On the Number of α-Orientations

Stefan Felsner and Florian Zickfeld

Technische Universität Berlin, Fachbereich Mathematik
Straße des 17. Juni 136, 10623 Berlin, Germany
{felsner,zickfeld}@math.tu-berlin.de

Abstract. We deal with the asymptotic enumeration of combinatorial structures on planar maps. Prominent instances of such problems are the enumeration of spanning trees, bipartite perfect matchings, and ice models. The notion of an α-orientation unifies many different combinatorial structures, including the afore mentioned. We ask for the number of α-orientations and also for special instances thereof, such as Schnyder woods and bipolar orientations. The main focus of this paper are bounds for the maximum number of such structures that a planar map with n vertices can have. We give examples of triangulations with 2.37^n Schnyder woods, 3-connected planar maps with 3.209^n Schnyder woods and inner triangulations with 2.91^n bipolar orientations. These lower bounds are accompanied by upper bounds of 3.56^n, 8^n, and 3.97^n, respectively. We also show that for any planar map M and any α the number of α-orientations is bounded from above by 3.73^n and we present a family of maps which have at least 2.598^n α-orientations for n big enough.

1 Introduction

A *planar map* is a planar graph together with a plane drawing. Many different structures on planar maps have attracted the attention of researchers. Among them are spanning trees, bipartite perfect matchings (or more generally bipartite f-factors), Eulerian orientations, Schnyder woods, bipolar orientations and 2-orientations of quadrangulations. The concept of α-orientations is a quite general one. Remarkably, all the above structures can be encoded by certain α-orientations. Let a planar map M with vertex set V and a function $\alpha : V \to \mathbb{N}$ be given. An orientation X of the edges of M is an α-orientation if every vertex v has out-degree $\alpha(v)$.

For some of the structures mentioned above it is not obvious how to encode them as α-orientations. For Schnyder woods on triangulations the encoding by 3-orientations goes back to de Fraysseix and de Mendez [6]. For bipolar orientations an encoding was proposed by Woods and independently by Tamassia and Tollis [22]. Bipolar orientations of M are one of the structures which cannot be encoded as α-orientations on M, an auxiliary map M' (the angle graph of M) has to be used instead. For Schnyder woods on 3-connected planar maps as well as bipartite f-factors and spanning trees Felsner [10] describes encodings as α-orientations. He also proves that the set of α-orientations of a planar map

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M is a distributive lattice. This structure on the set of α-orientations found applications in drawing algorithms for example in [3], and for enumeration and random sampling of graphs in [12].

Given the existence of a combinatorial structure on a class $\mathcal{M}_n$ of planar maps with n vertices, one of the questions of interest is how many such structures there are for a given map $M \in \mathcal{M}_n$. Especially, one is interested in the minimum and maximum that this number attains on the maps from $\mathcal{M}_n$. This question has been treated quite successfully for spanning trees and bipartite perfect matchings. For spanning trees the Kirchhoff Matrix Tree Theorem comes into the game and allows to bound the maximum number of spanning trees of a planar graph with n vertices between 5.02^n and 5.34^n, see [19]. Pfaffian orientations can be used to efficiently calculate the number of bipartite perfect matchings in the planar case, see for example [16]. Kasteleyn has shown, that the $k \times \ell$ square grid has about $e^{0.29 \cdot k\ell} \approx 1.34^{k\ell}$ perfect matchings. The number of Eulerian orientations is studied in statistical physics under the name of ice models. In particular Lieb [15] has shown that the square grid on the torus has $(8\sqrt{3}/9)^{k\ell} \approx 1.53^{k\ell}$ Eulerian orientations and Baxter [1] has worked out the asymptotics for the triangular grid on the torus as $(3\sqrt{3}/2)^{k\ell} \approx 2.598^{k\ell}$.

In many cases it is relatively easy to see which maps in a class $\mathcal{M}_n$ carry a unique object of a certain type, while the question about the maximum number is rather intricate. Therefore, we focus on finding the asymptotics or lower and upper bounds for the maximum number of α-orientations that a map from $\mathcal{M}_n$ can carry. The next table gives an overview of the results of this paper for different instances of $\mathcal{M}_n$ and α. The entry c in the "Upper Bound" column is to be read as $O(c^n)$, in the "Lower Bound" column as $\Omega(c^n)$ and for the "$\approx c$" entries the asymptotics are known.

