# Mixing 3-Colourings in Bipartite Graphs

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**Abstract.** For a 3-colourable graph $G$, the 3-colour graph of $G$, denoted $\mathcal{C}_3(G)$, is the graph with node set the proper vertex 3-colourings of $G$, and two nodes adjacent whenever the corresponding colourings differ on precisely one vertex of $G$. We consider the following question: given $G$, how easily can we decide whether or not $\mathcal{C}_3(G)$ is connected? We show that the 3-colour graph of a 3-chromatic graph is never connected, and characterise the bipartite graphs for which $\mathcal{C}_3(G)$ is connected. We also show that the problem of deciding the connectedness of the 3-colour graph of a bipartite graph is coNP-complete, but that restricted to planar bipartite graphs, the question is answerable in polynomial time.

## 1 Introduction

Throughout this paper a graph $G = (V, E)$ is simple, loopless and finite. We always regard a $k$-vertex-colouring of a graph $G$ as proper; that is, as a function $\alpha : V \to \{1, 2, \ldots, k\}$ such that $\alpha(u) \neq \alpha(v)$ for any $uv \in E$. For a positive integer $k$ and a graph $G$, we define the *$k$-colour graph of $G$*, denoted $\mathcal{C}_k(G)$, as the graph that has the $k$-colourings of $G$ as its node set, with two $k$-colourings joined by an edge in $\mathcal{C}_k(G)$ if they differ in colour on just one vertex of $G$. We say that $G$ is *$k$-mixing* if $\mathcal{C}_k(G)$ is connected.

Continuing a theme begun in an earlier paper [2], we investigate the connectedness of $\mathcal{C}_k(G)$ for a given $G$. The connectedness of the $k$-colour graph is an issue of interest when trying to obtain efficient algorithms for almost uniform sampling of $k$-colourings of a given graph. In particular, $\mathcal{C}_k(G)$ needs to be connected for the single-site Glauber dynamics of $G$ (a Markov chain defined on the $k$-colour graph of $G$) to be rapidly mixing. For further details, see, for example, [5,6] and references therein.

In [2] it was shown that if $G$ has chromatic number $k$ for $k = 2, 3$, then $G$ is not $k$-mixing, but that, on the other hand, for $k \geq 4$, there are $k$-chromatic graphs that are $k$-mixing and $k$-chromatic graphs that are not $k$-mixing. In this

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paper, we look further at the case $k = 3$: we know 3-chromatic graphs are not 3-mixing, but what about bipartite graphs? Examples of 3-mixing bipartite graphs include trees and $C_4$, the cycle on 4 vertices. On the other hand, all cycles except $C_4$ are not 3-mixing — see [2] for details. In Theorem 1, we distinguish between 3-mixing and non-3-mixing bipartite graphs in terms of their structure and the possible 3-colourings they may have. As $G$ is $k$-mixing if and only if every connected component of $G$ is $k$-mixing, we will take our "argument graph" $G$ to be connected.

Some terminology is required to state the result. If $v$ and $w$ are vertices of a bipartite graph $G$ at distance two, then a *pinch* on $v$ and $w$ is the identification of $v$ and $w$ ( and the removal of any double edges produced ). And $G$ is *pinchable* to a graph $H$ if there exists a sequence of pinches that transforms $G$ into $H$.

Given a 3-colouring $\alpha$, the *weight* of an edge $e = uv$ oriented from $u$ to $v$ is

$$w(\overrightarrow{uv}, \alpha) \;=\; \begin{cases} +1, \text{ if } \alpha(u)\alpha(v) \in \{12,\, 23,\, 31\}; \\ -1, \text{ if } \alpha(u)\alpha(v) \in \{21,\, 32,\, 13\}. \end{cases} \qquad (1)$$

To *orient* a cycle means to orient each edge on the cycle so that a directed cycle is obtained. If $C$ is a cycle, then by $\overrightarrow{C}$ we denote the cycle with one of the two possible orientations. The *weight* $W(\overrightarrow{C}, \alpha)$ of an oriented cycle $\overrightarrow{C}$ is the sum of the weights of its oriented edges.

**Theorem 1.** *Let $G$ be a connected bipartite graph. The following are equivalent :*

*(i) The graph $G$ is not 3-mixing.*
*(ii) There exists a cycle $C$ in $G$ and a 3-colouring $\alpha$ of $G$ with $W(\overrightarrow{C}, \alpha) \neq 0$.*
*(iii) The graph $G$ is pinchable to the 6-cycle $C_6$.*

We also determine the computational complexity of the following decision problem.

3-Mixing
*Instance* : A connected bipartite graph $G$.
*Question* : Is $G$ 3-mixing?

**Theorem 2.** *The decision problem* 3-Mixing *is* coNP*-complete.*

We also prove, however, that there is a polynomial algorithm for the restriction of 3-Mixing to planar graphs. We remark that this difference in complexity contrasts with many other well-known graph colouring problems where the planar case is no easier to solve.

**Theorem 3.** *Restricted to* planar *bipartite graphs, the decision problem* 3-Mixing *is in the complexity class* P*.*

Organization of the paper: we prove Theorems 1, 2 and 3 in Sections 2, 3 and 4 respectively.

