Lecture Notes of the Unione Matematica Italiana

Anatolii Korenovskii

4

Mean Oscillations and Equimeasurable Rearrangements of Functions

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Library of Congress Control Number: 2007933717

Mathematics Subject Classification (2000): 42B25, 46E30,

ISSN print edition: 1862-9113
ISSN electronic edition: 1862-9121
ISBN 978-3-540-74708-6 Springer Berlin Heidelberg New York

DOI 10.1007/978-3-540-74709-3

Springer is a part of Springer Science+Business Media
springer.com

Typesetting by the author and SPi using a Springer LaTeX macro package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 12112852 VA41/2141/SPi 5 4 3 2 1 0

Preface

The present book is devoted to the properties of functions that can be described in terms of mean integral oscillations. The starting point of this study was the work of F. John and L. Nirenberg [30], where the authors showed that the distribution function corresponding to a function of bounded mean oscillation is exponentially decreasing. This fact, which now is known as the John–Nirenberg inequality, plays an important role in various aspects of theory of functions. The fundamental work of C. Fefferman and E. Stein [11] gave a powerful impact to the further study of properties of functions of bounded mean oscillation (the BMO-functions). In this work it was proved that the BMO-class is conjugate to the class Re H^1 of the real parts of summable analytical functions. C. Bennett and R. Sharpley studied in detail the interpolation properties of the BMO-class. Their results, which later were published in monograph [3], widely enlarge the field of application of the BMO-functions. L. Gurov and Yu. Reshetnyak in [21, 22] considered the functions, satisfying one special condition, expressed in terms of mean oscillations. Afterwards this condition was called the Gurov–Reshetnyak inequality. The classes of functions satisfying the Gurov–Reshetnyak condition, have many applications in the theories of quasi-conformal maps, partial differential equations and weighted spaces.

There now exists a huge number of journal publications dealing with properties of functions expressed in terms of mean oscillations. Numerous facts can be found in monographs and overviews of various authors. The present work cannot pretend to describe all known results in this field. We have just attempted to give a systematic and detailed introduction to the study of the BMO-class, the Gurov–Reshetnyak class and some related classes of functions. We have paid particular attention to the finality of the results, though in certain cases we did not succeed in this way.

One of the main tools of our study were the sharp estimates of the equimeasurable rearrangements of functions. Equimeasurable rearrangements have been used by many authors for the analysis of the properties of function of the mentioned above classes. However recently, the problem was posed of the sharpness of the parameters, describing these properties, which shows that the estimates of the equimeasurable rearrangements of functions need to be improved. Usually these estimates are based on the application of the so-called covering lemmas. In the present monograph we give some new versions of these.

The choice of the subjects of my research was formed under the influence of my scientific supervisor professor V.I. Kolyada, who later became my colleague and a good friend. His permanent interest was the stimulus of my work, his personality was an example for me. I am sincerely grateful to him for this.

The translation by Russian into English was financially supported by the Accademia Pontaniana, Napoli. I would like to thank the Accademia and in particular Professor Carlo Sbordone for his great participation in the translation and publication of this book.

Odessa, 2007 *A. Korenovskii*

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Preliminaries and Auxiliary Results

1.1 Equimeasurable Rearrangements of Functions

The equimeasurable rearrangements of functions play an important role in various fields of mathematics. Their effectiveness comes from the fact that in certain cases they preserve the properties of the original functions and in the same time have a simpler form. Let us give the definitions.

The value

$$\lambda_f(y) = |\{x \in E : \ |f(x)| > y\}|, \quad 0 \le y < \infty$$

is called *the distribution function* of the function f, measurable on the set $E \subset \mathbb{R}^d$. If $|E| = \infty$, then in addition it is assumed that $\lambda_f(y) < \infty$ for all $y > 0$. The *non-increasing rearrangement* of the function f is a non-increasing on $(0, |E|]$ function f^* such that it is *equimeasurable* with $|f|$, i.e., for all $y > 0$

$$\lambda_{f^*}(y) \equiv |\{t \in [0, |E|] : \ f^*(t) > y\}| = |\{x \in E : \ |f(x)| > y\}| \equiv \lambda_f(y).$$

This property does not define the non-increasing rearrangement uniquely: it can take different values at points of discontinuity (the set of such points is at most countable). For definiteness let us assume in addition that the function f^* is continuous from the left on $(0, |E|]$. The relation between the distribution function and the non-increasing rearrangement is given by the following equality:

$$f^*(t) = \inf\{y > 0 : \ \lambda_f(y) < t\}, \quad 0 < t < |E|. \tag{1.1}$$

This formula shows that in a certain sense the non-increasing rearrangement is the inverse function to the distribution function.