The paper is organized as follows. In Section 2 we treat the most general case, where $\mathcal{M}_n$ is the class of all planar maps with n vertices and α can be any integer valued function. Apart from giving lower and upper bounds we briefly discuss the complexity of counting α-orientations and a reduction to counting perfect

Graph class and orientation type	**Lower bound**	**Upper bound**
α-orientations on planar maps	2.59	3.73
Eulerian orientations on planar maps	2.59	3.73
Schnyder woods on triangulations	2.37	3.56
Schnyder woods on the square grid	≈ 3.209	
Schnyder woods on 3-connected planar maps	3.209	8
2-orientations on quadrangulations	1.53	1.91
bipolar orientations on stacked triangulations	≈ 2	
bipolar orientations on outerplanar maps	≈ 1.618	
bipolar orientations on the square grid	2.18	2.62
bipolar orientations on planar maps	2.91	3.97

matchings of bipartite graphs. In Section 3 we consider Schnyder woods on plane triangulations and the more general case of Schnyder woods on 3-connected planar maps. We split the treatment of Schnyder woods because the more direct encoding of Schnyder woods on triangulations as α-orientations yields stronger bounds. We also discuss the asymptotic number of Schnyder woods on the square grid. In Section 4, we study bipolar orientations on the square grid and outerplanar maps as well as general planar maps. The upper bound for planar maps relies on a new encoding of bipolar orientations of inner triangulations. We conclude with some open problems.

In Sections 2 and 3 we include some proofs since many of the other proofs use these results or similar techniques. Proofs omitted due to space constraints are available in a full version of this paper on the authors' homepage.

2 Counting α-Orientations

A *planar map* M is a simple planar graph G together with a fixed crossing-free embedding of G in the Euclidean plane. In particular M has a designated outer (unbounded) face. We denote the sets of vertices, edges and faces of a given planar map by V, E, and $\mathcal{F}$, and their respective cardinalities by n, m and f. The degree of a vertex v will be denoted by $d(v)$.

Let M be a planar map and $\alpha : V \to \mathbb{N}$. An edge orientation X of M is an *α-orientation* if every $v \in V$ has exactly $\alpha(v)$ edges directed away from it in X.

Let X be an α-orientation of G and let C be a directed cycle in X. Define X^C as the orientation obtained from X by reversing all edges of C. Since the reversal of a directed cycle does not affect out-degrees the orientation X^C is also an α-orientation of M. The plane embedding of M allows us to classify a directed simple cycle as clockwise if the interior is to the right of C or as counterclockwise otherwise. If C is a ccw-cycle of X then we say that X^C is *left of* X. The set of α-orientations of M endowed with the transitive closure of the 'left of' relation is a distributive lattice [10].

The following observation is easy but very useful. Let M and $\alpha : V \to \mathbb{N}$ be given, $W \subset V$ and E_W the edges of M with one endpoint in W and one in $V \setminus W$. Suppose all edges of E_W are directed away from W in some α-orientation X_0 of M. Then, the demand of W for $\sum_{w \in W} \alpha(w)$ outgoing edges forces all edges in E_W to be directed away from W in every α-orientation of M. Edges with the same direction in every α-orientation are called *rigid*.

We denote the number of α-orientations of M by $r_\alpha(M)$. Most of this paper is concerned with lower and upper bounds for $\max_{M \in \mathcal{M}} r_{\alpha_M}(M)$ for some class $\mathcal{M}$ of planar maps. A planar map has at most 2^m α-orientations as every edge can be directed in at most two ways.

Lemma 1. *Let M be a planar map and $A \subset E$ be a cycle free subset of edges of M. Then, there are at most $2^{m-|A|}$ α-orientations of M. This holds for every function $\alpha : V \to \mathbb{N}$. Furthermore, M has less than 4^n α-orientations.*

Proof. Let X be an arbitrary but fixed orientation out of the $2^{m-|A|}$ orientations of the edges of $E \setminus A$. It suffices to show that X can be extended to an

α-orientation of M in at most one way. We proceed by induction over $|A|$. The base case $|A| = 0$ is trivial. If $|A| > 0$, then, as A is cycle free, there is a vertex v which is incident to exactly one edge e of A. If v has out-degree $\alpha(v)$ respectively $\alpha(v) - 1$ in X, then e must be directed towards respectively away from v. In either case the direction of e is determined by X, and by induction there is at most one way to extend the resulting orientation of $E \setminus (A - e)$ to an α-orientation of M. If v does not have $\alpha(v)$ or $\alpha(v) - 1$ outgoing edges, then there is no extension of X to an α-orientation of M. The bound $2^{m-n+1} < 4^n$ follows by choosing A to be a spanning forest and applying Euler's formula. □

To improve on this bound we state Lemma 2, but the proof is omitted here.

