4

The BMO-estimates of the Hardy-type Transforms

4.1 Estimates of Oscillations of the Hardy Transform

The operator $\mathcal{P}$ defined via the formula

$$\mathcal{P}f(t) = \frac{1}{t}\int_0^t f(u)\,du, \quad t \neq 0$$

is called the Hardy transform (operator) of the function $f \in L_{loc}(\mathbb{R})$. This operator has plenty of applications. We have seen some of them in Section 1.1. Namely, it easy to see that $f^{**}(t) = \mathcal{P}f^*(t),\ t \in \mathbb{R}_+$. The *Hardy inequality* [26]:

$$\int_0^\infty |\,\mathcal{P}f(x)|^p\,dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p\,dx$$

provides the boundedness of the Hardy operator in $L^p(\mathbb{R}_+)$. Here the constant $\left(\frac{p}{p-1}\right)^p$ is sharp. For $p = \infty$ the analogous inequality $\|\,\mathcal{P}f\,\|_\infty \leq \|f\|_\infty$ is trivial. For $p = 1$ the analog of the Hardy inequality is false, in order to see this it is enough to consider the function

$$f_0(x) = \frac{1}{x\ln^2\frac{1}{x}}\chi_{\left(0,\frac{1}{\mathrm{e}}\right)}(x), \quad x \in \mathbb{R}_+.$$

Indeed, $f_0 \in L(\mathbb{R}_+)$ and for $0 < x < \frac{1}{\mathrm{e}}$ we have $\mathcal{P}f_0(x) = \left(x\ln\frac{1}{x}\right)^{-1}$, so that $\mathcal{P}f_0 \notin L_{loc}(\mathbb{R}_+)$. It is easy to see that in general *the reverse Hardy inequality*

$$\int_0^\infty |\,\mathcal{P}f(x)|^p\,dx \geq c_p\int_0^\infty |f(x)|^p\,dx \tag{4.1}$$

fails for an arbitrary constant $c_p > 0$. However, if the non-negative function f is non-increasing on $\mathbb{R}_+$, then (4.1) is true for $c_p = \frac{p}{p-1}$ and this value cannot be increased (see [53, 64, 58]). If $p = \infty$ and f is a non-positive non-increasing function on $\mathbb{R}_+$, then obviously $\|\,\mathcal{P}f\,\|_\infty = \|f\|_\infty = \lim_{x\to 0+} f(x)$.

In this section we will study the behavior of the Hardy operator in the spaces BMO, BMO_p and BLO. The boundedness of $\mathcal{P}$ in BMO in different cases was proved by several authors in [73, 74, 71, 24, 19, 20, 80]. In particular, in [80] it was proved the following result.

Theorem 4.1 (Jie Xiao, [80]). *The operator $\mathcal{P}$ is bounded in $BMO(\mathbb{R}_+)$ and*

$$\| \mathcal{P}f \|_* \leq \|f\|_*.$$

On the other hand, if f is a positive non-increasing function on $\mathbb{R}_+$, then

$$\| \mathcal{P}f \|_* \geq \frac{1}{17}\|f\|_*.$$

Here we will prove some more general facts. Let us start with the direct estimate of the Hardy transform.

Theorem 4.2 ([40]). *Let $1 \leq p < \infty$. Then if f belongs to $BMO_p(\mathbb{R})$, then $\mathcal{P}f \in BMO_p(\mathbb{R})$ and*

$$\| \mathcal{P}f \|_{*,p} \leq \|f\|_{*,p}. \tag{4.2}$$

Moreover, in general the constant 1 in the right-hand side of (4.2) is sharp.

Proof. As in [80], we will use the equality

$$\mathcal{P}f(t) = \int_0^1 f(tu)\,du, \quad t \in \mathbb{R} \setminus \{0\}. \tag{4.3}$$

Fix the interval $[a, b] \equiv I \subset \mathbb{R}$. By the Fubini theorem,

$$(\mathcal{P}f)_I = \frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt = \int_0^1 \frac{1}{|I|}\int_I f(tu)\,dt\,du.$$

Denote $uI \equiv [ua, ub]$. Applying again the Fubini theorem and the Hölder inequality, we have

$$\Omega_p^p(\mathcal{P}f; I) = \frac{1}{|I|}\int_I \left|\int_0^1 f(\tau u)\,du - \int_0^1 \frac{1}{|I|}\int_I f(tu)\,dt\,du\right|^p d\tau \leq$$

$$\leq \int_0^1 \frac{1}{|I|}\int_I \left|f(\tau u) - \frac{1}{|I|}\int_I f(tu)\,dt\right|^p d\tau\,du =$$

$$= \int_0^1 \frac{1}{|uI|}\int_{uI} \left|f(v) - \frac{1}{uI}\int_{uI} f(\xi)\,d\xi\right|^p dv\,du = \int_0^1 \Omega_p^p(f; uI)\,du \leq \|f\|_{*,p}^p,$$

and (4.2) follows.

For the function $f(x) = \ln \frac{1}{|x|},\ x \in \mathbb{R}$, we have $\mathcal{P}f(x) = 1 + \ln \frac{1}{|x|}$. Hence for this choice of f inequality (4.2) becomes an equality, so that the constant in the right-hand side of (4.2) cannot be smaller than 1. □

If in the proof of Theorem 4.2 we choose $I \subset \mathbb{R}_+$, then $uI \subset \mathbb{R}_+$ for $u > 0$. Hence, repeating the proof of Theorem 4.2, we obtain the following statement.

Theorem 4.3 ([40]). *Let $1 \le p < \infty$. If $f \in BMO_p(\mathbb{R}_+)$, then $\mathcal{P}f \in BMO_p(\mathbb{R}_+)$ and*

$$\| \mathcal{P}f \|_{*,p} \le \|f\|_{*,p}.$$

Moreover, the constant 1 in the right-hand side is sharp.

Similarly one can obtain the estimates for the "norm" of the Hardy transform in BLO.

Theorem 4.4 ([41]). *Let $f \in BLO(\mathbb{R})$. Then $\mathcal{P}f \in BLO(\mathbb{R})$,*

$$\| \mathcal{P}f \|_{BLO} \le \|f\|_{BLO}, \tag{4.4}$$

and the constant 1 in the right-hand side is sharp.

Theorem 4.5 ([41]). *Let $f \in BLO(\mathbb{R}_+)$. Then $\mathcal{P}f \in BLO(\mathbb{R}_+)$,*

$$\| \mathcal{P}f \|_{BLO} \le \|f\|_{BLO},$$

and the constant 1 in the right-hand side is sharp.

As in the case of Theorems 4.2 and 4.3, the proofs of both Theorems 4.4 and 4.5 are similar. Here we give just one of them.