## 2 Characterising 3-Mixing Bipartite Graphs

To prove Theorem 1, we need some definitions, terminology and lemmas.

For the rest of this section, let $G = (V, E)$ denote a connected bipartite graph with vertex bipartition $X, Y$. We use $\alpha, \beta, \ldots$ to denote specific colourings, and, having defined the colourings as nodes of $\mathcal{C}_3(G)$, the meaning of, for example, the path between two colourings should be clear. We denote the cycle on $n$ vertices by $C_n$, and will often describe a colouring of $C_n$ by just listing the colours as they appear on consecutive vertices.

Given a 3-colouring $\alpha$ of $G$, we define a *height function for $\alpha$ with base $X$* as a function $h : V \to \mathbb{Z}$ satisfying the following conditions. (See [1,4] for other, similar height functions.)

H1 For all $v \in X$, $h(v) \equiv 0 \pmod 2$; for all $v \in Y$, $h(v) \equiv 1 \pmod 2$.

H2 For all $uv \in E$, $h(v) - h(u) = w(\overrightarrow{uv}, \alpha)$ $(\in \{-1, +1\})$.

H3 For all $v \in V$, $h(v) \equiv \alpha(v) \pmod 3$.

If $h : V \to \mathbb{Z}$ satisfies conditions H2, H3 and also

H1′ For all $v \in X$, $h(v) \equiv 1 \pmod 2$; while for $v \in Y$, $h(v) \equiv 0 \pmod 2$.

then $h$ is said to be a height function for $\alpha$ with base $Y$.

Observe that for a particular colouring of a given $G$, a height function might not exist. An example of this is the 6-cycle $C_6$ coloured 1-2-3-1-2-3.

Conversely, however, a function $h : V \to \mathbb{Z}$ satisfying conditions H1 and H2 induces a 3-colouring of $G$: the unique $\alpha : V \to \{1, 2, 3\}$ satisfying condition H3, and $h$ is in fact a height function for this $\alpha$. Observe also that if $h$ is a height function for $\alpha$ with base $X$, then so are $h + 6$ and $h - 6$; while $h + 3$ and $h - 3$ are height functions for $\alpha$ with base $Y$. Because we will be concerned solely with the question of *existence* of height functions, we assume henceforth that for a given $G$, all height functions have base $X$. Thus we let $\mathcal{H}_X(G)$ be the set of height functions with base $X$ corresponding to some 3-colouring of $G$, and define a metric $m$ on $\mathcal{H}_X(G)$ by setting

$$m(h_1, h_2) \;=\; \sum_{v \in V} |h_1(v) - h_2(v)|,$$

for $h_1, h_2 \in \mathcal{H}_X(G)$. Note that condition H1 above implies that $m(h_1, h_2)$ is always even.

For a given height function $h$, $h(v)$ is said to be a *local maximum* (respectively, *local minimum*) if $h(v)$ is larger than (respectively, smaller than) $h(u)$ for all neighbours $u$ of $v$. Following [4], we define the following *height transformations* on $h$.

- An *increasing height transformation* takes a local minimum $h(v)$ of $h$ and transforms $h$ into the height function $h'$ given by $h'(x) = \begin{cases} h(x) + 2, & \text{if } x = v; \\ h(x), & \text{if } x \neq v. \end{cases}$
- A *decreasing height transformation* takes a local maximum $h(v)$ of $h$ and transforms $h$ into the height function $h'$ given by $h'(x) = \begin{cases} h(x) - 2, & \text{if } x = v; \\ h(x), & \text{if } x \neq v. \end{cases}$

Notice that these height transformations give rise to transformations between the corresponding colourings. Specifically, if we let $\alpha'$ be the 3-colouring corresponding to $h'$, an increasing transformation yields $\alpha'(v) = \alpha(v) - 1$, while a decreasing transformation yields $\alpha'(v) = \alpha(v) + 1$, where addition is modulo 3.

The following lemma shows that colourings with height functions are connected in $\mathcal{C}_3(G)$. It is a simple extension of the range of applicability of a similar lemma appearing in [4].

**Lemma 1 ([4]).** *Let $\alpha, \beta$ be two 3-colourings of $G$ with corresponding height functions $h_\alpha, h_\beta$. Then there is a path between $\alpha$ and $\beta$ in $\mathcal{C}_3(G)$.*

*Proof.* We use induction on $m(h_\alpha, h_\beta)$. The lemma is trivially true when $m(h_\alpha, h_\beta) = 0$, since in this case $\alpha$ and $\beta$ are identical.

Suppose therefore that $m(h_\alpha, h_\beta) > 0$. We show that there is a height transformation transforming $h_\alpha$ into some height function $h$ with $m(h, h_\beta) = m(h_\alpha, h_\beta) - 2$, from which the lemma follows.

Without loss of generality, let us assume that there is some vertex $v \in V$ with $h_\alpha(v) > h_\beta(v)$, and let us choose $v$ with $h_\alpha(v)$ as large as possible. We show that such a $v$ must be a local maximum of $h_\alpha$. Let $u$ be any neighbour of $v$. If $h_\alpha(u) > h_\beta(u)$, then it follows that $h_\alpha(v) > h_\alpha(u)$, since $v$ was chosen with $h_\alpha(v)$ maximum, and $|h_\alpha(v) - h_\alpha(u)| = 1$. If, on the other hand, $h_\alpha(u) \le h_\beta(u)$, we have $h_\alpha(v) \ge h_\beta(v) + 1 \ge h_\beta(u) \ge h_\alpha(u)$, which in fact means $h_\alpha(v) > h_\alpha(u)$.