Lemma 2. *Let M be a planar map with n vertices that has an independent set of n_2 vertices, which have degree 2 in M. Then, M has at most $(3n-6)-(n_2-1)$ edges.*

Proposition 1. *Let M be a planar map, $\alpha : V \to \mathbb{N}$, and $I = I_1 \cup I_2$ an independent set of M, where I_2 is the set of vertices in I, which have degree 2 in M. Then, M has at most*

$$2^{2n-4-|I_2|} \cdot \prod_{v \in I_1} \left(\frac{1}{2^{d(v)-1}} \binom{d(v)}{\alpha(v)} \right) \tag{1}$$

α-orientations.

Proof. We may assume that M is connected. Let M_i, for $i = 1, \ldots c$, be the components of $M - I$. We claim that M has at most $(3n-6)-(c-1)-(|I_2|-1)$ edges. Note, that every component C of $M - I$ must be connected to some other component C' via a vertex $v \in I$ such that the edges vw and vw' with $w \in C$ and $w' \in C'$ form an angle at v. As w and w' are in different connected components the edge ww' is not in M and we can add it without destroying planarity. We can add at least $c - 1$ edges not incident to I in this fashion. Thus, by Lemma 2 we have that $m + (c - 1) \leq 3n - 6 - (I_2 - 1)$.

Let S' be a spanning forest of $M - I$, and let S be obtained from S' by adding one edge incident to every $v \in I$, S has $n - c$ edges. By Lemma 1 M has at most $2^{m-|S|}$ α-orientations and we note that $m - |S| \leq (3n - 6) - (c - 1) - (|I_2| - 1) - (n - c) = 2n - 4 - |I_2|$.

For every vertex $v \in I_1$ there are $2^{d(v)-1}$ possible orientations of the edges of $M - S$ at v. Only the orientations with $\alpha(v)$ or $\alpha(v) - 1$ outgoing edges at v can potentially be completed to an α-orientation of M. Since I_1 is an independent set it follows that M has at most

$$2^{m-|S|} \cdot \prod_{v \in I_1} \frac{\binom{d(v)-1}{\alpha(v)} + \binom{d(v)-1}{\alpha(v)-1}}{2^{d(v)-1}} \leq 2^{2n-4-|I_2|} \cdot \prod_{v \in I_1} \frac{\binom{d(v)}{\alpha(v)}}{2^{d(v)-1}} \tag{2}$$

α-orientations. □

Proposition 2. *For every α and M we have that $r_\alpha(M) \leq 3.73^n$. There are infinitely many graphs with more than 2.59^n Eulerian orientations.*

Proof. By the Four Color Theorem every planar map allows for an independent set I of size $n/4$. Edges incident to degree 1 vertices are always rigid, so we assume $d(v) \geq 3$ for $v \in I_1$. We use Proposition 1 and upper bound the right hand side of (2) to conclude that $r_\alpha(M) \leq 2^{2n-4-|I_2|} \cdot \left(\frac{3}{4}\right)^{|I_1|} \leq 3.73^n$. The lower bound uses a planarization of the triangular grid on the torus, which was mentioned in the introduction and is shown on the right in Figure 2. □

Given a planar map M and some $\alpha : V \to \mathbb{N}$, what is the complexity of computing the number of α-orientations of M? In some instances this number can be computed efficiently, e.g. for perfect matchings and spanning trees of general planar maps, as mentioned in the introduction.

Recently, Creed [4] has shown that counting Eulerian orientations is $\#P$-complete even for planar maps. In the full paper we use Creed's method and a reduction from perfect matchings of k-regular bipartite graphs, see [5], to show the following. Counting α-orientations is $\#P$-complete for 4-regular planar maps with $\alpha : V \to \{1, 2, 3\}$ as well as for planar maps with vertex degrees in $\{3, 4, 5\}$ and $\alpha(v) = 2$ for all $v \in V$.

