V \setminus V_i$. Hence,

$S_i^* \cup F_i \cup F_{i+k}$ can be partitioned into a set of connected vertex covers on each of the connected components of G_i (since F_i and F_{i+k} are made of cycles). Now, take an optimum connected vertex cover on each of these connected components, and define S_i as the union of these optimum solutions. Then, we have :

$$|S_i^* \cup F_i \cup F_{i+k}| \geq |S_i| \tag{7}$$

Now, let $p \in \{1, \ldots, k\}$. Let $V_0 = L_1 \cup L_2 \cup \ldots \cup L_p$, G_0 be the subgraph of G induced by V_0, $S_0^* = S^* \cap V_0$, and S_0 be an optimum vertex cover on G_0. With similar arguments as previously, we have:

$$|S_0^* \cup F_p| \geq |S_0| \tag{8}$$

We build a solution S^p on the whole graph G as follows. S^p is the union of S_0 and of all S_i's for $i = p \mod k$. Of course, S^p is a vertex cover of G, since any edge of G appears in at least one G_i (or G_0). Moreover, it is connected since:

- S_0 is connected, and each S_i is made of connected vertex covers on the connected components of G_i;
- each of these connected vertex covers in S_i is connected to S_{i-k} (or to S_0 if $i = p$): this is due to the fact that F_i belongs to V_i and to V_{i-k} (or V_0). Hence, a level i interior face f is common to S_{i-k} (or S_0) and to the connected vertex cover of S_i we are dealing with. Both partial solutions cover all the edges of this face f. Since f is a cycle, the two solutions are necessarily connected. In other words, each connected component of S_i is connected to S_{i-k} (or S_0) and, by recurrence, to S_0. Consequently, the whole solution S^p is connected.

Summing up equation (7) for each $i = p \bmod k$ and equation (8), we get:

$$|S_0^* \cup F_p| + \sum_{i=p \bmod k} |S_i^* \cup F_i \cup F_{i+k}| \geq |S_0| + \sum_{i=p \bmod k} |S_i| \tag{9}$$

By definition of S^p, we have $|S^p| \leq |S_0| + \sum_{i=p \bmod k} |S_i|$. On the other hand, since only vertices in F_i ($i = p \bmod k$) appear in two different V_i's ($i = 0$ or $i = p \bmod k$), we get that $|S_0^* \cup F_p| + \sum_{i=p \bmod k} |S_i^* \cup F_i \cup F_{i+k}| \leq |S^*| + 2\sum_{i=p \bmod k} |F_i|$. This leads to:

$$opt(G) + 2 \sum_{i=p \bmod k} |F_i| \geq |S^p| \tag{10}$$

If we consider the best solution S with value $apx(G)$ among the S^p's ($p \in \{1, \ldots, k\}$), we get :

$$opt(G) + \frac{2}{k} \sum_{i=1}^{q} |F_i| \geq apx(G) \tag{11}$$

To conclude, we observe that the following property holds:

Property 1. S^* takes at least one fourth of the vertices of each F_i.

To see this property of $S^* \cap F_i$, consider F_i and the set E_i of edges of G that belong to one and only one interior face of F_i (for example, in Figure 1, if there were edges $\{u_2, u_4\}$ and $\{u_3, v_1\}$, they would not be in E_i). Let n_i be the number of vertices in F_i, and m_i the number of edges in E_i. This graph is a collection of edge-disjoint (but not vertex-disjoint, as one can see in Figure 1) interior faces (cycles). Of course, $S^* \cap F_i$ is a vertex cover of this graph. Since this graph is a collection of interior faces (cycles), on each of these faces f $S^* \cap F_i$ cannot reject more than $|f|/2$ vertices. In all,

$$|S^* \cap F_i| \geq n_i - \sum_{f \in F_i} \frac{|f|}{2} \tag{12}$$

But since faces are edge-disjoint, $\sum_{f \in F_i} |f| = m_i$. On the other hand, if N_f denotes the number of interior faces in F_i, since each face contains at least 3 vertices, $m_i = \sum_{f \in F_i} |f| \geq 3N_f$. Since the graph is planar, using Euler formula we get $1 + m_i = n_i + N_f \leq n_i + m_i/3$. Hence, $m_i \leq 3n_i/2$. Finally, $|S^* \cap F_i| \geq n_i - m_i/2 \geq n_i/4$.

Based on this property, we get:

$$opt(G)\left(1 + \frac{8}{k}\right) \geq apx(G) \tag{13}$$

Taking k sufficiently large leads to a $1 + \varepsilon$ approximation. The polynomiality of this algorithm follows from the fact that each subgraph we deal with is (at most) $k + 1$-outerplanar, hence for a fixed k we can find an optimum solution in polynomial time.