Thus $h_\alpha(v) > h_\alpha(u)$ for all neighbours $u$ of $v$, and we can apply a decreasing height transformation to $h_\alpha$ at $v$ to obtain $h$. Clearly $m(h, h_\beta) = m(h_\alpha, h_\beta) - 2$. □

The next lemma tells us that for a given 3-colouring, non-zero weight cycles are, in some sense, the obstructing configurations forbidding the existence of a corresponding height function.

**Lemma 2.** *Let $\alpha$ be a 3-colouring of $G$ with no corresponding height function. Then $G$ contains a cycle $C$ for which $W(\overrightarrow{C}, \alpha) \neq 0$.*

*Proof.* For a path $P$ in $G$, let $\overrightarrow{P}$ denote one of the two possible directed paths obtainable from $P$, and let

$$W(\overrightarrow{P}, \alpha) = \sum_{\boldsymbol{e} \in E(\overrightarrow{P})} w(\boldsymbol{e}, \alpha),$$

where $w(\boldsymbol{e}, \alpha)$ takes values as defined in (1).

Notice that if a colouring does have a height function, it is possible to construct one by fixing a vertex $x \in X$, giving $x$ an appropriate height ( satisfying properties H1–H3 ) and then assigning heights to all vertices in $V$ by following a breadth-first ordering from $x$.

Whenever we attempt to construct a height function $h$ for $\alpha$ in such a fashion, we must come to a stage in the ordering where we attempt to give some vertex $v$ a height $h(v)$ and find ourselves unable to because $v$ has a neighbour $u$

with a previously assigned height $h(u)$ and $|h(u) - h(v)| > 1$. Letting $P$ be a path between $u$ and $v$ formed by vertices that have been assigned a height, and choosing the appropriate orientation of $P$, we have $w(\overrightarrow{P}, \alpha) = |h(u) - h(v)|$. The lemma now follows by letting $C$ be the cycle formed by $P$ and the edge $uv$. $\square$

The following lemma is obvious.

**Lemma 3.** *Let $u$ and $v$ be vertices on a cycle $C$ in a graph $G$, and suppose there is a path $P$ between $u$ and $v$ in $G$ internally disjoint from $C$. Let $\alpha$ be a 3-colouring of $G$. Let $C'$ and $C''$ be the two cycles formed from $P$ and edges of $C$, and let $\overrightarrow{C'}, \overrightarrow{C''}$ be the orientations of $C', C''$ induced by an orientation $\overrightarrow{C}$ of $C$ (so the edges of $P$ have opposite orientations in $\overrightarrow{C'}$ and $\overrightarrow{C''}$). Then $W(\overrightarrow{C}, \alpha) = W(\overrightarrow{C'}, \alpha) + W(\overrightarrow{C''}, \alpha)$.*

Note this tells us that $W(\overrightarrow{C}, \alpha) \neq 0$ implies $W(\overrightarrow{C'}, \alpha) \neq 0$ or $W(\overrightarrow{C''}, \alpha) \neq 0$.

*Proof of Theorem 1.* Let $G$ be a connected bipartite graph.

(i) $\Longrightarrow$ (ii). Suppose $\mathcal{C}_3(G)$ is not connected. Take two 3-colourings of $G$, $\alpha$ and $\beta$, in different components of $\mathcal{C}_3(G)$. By Lemma 1 we know at least one of them, say $\alpha$, has no corresponding height function, and, by Lemma 2, there is a cycle $C$ in $G$ with $W(\overrightarrow{C}, \alpha) \neq 0$.

(ii) $\Longrightarrow$ (iii). Let $G$ contain a cycle $C$ with $W(\overrightarrow{C}, \alpha) \neq 0$ for some 3-colouring $\alpha$ of $G$. Because $W(\overrightarrow{C_4}, \beta) = 0$ for any 3-colouring $\beta$ of $C_4$, it follows that $C = C_n$ for some even $n \geq 6$. If $G = C$, then it is easy to find a sequence of pinches that will yield $C_6$. If $G$ is $C$ plus some chords, then, by Lemma 3, there is a smaller cycle $C'$ with $W(\overrightarrow{C'}, \alpha) \neq 0$. Thus if $G \neq C$, we can assume that $V(G) \neq V(C)$, and we describe how to pinch a pair of vertices so that (ii) remains satisfied (for a specified cycle with $G$ replaced by the graph created by the pinch and $\alpha$ replaced by its restriction to that graph; also denoted $\alpha$); by repetition, we can obtain a graph that is a cycle and, by the previous observations, the implication is proved.