In general, computing the number of α-orientations can be reduced to counting f-factors in bipartite planar graphs and thus to counting perfect matchings in bipartite graphs [23]. This reduction is useful because bipartite perfect matchings have been the subject of extensive research (for example [16,13,18]). We mention two useful facts that follow from this relation. First, it can be tested in polynomial time if the α-orientations of a given map can be counted using Pfaffian orientations. If this is not the case, there is a fully polynomial randomized approximation scheme for approximating this number.

3 Counting Schnyder Woods

Schnyder woods for triangulations have been introduced as a tool for graph drawing and graph dimension theory in [20,21] and for 3-connected planar maps in [8]. Here we review the definitions and encodings as α-orientations, for a comprehensive introduction, see e.g. [9].

Let M be a planar map and a_1, a_2, a_3 be three vertices occurring in clockwise order on the outer face of M. A suspension M^σ of M is obtained by attaching a half-edge that reaches into the outer face to each of these *special vertices*. A *Schnyder wood* rooted at a_1, a_2, a_3 is an orientation and coloring of the edges of M^σ with the colors $1, 2, 3$ satisfying the following rules, see the left part of Figure **??**. Every edge e is oriented in one direction or in two opposite directions. The directions of edges are colored such that if e is bidirected the two directions have distinct colors. The half-edge at a_i is directed outwards and colored i. Every vertex v has out-degree one in each color. The edges e_1, e_2, e_3 leaving v in colors $1, 2, 3$ occur in clockwise order. Each edge entering v in color i enters v in the clockwise sector from e_{i+1} to e_{i-1}. There is no interior face the boundary of which is a monochromatic directed cycle. Note that for triangulations only the three outer edges are bidirected. The next theorem is from [6].

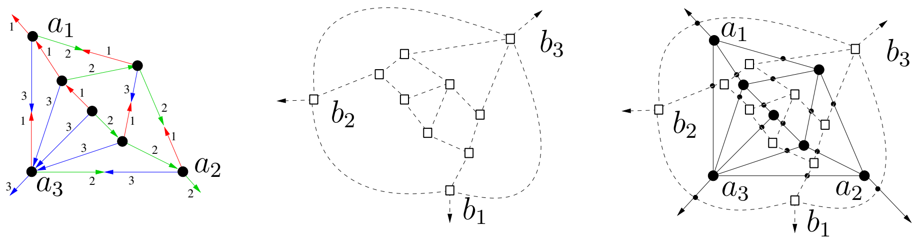

Fig. 1. Schnyder wood on a map M^σ, the suspension dual M^{σ^*}, the completion $\widetilde{M}$

Theorem 1. *Let T be a plane triangulation, let $\alpha_T(v) := 3$ if v is an internal vertex and $\alpha_T(v) := 0$ if v lies on the outer face. Then, there is a bijection between the Schnyder woods of T and the α_T-orientations of the interior edges of T.*

In the sequel we refer to an α_T-orientation simply as a 3-orientation. We now explain how Schnyder woods of non-triangular maps are encoded as α-orientations.

Let M^σ be a 3-connected planar map plus three rays emanating from three outer vertices a_1, a_2, a_3 into the unbounded face. The *suspension dual* M^{σ^*} of M^σ is obtained from the dual M^* of M as follows, see also Figure **??**. Replace the vertex v^*_∞, which represents the unbounded face of M in M^*, by a triangle on three new vertices b_1, b_2, b_3. Let P_i be the path from a_{i-1} to a_{i+1} on the outer face of M which avoids a_i. In M^{σ^*} the edges dual to those on P_i are incident to b_i instead of v^*_∞. Adding a ray to each of the b_i yields M^{σ^*}. The completion $\widetilde{M}$ of M^σ and M^{σ^*} is obtained by superimposing the two graphs such that exactly the primal dual pairs of edges cross. In the completion $\widetilde{M}$ the common subdivision of each crossing pair of edges is replaced by a new edge-vertex. Note that the rays emanating from the three special vertices of M^σ cross the three edges of the triangle induced by b_1, b_2, b_3 and thus produce edge vertices. The six rays emanating into the unbounded face of the completion end at a new vertex v_∞ placed in this unbounded face. Let a function α_M be defined on M^{σ^*} by $\alpha_M(v) = 3$ for every primal or dual vertex v, $\alpha_M(v_e) = 1$ for every edge vertex v_e, and $\alpha_M(v_\infty) = 0$ for the special closure vertex v_∞. For a proof of the next theorem see [10].