An equivalent definition of the non-increasing rearrangement can be written in the following way:

$$f^*(t) = \sup_{e \subset E, \ |e| = t} \ \inf_{x \in e} |f(x)|, \quad 0 < t < |E|. \tag{1.2}$$

Sometimes instead of the non-increasing rearrangement it is more convenient to use *the non-decreasing equimeasurable rearrangement.* For the function f, measurable on the set $E \subset \mathbb{R}^d$, the non-decreasing rearrangement is defined via the following equality:

$$f_*(t) = \inf_{e\subset E,\ |e|=t} \sup_{x\in e} |f(x)|, \quad 0 < t < |E|. \tag{1.3}$$

The function f_* is non-negative, it is equimeasurable with $|f|$ on E and it is non-decreasing on $[0, |E|)$. The connection between the non-increasing and non-decreasing rearrangements is given by the equality

$$f_*(t) = f^*\left(|E| - t\right),$$

which holds true at every point of continuity, i.e. almost everywhere on $(0, |E|)$.

The equimeasurability of functions f^*, f_* and $|f|$ implies that

$$\int_0^{|E|} \varphi(f^*(u))\, du = \int_0^{|E|} \varphi(f_*(u))\, du = \int_E \varphi(|f(x)|)\, dx$$

for every monotone on $[0, +\infty)$ function φ.

The most important properties of the equimeasurable rearrangements f^* and f_* follow directly from their definition and consist in the identities:

$$\sup_{e\subset E,\ |e|=t} \int_e |f(x)|\, dx = \int_0^t f^*(u)\, du, \quad 0 < t \le |E|, \tag{1.4}$$

$$\inf_{e\subset E,\ |e|=t} \int_e |f(x)|\, dx = \int_0^t f_*(u)\, du, \quad 0 < t \le |E|. \tag{1.5}$$

Moreover, the supremum and the infimum in the left-hand sides of both equalities are attained. Indeed, let us denote $e' = \{x \in E : \ |f(x)| > f^*(t)\}$, $0 < t \le |E|$. Then $t' \equiv |e'| \le t$ and $\int_{e'} |f(x)|\, dx = \int_0^{t'} f^*(u)\, du$. If $t' < t$, then $f^*(\tau) = f^*(t)$ for all $\tau \in (t', t]$, and there exists the set $e'' \subset E$ with $|e''| = t - t'$ such that $|f(x)| = f^*(t)$ for all $x \in e''$. Let $e = e' \cup e''$. Then $\int_e |f(x)|\, dx = \int_0^t f^*(u)\, du$, so that the supremum in the left-hand side of (1.4) is attained. Analogously one can prove the attainability of the infimum in the left-hand side of (1.5). Often it is useful to consider the following functions

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)\, du, \quad f_{**}(t) = \frac{1}{t}\int_0^t f_*(u)\, du, \quad t > 0.$$

In what follows we will need the notion of one more *non-increasing equimeasurable rearrangement* f_d. Namely, above we have considered the non-increasing rearrangement f^*, equimeasurable with $|f|$. By analogy with (1.2), for the function f, measurable on $E \subset \mathbb{R}^d$, we define the function

$$f_d(t) = \sup_{e\subset E,\ |e|=t} \inf_{x\in e} f(x), \quad 0 < t \le |E|.$$

Similarly to f^*, the function f_d does not increase on $(0, |E|]$, it is continuous from the left, but unlikely f^*, f_d is equimeasurable with f and not with $|f|$. Clearly, if f is a non-negative function on E, then its non-increasing rearrangements f^* and f_d coincide.

