3

Estimates of Rearrangements and the John–Nirenberg Theorem

3.1 Estimates of Rearrangements of the *BMO*-functions

The aim of the present section is to show that the non-increasing rearrangement f^* of a BMO-function f is also a BMO-function. First we will consider the case of the function f defined on the whole $\mathbb{R}^d$. As it was mentioned above, in addition we have to assume that $f^*(t)$ is defined for all $t > 0$.

Theorem 3.1 (Bennett, De Vore, Sharpley, [1]). *Let $f \in BMO(\mathbb{R}^d)$. Then*

$$f^{**}(t) - f^*(t) \le 2^{d+4}\|f\|_*, \quad 0 < t < \infty. \tag{3.1}$$

Proof. Since $\|\,|f|\,\|_* \le 2\|f\|_*$ it is enough to prove the inequality

$$f^{**}(t) - f^*(t) \le 2^{d+3}\|f\|_* \tag{3.2}$$

for a non-negative function f.

Fix $t > 0$ and denote $E = \{x \in \mathbb{R}^d : \ f(x) > f^*(t)\}$. Then $|E| \le t$. Let us construct an open set $G \supset E$ such that $|G| \le 2t$. Applying Lemma 1.12 to the set G we obtain a collection of cubes Q_j with pairwise disjoint interiors, which satisfy properties (1.11), (1.12) and (1.13) of the lemma. Then

$$t\,(f^{**}(t) - f^*(t)) = \int_0^t (f^*(u) - f^*(t))\,du =$$

$$= \int_0^{|E|} (f^*(u) - f^*(t))\,du + \int_{|E|}^t (f^*(u) - f^*(t))\,du =$$

$$= \int_0^{|E|} (f^*(u) - f^*(t))\,du = \int_E (f(x) - f^*(t))\,dx =$$

$$= \sum_j \int_{E\cap Q_j} (f(x) - f^*(t))\, dx =$$

$$= \sum_j \int_{E\cap Q_j} \big(f(x) - f_{Q_j}\big)\, dx + \sum_j \big(f_{Q_j} - f^*(t)\big)\, |E\cap Q_j| \le$$

$$\le \sum_j \int_{E\cap Q_j} \big|f(x) - f_{Q_j}\big|\, dx + {\sum_j}' \big(f_{Q_j} - f^*(t)\big)\, |E\cap Q_j|\,, \tag{3.3}$$

where ${\sum_j}'$ denotes the sum over all numbers j such that $f_{Q_j} > f^*(t)$. We have

$${\sum_j}' \big(f_{Q_j} - f^*(t)\big)\, |E\cap Q_j| \le {\sum_j}' \big(f_{Q_j} - f^*(t)\big)\, |G\cap Q_j| \le$$

$$\le {\sum_j}' \big(f_{Q_j} - f^*(t)\big)\, |Q_j \setminus G| = {\sum_j}' \int_{Q_j\setminus G} \big(f_{Q_j} - f^*(t)\big)\, dx \le$$

$$\le {\sum_j}' \int_{Q_j\setminus G} \big(f_{Q_j} - f(x)\big)\, dx \le {\sum_j}' \int_{Q_j} \big|f(x) - f_{Q_j}\big|\, dx \le$$

$$\le \sum_j \int_{Q_j} \big|f(x) - f_{Q_j}\big|\, dx.$$

Then (3.3) becomes

$$t\,(f^{**}(t) - f^*(t)) \le 2\sum_j \int_{Q_j} \big|f(x) - f_{Q_j}\big|\, dx \le$$

$$\le 2\|f\|_* \sum_j |Q_j| \le 2^{d+2}\|f\|_* \cdot |G| \le 2^{d+3}\|f\|_* \cdot t,$$

which is exactly (3.2). $\square$

In particular, from this lemma it follows that the rearrangement operator is bounded in BMO.

Theorem 3.2 (Garsia, Rodemich $(d = 1)$**, [17]; Bennett, De Vore, Sharpley** $(d \ge 1)$**, [1]).** *Let $f \in BMO(\mathbb{R}^d)$. Then $f^* \in BMO([0,\infty))$ and*

$$\|f^*\|_* \le c\|f\|_*,$$

where the constant c depends only on the dimension d of the space (one can take $c = 2^{d+5}$).

Proof. Since f^* is a non-increasing function on $[0,\infty)$

$$\|f^*\|_* = \sup_{t>0} \Omega(f^*;[0,t]).$$

But by the properties of oscillations

$$\Omega(f^*;[0,t]) \le 2\Omega'(f^*;[0,t]) = 2\inf_c \frac{1}{t}\int_0^t |f^*(u)-c|\,du \le$$

$$\le \frac{2}{t}\int_0^t (f^*(u)-f^*(t))\,du = 2\left(f^{**}(t)-f^*(t)\right),$$

and the result follows from the previous theorem. □

Now let us consider the case $f \in BMO(Q_0)$ for a fixed cube $Q_0 \subset \mathbb{R}^d$. In this case the presented proof of inequality (3.1) is valid only for t such that $0 < t \le \frac{1}{4}|Q_0|$ because it is based on the application of Lemma 1.13, which requires $|G| \le \frac{1}{2}|Q_0|$. Therefore the following theorem is valid.

Theorem 3.3 (Bennett, De Vore, Sharpley, [1]). *Let $f \in BMO(Q_0)$. Then*

$$f^{**}(t) - f^*(t) \le 2^{d+4}\|f\|_*, \quad 0 < t \le \frac{1}{4}|Q_0|. \tag{3.4}$$

Let us show that (3.4) fails as $t \to |Q_0|$ even if the coefficient in its right-hand side is arbitrarily big. Indeed, for $0 < h < 1$ set $f(x) = \ln\frac{1-x}{h}$, $x \in Q_0 \equiv [0, 1-h]$. Since f does not increase on $[0,1-h]$ it follows that $f^*(t) = f(t)$, $0 < t \le 1-h = |Q_0|$, and $f^*(1-h) = 0$. It is easy to see that $\|f\|_*$ does not exceed the BMO-norm of the function $\ln\frac{1}{x}$, $0 < x < \infty$, so that $\|f\|_* \le \frac{2}{\mathrm{e}}$. Thus it remains to show that $f^{**}(1-h) \to \infty$ as $h \to 0$. But this is indeed true, because

$$f^{**}(1-h) = \frac{1}{1-h}\int_0^{1-h} f^*(u)\,du = \frac{1}{1-h}\int_0^{1-h} \ln\frac{1-u}{h}\,du =$$

$$= \frac{h}{1-h}\int_h^1 \ln\frac{1}{z}\,\frac{dz}{z^2} = \frac{1}{1-h}\ln\frac{1}{h} - 1 \;\to\; \infty, \quad h \to 0.$$

We see that (3.4) fails for $t = |Q_0|$, and hence also for t close to $|Q_0|$. However, the analog of Theorem 3.2 for $BMO(Q_0)$ is true.