Proof of Theorem 4.4. Let $I \subset \mathbb{R}$ and $x \in I,\ x \ne 0$. By (4.3),

$$\frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt - \mathcal{P}f(x) = \int_0^1 \frac{1}{|I|}\int_I f(tu)\,du - \int_0^1 f(xu)\,du \le$$

$$\le \int_0^1 \left[\frac{1}{|I|}\int_I f(tu)\,dt - f(xu)\right] du = \int_0^1 \left[\frac{1}{|uI|}\int_{uI} f(v)\,dv - f(xu)\right] du.$$

Since $x \in I$ implies $ux \in uI$ for $u > 0$ we have

$$\frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt - \mathcal{P}f(x) \le \int_0^1 \left[\frac{1}{|uI|}\int_{uI} f(v)\,dv - \operatorname*{ess\,inf}_{y\in uI} f(y)\right] du =$$

$$= \int_0^1 L(f;uI)\,du \le \|f\|_{BLO}.$$

Hence, using the equality

$$L(\mathcal{P}f;I) = \frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt - \operatorname*{ess\,inf}_{x\in I} \mathcal{P}f(x) = \operatorname*{ess\,sup}_{x\in I}\left[\frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt - \mathcal{P}f(x)\right],$$

and taking the essential supremum over all $x \in I,\ x \neq 0$, we obtain

$$L(\mathcal{P}f; I) \leq \|f\|_{BLO}, \quad I \subset \mathbb{R}.$$

The same arguments as in the proof of Theorem 4.2 show that the constant 1 in the right-hand side of (4.4) is sharp. □

Now let us consider the lower bounds for the norm of the Hardy transform. It is easy to see that, similarly to (4.1), the inequality

$$\| \mathcal{P}f \|_* \geq c \|f\|_* \tag{4.5}$$

in general fails for arbitrary f and $c > 0$. But if we consider the functions f that are non-increasing and non-negative on $\mathbb{R}_+$, then, according to Theorem 4.1, inequality (4.5) holds true for $c = \frac{1}{17}$. In what follows we will derive (4.5) with the value of c grater than $\frac{1}{17}$ and find its upper bound (see Corollary 4.16).

Theorem 4.6 ([40]). *Let $1 \leq p < \infty$, and assume that the function f is non-decreasing on $(-\infty, 0)$ and non-increasing on $(0, +\infty)$. Then $f \in BMO_p(\mathbb{R})$ if and only if $\mathcal{P}f \in BMO_p(\mathbb{R})$ and*

$$\| \mathcal{P}f \|_{*,p} \leq \|f\|_{*,p} \leq \frac{2}{3 - \sqrt{7}} \| \mathcal{P}f \|_{*,p}. \tag{4.6}$$

Proof. The left inequality of (4.6) is contained in Theorem 4.2. Moreover, it is true even if f is not monotone. Now let us prove the right inequality.

Let $\lambda > 1$ (we will choose it later). Consider the function

$$g(t) = \frac{1}{\lambda} \mathcal{P}f(t) + \frac{\lambda - 1}{\lambda} f(t), \quad t \in \mathbb{R} \setminus \{0\}.$$

Then $f(t) = \frac{\lambda}{\lambda-1} g(t) - \frac{1}{\lambda-1} \mathcal{P}f(t)$, so that by Minkowski inequality

$$\|f\|_{*,p} \leq \frac{\lambda}{\lambda - 1} \|g\|_{*,p} + \frac{1}{\lambda - 1} \| \mathcal{P}f \|_{*,p}. \tag{4.7}$$

Let us estimate $\|g\|_{*,p}$. The monotonicity of f on $(-\infty, 0)$ and $(0, +\infty)$ implies

$$\mathcal{P}f(\lambda t) = \frac{1}{\lambda} \frac{1}{t} \int_0^t f(u)\, du + \frac{\lambda - 1}{\lambda} \frac{1}{\lambda t - t} \int_t^{\lambda t} f(u)\, du \leq \frac{1}{\lambda} \mathcal{P}f(t) + \frac{\lambda - 1}{\lambda} f(t)$$

for $t \neq 0$. Hence, using again the monotonicity of f, we obtain

$$\mathcal{P}f(\lambda t) \leq g(t) \leq \mathcal{P}f(t), \quad t \neq 0. \tag{4.8}$$

Assume $I \equiv [\alpha, \beta],\ \alpha < 0 < \beta$. By (4.8),

$$\frac{1}{|I|}\int_I \mathcal{P}f(\lambda t)\,dt = (\mathcal{P}f)_{\lambda I} \le g_I \le (\mathcal{P}f)_I.$$

The last inequality, together with (4.8), imply

$$g(t) - g_I \le \mathcal{P}f(t) - (\mathcal{P}f)_{\lambda I}, \quad t \ne 0, \tag{4.9}$$

$$g_I - g(t) \le (\mathcal{P}f)_I - \mathcal{P}f(\lambda t), \quad t \ne 0. \tag{4.10}$$

Denote $E_+ \equiv \{t \in I : \; g(t) \ge g_I\}$, $E_- \equiv \{t \in I : \; g(t) < g_I\}$. Multiplying (4.9) and (4.10) by $\chi_{E_+}(t)$ and $\chi_{E_-}(t)$ respectively and summing up the obtained inequalities for $t \ne 0$ we get

$$|g(t) - g_I| \le (\mathcal{P}f(t) - (\mathcal{P}f)_{\lambda I})\,\chi_{E_+}(t) + ((\mathcal{P}f)_I - \mathcal{P}f(\lambda t))\,\chi_{E_-}(t) =$$

$$= (\mathcal{P}f(t) - (\mathcal{P}f)_I)\,\chi_{E_+}(t) + ((\mathcal{P}f)_{\lambda I} - \mathcal{P}f(\lambda t))\,\chi_{E_-}(t) + ((\mathcal{P}f)_I - (\mathcal{P}f)_{\lambda I})\,.$$

Using Lemma 2.35 and the inclusion $\lambda I \supset I$, one can find the following estimate for the last term in the right hand side:

$$(\mathcal{P}f)_I - (\mathcal{P}f)_{\lambda I} = \frac{1}{|I|}\int_I (\mathcal{P}f(t) - (\mathcal{P}f)_{\lambda I})\;dt \le$$

$$\le \lambda\frac{1}{|\lambda I|}\int_{\{t\in\lambda I:\; \mathcal{P}f(t)>(\mathcal{P}f)_{\lambda I}\}} (\mathcal{P}f(t) - (\mathcal{P}f)_{\lambda I})\;dt = \frac{\lambda}{2}\Omega(\mathcal{P}f;\lambda I) \le \frac{\lambda}{2}\|\,\mathcal{P}f\,\|_*.$$

Thus, by Minkowski inequality,

$$\Omega_p(g;I) \le \left\{\frac{1}{|I|}\int_{E_+} |\mathcal{P}f(t) - (\mathcal{P}f)_I|^p\;dt\right\}^{\frac{1}{p}} +$$

$$+\left\{\frac{1}{|I|}\int_{E_-} |(\mathcal{P}f)_{\lambda I} - \mathcal{P}f(\lambda t)|^p\;dt\right\}^{\frac{1}{p}} + \frac{\lambda}{2}\|\,\mathcal{P}f\,\|_* \le$$

$$\le \Omega_p(\mathcal{P}f;I) + \Omega_p(\mathcal{P}f;\lambda I) + \frac{\lambda}{2}\|\,\mathcal{P}f\,\|_* \le \left(2 + \frac{\lambda}{2}\right)\|\,\mathcal{P}f\,\|_{*,p}.$$

Notice that both functions g and f are non-decreasing on $(-\infty,0)$ and non-increasing on $(0,+\infty)$. Since I is an arbitrary segment, which contains zero, the last inequality together with Lemma 2.23 imply

$$\|g\|_{*,p} \le \left(2 + \frac{\lambda}{2}\right)\|\,\mathcal{P}f\,\|_{*,p}.$$

Substituting this bound in (4.7), we obtain

$$\|f\|_{*,p} \le \frac{1}{\lambda - 1}\left[\lambda\left(2 + \frac{\lambda}{2}\right) + 1\right]\|\,\mathcal{P}f\,\|_{*,p}. \tag{4.11}$$

It remains to choose the constant $\lambda > 1$ which provides the minimal value to the function $\psi(\lambda) \equiv \frac{1}{\lambda-1}\left[\lambda\left(2+\frac{\lambda}{2}\right)+1\right]$. An easy calculation shows that

$$\min_{\lambda>1} \psi(\lambda) = \psi(1+\sqrt{7}) = \frac{2}{3-\sqrt{7}}.$$

Therefore, (4.11) implies the right inequality of (4.6). □

If in the proof of Theorem 4.6 instead of Lemma 2.23 we apply Lemma 2.22, then we get the following statement.

