On Minimum Area Planar Upward Drawings of Directed Trees and Other Families of Directed Acyclic Graphs

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Abstract. It has been shown in [9] that there exist planar digraphs that require exponential area in every upward *straight-line* planar drawing. On the other hand, upward *poly-line* planar drawings of planar graphs can be realized in $\Theta(n^2)$ area [9]. In this paper we consider families of DAGs that naturally arise in practice, like DAGs whose underlying graph is a tree (*directed trees*), is a bipartite graph (*directed bipartite graphs*), or is an outerplanar graph (*directed outerplanar graphs*). Concerning *directed trees*, we show that optimal $\Theta(n \log n)$ area upward straight-line/poly-line planar drawings can be constructed. However, we prove that if the order of the neighbors of each node is assigned, then exponential area is required for straight-line upward drawings and quadratic area is required for poly-line upward drawings, results surprisingly and sharply contrasting with the area bounds for planar upward drawings of undirected trees. After having established tight bounds on the area requirements of planar upward drawings of several families of directed trees, we show how the results obtained for trees can be exploited to determine asymptotic optimal values for the area occupation of planar upward drawings of *directed bipartite graphs* and *directed outerplanar graphs*.

1 Introduction

Upward drawings of directed acyclic digraphs (*DAGs* for short) have several applications in the visualization of hierarchical structures, as PERT diagrams, subroutine-call charts, Hasse diagrams, and is-a relationships, and hence they have been intensively studied from a theoretical point of view. It is known that testing the *upward planarity* of a graph is an *NP-complete* problem if the graph has a variable embedding [13], while it is polynomially solvable if the embedding of the graph is *fixed* [2], if the underlying graph is supposed to be an *outerplanar graph* ([15]), if the digraph has a *single source* ([14]), or if it's a *bipartite* DAG ([7]). Di Battista and Tamassia ([8]) showed that a graph is upward planar if and only if it's a subgraph of an *st-planar* graph. Moreover, some families of DAGs are always upward planar, like the *series-parallel* digraphs and the digraphs whose underlying graph is a *tree*.

Concerning algorithms for obtaining upward drawings of DAGs in small area, Di Battista et al. have shown in [9] that every upward planar embedding can

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be drawn with upward *poly-line* edges in optimal $\Theta(n^2)$ area, while there exist graphs that require exponential area in any planar *straight-line* upward drawing. Hence, it is natural to restrict the attention to interesting families of DAGs, searching for better area bounds. This research direction has been taken by Bertolazzi et al. in [1], where it is shown that *series-parallel* digraphs admit upward planar straight-line drawings in $\Theta(n^2)$ area, while exponential area is generally required if the embedding is chosen in advance.

In this paper we study classes of DAGs that commonly arise in practice, as DAGs whose underlying graph is a tree (*directed trees*), is a bipartite graph (*directed bipartite graphs*), or is an outerplanar graph (*directed outerplanar graphs*). All of such digraph classes exhibit simple and strong structural properties that allow to create planar upward drawings with less constraints and in a easier way with respect to general digraphs. Consequently, we are able to construct straight-line planar upward drawings of directed trees in $\Theta(n \log n)$ area, and to get $\Theta(n)$ area straight-line planar upward drawings for some sub-classes of directed trees. Surprisingly, we prove that when constraints are imposed on the drawings by forcing an ordering of the neighbors of each vertex, then again exponential area is required for constructing straight-line planar upward drawings and quadratic area is required for constructing poly-line planar upward drawings. Such negative results contrast with the fact that sub-quadratic area is sufficient for constructing straight-line order-preserving upward planar drawings of undirected trees ([3]). Furthermore, we prove that the lower bounds obtained for directed trees extend also to directed bipartite graphs and directed outerplanar graphs.

More in detail, we provide the following results: (i) straight-line and poly-line planar upward drawings of directed trees can be constructed in optimal $\Theta(n \log n)$ area (Sec. 3); (ii) straight-line order-preserving planar upward drawings of directed trees require (and can be constructed in) exponential area (Sec. 4); (iii) poly-line order-preserving planar upward drawings of directed trees require (and can be constructed in) quadratic area (Sec. 4); (iv) *directed binary trees* have the same area requirements of general directed trees (Sec. 5); (v) *directed caterpillars* and *directed spider trees* admit linear area straight-line drawings (Sec. 5); (vi) straight-line planar upward drawings of directed bipartite graphs require (and can be constructed in) exponential area (Sec. 5); (vii) poly-line planar upward drawings of directed bipartite graphs require (and can be constructed in) quadratic area (Sec. 5); (viii) straight-line outerplanar upward drawings of directed outerplanar graphs require (and can be constructed in) exponential area (Sec. 5); and

Table 1. A table summarizing the results on minimum area upward drawings of directed trees. Straight-line and poly-line non-order-preserving drawings are in the same columns, since they have the same area bounds. Constants b and c are greater than 1.