Theorem 3.4 (Garsia, Rodemich ($d = 1$), [17]; Bennett, De Vore, Sharpley ($d \ge 1$), [1]). *Let $f \in BMO(Q_0)$. Then $f^* \in BMO([0,|Q_0|])$ and*

$$\left\|(f - f_{Q_0})^*\right\|_* \le c\|f\|_*,$$

where the constant c depends only on the dimension d of the space.

Proof. We can assume that $f_{Q_0} = 0$. Fix the interval $[\alpha, \beta] \subset [0, |Q_0|]$. If $f^*_{[\alpha,\beta]} \le |f|_{Q_0}$, then

$$\Omega(f^*; [\alpha, \beta]) \le \frac{1}{\beta - \alpha} \int_\alpha^\beta \left| f^*(u) - f^*_{[\alpha,\beta]} \right| du \le$$

$$\le 2f^*_{[\alpha,\beta]} \le 2|f|_{Q_0} = \frac{2}{|Q_0|} \int_{Q_0} |f(x)|\, dx \le 2\|f\|_*.$$

In remains to consider the non-trivial case $f^*_{[\alpha,\beta]} > |f|_{Q_0}$. Choose β_0, $\beta \le \beta_0 \le |Q_0|$, such that $f^*_{[\alpha,\beta]} = f^*_{[0,\beta_0]}$. If $\beta_0 \ge \frac{1}{4}|Q_0|$, then

$$\Omega(f^*; [\alpha, \beta]) \le \Omega(f^*; [0, \beta_0]) = \frac{2}{\beta_0} \int_{\{u:\ f^*(u) > f^{**}(\beta_0)\}} (f^*(u) - f^{**}(\beta_0))\, du \le$$

$$\le \frac{2}{\beta_0} \int_{\{u:\ f^*(u) > f^{**}(\beta_0)\}} (f^*(u) - |f|_{Q_0})\, du \le$$

$$\le \frac{2}{\beta_0} \int_{\{u:\ f^*(u) > |f|_{Q_0}\}} (f^*(u) - |f|_{Q_0})\, du =$$

$$= \frac{2}{\beta_0} \int_{\{x \in Q_0:\ |f(x)| > |f|_{Q_0}\}} (|f(x)| - |f|_{Q_0})\, dx =$$

$$= \frac{|Q_0|}{\beta_0} \Omega(|f|; Q_0) \le 4 \cdot 2 \cdot \Omega(f; Q_0) \le 8\|f\|_*.$$

Otherwise, if $\beta_0 \le \frac{1}{4}|Q_0|$, then, by Theorem 3.3,

$$\Omega(f^*; [\alpha, \beta]) \le \Omega(f^*; [0, \beta_0]) \le 2\Omega'(f^*; [0, \beta_0]) \le$$

$$\le \frac{2}{\beta_0} \int_0^{\beta_0} (f^*(u) - f^*(\beta_0))\, du = 2\left(f^{**}(\beta_0) - f^*(\beta_0)\right) \le 2^{d+5}\|f\|_*.$$

Since the interval $[\alpha, \beta] \subset [0, |Q_0|]$ was arbitrary the theorem is proved. □

The estimates of the non-increasing rearrangement, which were obtained in Theorems 3.2 and 3.4, are based on the applications of Theorems 3.1 and 3.3 respectively, while for the proofs of Theorems 3.1 and 3.3 we used Lemma 1.12. Now we are going to consider another method of getting estimates for the BMO-norm of the non-increasing rearrangement, based on the application of "rising sun lemma" 1.16. For this we will use the non-increasing equimeasurable rearrangement f_d.

Theorem 3.5 (Klemes, [32]). *Let $f \in BMO([a_0, b_0])$. Then*

$$\|f_d\|_* \le \|f\|_*.$$

Proof. Fix the segment $J \subset [0, b_0 - a_0]$ and denote $\alpha = \frac{1}{|J|}\int_J f_d(u)\,du$. First let us consider the case $\alpha \geq \frac{1}{b_0 - a_0}\int_{a_0}^{b_0} f(x)\,dx$. Applying "rising sun lemma" 1.16 we construct the pairwise disjoint intervals $I_j \subset [a_0, b_0]$, $j = 1, 2, \ldots$, such that $f_{I_j} = \alpha$ and $f(x) \leq \alpha$ at almost every $x \in [a_0, b_0] \setminus E$ with $E = \bigcup_{j\geq 1} I_j$. If we prove the inequality

$$\frac{1}{|J|}\int_J |f_d(t) - \alpha|\;dt \leq \frac{1}{|E|}\int_E |f(x) - \alpha|\,dx, \tag{3.5}$$

then it will remain to use the fact, that

$$f_E = \frac{1}{|E|}\int_E f(x)\,dx = \frac{1}{\sum_j |I_j|}\sum_j \int_{I_j} f(x)\,dx = \alpha, \tag{3.6}$$

and

$$\frac{1}{|E|}\int_E |f(x) - \alpha|\,dx = \frac{1}{|E|}\sum_j \int_{I_j} |f(x) - \alpha|\,dx =$$

$$= \frac{1}{|E|}\sum_j |I_j| \frac{1}{|I_j|}\int_{I_j} \left|f(x) - f_{I_j}\right|\;dx = \frac{1}{|E|}\sum_j |I_j| \Omega(f; I_j) \leq \|f\|_*.$$

In order to prove (3.5), choose the maximal $t \in (0, b_0 - a_0]$ such that $J \subset [0, t]$ and $\frac{1}{t}\int_0^t f_d(u)\,du = \alpha$. The existence of such a t is guaranteed by the condition

$$\frac{1}{b_0 - a_0}\int_0^{b_0 - a_0} f_d(u)\,du = \frac{1}{b_0 - a_0}\int_{a_0}^{b_0} f(x)\,dx \leq \alpha = \frac{1}{|J|}\int_J f_d(u)\,du.$$

Using the monotonicity of the function f_d and applying Property 2.15, we obtain

$$\frac{1}{|J|}\int_J |f_d(u) - \alpha|\;du \leq \frac{1}{t}\int_0^t |f_d(u) - \alpha|\;du.$$

Now for the proof of (3.5) it is enough to show that

$$\frac{1}{t}\int_0^t |f_d(u) - \alpha|\;du \leq \frac{1}{|E|}\int_E |f(x) - \alpha|\,dx. \tag{3.7}$$

But (3.7) is a consequence of the following two relations

$$t \geq |E|, \tag{3.8}$$

$$\int_0^t |f_d(u) - \alpha|\;du = \int_E |f(x) - \alpha|\,dx. \tag{3.9}$$

Concerning (3.8), notice that by the definition of the non-increasing rearrangement we have

$$\int_0^{|E|} f_d(u)\,du \geq \int_E f(x)\,dx,$$

so that, by (3.6),

$$\frac{1}{|E|}\int_0^{|E|} f_d(u)\,du \geq \frac{1}{|E|}\int_E f(x)\,dx = \alpha = \frac{1}{t}\int_0^t f_d(u)\,du.$$