Theorem 4.7 ([40]). *Let $1 \le p < \infty$ and let $f \in L^p_{loc}(\mathbb{R}_+)$ be non-increasing on $\mathbb{R}_+$. Then $f \in BMO_p(\mathbb{R}_+)$ if and only if $\mathcal{P}f \in BMO_p(\mathbb{R}_+)$, and*

$$\|\mathcal{P}f\|_{*,p} \le \|f\|_{*,p} \le \frac{2}{3-\sqrt{7}}\|\mathcal{P}f\|_{*,p}. \tag{4.12}$$

In the particular case $p = 1$ the right inequality of (4.12) becomes

$$\|\mathcal{P}f\|_* \ge \frac{3-\sqrt{7}}{2}\|f\|_*, \quad f \in BMO(\mathbb{R}_+),\ f \text{ do not increase.} \tag{4.13}$$

Since $\frac{3-\sqrt{7}}{2} > \frac{1}{17}$ the new inequality is stronger than the inequality in the second part of Theorem 4.1.

The next theorem is an analog of Theorem 4.7 for the BLO-“norm”.

Theorem 4.8 ([41]). *Let $f \in L_{loc}(\mathbb{R}_+)$ be non-increasing on $\mathbb{R}_+$. Then*

$$\frac{1}{\mathrm{e}}\|f\|_{BLO} \le \|\mathcal{P}f\|_{BLO} \le \|f\|_{BLO}, \tag{4.14}$$

and in general the constants $\frac{1}{\mathrm{e}}$ and 1 in the left and right-hand sides are sharp.

Proof. The left inequality of (4.14) was already proved in Theorem 4.5. Let us show that the constant $\frac{1}{\mathrm{e}}$ in the left-hand side of (4.14) cannot be increased. For this consider the function $f_0(x) = \chi_{[0,1)}(x)$, $x \in \mathbb{R}_+$. By Lemma 2.34,

$$\mathcal{P}f_0(x) = \min\left(1, \frac{1}{x}\right), \quad \|f_0\|_{BLO} = \sup_{x>0}\left[\mathcal{P}f_0(x) - f_0(x)\right] = 1,$$

and for $x > 1$

$$L(\mathcal{P}f_0; [0,x]) = \frac{1}{x}\int_0^x \mathcal{P}f_0(t)\,dt - \mathcal{P}f_0(x) = \frac{1}{x}(1+\ln x) - \frac{1}{x} = \frac{\ln x}{x}.$$

Hence

$$\|\mathcal{P}f_0\|_{BLO} = \sup_{x>1}\frac{\ln x}{x} = \frac{1}{\mathrm{e}} = L(\mathcal{P}f_0; [0,\mathrm{e}]). \tag{4.15}$$

Therefore the constant $\frac{1}{\mathrm{e}}$ in the left-hand side of (4.14) cannot be increased.

It remains to prove the left inequality of (4.14) for an arbitrary f. By virtue of Lemma 2.34, it is enough to show that for any $x > 0$ there exists $y > 0$ such that

$$L(\mathcal{P}f;[0,y]) \geq \frac{1}{\mathrm{e}} L(f;[0,x]). \tag{4.16}$$

Without loss of generality we can assume that

$$x = 1, \quad f(1) = 0, \quad \int_0^1 f(t)\,dt = 1, \quad L(f;[0,1]) = 1. \tag{4.17}$$

As before, set $f_0(x) = \chi_{[0,1)}(x),\ x \in \mathbb{R}_+$. Let us show that

$$\mathcal{P}(f - f_0)(t) \geq (f - f_0)(t), \quad t > 0. \tag{4.18}$$

Indeed, if $0 < t \leq 1$, then (4.18) follows from the monotonicity of $f - f_0$. Otherwise, if $t > 1$, then $f(t) \leq 0$. Taking into account assumptions (4.17), from the monotonicity of f we obtain

$$\frac{1}{t}\int_0^t [f(u) - f_0(u)]\,du = \frac{1}{t}\int_0^1 [f(u) - 1]\,du + \frac{1}{t}\int_1^t f(u)\,du = \frac{1}{t}\int_1^t f(u)\,du =$$

$$= \left(1 - \frac{1}{t}\right)\frac{1}{t-1}\int_1^t f(u)\,du \geq \left(1 - \frac{1}{t}\right) f(t) \geq f(t) = (f - f_0)(t).$$

Now, by (4.18) and (4.15),

$$\frac{1}{\mathrm{e}}\int_0^{\mathrm{e}} \mathcal{P}f(t)\,dt - \mathcal{P}f(\mathrm{e}) - \frac{1}{\mathrm{e}} =$$

$$= \frac{1}{\mathrm{e}}\left[\int_0^{\mathrm{e}} \mathcal{P}f(t)\,dt - \int_0^{\mathrm{e}} \mathcal{P}f_0(t)\,dt\right] - \mathcal{P}f(\mathrm{e}) + \mathcal{P}f_0(\mathrm{e}) =$$

$$= \frac{1}{\mathrm{e}}\int_0^{\mathrm{e}} [\mathcal{P}f(t) - f(t) - \mathcal{P}f_0(t) + f_0(t)]\,dt =$$

$$= \frac{1}{\mathrm{e}}\int_0^{\mathrm{e}} [\mathcal{P}(f - f_0)(t) - (f - f_0)(t)]\,dt \geq 0.$$

Then

$$L(\mathcal{P}f;[0,\mathrm{e}]) \geq \frac{1}{\mathrm{e}}.$$

So, inequality (4.16) is proved and (4.14) follows. □

Let us come back to the estimate given by (4.13). One can improve this estimate using Theorem 4.8.

Corollary 4.9. *Let f be a non-increasing function on $\mathbb{R}_+$. Then*

$$\|\,\mathcal{P}f\,\|_* \geq \frac{1}{4}\|f\|_*. \tag{4.19}$$

Proof. Applying successively Theorems 2.36, 4.8 and again Theorem 2.36, we have

$$\| \mathcal{P}f \|_* \geq \frac{1}{2} \| \mathcal{P}f \|_{BLO} \geq \frac{1}{2}\frac{1}{\mathrm{e}} \|f\|_{BLO} \geq \frac{1}{2}\frac{1}{\mathrm{e}}\frac{\mathrm{e}}{2} \|f\|_* = \frac{1}{4}\|f\|_*. \quad \square$$

Let us show that inequality (4.19) can be also improved. For this we will need some auxiliary statements.