We shall choose vertices coloured alike to pinch so that the restriction of $\alpha$ to the graph obtained is well-defined and proper. If $C$ has three consecutive vertices $u, v, w$ with $\alpha(u) = \alpha(w)$, pinching $u$ and $w$ yields a graph containing a cycle $C' = C_{n-2}$ with $W(\overrightarrow{C'}, \alpha) = W(\overrightarrow{C}, \alpha)$. Otherwise $C$ is coloured 1-2-3-$\cdots$-1-2-3. We can choose $u, v, w$ to be three consecutive vertices of $C$, such that there is a vertex $x \notin V(C)$ adjacent to $v$. Suppose, without loss of generality, that $\alpha(x) = \alpha(u)$, and pinch $x$ and $u$ to obtain a graph in which $W(\overrightarrow{C}, \alpha)$ is unchanged.

(iii) $\Longrightarrow$ (i). Suppose $G$ is pinchable to $C_6$. Take two 3-colourings of $C_6$ not connected by a path in $\mathcal{C}_3(C_6)$ — 1-2-3-1-2-3 and 1-2-1-2-1-2, for example. Considering the appropriate orientation of $C_6$, note that the first colouring has weight 6 and the second has weight 0. We construct two 3-colourings of $G$ not connected by a path in $\mathcal{C}_3(G)$ as follows. Consider the reverse sequence of pinches that gives $G$ from $C_6$. Following this sequence, for each colouring of $C_6$, give every pair of new vertices introduced by an "unpinching" the same colour as the vertex from which they originated. In this manner we obtain two 3-colourings of $G$, $\alpha$ and $\beta$, say. Observe that every unpinching maintains a cycle in $G$ which has weight 6 with respect to the colouring induced by the first colouring of $C_6$ and weight 0 with respect to the second induced colouring. This means $G$ will contain a cycle $C$ for which $W(\overrightarrow{C}, \alpha) = 6$ and $W(\overrightarrow{C}, \beta) = 0$, showing that $\alpha$ and $\beta$ cannot possibly be in the same connected component of $\mathcal{C}_3(G)$.

This completes the proof of the theorem. □

## 3 The Complexity of 3-Mixing for Bipartite Graphs

Observing that Theorem 1 gives us two polynomial-time verifiable certificates for when $G$ is *not* 3-mixing, we immediately obtain that 3-Mixing is in the complexity class coNP. By the same theorem, the following decision problem is the complement of 3-Mixing.

Pinchable-to-$C_6$
*Instance*: A connected bipartite graph $G$.
*Question*: Is $G$ pinchable to $C_6$?

Our proof will in fact show that Pinchable-to-$C_6$ is NP-complete. We will obtain a reduction from the following decision problem.

Retractable-to-$C_6$
*Instance*: A connected bipartite graph $G$ with an induced 6-cycle $S$.
*Question*: Is $G$ *retractable* to $S$? That is, does there exist a homomorphism $r : V(G) \to V(S)$ such that $r(v) = v$ for all $v \in V(S)$?

In [7] it is mentioned, without references, that Tomás Feder and Gary MacGillivray have independently proved the following result: for completeness, we give a sketch of a proof.

**Theorem 4 (Feder, MacGillivray, see [7]).** Retractable-to-$C_6$ *is* NP-*complete.*

*Sketch of proof of Theorem 4.* It is clear that Retractable-to-$C_6$ is in NP.

Given a graph $G$, construct a new graph $G'$ as follows: subdivide every edge $uv$ of $G$ by inserting a vertex $y_{uv}$ between $u$ and $v$. Also add new vertices $a, b, c, d, e$ together with edges $za, ab, bc, cd, de, ez$, where $z$ is a particular vertex of $G$ (any one will do). The graph $G'$ is clearly connected and bipartite, and the vertices $z, a, b, c, d, e$ induce a 6-cycle $S$. We will prove that $G$ is 3-colourable if and only if $G'$ retracts to the induced 6-cycle $S$.

Assume that $G$ is 3-colourable and take a 3-colouring $\tau$ of $G$ with $\tau(z) = 1$. From $\tau$ we construct a 6-colouring $\sigma$ of $G'$. For this, first set $\sigma(x) = \tau(x)$, if $x \in V(G)$. For the new vertices $y_{uv}$ set $\sigma(y_{uv}) = \begin{cases} 4, \text{ if } \tau(u) = 1 \text{ and } \tau(v) = 2, \\ 5, \text{ if } \tau(u) = 2 \text{ and } \tau(v) = 3, \\ 6, \text{ if } \tau(u) = 3 \text{ and } \tau(v) = 1. \end{cases}$

And for the cycle $S$ we take $\sigma(a) = 4$, $\sigma(b) = 2$, $\sigma(c) = 5$, $\sigma(d) = 3$ and $\sigma(e) = 6$. Now define $r : V(G') \to V(S)$ by setting $r(x) = z$, if $\sigma(x) = 1$; $r(x) = a$, if $\sigma(x) = 4$; $r(x) = b$, if $\sigma(x) = 2$; $r(x) = c$, if $\sigma(x) = 5$; $r(x) = d$, if $\sigma(x) = 3$; and $r(x) = e$, if $\sigma(x) = 6$. It is easy to check that $r$ is a retraction of $G'$ to $S$.