References

1. Alimonti, P., Kann, V.: Some APX-completeness results for cubic graphs. Theor. Comput. Sci. 237(1-2), 123–134 (2000)
2. Arkin, E.M., Halldórsson, M.M., Hassin, R.: Approximating the tree and tour covers of a graph. Inf. Process. Lett. 47(6), 275–282 (1993)
3. Ausiello, G., Crescenzi, P., Gambosi, G., Kann, V., Marchetti-Spaccamela, A., Protasi, M.: Complexity and Approximation (Combinatorial Optimization Problems and Their Approximability Properties). Springer, Berlin (1999)
4. Baker, B.S.: Approximation algorithms for NP-complete problems on planar graphs. J. ACM 41(1), 153–180 (1994)
5. Bar-Yehuda, R., Even, S.: On approximating a vertex cover for planar graphs. In: STOC, pp. 303–309 (1982)
6. Bodlaender, H.L.: A partial -arboretum of graphs with bounded treewidth. Theor. Comput. Sci. 209(1-2), 1–45 (1998)
7. Brandstadt, A., Le, V.B., Spinrad, J.: Graph classes: a survey. Society for Industrial and Applied Mathematic, Philadelphia (1999)
8. Dawar, A., Grohe, M., Kreutzer, S., Schweikardt, N.: Approximation schemes for first-order definable optimisation problems. In: LICS, pp. 411–420 (2006)
9. Demaine, E.D., Hajiaghayi, M.T.: Bidimensionality: new connections between fpt algorithms and ptass. In: SODA, pp. 590–601 (2005)

10. Fernau, H., Manlove, D.: Vertex and edge covers with clustering properties: Complexity and algorithms. In: Algorithms and Complexity in Durham, pp. 69–84 (2006)
11. Fujito, T.: How to trim an mst: A 2-approximation algorithm for minimum cost tree cover. In: Bugliesi, M., Preneel, B., Sassone, V., Wegener, I. (eds.) ICALP 2006. LNCS, vol. 4051, pp. 431–442. Springer, Heidelberg (2006)
12. Garey, M.R., Johnson, D.S.: The rectilinear steiner tree problem in NP complete. SIAM Journal of Applied Mathematics 32, 826–834 (1977)
13. Garey, M.R., Johnson, D.S.: Computer and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, New York (1979)
14. Guo, J., Niedermeier, R., Wernicke, S.: Parameterized complexity of generalized vertex cover problems. In: Dehne, F., López-Ortiz, A., Sack, J.-R. (eds.) WADS 2005. LNCS, vol. 3608, pp. 36–48. Springer, Heidelberg (2005)
15. Khot, S., Regev, O.: Vertex cover might be hard to approximate to within $2 - \varepsilon$. In: IEEE Conference on Computational Complexity, pp. 379–386. IEEE Computer Society Press, Los Alamitos (2003)
16. Könemann, J., Konjevod, G., Parekh, O., Sinha, A.: Improved approximations for tour and tree covers. Algorithmica 38(3), 441–449 (2003)
17. Mölle, D., Richter, S., Rossmanith, P.: Enumerate and expand: Improved algorithms for connected vertex cover and tree cover. In: Grigoriev, D., Harrison, J., Hirsch, E.A. (eds.) CSR 2006. LNCS, vol. 3967, pp. 270–280. Springer, Heidelberg (2006)
18. Mölle, D., Richter, S., Rossmanith, P.: Enumerate and expand: New runtime bounds for vertex cover variants. In: Chen, D.Z., Lee, D.T. (eds.) COCOON 2006. LNCS, vol. 4112, pp. 265–273. Springer, Heidelberg (2006)
19. Moser, H.: Exact algorithms for generalizations of vertex cover. Master's thesis, Institut für Informatik, Friedrich-Schiller-Universität Jena (2005)
20. Savage, C.D.: Depth-first search and the vertex cover problem. Inf. Process. Lett. 14(5), 233–237 (1982)
21. Ueno, S., Kajitani, Y., Gotoh, S.: On the nonseparating independent set problem and feedback set problem for graphs with no vertex degree exceeding three. Discrete Mathematics 72, 355–360 (1988)
22. Wanatabe, T., Kajita, S., Onaga, K.: Vertex covers and connected vertex covers in 3-connected graphs. In: IEEE International Sympoisum on Circuits and Systems, pp. 1017–1020. IEEE Computer Society Press, Los Alamitos (1991)