From here and from the monotonicity of f_d inequality (3.8) follows. In order to prove (3.9) let us use the fact that $f(x) \leq \alpha$ almost everywhere on $[a_0, b_0] \setminus E$ (see (1.20)). Then, by (3.6) and the properties of mean oscillations,

$$\int_0^t |f_d(u) - \alpha|\; du = 2\int_{\{u\in[0,t]:\ f_d(u)>\alpha\}} (f_d(u) - \alpha)\; du =$$

$$= 2\int_{\{x\in[a_0,b_0]:\ f(x)>\alpha\}} (f(x) - \alpha)\,dx = 2\int_{\{x\in E:\ f(x)>\alpha\}} (f(x) - \alpha)\,dx =$$

$$= \int_E |f(x) - \alpha|\,dx.$$

This concludes the proof of inequality (3.7).

In the case $\alpha < \frac{1}{b_0-a_0}\int_{a_0}^{b_0} f(x)\,dx$ it is enough to apply the previous arguments to the function $-f$ and notice that the equality $(-f)_d(t) = -f_d(b_0 - a_0 - t)$ holds true for all $t \in [0, b_0 - a_0]$ except the set of measure zero of the points of discontinuity of the function $(-f)_d$. In this case again we have

$$\frac{1}{|J|}\int_J |f_d(u) - \alpha|\; du \leq \|f\|_*,$$

and this complete the proof of the theorem. □

Remark 3.6. As it was already noticed, if the function f is non-negative on $[a_0, b_0]$, then $f^* = f_d$. So, in this case Theorem 3.5 leads to the inequality

$$\|f^*\|_* \leq \|f\|_*, \tag{3.10}$$

which is sharp in the sense of constants. In this sense the estimate (3.10) for $d = 1$ is better than the one provided by Theorem 3.2.

If we drop the assumption that f is non-negative, then

$$f^* = |f|_d.$$

If in addition we take into account (Property 2.6) that

$$\Omega(|f|; I) \leq 2\Omega(f; I), \quad I \subset [a_0, b_0], \tag{3.11}$$

then, applying Theorem 3.5, we obtain

$$\|f^*\|_* = \|\ |f|_d\ \|_* \leq \|\ |f|\ \|_* \leq 2\|f\|_*. \tag{3.12}$$

However, the last inequality is not sharp despite of the fact that the constant 2 in (3.11) cannot be decreased (Property 2.6). Actually there holds true the following theorem.

Theorem 3.7 ([34]). *Let $f \in BMO([a_0, b_0])$. Then*

$$\| \, |f| \, \|_* \le \|f\|_*. \tag{3.13}$$

Proof. Fix the interval $I \subset [a_0, b_0]$ and denote by $g = f|I$ the restriction of f to I. Obviously then

$$\Omega(|f|; I) = \Omega(|g|; I) = \Omega(|g_d|; [0, |I|]).$$

But in view of Theorem 3.5, for any interval $J \subset [0, |I|]$ we have

$$\Omega(g_d; J) \le \sup_{K \subset I} \Omega(g; K) = \sup_{K \subset I} \Omega(f; K) \le \|f\|_*.$$

Hence in order to prove the theorem it is enough to prove the inequality

$$\Omega(|g_d|; [0, |I|]) \le \sup_{J \subset [0,|I|]} \Omega(g_d; J). \tag{3.14}$$

Without loss of generality we can assume that $|I| = 1$. Denote $K = [0, 1],\ h = g_d,\ \beta = |h|_K,\ \gamma = h_K$. Then (3.14) becomes

$$\int_K \big| \, |h(t)| - \beta \big| \, dt \le \sup_{J \subset K} \frac{1}{|J|} \int_J |h(t) - h_J| \, dt. \tag{3.15}$$

The proof of the theorem splits into the following three cases:

1. $\lim_{t \to 1-0} h(t) \ge -\beta$; obviously in this case $\lim_{t \to 0+} h(t) > \beta$;

2. $\lim_{t \to 0+} \le \beta$; obviously in this case $\lim_{t \to 1-0} h(t) < -\beta$;

3. $\lim_{t \to 1-0} h(t) < -\beta$ and $\lim_{t \to 0+} h(t) > \beta$.

In the first case, by properties of mean oscillations,

$$\int_K \big| \, |h(t)| - \beta \big| \, dt = 2 \int_{\{t \in K:\ |h(t)| > \beta\}} (|h(t)| - \beta) \, dt =$$

$$= 2 \int_{\{t \in K:\ h(t) > \beta\}} (h(t) - \beta) \, dt \le 2 \int_{\{t \in K:\ h(t) > \gamma\}} (h(t) - \gamma) \, dt =$$

$$= \int_K |h(t) - \gamma| \, dt = \int_K |h(t) - h_K| \, dt.$$

Similarly, in the second case we have

$$\int_K \big| \, |h(t)| - \beta \big| \, dt = 2 \int_{\{t \in K:\ |h(t)| > \beta\}} (|h(t)| - \beta) \, dt =$$

$$= 2 \int_{\{t \in K:\ h(t) < -\beta\}} (-h(t) - \beta)\, dt \leq 2 \int_{\{t \in K:\ h(t) < \gamma\}} (-h(t) - (-\gamma))\, dt =$$

$$= \int_K |h(t) - \gamma|\, dt = \int_K |h(t) - h_K|\, dt.$$

For the third case let us consider the function $\varphi(\tau) = \frac{1}{\tau} \int_0^\tau |h(t)|\, dt$. This function is continuous on $(0,1]$, $\lim_{\tau \to 0+} \varphi(\tau) > \beta$, $\varphi(1) = \beta$, and for $\varepsilon > 0$ small enough

$$\varphi(1-\varepsilon) = \frac{1}{1-\varepsilon} \int_0^{1-\varepsilon} |h(t)|\, dt = \frac{1}{1-\varepsilon} \int_0^1 |h(t)|\, dt - \frac{\varepsilon}{1-\varepsilon} \frac{1}{\varepsilon} \int_{1-\varepsilon}^1 |h(t)|\, dt < \beta.$$