Conversely, suppose $G'$ retracts to $S$. We can use this retraction to define a 6-colouring of $G'$ in a similar way to that in which we defined $r$ from $\sigma$ in the preceeding paragraph. The restriction of this 6-colouring to $G$ yields a 3-colouring of $G$, completing the proof. □

*Proof of Theorem 2.* We have established that it is sufficient to describe a polynomial reduction from Retractable-to-$C_6$ to Pinchable-to-$C_6$. We shall describe the reduction but leave the remainder of the proof — which is a simple matter of checking a number of cases and, though straightforward, is lengthy — to the reader.

The reduction we use follows that used in [7] to prove the NP-completeness of the following problem:

Compactable-to-$C_6$
*Instance*: A connected bipartite graph $G$.
*Question*: Is $G$ *compactable* to $C_6$? That is, does there exist an edge-surjective homomorphism $c : V(G) \to V(C_6)$?

Consider an instance of Retractable-to-$C_6$ : a connected bipartite graph $G$ and an induced 6-cycle $S$. From $G$ we construct, in time polynomial in the size of $G$, an instance $G'$ of Pinchable-to-$C_6$ such that

$$G \text{ retracts to } S \text{ if and only if } G' \text{ is pinchable to } C_6. \qquad (*)$$

Assume $G$ has vertex bipartition $(G_A, G_B)$. Let $V(S) = S_A \cup S_B$, where $S_A = \{h_0, h_2, h_4\}$ and $S_B = \{h_1, h_3, h_5\}$, and assume $E(S) = \{\, h_0h_1, \ldots, h_4h_5, h_5h_0 \,\}$.

The construction of $G'$ is as follows.

- For every vertex $a \in G_A \setminus S_A$, add to $G$ new vertices $u_1^a, u_2^a, w_1^a, y_1^a, y_2^a$, together with edges $u_1^a h_0, au_2^a, w_1^a h_3, aw_1^a, u_1^a w_1^a, y_1^a h_5, y_2^a h_2, u_1^a y_1^a, w_1^a y_2^a, u_1^a u_2^a, y_1^a y_2^a$.
- For every vertex $b \in G_B \setminus S_B$, add to $G$ new vertices $u_1^b, w_1^b, w_2^b, y_1^b, y_2^b$, together with edges $u_1^b h_0, bu_1^b, w_1^b h_3, bw_2^b, u_1^b w_1^b, y_1^b h_5, y_2^b h_2, u_1^b y_1^b, w_1^b y_2^b, w_1^b w_2^b, y_1^b y_2^b$.
- For every edge $ab \in E(G) \setminus E(S)$, with $a \in G_A \setminus S_A$ and $b \in G_B \setminus S_B$, add two new vertices: $x_a^{ab}$ adjacent to $a$ and $u_1^a$; and $x_b^{ab}$ adjacent to $b, w_1^b$ and $x_a^{ab}$.

It is clear that $G'$ is connected and bipartite and that $G'$ contains $G$ as an induced subgraph. Note also that the subgraphs constructed around a vertex $a \in G_A \setminus S_A$ and a vertex $b \in G_B \setminus S_B$ are isomorphic; these subgraphs are depicted below in Fig. 1 and Fig. 2.

It is now easy to prove $(*)$ by considering a number of cases. The details are omitted.

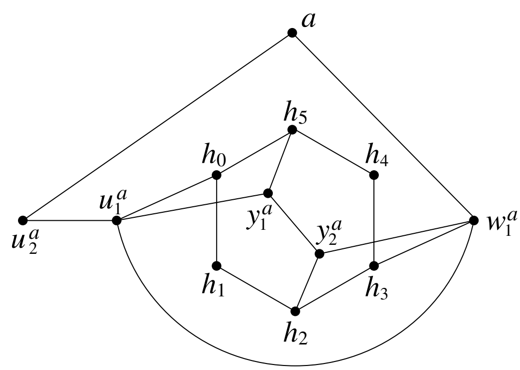


**Fig. 1.** The subgraph of $G'$ added around a vertex $a \in G_A \setminus S_A$, together with the 6-cycle $S$

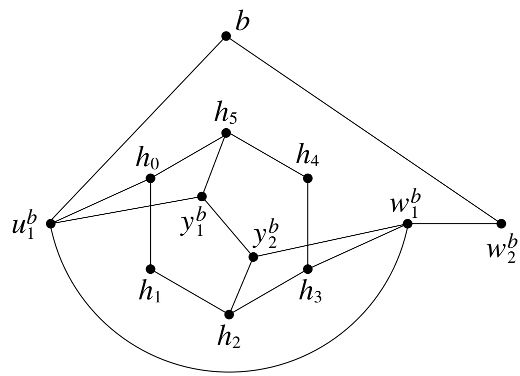


**Fig. 2.** The subgraph of $G'$ added around a vertex $b \in G_B \setminus S_B$, together with the 6-cycle $S$

## 4 A Polynomial-Time Algorithm for Planar Bipartite Graphs

Now let $G$ denote a bipartite *planar* graph. To prove Theorem 3 we need some technical results.