	Straight-line / Poly-line				Straight-line Order-Pres.				Poly-line Order-Pres.			
	UB	ref.	LB	ref.	UB	ref.	LB	ref	UB	ref.	LB	ref.
Dir. Trees	$O(n \log n)$	Th. 1	$\Omega(n \log n)$	Th. 1	$O(c^n)$	[12]	$\Omega(b^n)$	Th. 2	$O(n^2)$	[9]	$\Omega(n^2)$	Th. 3
Dir. Binary Trees	$O(n \log n)$	Th. 1	$\Omega(n \log n)$	Th. 1	$O(c^n)$	[12]	$\Omega(b^n)$	Th. 2	$O(n^2)$	[9]	$\Omega(n^2)$	Th. 3
Dir. Caterpillars	$O(n)$	Th. 4	$\Omega(n)$	trivial	$O(c^n)$	[12]	$\Omega(b^n)$	Th. 2	$O(n^2)$	[9]	$\Omega(n^2)$	Th. 3
Dir. Spider Trees	$O(n)$	Th. 5	$\Omega(n)$	trivial	$O(n)$	Th. 5	$\Omega(n)$	trivial	$O(n)$	Th. 5	$\Omega(n)$	trivial

(ix) poly-line planar upward drawings of directed outerplanar graphs require (and can be constructed in) quadratic area (Sec. 5). Table 1 summarizes the area requirements of planar upward drawings of directed trees, directed binary trees, directed caterpillars, and directed spider trees.

2 Preliminaries

We assume familiarity with graphs and their drawings (see also [5]).

A *grid drawing* of a graph is a mapping of each vertex to a distinct point of the plane with integer coordinates and of each edge to a Jordan curve between the endpoints of the edge. A *poly-line drawing* is such that the edges are sequences of rectilinear segments, with bends having integer coordinates. A *straight-line drawing* is such that all edges are rectilinear segments. A *planar drawing* is such that no two edges intersect. An *upward drawing* of a *digraph* is a planar drawing with each directed edge represented by a curve monotonically increasing in the vertical direction. In the following when we refer to upward drawings we always mean planar upward grid drawings. The graph obtained from a digraph G by considering its edges without orientation is called the *underlying graph* of G. An *embedding* of a graph is a circular ordering of the edges incident on each vertex. A drawing is *order-preserving* if the order of the edges incident on each vertex is the same as the one of an embedding specified in advance. The *bounding box* $B(\Gamma)$ of a drawing Γ is the smallest rectangle with sides parallel to the axes that covers Γ completely. We denote by $l(\Gamma)$, by $r(\Gamma)$, by $t(\Gamma)$, and by $b(\Gamma)$ the *left* side, the *right* side, the *top* side, the *bottom* side of $B(\Gamma)$, respectively. The *height* (*width*) of Γ is the height (width) of its bounding box plus one. The *area* of Γ is the height of Γ multiplied by its width. We denote by $y(v)$ the y-coordinate of a vertex v that is drawn on the plane.

An *outerplanar graph* is a graph that has a planar embedding in which all vertices are incident to the same face. Such an embedding is called *outerplanar embedding*. A *bipartite graph* is a graph G that has the vertices partitioned into two subsets such that G has edges only between vertices of different subsets. A *caterpillar* C is a tree such that the removal from C of all the leaves and of their incident edges turns C in a path. A *spider tree* is a tree having only one vertex of degree greater than two.

3 Upward Drawings of Trees

In this section we show that directed trees admit straight-line upward drawings in $\Theta(n \log n)$ area and that such an area is necessary in the worst case, even if bends are allowed on the edges. Concerning the lower bound, Crescenzi et al. in [4] showed a non-directed rooted binary tree T that requires $\Omega(n \log n)$ area in any *strictly upward* grid drawing. Now T can be turned in a directed binary tree T' by directing its edges away from the root. Since an upward drawing of T' is a strictly upward drawing of T, the lower bound on the area requirement of upward drawings of directed trees follows.

Now we show that every directed tree has an $O(n \log n)$ area straight-line upward drawing. This is done by means of an algorithm that consider a directed tree T, removes from T a path called *spine*, recursively draws each disconnected subtree, and finally puts the drawings of the subtrees together with a drawing of the spine, obtaining a drawing of T. This *divide et impera* strategy has been intensively used in algorithms for drawing undirected trees and outerplanar graphs ([3,11,10,6]). Let us describe the algorithm more formally.

Preprocessing: The input is a directed tree T with n nodes. We derive a non-directed rooted tree T' from T by removing the orientations from the edges of T and by choosing a node r in T as root of T'.

Divide: Let T^* be the current non-directed rooted tree and let r^* be its root (at the first step the current tree is T' rooted at r).