Theorem 2. *The Schnyder woods of a suspended planar map M^σ are in bijection with the α_M-orientations of $\widetilde{M}$.*

The full paper contains a constructive characterization of all maps with a unique Schnyder wood, thus generalizing the known characterization for triangulations.

Bonichon [2] found a bijection between Schnyder woods on triangulations with n vertices and pairs of non-crossing Dyck-paths, which implies that there are $C_{n+2}C_n - C^2_{n+1}$ Schnyder woods on triangulations with n vertices, where C_n denotes the nth Catalan number. Thus, there are asymptotically about 16^n Schnyder woods on triangulations with n vertices. Tutte's classic result says, that there are asymptotically about 9.48^n plane triangulations on n vertices. See [17] for a proof of Tutte's formula using Schnyder woods. The two results together imply, that on average there are about 1.68^n Schnyder woods on a triangulation with n vertices. The next theorem is concerned with the maximum number of Schnyder woods on a fixed triangulation.

Theorem 3. *Let $\mathcal{T}_n$ denote the set of all plane triangulations with n vertices and $\mathcal{S}(T)$ the set of Schnyder woods of $T \in \mathcal{T}_n$. Then, $2.37^n \leq \max_{T\in\mathcal{T}_n} |\mathcal{S}(T)| \leq 3.56^n$.*

Proof. The upper bound follows from Proposition 1 by using that $\binom{d(v)}{3}\cdot 2^{1-d(v)} \leq \frac{5}{8}$ for $d(v) \geq 3$. For the proof of the lower bound we introduce the *triangular grid*, which is derived from the *square grid* $G_{k,\ell}$. The graph $G_{k,\ell}$ has vertex set $\{(i,j) \mid 1 \leq i \leq k,\ 1 \leq j \leq \ell\}$ and all edges of the form $\{(i,j),(i,j+1)\}$ and $\{(i,j),(i+1,j)\}$. The triangular grid $T_{k,\ell}$ is obtained by adding the edges of the form $\{(i,j),(i-1,j+1)\}$. The grid triangulation $T^*_{k,\ell}$ is derived from $T_{k,\ell}$ by augmenting it with a triangle as shown in Figure 2.

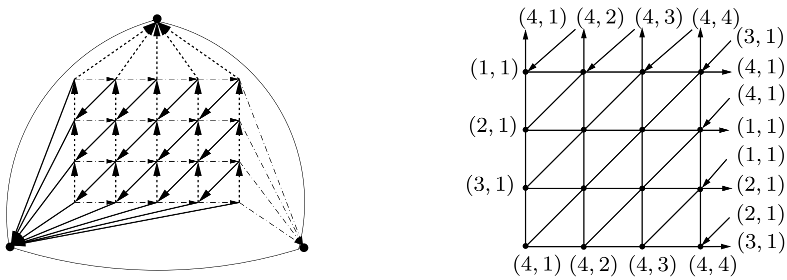

Fig. 2. The graphs $T^*_{4,5}$ with a canonical Schnyder wood and $T_{4,4}$ with the additional edges simulating Baxter's boundary conditions

Intuitively, $T^*_{k,\ell}$ promises to be a good candidate for a lower bound because the canonical orientation shown in Figure 2 on the left has many directed cycles. We formalize this now by showing that $T_{k,k}$ has at least $2^{5/4(k-1)^2}$ Schnyder woods, which yields the claimed bound for k big enough. Instead of working with the 3-orientations of $T^*_{k,\ell}$ we use α^*-orientations of $T_{k,\ell}$ where $\alpha^*(i,j) = 3$ if $2 \leq i \leq k-1$ and $2 \leq i \leq \ell - 1$ $\alpha^*(i,j) = 1$ if $(i,j) \in \{(1,1),(1,\ell),(k,\ell)\}$ and $\alpha^*(i,j) = 2$ otherwise. For simplicity, we refer to α^*-orientations of $T_{k,\ell}$ as 3-orientations.

The boundaries of the triangles of $T_{k,k}$ can be partitioned into two classes $\mathcal{C}$ and $\mathcal{C}'$ of directed cycles of cardinality $(k-1)^2$ each. No two cycles from the same class share an edge and $C \in \mathcal{C}'$ shares an edge with three cycles from $\mathcal{C}$ if it does not include a boundary edge.