Lemma 4.10. *The equation*

$$\psi(x) \equiv \ln \frac{x}{1+\ln x} - \frac{\ln x}{1+\ln x} = 0 \tag{4.20}$$

has a unique root γ_1 *on* $(1, +\infty)$.

Proof. One can easily check that $\psi(1) = 0$,

$$\lim_{x \to +\infty} \psi(x) = +\infty, \quad \psi'(x) = \frac{\ln^2 x + \ln x - 1}{x(1+\ln x)^2},$$

$\psi'(1) = -1$, and the equation $\psi'(x) = 0$ has a unique root on the interval $(1, +\infty)$. Since the function ψ is differentiable on $[1, +\infty)$ the listed properties imply the statement of the lemma. $\square$

Let $\gamma_1 > 1$ be the root of equation (4.20). In what follows we will use the following constants:

$$\beta_0 = \frac{1 + \ln \gamma_1}{\gamma_1}, \quad \gamma_0 = \frac{1}{\beta_0}, \quad \alpha_0 = \frac{4}{\gamma_1} \ln \gamma_0. \tag{4.21}$$

The approximate values of these constants are

$$\alpha_0 \approx 0.52, \quad \beta_0 \approx 0.546, \quad \gamma_0 \approx 1.83, \quad \gamma_1 \approx 4.65.$$

The next lemma explains the original meaning of equation (4.20) and of constants $\alpha_0,\ \beta_0,\ \gamma_0$, defined by (4.21).

Lemma 4.11. *For the function* $f_0(x) = \chi_{[0,1)}(x),\ x \in \mathbb{R}_+$,

$$\|f_0\|_* = \Omega(f_0; [0,2]) = \frac{1}{2}, \tag{4.22}$$

$$\| \mathcal{P}f_0 \|_* = \Omega\left(\mathcal{P}f_0; [0, \gamma_1]\right) = \frac{1}{2}\alpha_0, \tag{4.23}$$

where γ_1 *is a root of equation* (4.20)*, and* α_0 *is defined by* (4.21).

Proof. According to Property 2.7, $\Omega(f_0; I) \leq \frac{1}{2}$ for any interval $I \subset \mathbb{R}_+$ and $\Omega(f_0; [0,2]) = \frac{1}{2}$. Thus equality (4.22) follows.

In order to prove (4.23) we use the equality $\mathcal{P}f_0(x) = \min\left(1, \frac{1}{x}\right),\ x \in \mathbb{R}_+$. Then for $x > 1$ and $I \equiv [0, x]$

$$(\mathcal{P}f_0)_I = \frac{1}{x}\int_0^1 \mathcal{P}f_0(t)\,dt + \frac{1}{x}\int_1^x \mathcal{P}f_0(t)\,dt = \frac{1+\ln x}{x}.$$

Set $x_0 = \frac{x}{1+\ln x}$. Then $\mathcal{P}f_0(t) \geq (\mathcal{P}f_0)_I$ if $t \leq x_0$, and $\mathcal{P}f_0(t) \leq (\mathcal{P}f_0)_I$ if $t \geq x_0$. Hence, by Property 2.1,

$$\Omega\left(\mathcal{P}f_0; I\right) = \frac{2}{x}\int_{x_0}^{x} \left[(\mathcal{P}f_0)_I - \mathcal{P}f_0(t)\right]\,dt =$$

$$= \frac{2}{x}\left[\frac{1+\ln x}{x}\left(x - \frac{x}{1+\ln x}\right) - \ln(1+\ln x)\right] = \frac{2}{x}[\ln x - \ln(1+\ln x)] \equiv \varphi(x).$$

We have

$$\varphi'(x) = \frac{2}{x^2}\left[\frac{\ln x}{1+\ln x} - \ln\frac{x}{1+\ln x}\right] = \frac{2}{x^2}\psi(x),$$

where the function ψ was defined in Lemma 4.10. Applying Lemmas 4.10 and 2.34 it is easy to see that

$$\|\,\mathcal{P}f_0\,\|_* = \sup_{x>1}\Omega\left(\mathcal{P}f_0; [0,x]\right) = \max_{x>1}\varphi(x) = \varphi(\gamma_1) = \Omega\left(\mathcal{P}f_0; [0,\gamma_1]\right) = \frac{1}{2}\alpha_0,$$

which proves (4.23). □

Remark 4.12. The following formula explains the meaning of the constants β_0 and γ_0, defined by (4.21),

$$(\mathcal{P}f_0)_{[0,\gamma_1]} = \mathcal{P}f_0\left(\gamma_0\right) = \beta_0.$$

Remark 4.13. Let us denote the maximal value of the constant c in (4.5) by

$$c_* = \sup\left\{c:\ \frac{\|\,\mathcal{P}f\,\|_*}{\|f\|_*} \geq c \quad \forall\, f\downarrow \quad \text{on } \mathbb{R}_+,\ f \in BMO,\ f \neq Const\right\} =$$

$$= \inf\left\{\frac{\|\,\mathcal{P}f\,\|_*}{\|f\|_*}:\ f \in BMO,\ f\downarrow \quad \text{on } \mathbb{R}_+,\ f \neq Const\right\}. \tag{4.24}$$

Lemma 4.11 implies that $c_* \leq \alpha_0$. On the other hand, according to (4.19), we have that $c_* \geq \frac{1}{4}$.

The next theorem allows to improve the lower bound for c_*. We hope that this result could be of interest also outside of the present context.

Theorem 4.14 ([41]). *Let f be a non-increasing function on $\mathbb{R}_+$. Then*

$$\| \mathcal{P}f \|_* \geq \frac{\alpha_0}{2} \|f\|_{BLO}, \tag{4.25}$$

where α_0 *is defined by* (4.21), *and in general the constant* $\frac{\alpha_0}{2}$ *in the right-hand side of* (4.25) *is sharp.*

In order to prove Theorem 4.14 we need the following statement.

Lemma 4.15. *Let* $g \in L_{loc}(\mathbb{R}_+)$ *be such that* $g(x) \geq g(y),\ 0 \leq x \leq y \leq 1$, *and*

$$\int_0^1 g(t)\,dt \geq g(x) \geq g(y), \quad 1 \leq x \leq y. \tag{4.26}$$

Then the function $\mathcal{P}g$ *does not increase on* $\mathbb{R}_+$.