**Lemma 4.** *Let $P$ be a shortest path between distinct vertices $u$ and $v$ in a bipartite graph $H$. Then $H$ is pinchable to $P$.*

*Proof.* Let $P$ have vertices $u = v_0, v_1, \ldots, v_{k-1}, v_k = v$, and let $T$ be a breadth-first spanning tree of $H$ rooted at $u$ that contains $P$ ( we can choose $T$ so that it contains $P$ since $P$ is a shortest path ). Now, working in $T$, pinch all vertices at distance one from $u$ to $v_1$. Next pinch all vertices at distance two from $u$ to $v_2$. Continue until all vertices at distance $k$ from $u$ are pinched to $v_k = v$. If necessary, arbitrary pinches on the vertices at distance at least $k+1$ from $u$ will yield $P$. □

**Lemma 5.** *Let $H$ be a bipartite graph.*

*(i) Let $u$ and $v$ be two vertices in $H$ properly pre-coloured with colours from $1, 2, 3$. Then this colouring can be extended to a proper 3-colouring of $H$.*
*(ii) Let $u, v$ and $w$ be three vertices in $H$ with $uv, vw \in E(H)$. Suppose $u, v, w$ are properly pre-coloured with colours from $1, 2, 3$. Then this colouring can be extended to a proper 3-colouring of $H$.*
*(iii) Suppose the vertices of a 4-cycle in $H$ are properly 3-coloured. Then this 3-colouring can be extended to a proper 3-colouring of $H$.*

*Proof.* (i) is trivial.

(ii) Without loss of generality we can assume that the colouring of $u, v, w$ is 1-2-1 or 1-2-3. In the first instance, since $H$ is bipartite, we can extend the colouring of $u, v, w$ to a colouring of $H$ using colours 1 and 2 only. For the second case, we can use the same 1,2-colouring, except leaving $w$ with colour 3.

(iii) Since any 3-colouring of a $C_4$ has two vertices with the same colour, without loss of generality we can assume the 4 vertices are coloured 1-2-1-2 or 1-2-1-3. Colourings similar to those used in (ii) above will immediately lead to the appropriate 3-colourings of $H$. □

*Proof of Theorem 3.* The sequence of claims below outlines an algorithm that, given $G$ as input, determines in polynomial time whether or not $G$ is 3-mixing.

The first claim is a simple observation.

**Claim 1.** *If $G$ is not connected, then $G$ is 3-mixing if and only if every component of $G$ is 3-mixing.*

We next show how we can reduce the case to 2-connected graphs.

**Claim 2.** *Suppose $G$ has a cut-vertex $v$. Let $H_1$ be a component of $G-v$. Denote by $G_1$ the subgraph of $G$ induced by $V(H_1) \cup \{v\}$, and let $G_2$ be the subgraph induced by $V(G) \setminus V(H_1)$. Then $G$ is 3-mixing if and only if both $G_1$ and $G_2$ are 3-mixing.*

*Proof.* If $G$ is 3-mixing, then clearly so are $G_1$ and $G_2$. Conversely, if $G$ is not 3-mixing, we know by Theorem 1 that there must exist a 3-colouring $\alpha$ of $G$ and a cycle $C$ in $G$ such that $W(\overrightarrow{C}, \alpha) \neq 0$. But because $C$ must lie completely in $G_1$ or $G_2$, we have that $G_1$ or $G_2$ is not 3-mixing. □

Now we can assume that $G$ is 2-connected. In the next claim we will show that we can actually assume $G$ to be 3-connected.

**Claim 3.** *Suppose $G$ has a 2-vertex-cut $\{u, v\}$. Let $H_1$ be a component of $G - \{u, v\}$. Denote by $G_1$ the subgraph of $G$ induced by $V(H_1) \cup \{u, v\}$, and let $G_2$ be the subgraph induced by $V(G) \setminus V(H_1)$. For $i = 1, 2$, let $\ell_i$ be the distance between $u$ and $v$ in $G_i$.*

*Then only the following cases can occur:*

*(i) We have $\ell_1 = \ell_2 = 1$. Then $G$ is 3-mixing if and only if both $G_1$ and $G_2$ are 3-mixing.*

*(ii) We have* $\ell_1 = \ell_2 = 2$. *(So for* $i = 1, 2$, *there is a vertex* $w_i \in V(G_i)$ *so that* $uw_i, vw_i \in E(G_i)$. *) Let* $G_1^*$ *be the subgraph of* $G$ *induced by* $V(G_1) \cup \{w_2\}$ *and let* $G_2^*$ *be the subgraph induced by* $V(G_2) \cup \{w_1\}$. *Then* $G$ *is 3-mixing if and only if both* $G_1^*$ *and* $G_2^*$ *are 3-mixing.*

*(iii) We have* $\ell_1 + \ell_2 \geq 6$. *Then* $G$ *is not 3-mixing.*

*Proof.* Because $G$ is bipartite, $\ell_1$ and $\ell_2$ must have the same parity. If $\ell_1 = 1$ or $\ell_2 = 1$, then there is an edge $uv$ in $G$, and this same edge must appear in both $G_1$ and $G_2$. This guarantees that both $\ell_1 = \ell_2 = 1$, and shows that we always have one of the three cases.