If the number of nodes in T^* is greater than one, then select a spine $S^* = (v_0, v_1, \ldots, v_k)$ in T^* with the following properties: (i) $v_0 = r^*$, (ii) for $1 \leq i \leq k$, v_i is the root of the heaviest (i.e. with the greatest number of nodes) subtree among the subtrees rooted at the children of v_{i-1}, and (iii) each edge (v_{i-1}, v_i) is directed from v_i to v_{i-1} in T, for $1 \leq i < k$, and (iv) edge (v_{k-1}, v_k) is directed from v_{k-1} to v_k in T, or v_k is a leaf. Remove from T^* the nodes of S^*, but for v_k, disconnecting T^* in several non-directed subtrees. We classify such subtrees into sets $T^*(\uparrow, v_i)$ and $T^*(\downarrow, v_i)$, with $0 \leq i < k$, so that a tree rooted at a vertex v goes into set $T^*(\uparrow, v_i)$ (resp. $T^*(\downarrow, v_i)$) if in the directed tree T there is an edge directed from v to v_i (resp. there is an edge directed from v_i to v). Notice that each set could contain several trees. We denote by $T^*(v_k)$ the tree rooted at v_k and by $r(T^*)$ the root of a non-directed tree T^*.

Impera: Assume that in the *Divide* step a tree T^* has been disconnected in a spine S^*, in a subtree $T^*(v_k)$, and in several subtrees in $T^*(\uparrow, v_i)$ and in $T^*(\downarrow, v_i)$, with $0 \leq i < k$. Introduce again the direction on the edges of T^*, obtaining a directed tree $T(v_k)$ from $T^*(v_k)$, obtaining a set of directed trees $T(\uparrow, v_i)$ from the trees in $T^*(\uparrow, v_i)$, and obtaining a set of directed trees $T(\downarrow, v_i)$ from the trees in $T^*(\downarrow, v_i)$. Assume to have for each of such directed trees a drawing with the following properties: ($\mathbf{P_1}$) the drawing is planar, upward, and straight-line; ($\mathbf{P_2}$) the root of the tree is placed on the left side of the bounding box of the drawing; and ($\mathbf{P_3}$) no node of the tree is placed in the drawing below and on the same vertical line of the root of the tree.

Notice that such a drawing can be trivially constructed for a tree with at most one node. Now we show how to construct a drawing Γ satisfying properties P_1, P_2, and P_3 for the directed tree $\overline{T}$ obtained from T^* by introducing again the directions on the edges. Notice that, in the last *Impera* step, Γ will be a drawing of the whole directed tree T. We distinguish two cases:

$\mathbf{k = 1}$: Place the drawings of the trees in $T(\downarrow, v_0)$ stacked one above the other at one unit of vertical distance, with the left side of their bounding boxes on the same vertical line l, obtaining a drawing Γ'. Place v_0 one unit to the left of l and one unit below $b(\Gamma')$. Place the drawings of the trees in $T(\uparrow, v_0)$ stacked one above the other at one unit of vertical distance, with the left side of their bounding boxes on l, and so that the highest horizontal line intersecting a drawing of a tree in $T(\uparrow, v_0)$ is one unit below v_0, obtaining a drawing Γ''. If (v_0, v_1) is directed from v_0 to v_1, then place the drawing of $T(v_1)$ so that the left side of its bounding box is on the same vertical line of v_0 and so that the bottom side of its bounding box is one unit above $t(\Gamma'')$ (see Fig. 1.a). Otherwise, that is v_1 is a leaf and (v_0, v_1) is directed from v_1 to v_0, place v_1 on the same vertical line of v_0 and one unit below $b(\Gamma'')$.