Proof. The monotonicity of $\mathcal{P}g$ on $[0,1]$ follows immediately from the monotonicity of the function g on $[0,1]$. If we prove that

$$\mathcal{P}g(x) \geq \mathcal{P}g(y), \quad 1 < x < y, \tag{4.27}$$

then, due to the continuity of $\mathcal{P}g$, we immediately obtain the statement of the lemma. For $1 < x < y$

$$\mathcal{P}g(y) - \mathcal{P}g(x) = \left(1 - \frac{x}{y}\right)\left\{\frac{1}{x}\left[\frac{1}{x-1}\int_1^x g(t)\,dt - \int_0^1 g(t)\,dt\right] + \right.$$

$$\left. + \left[\frac{1}{y-x}\int_x^y g(t)\,dt - \frac{1}{x-1}\int_1^x g(t)\,dt\right]\right\}.$$

Now it is easy to see that (4.27) follows from (4.26). □

Proof of Theorem 4.14. According to Lemmas 2.22 and 2.34, it is enough to show that for any $\gamma > 0$ there exists $\gamma' > 0$ such that

$$\Omega\left(\mathcal{P}f; [0, \gamma']\right) \geq \frac{\alpha_0}{2}\left[\mathcal{P}f(\gamma) - f(\gamma)\right]. \tag{4.28}$$

We can assume $\gamma = 1,\ f(1) = 0,\ \mathcal{P}f(1) = 1$. Set $f_0(x) = \chi_{[0,1)}(x),\ x \in \mathbb{R}_+$. Then the function $g \equiv f - f_0$ obviously satisfies the conditions of Lemma 4.15. By this lemma, the function $\mathcal{P}g$ does not increase on $\mathbb{R}_+$. Let us denote $h(x) = g(x) - \mathcal{P}g(\gamma_0)$, with γ_0 defined by (4.21). Then $\mathcal{P}h$ is also non-increasing on $\mathbb{R}_+$ and $\mathcal{P}h(\gamma_0) = 0$. Therefore

$$0 \leq \int_0^{\gamma_0} \mathcal{P}h(t)\,dt = \int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f_0(t)\right]\,dt - \mathcal{P}f(\gamma_0) + \mathcal{P}f_0(\gamma_0),$$

or, equivalently,

$$\frac{1}{\gamma_0}\int_0^{\gamma_0} \left[\mathcal{P}f_0(t) - \mathcal{P}f_0(\gamma_0)\right]\,dt \leq \frac{1}{\gamma_0}\int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f(\gamma_0)\right]\,dt. \tag{4.29}$$

Now choose γ' such that

$$\frac{1}{\gamma'}\int_0^{\gamma'} \mathcal{P}f(t)\,dt = \mathcal{P}f(\gamma_0).$$

Clearly, $\gamma' > \gamma_0$. Comparing γ' with γ_1, which was defined in Lemma 4.10 as a root of equation (4.20), we see that the following two situations are possible:

1. $\gamma' \le \gamma_1$; in this case, by Property 2.1, (4.29) implies

$$\Omega\left(\mathcal{P}f;[0,\gamma']\right) = \frac{2}{\gamma'}\int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f(\gamma_0)\right]\,dt \ge$$

$$\ge \frac{2}{\gamma_1}\int_0^{\gamma_0} \left[\mathcal{P}f_0(t) - \mathcal{P}f_0(\gamma_0)\right]\,dt = \Omega\left(\mathcal{P}f_0;[0,\gamma_1]\right) = \frac{\alpha_0}{2},$$

so that (4.28) holds true.

2. If $\gamma' > \gamma_1$, then the listed above properties of function $\mathcal{P}h$ imply

$$0 \ge \int_{\gamma_0}^{\gamma_1} \mathcal{P}h(t)\,dt = \int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f(t) - \mathcal{P}f_0(t)\right]\,dt - \mathcal{P}f(\gamma_0) + \mathcal{P}f_0(\gamma_0),$$

i.e.,

$$\frac{1}{\gamma_1-\gamma_0}\int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f(\gamma_0) - \mathcal{P}f(t)\right]\,dt \ge \frac{1}{\gamma_1-\gamma_0}\int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f_0(\gamma_0) - \mathcal{P}f_0(t)\right]\,dt.$$

But since the function $\mathcal{P}f(\gamma_0) - \mathcal{P}f(t),\ t > \gamma_0$, is non-decreasing and $\gamma' > \gamma_1$

$$\frac{1}{\gamma'-\gamma_0}\int_{\gamma_0}^{\gamma'} \left[\mathcal{P}f(\gamma_0) - \mathcal{P}f(t)\right]\,dt \ge \frac{1}{\gamma_1-\gamma_0}\int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f_0(\gamma_0) - \mathcal{P}f_0(t)\right]\,dt. \tag{4.30}$$

One can rewrite inequalities (4.29) and (4.30) in the following form

$$\frac{\gamma_0}{\int_0^{\gamma_0}\left[\mathcal{P}f_0(t) - \mathcal{P}f_0(\gamma_0\right]\,dt} \ge \frac{\gamma_0}{\int_0^{\gamma_0}\left[\mathcal{P}f(t) - \mathcal{P}f(\gamma_0)\right]\,dt},$$

$$\frac{\gamma_1-\gamma_0}{\int_{\gamma_0}^{\gamma_1}\left[\mathcal{P}f_0(\gamma_0) - \mathcal{P}f_0(t)\right]\,dt} \ge \frac{\gamma'-\gamma_0}{\int_{\gamma_0}^{\gamma'}\left[\mathcal{P}f(\gamma_0) - \mathcal{P}f(t)\right]\,dt}.$$

Notice that, by Property 2.1, the denominators of the fractions in the right and left-hand sides are the same. Summing up, we obtain

$$\frac{1}{\gamma_1}\int_0^{\gamma_0} \left[\mathcal{P}f_0(t) - \mathcal{P}f_0(\gamma_0)\right]\,dt \le \frac{1}{\gamma'}\int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f(\gamma_0)\right]\,dt.$$

Then, by Property 2.1,

$$\Omega\left(\mathcal{P}f;[0,\gamma']\right) \ge \Omega\left(\mathcal{P}f_0;[0,\gamma_1]\right) = \frac{\alpha_0}{2}$$

(see also the proof of Lemma 4.11). Therefore, in this case (4.28) holds true, too.

It remains to show that the constant $\frac{\alpha_0}{2}$ in the right-hand side of (4.25) cannot be increased. But, according to Lemma 4.11, for the function $f_0(x) = \chi_{[0,1)}(x)$, $x \in \mathbb{R}_+$, we have $\| \mathcal{P} f_0 \|_* = \frac{\alpha_0}{2}$, and hence $\|f_0\|_{BLO}$ is obviously equal to 1. □

From Theorem 4.14 we immediately get

Corollary 4.16 ([41]).

$$\frac{e\alpha_0}{4} \le c_* \le \alpha_0 \tag{4.31}$$

where c_* is the constant defined by (4.24).

Proof. As we have already mentioned, the right inequality of (4.31) follows from Lemma 4.11. On the other hand, if f is an arbitrary non-increasing function on $\mathbb{R}_+$, then by Theorems 4.14 and 2.36

$$\| \mathcal{P} f \|_* \ge \frac{\alpha_0}{2} \|f\|_{BLO} \ge \frac{\alpha_0}{2} \frac{e}{2} \|f\|_*,$$

which implies the left inequality of (4.31). □

Remark 4.17. We do not know the value of c_*, defined by equality (4.24).

4.2 Estimates of the Oscillations of the Conjugate Hardy Transform and the Calderón Transform

In this section we consider the non-negative summable functions f on $\mathbb{R}_+$ such that the integral $\int_1^{+\infty} f(x) \frac{dx}{x}$ converges. The following formulas define *the conjugate Hardy operator* $\mathcal{P}^*$ and *the Calderón operator* $\mathcal{S}$ respectively (see [51, 3]):

$$\mathcal{P}^* f(t) = \int_t^{+\infty} f(x) \frac{dx}{x}, \quad t > 0,$$

$$\mathcal{S} f(t) = \frac{1}{t} \int_0^t f(x)\, dx + \int_t^{+\infty} f(x) \frac{dx}{x} = \mathcal{P} f(t) + \mathcal{P}^* f(t), \quad t > 0.$$

The operators $\mathcal{P}^*$ and $\mathcal{S}$, together with the operator $\mathcal{P}$, are often used in various fields of mathematics.