(i) In this case we have an edge $uv$ in all of $G, G_1, G_2$. If one of $G_1$ and $G_2$ is not 3-mixing, say $G_1$, we must have a 3-colouring $\alpha$ of $G_1$ and a cycle $C$ in $G_1$ for which $W(\overrightarrow{C}, \alpha) \neq 0$. By Lemma 5 (i) we can easily extend $\alpha$ to the whole of $G$, showing that $G$ is not 3-mixing. On the other hand, if $G$ is not 3-mixing, we know we must have a 3-colouring $\beta$ of $G$ and a cycle $D$ in $G$ for which $W(\overrightarrow{D}, \beta) \neq 0$. If $D$ is contained entirely in one of $G_1$ or $G_2$, we are done. If not, $D$ must pass through $u$ and $v$. For $i = 1, 2$, consider the cycle $D^i$ formed from the part of $D$ that is in $G_i$ together with the edge $uv$. From Lemma 3 it follows that one of $D^1$ and $D^2$ has non-zero weight under $\beta$, showing that $G_1$ or $G_2$ is not 3-mixing.

(ii) If one of $G_1^*$ and $G_2^*$ is not 3-mixing, we can use a similar argument as in (i) (now using Lemma 5 (ii)) to conclude that $G$ is not 3-mixing. For the converse we assume $G$ is not 3-mixing. So there is a 3-colouring $\alpha$ of $G$ and a cycle $C$ in $G$ for which $W(\overrightarrow{C}, \alpha) \neq 0$. If $C$ is contained entirely in one of $G_1^*$ or $G_2^*$, we are done. If not, $C$ must pass through $u$ and $v$. If $C$ does not contain $w_1$, then for $i = 1, 2$, consider the cycle $C^i$ formed by the part of $C$ that is in $G_i^*$ plus the path $uw_1v$. From Lemma 3 it follows that one of $C^1, C^2$ has non-zero weight under $\alpha$, showing that $G_1^*$ or $G_2^*$ is not 3-mixing. If $w_1$ is contained in $C$, then we can use the same argument but now using the edge $uw_1$ or $vw_1$ as the path (at least one of these edges is not on $C$ since $C$ is not contained entirely in $G_2^*$).

(iii) For $i = 1, 2$, let $P_i$ be a shortest path between $u$ and $v$ in $G_i$, so $P_i$ has length $\ell_i$. Then, by Lemma 4, we can see that $G$ is pinchable to $C_{\ell_1+\ell_2}$ (follow, in $G$, the sequence of pinches that transforms $G_1$ into $P_1$ and $G_2$ into $P_2$). Since $\ell_1 + \ell_2 \geq 6$, $C_{\ell_1+\ell_2}$ is of course pinchable to $C_6$, and hence $G$ is not 3-mixing. □

From now on we consider $G$ to be 3-connected, and can therefore use the following result of Whitney — for details, see, for example, [3] pp. 78–80.

**Theorem 5 (Whitney).** *Any two planar embeddings of a 3-connected graph are equivalent.*

Henceforth, we identify $G$ with its (essentially unique) planar embedding. For a cycle $D$ in $G$, denote by $\mathrm{Int}(D)$ and $\mathrm{Ext}(D)$ the set of vertices inside and outside of $D$, respectively. If both $\mathrm{Int}(D)$ and $\mathrm{Ext}(D)$ are non-empty, $D$ is *separating* and we define $G_{\mathrm{Int}}(D) = G - \mathrm{Ext}(D)$ and $G_{\mathrm{Ext}}(D) = G - \mathrm{Int}(D)$.

We next consider the case that $G$ has a separating 4-cycle.

**Claim 4.** *Suppose* $G$ *has a separating 4-cycle* $D$. *Then* $G$ *is 3-mixing if and only if* $G_{\mathrm{Int}}(D)$ *and* $G_{\mathrm{Ext}}(D)$ *are both 3-mixing.*

*Proof.* To prove necessity, we show that if one of $G_{\rm Int}(D)$ or $G_{\rm Ext}(D)$ is not 3-mixing, then $G$ is not 3-mixing. Without loss of generality, suppose that $G_{\rm Int}(D)$ is not 3-mixing, so there exists a 3-colouring $\alpha$ of $G_{\rm Int}(D)$ and a cycle $C$ in $G_{\rm Int}(D)$ with $W(\overrightarrow{C}, \alpha) \neq 0$. The 3-colouring of the vertices of the 4-cycle $D$ can be extended to a 3-colouring of $G_{\rm Ext}(D)$ ( use Lemma 5 (iii) ). The combination of the 3-colourings of $G_{\rm Int}(D)$ and $G_{\rm Ext}(D)$ gives a 3-colouring of $G$ with a non-zero weight cycle, showing $G$ is not 3-mixing.

To prove sufficiency, we show that if $G$ is not 3-mixing, then at least one of $G_{\rm Int}(D)$ and $G_{\rm Ext}(D)$ must fail to be 3-mixing. Suppose that $\alpha$ is a 3-colouring of $G$ for which there is a cycle $C$ with $W(\overrightarrow{C}, \alpha) \neq 0$. If $C$ is contained entirely within $G_{\rm Int}(D)$ or $G_{\rm Ext}(D)$ we are done; so let us assume that $C$ has some vertices in $\mathrm{Int}(D)$ and some in $\mathrm{Ext}(D)$. Then applying Lemma 3 ( repeatedly, if necessary ) we can find a cycle $C'$ contained entirely in $G_{\rm Int}(D)$ or $G_{\rm Ext}(D)$ for which $W(\overrightarrow{C'}, \alpha) \neq 0$, completing the proof. □

We call a face of $G$ with $k$ edges in its boundary a *$k$-face*, and a face with at least $k$ edges in its boundary a *$\geq k$-face.* The number of $\geq 6$-faces in $G$ — which now we can assume is a 3-connected bipartite planar graph with no separating 4-cycle — will lead to our final claim.