From the properties of the function φ it follows that there exists $\tau_0 \in (0,1)$ such that $\varphi(\tau_0) = \beta$. Denote $K_1 = [0, \tau_0],\ K_2 = [\tau_0, 1]$. Then

$$|h|_{K_1} = \varphi(\tau_0) = \beta,$$

$$|h|_{K_2} = \frac{1}{1-\tau_0} \int_{\tau_0}^1 |h(t)|\, dt = \frac{1}{1-\tau_0} \left(\int_0^1 |h(t)|\, dt - \int_0^{\tau_0} |h(t)|\, dt \right) =$$

$$= \frac{1}{1-\tau_0} (\beta - \tau_0 \beta) = \beta,$$

$$\int_K \big| |h(t)| - \beta \big|\, dt = \tau_0 \frac{1}{|K_1|} \int_{K_1} \big| |h(t)| - \beta \big|\, dt + (1-\tau_0) \frac{1}{|K_2|} \int_{K_2} \big| |h(t)| - \beta \big|\, dt \leq$$

$$\leq \max_{i=1,2} \frac{1}{|K_i|} \int_{K_i} \big| |h(t)| - \beta \big|\, dt.$$

If we show that

$$\int_{K_i} \big| |h(t)| - \beta \big|\, dt \leq \int_{K_i} |h(t) - h_{K_i}|\, dt, \quad i = 1, 2, \tag{3.16}$$

then

$$\int_K \big| |h(t)| - \beta \big|\, dt \leq \max_{i=1,2} \frac{1}{|K_i|} \int_{K_i} \big| |h(t)| - \beta \big|\, dt \leq$$

$$\leq \max_{i=1,2} \frac{1}{|K_i|} \int_{K_i} |h(t) - h_{K_i}|\, dt \leq \sup_{J \subset K} \frac{1}{|J|} \int_J |h(t) - h_J|\, dt,$$

i.e., (3.15). So, it remains to prove (3.16).

For $i = 1$

$$\int_{K_1} \left| |h(t)| - \beta \right| dt = 2 \int_{\{t\in K_1:\ |h(t)|>\beta\}} (|h(t)| - \beta)\, dt \le$$

$$\le 2 \int_{\left\{t\in K_1:\ h(t)>h_{K_1}\right\}} \left(h(t) - h_{K_1}\right)\, dt = \int_{K_1} \left|h(t) - h_{K_1}\right|\, dt.$$

Similarly, for $i = 2$

$$\int_{K_2} \left| |h(t)| - \beta \right| dt = 2 \int_{\{t\in K_2:\ |h(t)|>\beta\}} (|h(t)| - \beta)\, dt =$$

$$= 2 \int_{\{t\in K_2:\ h(t)<-\beta\}} (-h(t) - \beta)\, dt \le$$

$$\le 2 \int_{\left\{t\in K_2:\ h(t)<h_{K_2}\right\}} \left(-h(t) + h_{K_2}\right)\, dt = \int_{K_2} \left|h(t) - h_{K_2}\right|\, dt.$$

This proves (3.16) and completes the proof of (3.15). $\square$

Remark 3.8. We have proved (3.13) in the one-dimensional case. If the dimension of the space $d \ge 2$, then Property 2.6 immediately implies that

$$\| \, |f| \, \|_* \le 2\|f\|_*.$$

For $d \ge 2$ we do not know the minimal constant c (which possibly depends on d) for the inequality

$$\| \, |f| \, \|_* \le c\|f\|_*.$$

By means of Theorem 3.7 one can improve the last inequality in (3.12) and obtain the following

Corollary 3.9 ([34]). *Let $f \in BMO([a_0, b_0])$. Then*

$$\|f^*\|_* \le \|f\|_*. \tag{3.17}$$

Remark 3.10. For $d \ge 2$ we do not know the minimal constant c in the inequality

$$\|f^*\|_* \le c\|f\|_*.$$

Now let us consider the estimates of the BMO-norm of the non-increasing equimeasurable rearrangement of a BMO^R-function. Recall that BMO^R differs from BMO if $d \ge 2$ because the oscillations must be calculated over all possible rectangles, not only the cubes. First of all we prove the multidimensional analog of Theorem 3.5.

Theorem 3.11 ([45]). *Let $f \in BMO^R(R_0)$, where $R_0 \subset \mathbb{R}^d$ is a multidimensional segment. Then*

$$\|f_d\|_* \le \|f\|_{*,R}. \tag{3.18}$$

Proof. Essentially we will repeat the proof of Theorem 3.5. Fix the interval $J \subset [0, |R_0|]$ and denote $\alpha = \frac{1}{|J|}\int_J f_d(u)\,du$. Let $\alpha \geq \frac{1}{|R_0|}\int_{R_0} f(x)\,dx$. Applying the multidimensional analog of the Riesz "rising sun lemma" (Lemma 1.30) [1], we construct the pairwise disjoint segments $I_j \subset R_0,\ j = 1, 2, \ldots$ such that $f_{I_j} = \alpha,\ j = 1, 2, \ldots$, and $f(x) \leq \alpha$ for almost all $x \in R_0 \setminus E$ with $E = \bigcup_{j\geq 1} I_j$. If we prove the inequality

$$\frac{1}{|J|}\int_J |f_d(t) - \alpha|\ dt \leq \frac{1}{|E|}\int_E |f(x) - \alpha|\,dx, \tag{3.19}$$

then in order to complete the proof it will remain to use the relations

$$f_E = \frac{1}{|E|}\int_E f(x)\,dx = \frac{1}{\sum_j |I_j|}\sum_j \int_{I_j} f(x)\,dx = \alpha, \tag{3.20}$$

$$\frac{1}{|E|}\int_E |f(x) - \alpha|\,dx = \frac{1}{|E|}\sum_j \int_{I_j} |f(x) - \alpha|\,dx =$$

$$= \frac{1}{|E|}\sum_j |I_j| \frac{1}{|I_j|}\int_{I_j} \left|f(x) - f_{I_j}\right|\ dx = \frac{1}{|E|}\sum_j |I_j| \Omega(f; I_j) \leq \|f\|_{*,R}.$$

For the proof of (3.19) let us choose the maximal $t \in (0, |R_0|]$ such that $J \subset [0, t]$ and $\frac{1}{t}\int_0^t f_d(u)\,du = \alpha$. The existence of such a t follows from the condition

$$\frac{1}{|R_0|}\int_0^{|R_0|} f_d(u)\,du = \frac{1}{|R_0|}\int_{R_0} f(x)\,dx \leq \alpha = \frac{1}{|J|}\int_J f_d(u)\,du.$$

Using the monotonicity of the function f_d and applying Property 2.15, we obtain

$$\frac{1}{|J|}\int_J |f_d(u) - \alpha|\ du \leq \frac{1}{t}\int_0^t |f_d(u) - \alpha|\ du.$$

So, in order to prove (3.19) it is enough to show that

$$\frac{1}{t}\int_0^t |f_d(u) - \alpha|\ du \leq \frac{1}{|E|}\int_E |f(x) - \alpha|\,dx. \tag{3.21}$$

In its own turn the inequality (3.21) is a consequence of the following two statements:

[1] We could also use Lemma 1.21 and Remark 1.24. But in order to use Remark 1.24 one should prove that $f \in L^p(R_0)$ for some $p > 1$. Indeed, from the John–Nirenberg inequality, which will be proved in the next section, it follows that $f \in L^p(R_0)$ for every $p < \infty$. This is not a vicious circle, because we do not need Theorem 3.11 to prove the John–Nirenberg inequality (for $d = 2$ it is enough to use Lemma 1.22).

$$t \geq |E|, \tag{3.22}$$

$$\int_0^t |f_d(u) - \alpha| \, du = \int_E |f(x) - \alpha| \, dx. \tag{3.23}$$

For the proof of (3.22) notice, that by the definition of the non-increasing rearrangement

$$\int_0^{|E|} f_d(u) \, du \geq \int_E f(x) \, dx,$$

so that, by (3.20),

$$\frac{1}{|E|} \int_0^{|E|} f_d(u) \, du \geq \frac{1}{|E|} \int_E f(x) \, dx = \alpha = \frac{1}{t} \int_0^t f_d(u) \, du.$$

Taking into account the monotonicity of f_d, from here we obtain (3.22).