Example 4.18. Let $f_0(x) = \ln \frac{1}{x} \chi_{[0,1)}(x)$, $x \in \mathbb{R}_+$. Then according to Example 2.24, $f_0 \in BMO(\mathbb{R}_+)$. In the same time, $\mathcal{P}^* f_0(x) = \frac{1}{2} \ln^2 x \chi_{[0,1)}(x)$, and so $\mathcal{P}^* f_0 \notin BMO(\mathbb{R}_+)$. Indeed, the assumption $\mathcal{P}^* f_0 \in BMO$ contradicts to John–Nirenberg inequality (3.33). Similarly, it can be shown that $\mathcal{S} f_0 \notin BMO(\mathbb{R}_+)$. □

So, unlikely the operator $\mathcal{P}$, the operators $\mathcal{P}^*$ and $\mathcal{S}$ do not act from BMO into BMO. However, it is easy to see that $\mathcal{P}^* f \in BMO(\mathbb{R}_+)$ and $\mathcal{S} f \in BMO(\mathbb{R}_+)$ for $f \in L^\infty(\mathbb{R}_+)$. Moreover, the following theorem holds true.

Theorem 4.19 ([39]). *Let f be a non-negative locally summable function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\,\frac{dx}{x} < \infty$. Then*

$$\|\,\mathcal{P}^*f\,\|_{BLO} = \|\,\mathcal{P}f\,\|_\infty, \tag{4.32}$$

$$\|\,\mathcal{S}f\,\|_{BLO} = \|\mathcal{P}(\mathcal{P}f)\|_\infty. \tag{4.33}$$

Proof. Since $f(x) \geq 0$ for $x \in \mathbb{R}_+$ it is clear that the function $\mathcal{P}^*f$ does not increase on $\mathbb{R}_+$. Further, for $0 < t < s$

$$\mathcal{S}f(t) - \mathcal{S}f(s) = \left(\frac{1}{t} - \frac{1}{s}\right)\int_0^t f(x)\,dx + \int_t^s f(x)\left(\frac{1}{x} - \frac{1}{s}\right)\,dx \geq 0,$$

so that also $\mathcal{S}f$ does not increase on $\mathbb{R}_+$. Hence, by Lemma 2.34,

$$\|\,\mathcal{P}^*f\,\|_{BLO} = \sup_{t>0}\left(\frac{1}{t}\int_0^t \mathcal{P}^*f(u)\,du - \mathcal{P}^*f(t)\right) =$$

$$= \sup_{t>0}\left(\frac{1}{t}\int_0^t\int_u^{+\infty} f(x)\,\frac{dx}{x}\,du - \int_t^{+\infty} f(x)\frac{dx}{x}\right) = \sup_{t>0}\frac{1}{t}\int_0^t\int_u^t f(x)\,\frac{dx}{x}\,du =$$

$$= \sup_{t>0}\frac{1}{t}\int_0^t f(x)\,dx = \sup_{t>0}\mathcal{P}f(t) = \|\,\mathcal{P}f\,\|_\infty.$$

Similarly,

$$\|\,\mathcal{S}f\,\|_{BLO} = \sup_{t>0}\left(\frac{1}{t}\int_0^t \mathcal{S}f(u)\,du - \mathcal{S}f(t)\right) =$$

$$= \sup_{t>0}\left(\frac{1}{t}\int_0^t\frac{1}{u}\int_0^u f(x)\,dx\,du - \frac{1}{t}\int_0^t f(x)\,dx + \frac{1}{t}\int_0^t\int_u^t f(x)\,\frac{dx}{x}\,du\right) =$$

$$= \sup_{t>0}\frac{1}{t}\int_0^t\frac{1}{u}\int_0^u f(x)\,dx\,du = \sup_{t>0}\mathcal{P}(\mathcal{P}f)(t) = \|\mathcal{P}(\mathcal{P}f)\|_\infty.$$

□

Theorem 4.20 ([39]). *Let f be a non-negative locally summable function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\,\frac{dx}{x} < \infty$. Then*

$$\frac{1}{2}\|\,\mathcal{P}f\,\|_\infty \leq \|\,\mathcal{P}^*f\,\|_* \leq \frac{2}{\mathrm{e}}\|\,\mathcal{P}f\,\|_\infty, \tag{4.34}$$

$$\frac{1}{2}\|\mathcal{P}(\mathcal{P}f)\|_\infty \leq \|\,\mathcal{S}f\,\|_* \leq \frac{2}{\mathrm{e}}\|\mathcal{P}(\mathcal{P}f)\|_\infty. \tag{4.35}$$

Proof. As it was shown in the proof of Theorem 4.19, both functions $\mathcal{P}^* f$ and $\mathcal{S}f$ do not increase on $\mathbb{R}_+$. Then, by virtue of (4.32) and (4.33), Theorem 2.36 applied to these functions immediately yields (4.34) and (4.35) respectively.

Let us show that the constant $\frac{1}{2}$ in the left-hand side of (4.34) cannot be increased. For $0 < \varepsilon < 1$ let $f_\varepsilon(x) = \frac{1}{\varepsilon}\chi_{[1-\varepsilon,1]}(x)$ $x \in \mathbb{R}_+$. Then $\| \mathcal{P} f_\varepsilon \|_\infty = 1$, $\mathcal{P}^* f_\varepsilon(t) = \frac{1}{\varepsilon} \min\left(\ln \frac{1}{1-\varepsilon}, \ln \frac{1}{t}\right) \chi_{[0,1]}(t)$, $t \in \mathbb{R}_+$. Hence, by Property 2.7,

$$\| \mathcal{P}^* f_\varepsilon \|_* \le \frac{1}{2}\frac{1}{\varepsilon} \ln \frac{1}{1-\varepsilon} \le \frac{1}{2}\frac{1}{1-\varepsilon} \to \frac{1}{2}, \quad \varepsilon \to 0+ .$$

Therefore the constant $\frac{1}{2}$ in the left-hand side of (4.34) is sharp.

For $\varepsilon = 1$ we have $f_1(x) = \chi_{[0,1]}(x)$, $x \in \mathbb{R}_+$, $\| \mathcal{P} f_1 \|_\infty = 1$, $\mathcal{P}^* f_1(t) = \ln \frac{1}{t}\chi_{[0,1]}(t)$, $t \in \mathbb{R}_+$. As it was shown in Example 2.28, this implies

$$\| \mathcal{P}^* f_1 \|_* \ge \Omega\left(\mathcal{P}^* f_1; [0,1]\right) = \frac{2}{\mathrm{e}},$$

so that the constant $\frac{2}{\mathrm{e}}$ in the right-hand side of (4.34) is sharp, too.