For any subset D of $\mathcal{C}$ reversing all the cycles in D yields a 3-orientation of $T_{k,k}$, and we can encode this orientation as a 0-1-sequence of length $(k-1)^2$. After performing the flips of a given 0-1-sequence a a cycle $C' \in \mathcal{C}'$ is directed if and only if either all or none of the three cycles sharing an edge with C' have been reversed. Thus the number of different cycle flip sequences on $\mathcal{C} \cup \mathcal{C}'$ is bounded from below by $\sum_{a\in\{0,1\}^{(k-1)^2}} 2^{\sum_{C'\in\mathcal{C}'} X_{C'}(a)}$. Here $X_{C'}(a)$ is an indicator function, which takes value 1 if C' is directed after performing the flips of a and 0 otherwise.

We now assume that every $a \in \{0,1\}^{(k-1)^2}$ is chosen uniformly at random. The expected value of the above function is then

$$\mathbb{E}[2^{\sum X_{C'}}] = \frac{1}{2^{(k-1)^2}} \sum_{a\in\{0,1\}^{(k-1)^2}} 2^{\sum_{C'\in\mathcal{C}'} X_{C'}(a)}.$$

Jensen's inequality $\mathbb{E}[\varphi(X)] \geq \varphi(\mathbb{E}[X])$ holds for a random variable X and a convex function φ. We derive that $\mathbb{E}[2^{\sum X_{C'}}] \geq 2^{\mathbb{E}[\sum X_{C'}]} = 2^{\sum \mathbb{P}[C' \ \mathit{flippable}]}$. The probability that C' is flippable is at least 1/4. For C' which does not include a boundary edge the probability depends only on the three cycles from $\mathcal{C}$ that share an edge with C' and two out of the eight flip vectors for these three cycles make C' flippable. A similar reasoning applies for C' including a boundary edge. Altogether this yields that $\sum_{a\in\{0,1\}^{(k-1)^2}} 2^{\sum_{C'\in\mathcal{C}'} X_{C'}(a)} \geq 2^{(k-1)^2} \cdot 2^{1/4\cdot(k-1)^2}$. Different cycle flip sequences yield different Schnyder woods. The orientation of an edge is determined by whether the two cycles on which it lies are both flipped or not. We can tell a flip sequence apart from its complement by looking at the boundary edges. □

Remark. We relate Baxter's result [1] for Eulerian orientations of the triangular grid on the torus to Schnyder woods on $T^*_{k,\ell}$. Every 3-orientation of $T_{k,\ell}$ plus the wrap-around edges oriented as shown in Figure 2 on the right yields a Eulerian orientation on the torus. We deduce that $T^*_{k,\ell}$ has at most 2.599^n Schnyder woods. There are only $2^{2(k+\ell)-1}$ different orientations of these wrap-around edges. By the pigeon hole principle there is an orientation $\alpha_{k,\ell}$ of these edges which can be extended to a Eulerian orientation in asymptotically $2.598^{k\ell}$ ways. Thus, there is an $\alpha_{k,\ell}$ for $T^*_{k,\ell}$, which deviates from a 3-orientation only on the special vertices and the border of the grid such that there are asymptotically 2.598^n $\alpha_{k,\ell}$-orientations. Note, however, that directing all the wrap-around edges away from the vertex to which they are attached in Figure 2 induces a unique Eulerian orientation of $T^*_{k,\ell}$. We have not been able to show that $T_{k,\ell}$ has $2.598^{k\ell}$ 3-orientations, i.e. to verify that Baxter's result also gives a lower bound for the number of 3-orientations.

Now we discuss bounds on the number of Schnyder woods on 3-connected planar maps. The lower bound comes from the grid. The upper bound for this case is much larger than the one for triangulations. This is due to the encoding of Schnyder woods by 3-orientations on the primal dual completion graph, which has more vertices.