$\mathbf{k \geq 2}$: Place the drawings of the trees in $T(\downarrow, v_0)$ stacked one above the other at one unit of vertical distance, with the left side of their bounding boxes on the same vertical line l, obtaining a drawing Γ'. Place v_0 two units to the left of l and one unit below $b(\Gamma')$. Place the drawings of the trees in $T(\uparrow, v_0)$ stacked one above the other at one unit of vertical distance, with the left side of their bounding boxes on l, and so that the highest horizontal line intersecting a drawing of a tree in $T(\uparrow, v_0)$ is one unit below v_0, obtaining a drawing Γ_0. For $i = 1, 2, \dots, k-2$, place the drawings of the trees in $T(\downarrow, v_i)$ stacked one above the other at one unit of vertical distance, with the left side of their bounding boxes on l, and so that the highest horizontal line intersecting a drawing of a tree in $T(\downarrow, v_i)$ is one unit below $b(\Gamma_{i-1})$, obtaining a drawing Γ'. Place v_i one unit to the left of l and and one unit below $b(\Gamma')$. Place the drawings of the trees in $T(\uparrow, v_i)$ stacked one above the other at one unit of vertical distance, with the left side of their bounding boxes on l, and so that the highest horizontal line intersecting a drawing of a tree in $T(\uparrow, v_i)$ is one unit below v_i, obtaining a drawing Γ_i. Let W be the maximum between the width of the drawing of $T(v_k)$ minus 1 and the maximum width of a drawing of a tree in $T(\uparrow, v_i)$ or in $T(\downarrow, v_i)$ plus 2, with $0 \leq i < k$. Let l' be the vertical line W units to the right of v_0. Mirror the drawings of the trees in $T(\uparrow, v_{k-1})$ with respect to a vertical line and place them stacked one above the other at one unit of vertical distance, with the right side of their mirrored bounding boxes one unit to the left of l' and so that the highest horizontal line intersecting a drawing of a tree in $T(\uparrow, v_{k-1})$ is one unit below $b(\Gamma_{k-2})$. Mirror the drawings of the trees in $T(\downarrow, v_{k-1})$ with respect to a vertical line and place them stacked one above the other at one unit of vertical distance, with the right side of their mirrored bounding boxes one unit to the left of l', and so that the lowest horizontal line intersecting a drawing of a tree in $T(\downarrow, v_{k-1})$ is one unit above $t(\Gamma_{k-2})$. Place v_{k-1} on l' one unit below v_{k-2}, obtaining a drawing Γ_{k-1}. Finally, if edge (v_{k-1}, v_k) is directed from v_{k-1} to v_k, mirror the drawing of $T(v_k)$ with respect to a vertical line and place it with the right side of its mirrored bounding box on l' so that the bottom side of its bounding box is one unit above $t(\Gamma_{k-1})$; otherwise, that is v_k is a leaf and edge (v_{k-1}, v_k) is directed from v_k to v_{k-1}, place v_k on l' one unit below $b(\Gamma_{k-1})$.

The planarity and the upwardness of the final drawing Γ of T can be easily verified. Concerning the area requirements of Γ, the *height* of Γ is $O(n)$, since there is at least one node of the tree for each horizontal line intersecting Γ. Denote by $w(T(\uparrow, v_i))$, by $w(T(\downarrow, v_i))$, by $w(T(v_i))$, and by $w(n)$ the width of the drawing of a tree in $T(\uparrow, v_i)$, of a tree in $T(\downarrow, v_i)$, of a tree $T(v_i)$, and of

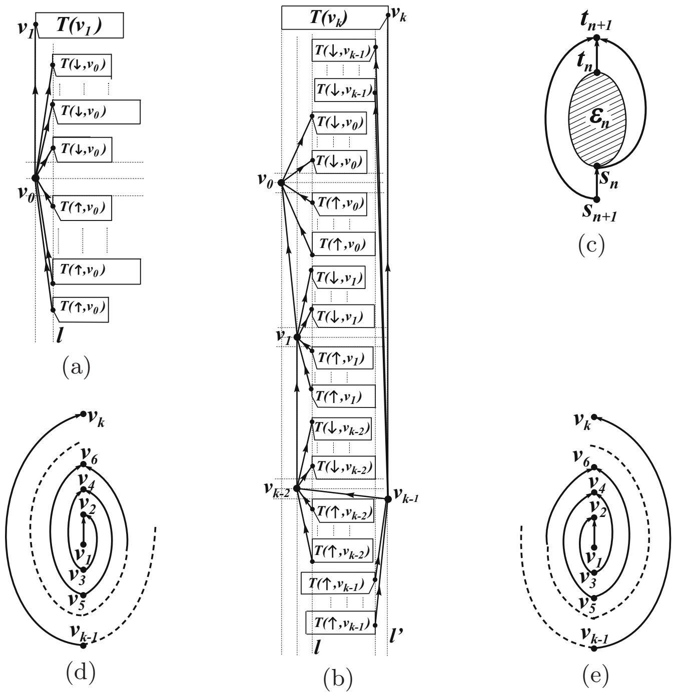

Fig. 1. (a) and (b) *Impera* step of the algorithm for obtaining straight-line non-order preserving upward drawings of trees, in the case $k = 1$ and $k \geq 2$. (c) Embedding $\mathcal{E}_{n+1}$ of the series-parallel digraph presented in [1]. (d) A clockwise coil. (e) A counter-clockwise coil.

an n-nodes tree constructed by the above described algorithm, respectively. In case $k = 1$ we have $w(T) = \max\{w(T(v_1)), 1 + w(T(\uparrow, v_0)), 1 + w(T(\downarrow, v_0))\}$, and in case $k \geq 2$ we have $w(T) = \max_{0 \leq i < k}\{w(T(v_k)), 3 + w(T(\uparrow, v_i)), 3 + w(T(\downarrow, v_i))\}$. By the definition of S, each tree in $T(\uparrow, v_i)$ and each tree in $T(\downarrow, v_i)$ has at most $n/2$ nodes, and $T(v_k)$ has at most $n - k$ nodes. It follows that $w(n) = \max\{w(n-1), 3 + w(n/2)\}$, that easily solves to $w(n) = O(\log n)$. So we have the following:

Theorem 1. *Every n-nodes directed tree admits an upward straight-line drawing in optimal $\Theta(n \log n)$ area.*

4 Upward Drawings of Trees with Fixed Embedding

We discuss the area requirement of order-preserving upward drawings of directed trees. Garg and Tamassia ([12]) proved that any upward planar embedding can be realized with straight-line edges in exponential area. Hence, exponential area straight-line upward drawings of embedded directed trees are feasible.