It remains to show that the constant $\frac{2}{\mathrm{e}}$ in the right-hand side of (4.35) cannot be decreased. For the function $f_1(x) = \chi_{[0,1]}(x)$, $x \in \mathbb{R}_+$, we have

$$\|\mathcal{P}(\mathcal{P}f_1)\|_\infty = 1, \quad \mathcal{S}f_1(t) = \left(1 + \ln \frac{1}{t}\right) \chi_{[0,1]}(t) + \frac{1}{t}\chi_{(1,+\infty)}(t), \quad t \in \mathbb{R}_+.$$

So, it is enough to show that for the function $g \equiv \mathcal{S}f_1$

$$\|g\|_* = \frac{2}{\mathrm{e}}. \tag{4.36}$$

Since g is non-increasing on $\mathbb{R}_+$ according to Lemma 2.22 the last relation follows from the equality

$$\sup_{t>0} \Omega(g; [0,t]) = \frac{2}{\mathrm{e}}. \tag{4.37}$$

But for $0 < t \le 1$

$$\Omega(g; [0,t]) = \frac{2}{t} \int_0^{t/\mathrm{e}} \left(\ln \frac{1}{u} - \ln \frac{\mathrm{e}}{t}\right) du = \frac{2}{\mathrm{e}},$$

so that in order to prove (4.37) it remains to show that

$$\Omega(g; [0,t]) \le \frac{2}{\mathrm{e}}, \quad 1 < t < \infty. \tag{4.38}$$

Let t_0, $t_0 > \mathrm{e}$, be the root of the equation $\ln x = x - 2$. We have to consider the following two cases.

1. If $1 < t \le t_0$, then $g_{[0,t]} \ge 1$. Denote $h(u) = 1 + \ln \frac{1}{u}$, $u \in \mathbb{R}_+$. Since $h(u) \le \frac{1}{u}$, $u \ge 1$, there exists t_1, $1 \le t_1 < t$ such that $g_{[0,t]} = \frac{1}{t_1}\int_0^{t_1}\left(1+\ln\frac{1}{u}\right)du = h_{[0,t_1]}$. Now, by Property 2.1,

$$\Omega(g;[0,t]) = \frac{2}{t}\int_{\left\{u:\ 1+\ln\frac{1}{u} > g_{[0,t]}\right\}}\left(1+\ln\frac{1}{u} - g_{[0,t]}\right)du \le$$

$$\le \frac{2}{t_1}\int_{\left\{u:\ 1+\ln\frac{1}{u} \ge g_{[0,t]}\right\}}\left(1+\ln\frac{1}{u} - h_{[0,t_1]}\right)du = \Omega(h;[0,t_1]) = \frac{2}{\mathrm{e}},$$

so that (4.38) holds true in this case.

2. Let $t > t_0$, i.e., $g_{[0,t]} = \frac{1}{t}(2+\ln t) < 1$. In this case, applying Property 2.1, we obtain

$$\Omega(g;[0,t]) = \frac{2}{t}\int_0^{t_2}\left(g(u) - g_{[0,t]}\right)du = 2\frac{1+\ln t - \ln(2+\ln t)}{t},$$

where t_2 is to be defined from the condition $\frac{1}{t_2} = g_{[0,t]}$, i.e. $t_2 = \frac{t}{2+\ln t}$. Denote $\psi(t) = \frac{1+\ln t-\ln(2+\ln t)}{t}$, $t \ge t_0$. Since $t_0 > \mathrm{e}$ we have $\psi(t_0) = \frac{1}{t_0} < \frac{1}{\mathrm{e}}$. It is easy to see that $\psi'(t) \le 0$ for $t \ge t_0$. Hence $\psi(t) \le \frac{1}{\mathrm{e}}$ for $t \ge t_0$, and in this case (4.38) holds true as well. □

Remark 4.21. We do not know whether the constant $\frac{1}{2}$ in the left-hand side of (4.35) is sharp.

Remark 4.22. Clearly, $\|\mathcal{P}f\|_\infty \le \|f\|_\infty$, though the condition $\|f\|_\infty < \infty$ is not necessary for the boundedness of $\mathcal{P}f$. On the other hand, is f is non-negative on $\mathbb{R}_+$, then obviously $u\,\mathcal{P}f(u) \ge t\,\mathcal{P}f(t)$, $u \ge t > 0$, so that

$$\mathcal{P}(\mathcal{P}f)(2t) \ge \frac{1}{2t}\int_t^{2t}\mathcal{P}f(u)\,du \ge \frac{\ln 2}{2}\,\mathcal{P}f(t), \quad t > 0.$$

This means that the conditions $\mathcal{P}f \in L^\infty$ and $\mathcal{P}(\mathcal{P}f) \in L^\infty$ are equivalent. In other words, Theorems 4.19 and 4.20 show, that for a non-negative on $\mathbb{R}_+$ function f the boundedness of $\mathcal{P}f$ (and not the essential boundedness of f) is the necessary and sufficient condition for $\mathcal{P}^*f$ and $\mathcal{S}f$ to belong to BLO and BMO.

The next theorem provides the lower bound of the BMO-norms of $\mathcal{P}^*f$ and $\mathcal{S}f$, which reflect the behavior of the function f in the neighborhood of zero.

Theorem 4.23 ([39]). *Let f be a non-negative locally summable function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\,\frac{dx}{x} < \infty$. Then*

$$\|\mathcal{P}^*f\|_* \ge \frac{2}{\mathrm{e}}\lim_{t\to 0+}\ \operatorname*{ess\,inf}_{u\in(0,t)} f(u), \tag{4.39}$$

$$\| \mathcal{S}f \|_* \geq \frac{2}{\mathrm{e}} \lim_{t\to 0+} \operatorname*{ess\,inf}_{u\in(0,t)} f(u), \tag{4.40}$$

and in general the constants $\frac{2}{\mathrm{e}}$ *in the right-hand sides of* (4.39) *and* (4.40) *are sharp.*

Proof. Let us denote $A = \lim_{t\to 0+} \operatorname{ess\,inf}_{u\in(0,t)} f(u)$. If $A = 0$, then (4.39) and (4.40) are trivial. Let $A > 0$. Fix a, $0 < a < A$, and choose $\varepsilon > 0$ such that $f(u) > a$ for almost all $u \in (0, \varepsilon)$. Then for $t < \varepsilon$

$$\mathcal{P}^* f(t) \geq \int_t^{\varepsilon} f(u) \frac{du}{u} \geq a \ln \frac{\varepsilon}{t}. \tag{4.41}$$

Now let us use the John–Nirenberg inequality with exact exponent (Theorem 3.21). Assuming $\mathcal{P}^* f \in BMO$ and taking into account the monotonicity of $\mathcal{P}^* f$, one can rewrite the John-Nirenberg inequality (3.39) in the following way

$$\mathcal{P}^* f(t) \leq (\mathcal{P}^* f)_{[0,1]} + \frac{\mathrm{e}}{2} \| \mathcal{P}^* f \|_* \left(\ln \frac{1}{t} + \ln B \right). \tag{4.42}$$

Here the constant $B = \exp\left(1 + \frac{2}{\mathrm{e}}\right)$ is taken from Theorem 3.21, and $t > 0$ is small. Comparing (4.41) and (4.42), we have

$$a \ln \frac{\varepsilon}{t} \leq (\mathcal{P}^* f)_{[0,t]} + \frac{\mathrm{e}}{2} \| \mathcal{P}^* f \|_* \left(\ln \frac{1}{t} + \ln B \right)$$

for $t > 0$ small enough. This immediately implies $\| \mathcal{P}^* f \|_* \geq a$. As a was an arbitrary number smaller than A, inequality (4.39) is proved.