Theorem 4. *Let $\mathcal{M}_n$ be the set of 3-connected planar maps with n vertices and $\mathcal{S}(M)$ denote the set of Schnyder woods of $M \in \mathcal{M}_n$. Then,* $3.209^n \leq |\mathcal{S}(M)| \leq 8^n$*.*

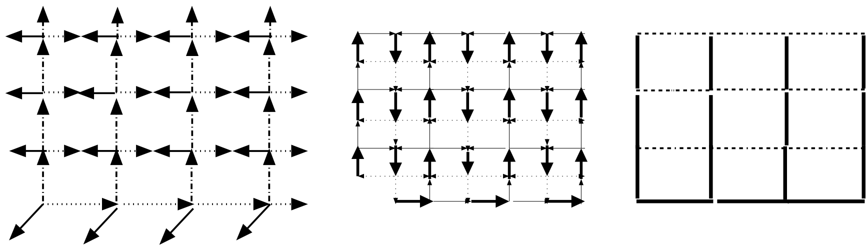

Fig. 3. A Schnyder wood on the map $G^*_{4,4}$, the reduced primal dual completion $G_{7,7} - (7,1)$ with the corresponding orientation and the associated spanning tree

The example used for the lower bound is the square grid graph $G_{k,\ell}$. Enumeration and counting of different combinatorial structures on the grid graph has received a lot of attention in the literature, see e.g. [15].

As Schnyder woods are defined on 3-connected graphs we augment $G_{k,\ell}$ by an outer triangle $\{a_1, a_2, a_3\}$ and edges from the boundary vertices of the grid to the vertices of this triangle. Figure 2 shows an augmented triangular grid; removing the diagonals in the squares yields $G^*_{4,5}$.

Theorem 5. *For k, ℓ big enough the number of Schnyder woods of the augmented grid $G^*_{k,\ell}$, is $|\mathcal{S}(G^*_{k,\ell})| \approx 3.209^{k\ell}$.*

The proof uses a bijection between Schnyder woods on $G^*_{k,\ell}$ and perfect matchings of $G_{2k-1,2\ell-1}$ minus on corner. This bijection also shows that Schnyder woods of $G^*_{k,\ell}$ are in bijection with spanning trees of $G_{k,\ell}$ by a result of Temperley, see [14] for more on the topic.

The proof of the upper bound stated in Theorem 4 uses the upper bound for Schnyder woods on plane triangulations. A triangulation T_M is derived from a planar map M by taking barycentric subdivisions of all non-triangular bounded faces. The number of Schnyder woods of T_M is shown to be an upper bound for the number of Schnyder woods of M and a specialization of the techniques from Proposition 1 yields the claimed bound.

Felsner et al. [11] present a theory of 2-orientations of plane quadrangulations, which shows many similarities with the theory of Schnyder woods for triangulations. We have studied the number of 2-orientations of quadrangulations and obtained the following results.

Theorem 6. *Let $\mathcal{Q}_n$ denote the set of all plane quadrangulations with n vertices and $\mathcal{Z}(Q)$ the set of 2-orientations of $Q \in \mathcal{Q}_n$. Then, for n big enough $1.53^n \leq \max_{Q \in \mathcal{Q}_n} |\mathcal{Z}(Q)| \leq 1.91^n$.*

4 Counting Bipolar Orientations

A good starting point for reading about bipolar orientations is [7]. Let G be a connected graph and $e = st$ a distinguished edge of G. An orientation X of the edges of G is an *e-bipolar orientation* of G if it is acyclic, s is the only vertex without incoming edges and t is the only vertex without outgoing edges. We call s and t the source respectively sink of X. There are many equivalent definitions of bipolar orientations, c.f. [7]. For our considerations any choice of two vertices s, t on the outer face will do, they need not be adjacent. We will simply refer to bipolar orientations instead of e-bipolar orientations. At this point we restrict ourselves to giving the encoding of bipolar orientations as α-orientations, which is introduced in [7].

Theorem 7. *Let M be a planar map and $\widehat{M}$ its angle graph. Let $\hat{\alpha} : V \cup \mathcal{F} \to \mathbb{N}$ be such that $\hat{\alpha}(F) = 2 = \hat{\alpha}(v)$ for $F \in \mathcal{F}$ and $v \in V \setminus \{s, t\}$. The source s and sink t have $\hat{\alpha}(s) = \hat{\alpha}(t) = 0$. Then, the bipolar orientations of M are in bijection with the $\hat{\alpha}$-orientations of $\widehat{M}$.*

Below, in Theorem 10, we give another encoding of bipolar orientations, which will turn out to be useful to upper bound the number of bipolar orientations.