Now we prove the claimed exponential lower bound. Bertolazzi et al. showed in [1] an embedding $\mathcal{E}_n$ of a $2n$-vertex *series-parallel* digraph requiring $\Omega(4^n)$ area in any order-preserving upward straight-line drawing. Such an embedding is recursively defined as follows: $\mathcal{E}_0$ consists of a single edge (s_0, t_0); $\mathcal{E}_{n+1}$ is obtained from $\mathcal{E}_n$ by adding (i) two new nodes s_{n+1} and t_{n+1}, (ii) an edge from s_{n+1} to s_n, (iii) an edge from t_n to t_{n+1}, (iv) an edge from s_n to t_{n+1} on the right of $\mathcal{E}_n$, and (v) an edge from s_{n+1} to t_{n+1} on the left of $\mathcal{E}_n$ (see Fig. 1.c).

We define a *clockwise coil* S to be an upward planar drawing of a directed path $P = (v_1, v_2, \ldots, v_k)$ that respects three properties: **property (i)** the edges (v_i, v_{i+1}) of P, with i odd (with i even), are directed from v_i to v_{i+1} (resp. from v_{i+1} to v_i), **property (ii)** $y(v_i) < y(v_j)$ $(y(v_i) > y(v_j))$, for every i odd (resp. for every i even) and every j such that $j < i$, and **property (iii)** for i odd (for i even) every vertex v_j, with $j < i$, is contained in the region $R(v_i, v_{i+1})$ delimited by the edge (v_i, v_{i+1}) and by the horizontal half-lines starting at v_i and at v_{i+1} and directed toward increasing x-coordinates (resp. toward decreasing x-coordinates) (see Fig. 1.d). A *counter-clockwise coil* is defined analogously, with *odd* replaced by *even* and vice-versa in property (iii) (see Fig. 1.e). We have:

Lemma 1. *A straight-line n-vertex clockwise or counter-clockwise coil S requires $\Omega(2^n)$ area.*

Proof. Consider any straight-line clockwise coil S. We show that adding segments (v_i, v_{i+2}), for $i = 1, 2 \ldots, n-2$, augments S in a planar drawing S'. Namely, we prove that a segment (v_i, v_{i+2}) does not intersect (a) any segment (v_j, v_{j+1}) of S, with $j \le i$, (b) segment (v_{i+1}, v_{i+2}) of S, (c) segment (v_{i+2}, v_{i+3}) of S, (d) any segment (v_j, v_{j+1}) of S, with $j > i+2$, and (e) any segment (v_j, v_{j+2}), with $j \ne i$ added to S.

(a) Suppose i is odd (is even). By property (ii) no vertex v_j of S, with $j < i+2$ and $j \ne i$, lies in the open half-plane $\mathcal{H}$ below (resp. above) the horizontal line passing through v_i. Moreover, v_{i+2} is contained in $\mathcal{H}$. Hence, (v_i, v_{i+2}) does not create crossings with any segment (v_j, v_{j+1}) of S, with $j \le i$. (b) Since they are adjacent, (v_i, v_{i+2}) and (v_{i+1}, v_{i+2}) cross only if they overlap. But in such a case (v_i, v_{i+1}) and (v_{i+1}, v_{i+2}) overlap, too. However, this is not possible by the supposed planarity of S. (c) By property (iii) v_i is contained inside $R(v_{i+2}, v_{i+3})$. Hence (v_i, v_{i+2}) is internal to $R(v_{i+2}, v_{i+3})$ and can not cross (v_{i+2}, v_{i+3}) that is on the border of $R(v_{i+2}, v_{i+3})$. (d) By property (iii) v_i and v_{i+2} are contained inside $R(v_j, v_{j+1})$, so (v_i, v_{i+2}) is internal to $R(v_{i+2}, v_{i+3})$ and can not cross (v_{i+2}, v_{i+3}) that is on the border of $R(v_{i+2}, v_{i+3})$. (e) It's easy to see that segments (v_i, v_{i+2}), for $i = 1, 2 \ldots, n-2$, form a directed path with increasing y-coordinate and so they don't cross each other.