The same arguments lead the following inequality

$$a \ln \frac{\varepsilon}{t} \leq (\mathcal{S}f)_{[0,t]} + \frac{\mathrm{e}}{2} \| \mathcal{S}f \|_* \left(\ln \frac{1}{t} + \ln B \right),$$

with $a < A$ being an arbitrary number. This inequality implies (4.40).

It remains to prove that the constant $\frac{2}{\mathrm{e}}$ in the right-hand sides of (4.39) and (4.40) cannot be increased. In the proof of Theorem 4.20 we showed that for the function $f_1(x) = \chi_{[0,1]}(x)$, $x \in \mathbb{R}_+$, one has $\| \mathcal{S}f_1 \|_* = \frac{2}{\mathrm{e}}$. Hence for the function f_1 the inequality (4.40) becomes an equality, so that the constant $\frac{2}{\mathrm{e}}$ in (4.40) is sharp. In order to proof that $\frac{2}{\mathrm{e}}$ in (4.39) is also sharp, obviously it is enough to show that

$$\| \mathcal{P}^* f_1 \|_* = \frac{2}{\mathrm{e}}. \tag{4.43}$$

Denote $g(t) \equiv \mathcal{P}^* f_1(t) = \ln \frac{1}{t} \chi_{[0,1]}(t)$, $t \in \mathbb{R}_+$. If $0 < t \leq 1$, then it is easy to see that $\Omega(g; [0,t]) = \frac{2}{\mathrm{e}}$. Otherwise, if $t > 1$, then for the function $h(x) = \ln \frac{1}{x}$, $x \in \mathbb{R}_+$, there exists t_1, $1 < t_1 \leq t$, such that $g_{[0,t]} = h_{[0,t_1]}$. Therefore, by Property 2.1,

$$\Omega(g; [0,t]) = \frac{2}{t} \int_{\left\{u:\ g(u) > g_{[0,t]}\right\}} \left(g(u) - g_{[0,t]} \right) du \leq$$

$$\leq \frac{2}{t_1} \int_{\{u:\ h(u)>h_{[0,t_1]}\}} \big(h(u) - h_{[0,t_1]}\big)\, du = \Omega(h; [0,t_1]) = \frac{2}{\mathrm{e}}.$$

So, we have proved (4.43), and this completes the proof of the theorem. □

Remark 4.24. We cannot substitute the ess inf in the left-hand sides of (4.39) and (4.40) by ess sup. Indeed, for the function

$$f(x) = \sum_{k=0}^{\infty} 2^{k+1} \chi_{[2^{-k}-2^{-2k-2}, 2^{-k}]}(x), \quad x \in \mathbb{R}_+,$$

we obviously have $\lim_{t\to 0+} \operatorname{ess\,sup}_{u\in(0,t)} f(u) = +\infty$. Let us show that $\| \mathcal{P}f \|_\infty \leq 2$. Indeed, if $x > 1$, then

$$\mathcal{P}f(x) \leq \int_0^1 f(t)\, dt = \sum_{k=0}^{\infty} 2^{k+1} \cdot 2^{-2k-2} = \sum_{k=0}^{\infty} 2^{-k-1} = 1.$$

If $0 < x \leq 1$, we can find an integer n such that $2^{-n-1} < x \leq 2^{-n}$. Then

$$\mathcal{P}f(x) \leq \frac{1}{2^{-n-1}} \int_0^{2^{-n}} f(t)\, dt = 2^{n+1} \sum_{k=n}^{\infty} 2^{k+1} \cdot 2^{-2k-2} = 2.$$

Hence

$$\|\mathcal{P}(\mathcal{P}f)\|_\infty \leq \| \mathcal{P}f \|_\infty \leq 2,$$

and, according to Theorem 4.20,

$$\| \mathcal{P}^* f \|_* \leq \frac{4}{\mathrm{e}}, \quad \| \mathcal{S}f \|_* \leq \frac{4}{\mathrm{e}}.$$

This shows that (4.39) and (4.40) fail if we substitute ess inf by ess sup.

Now let f be a non-negative non-increasing function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\, \frac{dx}{x} < \infty$. Clearly, in this case

$$\|f\|_\infty = \| \mathcal{P}f \|_\infty = \lim_{t\to 0+} f(t).$$

Thus Theorems 4.20 and 4.23 immediately lead to the following results.

Corollary 4.25 ([39]). *If f is a non-negative non-increasing function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\, \frac{dx}{x} < \infty$, then*

$$\| \mathcal{P}^* f \|_* = \| \mathcal{S}f \|_* = \frac{2}{\mathrm{e}} \|f\|_\infty. \tag{4.44}$$

Corollary 4.26 ([39]). *If f is a non-negative non-increasing function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\,\frac{dx}{x} < \infty$, then*

$$\|\,\mathcal{P}^* f\,\|_* \geq \alpha_0 \|f\|_*, \tag{4.45}$$

$$\|\,\mathcal{S} f\,\|_* \geq \alpha_0 \|f\|_*, \tag{4.46}$$

where the constant α_0 is defined by (4.21).

Proof. Applying successively (4.44), the inequality $\|\,\mathcal{P} f\,\|_* \leq \frac{1}{2}\|\,\mathcal{P} f\,\|_\infty$ (which follows from Property 2.7), and (4.31), we obtain

$$\|\,\mathcal{P}^* f\,\|_* = \frac{2}{\mathrm{e}}\|\,\mathcal{P} f\,\|_\infty \geq \frac{4}{\mathrm{e}}\|\,\mathcal{P} f\,\|_* \geq \frac{4}{\mathrm{e}} c_* \|f\|_* \geq \alpha_0 \|f\|_*,$$

where the constant c_* is defined by (4.24).

Analogously one can prove (4.46). □

Remark 4.27. Without the monotonicity assumption on f the inequalities (4.45) and (4.46) fail even if the constants α_0 in their right-hand sides are arbitrarily small. It can be easily seen from the following example. Take $f_0(x) = \chi_{[1-\varepsilon,1]}(x),\ x \in \mathbb{R}_+$, with $0 < \varepsilon < 1$. Then $\|f_0\|_* = \frac{1}{2}$ and

$$\max\left(\|\,\mathcal{P}^* f_0\,\|_*, \|\,\mathcal{S} f_0\,\|_*\right) \leq \|\,\mathcal{S} f_0\,\|_\infty \leq \varepsilon + \ln \frac{1}{1-\varepsilon} \to 0, \quad \varepsilon \to 0+\,.$$

On the other hand, the boundedness condition in Corollary 4.26 can be neglected. Indeed, if f is unbounded, then, by (4.44), the left-hand sides of (4.45) and (4.46) are infinite.

Remark 4.28. Equality (4.44) implies that it is impossible to get the upper bounds of $\|\,\mathcal{P}^* f\,\|_*$ and $\|\,\mathcal{S} f\,\|_*$ in terms of $\|f\|_*$ even for the monotone bounded function f. Indeed, if such upper bounds exist, equality (4.44) would imply $\|f\|_\infty \leq c\|f\|_*$ with some constant $c > 0$, which is wrong. In order to see this it is enough to consider the function

$$f_N(x) = \frac{1}{N} \min\left(N, \ln \frac{1}{x}\right) \chi_{[0,1)}(x), \quad x \in \mathbb{R}_+.$$

We have $\|f_N\|_\infty = 1$ and one can easily check that $\|f_N\|_* \to 0$ as $N \to \infty$.