**Claim 5.** *Let $G$ be a 3-connected bipartite planar graph with no separating 4-cycle. Then $G$ is 3-mixing if and only if it has at most one $\geq 6$-face.*

*Proof.* We first prove sufficiency. Suppose $G$ has no $\geq 6$-faces, so has only 4-faces. Let $\alpha$ be any 3-colouring of $G$ and let $C$ be any cycle in $G$. We show $W(\overrightarrow{C}, \alpha) = 0$ by induction on the number of faces inside $C$. If there is just one face inside $C$, $C$ is in fact a facial 4-cycle and $W(\overrightarrow{C}, \alpha) = 0$. For the inductive step, let $C$ be a cycle with $r \geq 2$ faces in its interior. If, for two consecutive vertices $u, v$ of $C$, we have vertices $a, b \in \mathrm{Int}(C)$ together with edges $ua, ab, bv$ in $G$, let $C'$ be the cycle formed from $C$ by the removal of the edge $uv$ and the addition of edges $ua, ab, bv$. If not, check whether for three consecutive vertices $u, v, w$ of $C$, there is a vertex $a \in \mathrm{Int}(C)$ with edges $ua, aw$ in $G$. If so, let $C'$ be the cycle formed from $C$ by the removal of the vertex $v$ and the addition of the edges $ua, aw$. If neither of the previous two cases apply, we must have, for $u, v, w, x$ four consecutive vertices of $C$, an edge $ux$ inside $C$. In such a case, let $C'$ be the cycle formed from $C$ by the removal of vertices $v, w$ and the addition of the edge $ux$. In all cases we have that $C'$ has $r - 1$ faces in its interior, so, by induction, we can assume $W(\overrightarrow{C'}, \alpha) = 0$. From Lemma 3 we then obtain $W(\overrightarrow{C}, \alpha) = 0$.

Suppose now that $G$ contains exactly one $\geq 6$-face. Without loss of generality we can assume that this face is the outside face, and hence the argument above will work exactly the same to show that $G$ is 3-mixing.

To prove necessity we show that if $G$ contains at least two $\geq 6$-faces, then $G$ is pinchable to $C_6$. For $f$ a $\geq 6$-face in $G$, a separating cycle $D$ is said to be *$f$-separating* if $f$ lies inside $D$. Let $f$ and $f_o$ be two $\geq 6$-faces in $G$, where we can assume $f_o$ is the outer face of $G$, and let $C$ be the cycle bounding $f$. Our claim is that we can successively pinch vertices into a cycle of length at least 6 without ever introducing an $f$-separating 4-cycle — we will initially do this around $C$.

Let $x, y, z$ be any three consecutive vertices of $C$ with $y$ having degree at least 3 — if there is no such vertex $y$, then $G$ is simply a cycle of length at least 6 and we are done. Let $a$ be a neighbour of $y$ distinct from $x$ and $z$, such that the edges $ya$ and $yz$ form part of the boundary of a face adjacent to $f$. If the result of pinching $a$ and $z$ introduces no $f$-separating 4-cycle, then pinch $a$ and $z$ and repeat the process. If pinching $a$ and $z$ *does* result in the creation of an $f$-separating 4-cycle, this must be because the path $ay, yz$ forms part of an $f$-separating 6-cycle $D$. We now show how we can find alternative pinches which do not introduce an $f$-separating 4-cycle. The fact that $D$ is $f$-separating means there is a path $P \subseteq D$ of length 4 between $a$ and $z$. Note that $P$ cannot contain $y$, for this would contradict the fact that $G$ has no separating 4-cycle. Consider the graph $G' = G_{\text{Int}}(D) - \{yz\}$. We claim that the path $P' = P \cup \{ay\}$ is a shortest path between $y$ and $z$ in $G'$. To see this, remember that $G$ is bipartite, so any path between $y$ and $z$ in $G$ has to have odd length. We cannot have another edge $yz \in E(G')$ since $G$ is simple. Finally, any path between $y$ and $z$ in $G'$ would, together with the edge $yz$, form an $f$-separating cycle in $G$. Hence a path of length 3 between $y$ and $z$ would contradict the fact that $G$ has no separating 4-cycle. By Lemma 4, we see $G'$ is pinchable to $P'$. Using the same sequence of pinches in $G$ will pinch $G_{\text{Int}}(D)$ into $D$. Note this introduces no separating 4-cycle into the resulting graph. If necessary, we can repeat the process by pinching vertices into $D$, which now bounds a 6-face. This completes the proof. □

The sequence of Claims 1 – 5 can easily be used to obtain a polynomial-time algorithm to check if a given planar bipartite graph $G$ is 3-mixing. This completes the proof of Theorem 3. □

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