Now one can observe that S' is an upward drawing of $\mathcal{E}_{n/2}$ (see [1] and the beginning of the section). Hence, an n-vertex straight-line clockwise coil S requires the same area of a straight-line drawing of $\mathcal{E}_{n/2}$, that is $\Omega(4^{n/2}) = \Omega(2^n)$. If S is a counter-clockwise straight-line coil a straightforward modification of the previous proof shows that S requires $\Omega(2^n)$ area. □

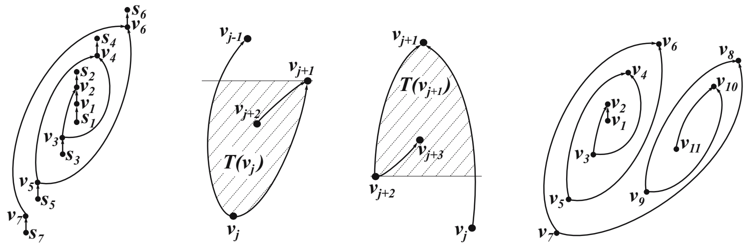

Fig. 2. (a) An upward drawing of T^* with embedding $\mathcal{E}^*$. (b) $T(v_j)$. (c) $T(v_{j+1})$. (d) An upward drawing of P^*. Notice that $(v_1, v_2, \ldots, v_7)$ is a counter-clockwise coil, while $(v_{11}, v_{10}, \ldots, v_7)$ is a clockwise coil.

Now let T^* be a tree composed by an $n/2$-nodes path $P^* = (v_1, v_2, \ldots, v_{n/2})$ and by $n/2$ leaves s_i, $1 \leq i \leq n/2$, such that s_i adjacent to v_i, with n even and $n/2$ odd. Edges (v_i, v_{i+1}), with i odd (with i even), are directed from v_i to v_{i+1} (resp. from v_{i+1} to v_i). Edges (v_i, s_i), with i odd (with i even), are directed from s_i to v_i (resp. from v_i to s_i). We fix for T^* an embedding $\mathcal{E}^*$ such that for each node v_i, $2 \leq i \leq n/2$, the clockwise order of the edges incident in v_i is $[s_i, v_{i-1}, v_{i+1}]$ (see Fig. 2.a). We claim the following:

Lemma 2. *Every upward drawing Γ^* of T^* with embedding $\mathcal{E}^*$ contains a clockwise or a counter-clockwise coil of at least $n/4$ nodes.*

Proof. Observe that, by the embedding constraints of $\mathcal{E}^*$ and by the upwardness of Γ^*, path P^* turns in clockwise direction at every edge (v_{i-1}, v_i), for $i = 2, 3, \ldots, n/2$, i. e. considering the half-lines t_1 and t_2 starting at v_i and tangent to the curves representing edges (v_{i-2}, v_{i-1}) and (v_{i_1}, v_i), respectively, the angle described by a clockwise movement that leads t_1 to overlap with t_2 is less than π. Let j be the highest index such that the drawing S_1^* of the subpath $(v_1, v_2, \ldots, v_j)$ of P^* is a counter-clockwise coil. If $j \geq n/4$ or if such a j doesn't exist, i.e. P^* is entirely drawn as a counter-clockwise coil, the lemma follows. Otherwise, we claim that the drawing S_2^* of the subpath $(v_{n/2}, v_{n/2-1}, \ldots, v_{j+1}, v_j)$ of P^* is a clockwise coil. Property (i) follows from the upwardness of Γ^*. Consider three vertices v_{i-1}, v_i, and v_{i+1} that are consecutive in P^*. Let v_t be the one between v_{i-1} and v_{i+1} such that $|y(v_i) - y(v_t)|$ is minimum. Denote by $T(v_i)$, with $i = j, j+1, \ldots, n/2-1$ the triangle with curved edges delimited by (v_i, v_{i-1}), by (v_i, v_{i+1}), and by the horizontal line through v_t. Assume j is odd. Since $(v_1, v_2, \ldots, v_j, v_{j+1})$ is not a coil, then $y(v_{j-1}) \geq y(v_{j+1})$. Since (v_{j+1}, v_{j+2}) turns in clockwise direction with respect to (v_j, v_{j+1}), the planarity and the upwardness of Γ^* imply that v_{j+2} is inside $T(v_j)$, and so $y(v_{j+2}) > y(v_j)$ (see Fig. 2.b). Since (v_{j+2}, v_{j+3}) turns in clockwise direction with respect to (v_{j+1}, v_{j+2}), the planarity and the upwardness of Γ^* imply that v_{j+3} is inside $T(v_{j+1})$, and so $y(v_{j+3}) > y(v_{j+1})$ (see Fig. 2.c). Proceeding in the same way, it follows that, for all $i = j, j+1, \ldots, n/2-2$, $y(v_{i+2}) > y(v_i)$