For the proof of (3.23) we will use the fact that $f(x) \leq \alpha$ almost everywhere on $R_0 \setminus E$. Then, applying (3.20) and the properties of mean oscillations, we get

$$\int_0^t |f_d(u) - \alpha| \, du = 2 \int_{\{u \in [0,t]:\ f_d(u) > \alpha\}} (f_d(u) - \alpha) \, du =$$

$$= 2 \int_{\{x \in R_0:\ f(x) > \alpha\}} (f(x) - \alpha) \, dx = 2 \int_{\{x \in E:\ f(x) > \alpha\}} (f(x) - \alpha) \, dx =$$

$$= \int_E |f(x) - \alpha| \, dx.$$

This concludes the proof of (3.21).

In the case $\alpha < \frac{1}{|R_0|} \int_{R_0} f(x) \, dx$ it is enough to apply the preceding arguments to the function $-f$ and to note that for all $t \in [0, |R_0|]$, except the set of zero measure of the points of discontinuity of $(-f)_d$, we have $(-f)_d(t) = -f_d(|R_0| - t)$. In addition, in this case

$$\frac{1}{|J|} \int_J |f_d(u) - \alpha| \, du \leq \|f\|_{*,R},$$

and this completes the proof of the theorem. □

The next theorem is the multidimensional analog of Theorem 3.7 (it is interesting to compare it with Remark 3.10).

Theorem 3.12 ([45]). *Let $f \in BMO^R(R_0)$, where $R_0 \subset \mathbb{R}^d$ is a multidimensional segment. Then*

$$\| \, |f| \, \|_{*,R} \leq \|f\|_{*,R}.$$

Proof. Fix the segment $I \subset R_0$ and denote by $g = f|I$ the restriction of the function f to I. Obviously then

$$\Omega(|f|;I) = \Omega(|g|;I) = \Omega(|g_d|;[0,|I|]).$$

On the other hand, according to Theorem 3.11, for every interval $J \subset [0,|I|]$

$$\Omega(g_d;J) \le \sup_{K\subset I} \Omega(g;K) = \sup_{K\subset I} \Omega(f;K) \le \|f\|_{*,R}.$$

Hence, in order to prove the theorem it is enough to prove the inequality

$$\Omega(|g_d|;[0,|I|]) \le \sup_{J\subset[0,|I|]} \Omega(g_d;J). \tag{3.24}$$

But we have already obtained (3.24) while proving Theorem 3.7 (inequality (3.14)). Indeed, formulas (3.14) and (3.24) express the relation between the oscillation of the function g_d and its absolute value independently on the dimension of the space. □

Now we can easily get the multidimensional analog of inequality (3.17).

Theorem 3.13 ([45]). *Let $f \in BMO^R(R_0)$, where $R_0 \subset \mathbb{R}^d$ is a multidimensional segment. Then*

$$\|f^*\|_* \le \|f\|_{*,R}.$$

Proof. Using the trivial equality $f^* = |f|_d$ and applying Theorems 3.11 and 3.12 we obtain

$$\|f^*\|_* = \|\,|f|_d\,\|_* \le \|\,|f|\,\|_{*,R} \le \|f\|_{*,R}. \quad \square$$

3.2 The John–Nirenberg Inequality

We have already mentioned (see p. 41), that the logarithmic function is a typical representative of the BMO-class. This means that the distribution function of the BMO-function decreases exponentially.

Theorem 3.14 (John, Nirenberg, [30]). *There exist constants b and B (possibly depending on the dimension d of the space) such that for any function $f \in BMO(\mathbb{R}^d)$ and any cube $Q_0 \subset \mathbb{R}^d$*

$$|\{x \in Q_o :\ |f(x) - f_{Q_0}| > \lambda\}| \le B \cdot |Q_0| \cdot \exp\left(-\frac{b\lambda}{\|f\|_*}\right), \quad \lambda > 0. \tag{3.25}$$

Proof. Since inequality (3.25) is homogeneous with respect to the multiplication of the function f by a constant we can assume that $\|f\|_* = 1$. Let us apply the Calderón–Zygmund lemma (Lemma 1.14) with $\alpha = \frac{3}{2}$ to the function $|f - f_{Q_0}|$. As the result we obtain a collection of cubes $\left\{Q_j^{(1)}\right\}_{j\geq 1}$ with pairwise disjoint interiors and verifying the following properties:

$$\frac{3}{2} < \frac{1}{\left|Q_j^{(1)}\right|} \int_{Q_j^{(1)}} |f(x) - f_{Q_0}|\, dx \leq 2^d \cdot \frac{3}{2}, \quad j = 1, 2, \ldots, \tag{3.26}$$

$$|f(x) - f_{Q_0}| \leq \frac{3}{2} \quad \text{for a.e. } x \in Q_0 \setminus \left(\bigcup_{j\geq 1} Q_j^{(1)} \right).$$

The left inequality of (3.26) implies

$$\sum_{j\geq 1} \left|Q_j^{(1)}\right| \leq \frac{1}{3/2} \sum_{j\geq 1} \int_{Q_j^{(1)}} |f(x) - f_{Q_0}|\, dx \leq$$

$$\leq \frac{2}{3} \int_{Q_0} |f(x) - f_{Q_0}|\, dx \leq \frac{2}{3}|Q_0| \cdot \|f\|_* = \frac{2}{3}|Q_0|,$$

while from the right inequality we have

$$\left|f_{Q_j^{(1)}} - f_{Q_0}\right| = \left| \frac{1}{\left|Q_j^{(1)}\right|} \int_{Q_j^{(1)}} (f(x) - f_{Q_0})\, dx \right| \leq$$

$$\leq \frac{1}{\left|Q_j^{(1)}\right|} \int_{Q_j^{(1)}} |f(x) - f_{Q_0}|\, dx \leq 2^d \cdot \frac{3}{2}, \quad j = 1, 2, \ldots$$

To every cube $Q_j^{(1)}$ we apply again Calderón–Zygmund lemma 1.14.