Theorem 8. *Let $\mathcal{B}(G_{k,\ell})$ denote the set of bipolar orientations of $G_{k,\ell}$ with source $(1, 1)$ and sink (k, ℓ) respectively $(k, 1)$. For k, ℓ big enough the number of bipolar orientations of the grid $G_{k,\ell}$ is bounded by $2.18^{k\ell} \leq |\mathcal{B}(G_{k,\ell})| \leq 2.619^{k\ell}$.*

The lower bound uses reorientations of a canonical 2-orientation of the angle graph of $G_{k,\ell}$, which looks like a tilted grid. For the upper bound we use Lieb's bijection [15] between 2-orientations of the grid and 3-colorings of the squares of the grid. We encode such a 3-coloring by a sparse sequence, that is a 0-1-sequence without consecutive 1s.

Theorem 9. *Let $\mathcal{M}_n$ denote the set of all planar maps with n vertices and $\mathcal{B}(M)$ the set of all bipolar orientations of $M \in \mathcal{M}_n$. Then, for n big enough $2.91^n \leq \max_{M \in \mathcal{M}_n} |\mathcal{B}(M)| \leq 3.97^n$.*

Since it is not hard to prove that adding edges to non-triangular faces of a planar map M can only increase the number of bipolar orientations we restrict our considerations to plane inner triangulations. In the full paper it is shown that the set of all outerplanar maps with n vertices has $\max_{M \in \mathcal{O}_n} |\mathcal{B}(M)| = F_{n-1}$, where F_n is the nth Fibonacci number, and that $|\mathcal{B}(T_{2,\ell})|$ attains this value. The proof of the lower bound makes use of this by glueing together $k-1$ copies of $T_{2,\ell}$, which yields again a triangular grid.

The following relation is useful to upper bound the number of bipolar orientations for general plane inner triangulations. Let $\mathcal{F}_b$ be the set of bounded faces of M and $\mathcal{B}$ the set of bipolar orientations of M. Fix a bipolar orientation B. The boundary of every triangle $\Delta \in \mathcal{F}_b$ consists of a path of length 2 and an edge from the source to the sink of Δ. We say that Δ is a $+$ triangle of B if looking along the direct source-sink edge the triangle is on the left. Otherwise, if the triangle is on the right of the edge we speak of a $-$ triangle. We use this notation to define a mapping $G_B : \mathcal{F}_b \to \{-, +\}$.

The next result immediately yields an upper bound of 4^n for the number of bipolar orientations. The improvement can be made using the observation that every vertex is incident to faces of both types.

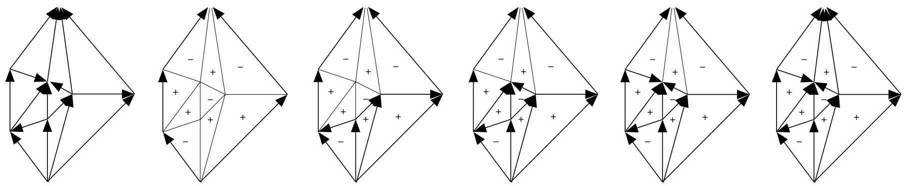

Fig. 4. A bipolar orientation, the corresponding +/− encoding and an illustration of the decoding algorithm

Theorem 10. *Let M be a plane inner triangulation and B a bipolar orientation of M. Given G_B, i.e., the signs of bounded faces, it is possible to recover B. In other words the function $B \to G_B$ is injective from $\mathcal{B}(M) \to \{-,+\}^{|\mathcal{F}_b|}$.*

5 Conclusions

In this paper we have studied the maximum number of α-orientations for different classes of planar maps and different α. In most cases we have exponential upper and lower bounds c_L^n and c_U^n for this number. The obvious problem is to improve on the constants c_L and c_U for the different instances. We think, that in particular improving the upper bound of 8^n for the number of Schnyder woods on 3-connected planar maps is worth further efforts.

Results by Lieb [15] and Baxter [1] yield the exact asymptotic behavior of the number of Eulerian orientations for the square and triangular grid on the torus. This yields upper bounds for the number of 2-orientations on the square grid and the number of Schnyder woods on triangular grids of specific dimensions. We could not yet utilize these results for improving the lower bounds for the number of 2-orientations respectively Schnyder woods.

We mentioned several #P-completeness results in Section 2. This contrasts with spanning trees and planar bipartite perfect matchings for which polynomial algorithms are available. It remains open to determine the complexity of counting Schnyder woods and bipolar orientations on planar maps.

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