$(y(v_{i+2}) > y(v_i))$ with i odd (resp. with i even). Hence, property (ii) is satisfied by S_2^*. Further, property (iii) is satisfied by S_2^*, since every vertex v_k, with $k \geq i+2$ is contained inside $T(v_i)$ and, consequently, inside $R(v_i, v_{i+1})$, that encloses $T(v_i)$. If j is even an analogous proof shows that S_2^* is a clockwise coil. Finally, since $j < n/4$, S_2^* contains at least $n/2 - j > n/4$ nodes. □

Theorem 2. *There exists an n-nodes embedded directed tree requiring $\Omega(b^n)$ area, with b greater than* 1, *in any upward straight-line order-preserving drawing.*

Proof. Consider T^* and its embedding $\mathcal{E}^*$ described in this section. By Lemma 2 every upward drawing of T^* with embedding $\mathcal{E}^*$ contains a coil of at least $n/4$ nodes that, by Lemma 1, requires $\Omega(2^{n/4}) = \Omega((\sqrt[4]{2})^n) = \Omega(b^n)$, with $b = \sqrt[4]{2}$. □

Now we turn to poly-line drawings. Di Battista et al. have shown in [9] that every upward planar embedding can be drawn with poly-line edges in $O(n^2)$ area. It follows that quadratic area poly-line upward drawings of embedded directed trees are feasible. Concerning the lower bound, we have the following:

Lemma 3. *An n-vertex poly-line clockwise or counter-clockwise coil S requires $\Omega(n^2)$ area.*

Proof. By property (ii) vertex v_i, with i odd, has y-coordinate less than the one of every vertex v_j, with $j < i$. This implies that $n/2$ vertices v_i such that i is odd occupy $n/2$ distinct horizontal lines and so the height of S is $\Omega(n)$. Concerning the width of S, suppose w.l.o.g. to draw S starting from a drawing Γ_1 of v_1, and then iteratively constructing a drawing Γ_i by adding vertex v_i and edge (v_{i-1}, v_i) to Γ_{i-1}, for $i = 2, \ldots, n$. We claim that the width of Γ_i is at least the width of Γ_{i-1} plus one. Suppose that the width of Γ_i is equal to the width of Γ_{i-1}. Then edge (v_{i-1}, v_i) can not be on the left or on the right of Γ_{i-1} and so property (iii) can not be satisfied. It follows that the width of S is $\Omega(n)$. □

Hence, we can again consider directed tree T^* with fixed embedding $\mathcal{E}^*$. By Lemma 2 every upward drawing of T^* with embedding $\mathcal{E}^*$ contains a clockwise or a counter-clockwise coil S of at least $n/4$ nodes. By Lemma 3 $\Omega(n^2)$ area is required for S.

Theorem 3. *There exists an n-nodes directed tree T^* and an embedding of T^* requiring $\Omega(n^2)$ area in any upward poly-line order-preserving drawing.*

5 Upward Drawings of Some Families of DAGs

In the first part of this section we study the area requirement of planar upward drawings of some families of directed trees, like *directed binary trees*, *directed caterpillars*, and *directed spider trees*, searching for better area bounds with respect to those obtained for general trees. In the second part of this section we show that the results obtained for directed trees can be exploited to obtain area bounds for several others families of DAGs, like *directed bipartite graphs* and *directed outerplanar graphs*. The proofs of the theorems claimed in this section are omitted, for reasons of space.

Concerning *directed binary trees*, one can observe that the lower bounds on the area requirement of planar upward drawings of directed trees presented in Sections 3 and 4 are obtained by considering directed *binary* trees. Hence such lower bounds are still valid here. Moreover, the algorithms for drawing directed trees clearly apply also to directed binary trees, hence the optimal bounds on the area requirement of planar upward drawings of directed binary trees are the same of the ones of general trees.

Analogously, concerning *directed caterpillars*, we notice that the lower bound on the area requirement of order-preserving upward drawings of directed trees presented in Section 4 was obtained by considering a directed caterpillar. Hence such a lower bound is still valid here. On the other hand, for non-order-preserving drawings one can obtain better results with respect to those for general trees, as shown by the following:

Theorem 4. *Every n-nodes directed caterpillar tree admits an upward straight-line drawing in optimal $\Theta(n)$ area.*

For *directed spider trees* linear area is achievable also for order-preserving drawings:

Theorem 5. *Every n-nodes directed spider tree admits an upward order-preserving straight-line drawing in optimal $\Theta(n)$ area.*