In the k-th step, applying Calderón–Zygmund lemma 1.14 with $\alpha = \frac{3}{2}$ to the function $\left|f - f_{Q_j^{(k-1)}}\right|$ on every cube $Q_j^{(k-1)}$, $j = 1, 2 \ldots$, we obtain a family of cubes $Q_{i,j}^{(k)} \subset Q_j^{(k-1)}$, $i = 1, 2, \ldots$, with pairwise disjoint interiors, and such that

$$\frac{3}{2} < \frac{1}{\left|Q_{i,j}^{(k)}\right|} \int_{Q_{i,j}^{(k)}} \left|f(x) - f_{Q_j^{(k-1)}}\right| dx \leq 2^d \cdot \frac{3}{2}, \tag{3.27}$$

$$\left|f(x) - f_{Q_j^{(k-1)}}\right| \leq \frac{3}{2} \quad \text{for a.e.} \quad x \in Q_j^{(k-1)} \setminus \left(\bigcup_{i\geq 1} Q_{i,j}^{(k)} \right). \tag{3.28}$$

Inequality (3.27) implies

$$\sum_{i\geq 1}\left|Q_{i,j}^{(k)}\right| \leq \frac{2}{3}\sum_{i\geq 1}\int_{Q_{i,j}^{(k)}}\left|f(x)-f_{Q_j^{(k-1)}}\right|\,dx \leq$$

$$\leq \frac{2}{3}\int_{Q_j^{(k-1)}}\left|f(x)-f_{Q_j^{(k-1)}}\right|\,dx \leq \frac{2}{3}\left|Q_j^{(k-1)}\right|\cdot\|f\|_* = \frac{2}{3}\left|Q_j^{(k-1)}\right|, \quad (3.29)$$

$$\left|f_{Q_{i,j}^{(k)}} - f_{Q_j^{(k-1)}}\right| \leq \frac{1}{\left|Q_{i,j}^{(k)}\right|}\int_{Q_{i,j}^{(k)}}\left|f(x)-f_{Q_j^{(k-1)}}\right|\,dx \leq 2^d\cdot\frac{3}{2}. \quad (3.30)$$

Numbering all cubes $Q_{i,j}^{(k)},\ i,j=1,2,\dots$ we get the collection $\left\{Q_j^{(k)}\right\}_{j\geq 1}$. In addition, by (3.29),

$$\sum_{j\geq 1}\left|Q_j^{(k)}\right| \leq \frac{2}{3}\sum_{j\geq 1}\left|Q_j^{(k-1)}\right| \leq \dots \leq \left(\frac{2}{3}\right)^{k-1}\sum_{j\geq 1}\left|Q_j^{(1)}\right| \leq \left(\frac{2}{3}\right)^{k}|Q_0|\,, \quad (3.31)$$

while from (3.28) and (3.30) it follows that

$$|f(x)-f_{Q_0}| \leq \left|f(x)-f_{Q_{j_{k-1}}^{(k-1)}}\right| + \left|f_{Q_{j_{k-1}}^{(k-1)}} - f_{Q_{j_{k-2}}^{(k-2)}}\right| + \dots + \left|f_{Q_{j_1}^{(1)}} - f_{Q_0}\right| \leq$$

$$\leq \frac{3}{2}+2^d(k-1)\frac{3}{2} \leq k\cdot 2^d\cdot\frac{3}{2} \quad \text{for a.e. } x\in\left(\cup_{j\geq 1}Q_j^{(k-1)}\right)\setminus\left(\cup_{j\geq 1}Q_j^{(k)}\right), \quad (3.32)$$

where $Q_{j_{i+1}}^{(i+1)} \subset Q_{j_i}^{(i)},\ i=1,\dots,k-2$. Then we pass to the next, $(k+1)$-th step.

Take an arbitrary number $\lambda>0$. If $k\cdot 2^d\cdot\frac{3}{2} < \lambda \leq (k+1)\cdot 2^d\cdot\frac{3}{2}$ for some $k\in\mathbb{N}$, then, by (3.31) and (3.32),

$$|\{x\in Q_0 :\ |f(x)-f_{Q_0}|>\lambda\}| \leq \left|\left\{x\in Q_0 :\ |f(x)-f_{Q_0}| > k\cdot 2^d\cdot\frac{3}{2}\right\}\right| \leq$$

$$\leq \sum_{j\geq 1}\left|Q_j^{(k)}\right| \leq \left(\frac{2}{3}\right)^k\cdot|Q_0| = |Q_0|\exp\left(-k\ln\frac{3}{2}\right) \leq$$

$$\leq |Q_0|\exp\left(\left(1-\frac{\lambda}{2^d\cdot\frac{3}{2}}\right)\ln\frac{3}{2}\right) = \frac{3}{2}|Q_0|\exp(-b\lambda),$$

where $b=\frac{2}{3}\cdot\ln\frac{3}{2}\cdot 2^{-d}$. Otherwise, if $\lambda\leq 2^d\cdot\frac{3}{2}$, then

$$|\{x\in Q_o :\ |f(x)-f_{Q_0}|>\lambda\}| \leq |Q_0|\exp(-b\lambda)\cdot\exp\left(b\cdot 2^d\frac{3}{2}\right) \equiv$$

$$\equiv |Q_0|B_1\exp(-b\lambda),$$

where $B_1=\exp\left(b\cdot 2^d\cdot\frac{3}{2}\right)$. Setting $B=B_1+\frac{3}{2}$, we obtain (3.25). $\square$

Remark 3.15. In terms of equimeasurable rearrangements inequality (3.25) can be rewritten in the following form:

$$(f - f_{Q_0})^*(t) \le \frac{\|f\|_*}{b} \ln \frac{B|Q_0|}{t}, \quad 0 < t \le |Q_0|. \tag{3.33}$$

So, if $f \in BMO$, then its equimeasurable rearrangement do not grow faster than the logarithmic function as the argument tends to zero.

Remark 3.16. In a certain sense the John–Nirenberg theorem is invertible. Namely, if f is a locally summable on $\mathbb{R}^d$ function such that for any cube $Q_0 \subset \mathbb{R}^d$

$$|\{x \in Q_0 : \; |f(x) - f_{Q_0}| > \lambda\}| \le B|Q_0| \cdot \exp(-b\lambda), \quad \lambda > 0, \tag{3.34}$$

where the constants B and b do not depend on Q_0, then $f \in BMO(\mathbb{R}^d)$. Indeed, let us rewrite (3.34) in the form

$$(f - f_{Q_0})^*(t) \le \frac{1}{b} \ln \frac{B|Q_0|}{t}, \quad 0 < t \le |Q_0|. \tag{3.35}$$

Then

$$\frac{1}{|Q_0|} \int_{Q_0} |f(x) - f_{Q_0}| \, dx = \frac{1}{|Q_0|} \int_0^{|Q_0|} (f - f_{Q_0})^*(t) \, dt \le$$

$$\le \frac{1}{b} \frac{1}{|Q_0|} \int_0^{|Q_0|} \ln \frac{B|Q_0|}{t} \, dt = \frac{1}{b} \int_0^1 \ln \frac{B}{u} \, du = \frac{1}{b}(1 + \ln B).$$

Taking the supremum over all cubes $Q_0 \subset \mathbb{R}^d$, we obtain $\|f\|_* \le \frac{1}{b}(1 + \ln B)$.

Remark 3.17. Now let $f \in BMO(Q_0)$ for some fixed cube $Q_0 \subset \mathbb{R}^d$. Obviously then the proof of inequality (3.25) holds true, and so do (3.33). However, (3.34), as well as its equivalent form (3.35), does not imply $f \in BMO(Q_0)$. One can easily construct the corresponding example, we omit this point here.

The John–Nirenberg theorem implies

Corollary 3.18. *If $f \in BMO(\mathbb{R}^d)$, then $f \in L^p_{loc}$ for any $p < \infty$.*

Proof. It is enough to prove that $f - f_{Q_0} \in L^p(Q_0)$ for any cube $Q_0 \subset \mathbb{R}^d$. The John–Nirenberg inequality in the form (3.33) yields

$$\int_{Q_0} |f(x) - f_{Q_0}|^p \, dx = \int_0^{|Q_0|} \left[(f - f_{Q_0})^*(t)\right]^p \, dt \le$$

$$\le \left(\frac{\|f\|_*}{b}\right)^p \int_0^{|Q_0|} \ln^p \frac{B|Q_0|}{t} \, dt =$$

$$= \left(\frac{\|f\|_*}{b}\right)^p |Q_0| \cdot B \int_0^{1/B} \ln^p \frac{1}{u} \, du < \infty. \quad \square \tag{3.36}$$

Remark 3.19. Inequality (3.36) can be rewritten as follows:

$$\Omega_p(f; Q_0) \le c_{p,d} \|f\|_*, \quad Q_0 \subset \mathbb{R}^d,$$

where the constant $c_{p,d}$ depends only on p and d. Hence

$$\|f\|_{*,p} \le c_{p,d} \|f\|_*.$$

As we have already mentioned, the inequality $\|f\|_* \le \|f\|_{*,p}$ for $1 < p < \infty$ is a direct consequence of the Hölder inequality. Therefore all the classes $BMO_p(\mathbb{R}^d)$ coincide for all p, $1 \le p < \infty$. Analogously, for any fixed cube $Q_0 \subset \mathbb{R}^d$ all the classes $BMO_p(Q_0)$ coincide.

The John–Nirenberg inequality in the form (3.33) can be easily derived from the estimate of the rearrangement, provided by Theorem 3.4. Indeed, let $Q_0 \subset \mathbb{R}^d$, $f \in BMO(Q_0)$ and $f_{Q_0} = 0$. Then

$$f^{**}\left(\frac{t}{2}\right) - f^{**}(t) = \frac{2}{t} \int_0^{t/2} \left(f^*(u) - f^{**}(t)\right) du \le$$

$$\le \frac{2}{t} \int_0^t |f^*(u) - f^{**}(t)| \, du \le 2\|f^*\|_*, \quad 0 < t \le |Q_0|.$$

According to Theorem 3.4,

$$f^{**}\left(\frac{t}{2}\right) - f^{**}(t) \le 2c\|f\|_*, \quad 0 < t \le |Q_0|, \tag{3.37}$$

where the constant c depends only on the dimension d of the space.

Fix some $t \in (0, |Q_0|]$ and choose n such that $2^{-n-1}|Q_0| < t \le 2^{-n}|Q_0|$. Applying (3.37) we obtain

$$f^{**}(t) \le f^{**}\left(2^{-n-1}|Q_0|\right) \le f^{**}\left(2^{-n}|Q_0|\right) + 2c\|f\|_* \le$$

$$\le f^{**}\left(2^{-n+1}|Q_0|\right) + 2 \cdot (2c\|f\|_*) \le \cdots \le f^{**}\left(|Q_0|\right) + (n+1)(2c\|f\|_*).$$

Taking into account that $f_{Q_0} = 0$ implies

$$f^{**}\left(|Q_0|\right) = \frac{1}{|Q_0|} \int_0^{|Q_0|} f^*(u) \, du = \frac{1}{|Q_0|} \int_{Q_0} |f(x)| \, dx \le \|f\|_*,$$

we get

$$f^*(t) \le f^{**}(t) \le 2c(n+2)\|f\|_* \le 2c \left(\frac{\ln \frac{|Q_0|}{t}}{\ln 2} + 2 \right) \|f\|_* =$$

$$= \frac{2c}{\ln 2} \|f\|_* \ln \frac{4|Q_0|}{t}, \quad 0 < t \le |Q_0|,$$

which for $f_{Q_0} = 0$ yields (3.33) with $B = 4$, $b = \frac{\ln 2}{2c}$.

Notice, that the assumption $f_{Q_0} = 0$ is not restrictive, and one could obtain (3.33) without this additional condition. $\square$

3.2.1 One-Dimensional Case

For the proof of the John–Nirenberg inequality we were using the arguments, based on the estimates of the equimeasurable rearrangements of functions. These arguments were used in the original work [1]. One can improve this result and get the exact exponent in the John–Nirenberg inequality (3.25) for the one-dimensional case. Indeed, Lemma 2.2 has the following

Corollary 3.20. *Let* $f \in BMO([a_0, b_0])$ *with* $[a_0, b_0] \subset \mathbb{R}$ *and* $f_{[a_0,b_0]} = 0$. *Then for any* $a > 1$

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \leq \frac{a}{2}\|f\|_*, \quad 0 < t \leq b_0 - a_0. \tag{3.38}$$

Proof. Taking $\varphi = f^*$ in Lemma 2.2, we have $F = f^{**}$. Then, by (2.2),

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \leq \frac{a}{2}\|f^*\|_*, \quad 0 < t \leq b_0 - a_0.$$

This, together with inequality (3.17) (see Corollary 3.9), implies (3.38). □

Using (3.38) it is easy to derive the John–Nirenberg inequality with the exact exponent in the one-dimensional case.