Considering families of DAGs richer than directed trees, exponential area is sometimes necessary even without forcing an order of the neighbors of each vertex. In the following we show the inductive construction of an n-vertex directed bipartite graph B_n. Such a digraph contains an $O(n)$ nodes coil in any upward planar drawing, hence it requires exponential area in any straight-line upward drawing and quadratic area in any poly-line upward drawing. Such lower bounds are again matched by the upper bounds in [12,9]. We define B_n as the directed bipartite graph with vertex sets V and U, inductively defined as follows: (i) B_8 has vertices $v_{-2}, v_{-1}, v_1, v_2 \in V$ and $u_{-2}, u_{-1}, u_1, u_2 \in U$, the edges of a directed path $(v_{-2}, u_{-2}, v_{-1}, u_{-1}, v_1, u_1, v_2, u_2)$, and the directed edges (v_1, u_2), (v_{-1}, u_1), (v_{-2}, u_1) and (v_{-1}, u_2) (see Fig. 3.a); (ii) B_n, with n multiple of 4, is done by B_{n-4}, by four new vertices $v_{n/4}$, $u_{n/4}$, $v_{-n/4}$, and $u_{-n/4}$ and by eight directed edges $(v_{-n/4}, u_{-n/4})$, $(u_{-n/4}, v_{-n/4+1})$, $(u_{n/4-1}, v_{n/4})$, $(v_{n/4}, u_{n/4})$, $(v_{-n/4+2}, u_{n/4})$, $(v_{-n/4+1}, u_{n/4-1})$, $(v_{-n/4}, u_{n/4-1})$, and $(v_{-n/4+1}, u_{n/4})$ (see Fig. 3.b). An extensive study of the properties of B_n leads to the followings:

Theorem 6. *There exists an n-vertex directed bipartite graph requiring $\Omega(b^n)$ area, with b greater than 1, in any upward straight-line drawing.*

Theorem 7. *There exists an n-vertex directed bipartite graph requiring $\Omega(n^2)$ area in any upward poly-line drawing.*

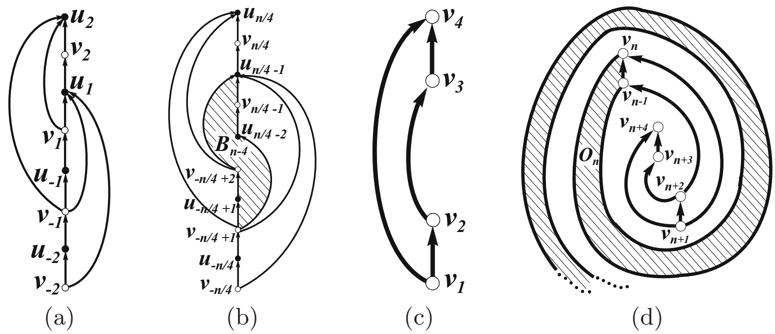

Fig. 3. (a) B_8. (b) B_n. (c) O_4. (d) O_n.

Again using arguments based on the results obtained for directed trees, it can be shown that *directed outerplanar graphs* generally require exponential area in any outerplanar straight-line upward drawing and quadratic area in any poly-line upward drawing. These results are achieved by considering the n-vertex directed outerplanar graph O_n inductively defined as follows: (i) O_4 has four vertices v_1, v_2, v_3, and v_4 and four directed edges (v_1, v_2), (v_1, v_4), (v_2, v_3), and (v_3, v_4) (see Fig. 3.c); (ii) O_{n+4} is composed by O_n, by four new vertices v_{n+1}, v_{n+2}, v_{n+3}, and v_{n+4}, and by six new directed edges (v_{n+1}, v_n), (v_{n+2}, v_{n-1}), (v_{n+1}, v_{n+2}), (v_{n+2}, v_{n+3}), (v_{n+1}, v_{n+4}), and (v_{n+3}, v_{n+4}) (see Fig. 3.d). Studying the properties of upward drawings of O_n the followings can be proved:

Theorem 8. *There exists an n-vertex directed outerplanar graph requiring $\Omega(b^n)$ area, with b greater than 1, in any upward outerplanar straight-line drawing.*

Theorem 9. *There exists an n-vertex directed outerplanar graph requiring $\Omega(n^2)$ area in any upward poly-line drawing.*

6 Conclusions and Open Problems

In this paper we have studied the area requirement of upward drawings of several classes of DAGs that frequently arise in theory and in practice.

We provided tight bounds on the area requirement of straight-/poly-line order-/non-order-preserving upward drawings of general directed trees and of several families of directed trees. However, the following problem is still open:

Problem 1. Which is the minimum area of upward straight/poly-line order/non order-preserving drawings of *complete* and *balanced* trees?

Concerning directed bipartite graphs, we have shown an exponential area lower bound for straight-line upward drawings, but the following is still open:

Problem 2. Which is the minimum area of an upward drawing of a *bipartite DAG*? Bipartite DAGs [7] are those DAGs having a vertex set partitioned into two subsets V_1 and V_2 with each edge directed from a vertex of V_1 to a vertex of V_2. Consequently, bipartite DAGs form a subclass of the digraphs whose underlying graph is bipartite, that was considered in this paper.

Further, we have shown an outerplanar graph requiring exponential area in any straight-line *outerplanar* upward drawing. However, when considering *non-outerplanar* drawings, one could obtain better area bounds, so we ask:

Problem 3. Which is the minimum area of straight-line non-outerplanar upward drawings of directed outerplanar graphs?

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