Theorem 3.21 ([34]). *Let* $f \in BMO([a_0, b_0])$. *Then*

$$\left(f - f_{[a_0,b_0]}\right)^*(t) \leq \frac{\|f\|_*}{2/\mathrm{e}} \ln \frac{B(b_0 - a_0)}{t}, \quad 0 < t \leq b_0 - a_0, \tag{3.39}$$

with $B = \exp\left(1 + \frac{2}{\mathrm{e}}\right)$. *Moreover, in general the denominator* $2/\mathrm{e}$ *in the fraction, preceding to the logarithm, cannot be increased.*

Proof. Without loss of generality, we can assume that $f_{[a_0,b_0]} = 0$. Let $a > 1$ (we will choose this constant later). Summing up the inequalities

$$f^{**}\left(\frac{b_0 - a_0}{a^i}\right) - f^{**}\left(\frac{b_0 - a_0}{a^{i-1}}\right) \leq \frac{a}{2}\|f\|_*, \quad i = 1, \ldots, k+1,$$

which follow from (3.38), we get

$$f^{**}\left(\frac{b_0 - a_0}{a^{k+1}}\right) \leq (k+1)\frac{a}{2}\|f\|_* + f^{**}(b_0 - a_0). \tag{3.40}$$

Since $f_{[a_0,b_0]} = 0$ we have

$$f^{**}(b_0 - a_0) = \frac{1}{b_0 - a_0}\int_0^{b_0 - a_0} f^*(u)\, du = \frac{1}{b_0 - a_0}\int_{a_0}^{b_0} |f(x)|\, dx =$$

$$= \Omega\left(f;[a_0,b_0]\right) \le \|f\|_*,$$

so that

$$f^{**}\left(\frac{b_0-a_0}{a^{k+1}}\right) \le \left((k+1)\frac{a}{2}+1\right)\|f\|_*, \quad k=0,1,\dots \tag{3.41}$$

Choose some $t \in (0, b_0-a_0]$ and k such that

$$\frac{b_0-a_0}{a^{k+1}} < t \le \frac{b_0-a_0}{a^k}.$$

Then $k \le \frac{1}{\ln a}\ln\frac{b_0-a_0}{t}$ and from (3.41) we obtain

$$f^*(t) \le f^{**}(t) \le f^{**}\left(\frac{b_0-a_0}{a^{k+1}}\right) \le \left((k+1)\frac{a}{2}+1\right)\|f\|_* \le$$

$$\le \left(\left(\frac{1}{\ln a}\cdot\ln\frac{b_0-a_0}{t}+1\right)\frac{a}{2}+1\right)\|f\|_* =$$

$$= \left(\frac{1}{2}\frac{a}{\ln a}\cdot\ln\frac{b_0-a_0}{t}+\frac{a}{2}+1\right)\|f\|_*. \tag{3.42}$$

The function $\frac{a}{\ln a}$ for $a>1$ takes its minimal value at $a=\mathrm{e}$. Substituting $a=\mathrm{e}$ in (3.42) for $0<t\le b_0-a_0$ we have

$$f^*(t) \le \left(\frac{\mathrm{e}}{2}\ln\frac{b_0-a_0}{t}+\frac{\mathrm{e}}{2}+1\right)\|f\|_* = \frac{\|f\|_*}{2/\mathrm{e}}\ln\frac{\exp\left(1+\frac{2}{\mathrm{e}}\right)(b_0-a_0)}{t},$$

and (3.39) follows.

It remains to show that the constant $2/\mathrm{e}$ in the denominator in the right-hand side of (3.39) cannot be increased. Indeed, for the function $f(x)=\ln\frac{1}{x}-1$, $0\le x\le 1$, we have $f_{[0,1]}=0$. Moreover, as it was already shown (see Example 2.24), $\|f\|_*=\frac{2}{\mathrm{e}}$. Hence (3.39) becomes

$$f^*(t) \le \ln\frac{1}{t}+1+\frac{2}{\mathrm{e}}, \quad 0<t\le 1.$$

On the other hand, it is easy to see that $f^*(t)=\ln\frac{1}{\mathrm{e}t}$, $0<t\le \mathrm{e}^{-2}$. Therefore the coefficient of the logarithm in the right-hand side of (3.39) cannot be decreased. □

In terms of the distribution function, inequality (3.39) can be rewritten in the following way.

Corollary 3.22 ([34]). *Let $f\in BMO\left([a_0,b_0]\right)$. Then*

$$\left|\left\{x\in[a_0,b_0]:\ \left|f(x)-f_{[a_0,b_0]}\right|>\lambda\right\}\right| \le B\,(b_0-a_0)\exp\left(-\frac{2/\mathrm{e}}{\|f\|_*}\lambda\right), \quad \lambda>0, \tag{3.43}$$

where $B=\exp\left(1+\frac{2}{\mathrm{e}}\right)$, and in general the constant $2/\mathrm{e}$ in the exponent cannot be increased.

3.2.2 Anisotropic Case

To our knowledge in the multidimensional case the problem of finding the upper bound of those b that provide John–Nirenberg inequality (3.25) is still open. If instead of $\|f\|_*$ in (3.25) we consider $\|f\|_{*,R}$, then the maximal value of the constant b in the John–Nirenberg inequality is equal to $\frac{2}{e}$ as in the one-dimensional case. Namely, we have

Theorem 3.23 ([45]). *Let $f \in BMO^R(\mathbb{R}^d)$. Then for any segment $R_0 \subset \mathbb{R}^d$*

$$|\{x \in R_0 : \ |f(x) - f_{R_0}| > \lambda\}| \le B \cdot |R_0| \cdot \exp\left(-\frac{2/\mathrm{e}}{\|f\|_{*,R}}\lambda\right), \quad \lambda > 0, \tag{3.44}$$

where $B = \exp\left(1 + \frac{2}{e}\right)$, and in general the constant $\frac{2}{e}$ in the exponent cannot be increased.

Proof. Without loss of generality we can assume that $f_{R_0} = 0$. Then rewriting inequality (3.44) in terms of equimeasurable rearrangements we have

$$f^*(t) \le \frac{\|f\|_{*,R}}{2/\mathrm{e}} \ln \frac{B|R_0|}{t}, \quad 0 < t \le |R_0|. \tag{3.45}$$

Essentially in order to prove (3.45) we have to repeat the same arguments as in the proof of Theorem 3.21. Indeed, setting $\varphi = f^*$ in Lemma 2.2, we have

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \le \frac{a}{2}\|f^*\|_*, \quad 0 < t \le |R_0|,$$

which together with Theorem 3.13 leads to the following multidimensional analog of inequality (3.38):

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \le \frac{a}{2}\|f\|_{*,R}, \quad 0 < t \le |R_0|$$

for an arbitrary $a > 1$. Now it remains to repeat completely the proof of Theorem 3.21, taking R_0 instead of $[a_0, b_0]$, and $\|f\|_{*,R}$ instead of $\|f\|_*$.

The fact that the denominator $\frac{2}{e}$ in the right-hand side of (3.45) cannot be increased can be easily checked on the following function:

$$f(x_1, \ldots, x_d) = \ln \frac{1}{x_1} - 1, \quad x \equiv (x_1, \ldots, x_d) \in R_0 \equiv [0,1]^d. \quad \square$$