1.2 Covering Lemmas

We will say that the *collection of sets* $E_\alpha \subset \mathbb{R}^d$ *is contractible to the point* x, if $x \in E_\alpha$ for all α and the *diameters* of these sets: $\mathrm{diam} E_\alpha \equiv \sup_{x,y\in E_\alpha} |x-y|$ tend to zero. Below we will consider the following sets E_α: an *open ball* of radius r centered at the point x_0: $B(x_0, r) \equiv \{x \in \mathbb{R}^d : \ |x - x_0| < r\}$; *a segment* (or *rectangle*) with the sides parallel to the coordinate axes:

$$R \equiv \left\{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \ a_i \le x_i \le b_i, \ i = 1, \dots, d\right\} \equiv \prod_{i=1}^{d} [a_i, b_i],$$

where $a_i \le b_i, \ i = 1, \dots, d$; *a cube* $Q \equiv \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \ a_i \le x_i \le b_i, \ i = 1, \dots, d\}$, where $0 \le b_i - a_i = Const, \ i = 1, \dots, d$. We will consider only the segments and cubes with the sides parallel to the coordinate axes. For the cube $Q \subset \mathbb{R}^d$ we will denote by $l(Q)$ *the side-length* $l(Q) = b_i - a_i$. The numbers $l_i \equiv l_i(R) = b_i - a_i, \ i = 1, \dots,$ are called *the side-lengths of the segment* R. Clearly, any segment in $\mathbb{R}^d$ is a Cartesian product of d one-dimensional segments and any cube is a Cartesian product of one-dimensional segments of equal length. The union of all inner points of a segment is called *the interval* (multidimensional) or *the interior* of the segment R and denoted by $\mathrm{int} R$.

The function f on $\mathbb{R}^d$ is called *locally summable* if it is summable on every cube $Q \subset \mathbb{R}^d$. The class of all locally summable functions is denoted by L_{loc}. For $1 \le p < \infty$ by L^p_{loc} is denoted the class of measurable on $\mathbb{R}^d$ functions f such that $|f|^p \in L_{loc}$. From the Hölder inequality it follows immediately that $L^p_{loc} \subset L^q_{loc}, \ 1 \le q < p < \infty$.

The value

$$\operatorname*{ess\,sup}_{x\in E} f(x) \equiv \sup\{\alpha \in \mathbb{R} : \ |\{x \in E : \ f(x) > \alpha\}| > 0\} \tag{1.6}$$

is called the *essential supremum* of the function f, measurable on the set E, $|E| > 0$. Analogously,

$$\operatorname*{ess\,inf}_{x\in E} f(x) \equiv \inf\{\beta \in \mathbb{R} : \ |\{x \in E : \ f(x) < \beta\}| > 0\} \tag{1.7}$$

is called the *essential infimum* of the function f. It is easy to show that [1]

$$\operatorname*{ess\,sup}_{x\in E} f(x) = \inf\{\alpha' \in \mathbb{R} : \ |\{x \in E : \ f(x) > \alpha'\}| = 0\}, \tag{1.8}$$

$$\operatorname*{ess\,inf}_{x\in E} f(x) = \sup\{\beta' \in \mathbb{R} : \ |\{x \in E : \ f(x) < \beta'\}| = 0\}. \tag{1.9}$$

The function f is said to be *essentially bounded* on E if

$$\|f\|_\infty \equiv \operatorname*{ess\,sup}_{x\in E} |f(x)| < \infty.$$

The class of all essentially bounded functions on E is denoted by $L^\infty \equiv L^\infty(E)$. The function f, which is essentially bounded from below (from above) on every cube $Q \subset \mathbb{R}^d$, is called *locally essentially bounded from below (from above).*

Recall, that the function $\chi_E(x)$, which is equal to 1 for all $x \in E$ and to zero for $x \notin E$, is called the *characteristic function* of the set $E \subset \mathbb{R}^d$.

In what follows we will need the following four theorems and a corollary from the theory of differentiation of integrals. We give these results without proofs.

Theorem 1.1 (Lebesgue, [70]). *Let $f \in L_{loc}(\mathbb{R}^d)$. Then for almost all $x \in \mathbb{R}^d$*

$$\lim_{\operatorname{diam} Q \to 0} \frac{1}{|Q|} \int_Q f(y)\, dy = f(x),$$

where the limit is taken over the cubes $Q \subset \mathbb{R}^d$, contractible to x.

This theorem can be generalized in the following way.

Theorem 1.2 (Lebesgue, [70]). *Let $f \in L_{loc}(\mathbb{R}^d)$. Then for almost all x*

$$\lim_{\operatorname{diam} R \to 0} \frac{1}{|R|} \int_R f(y)\, dy = f(x), \tag{1.10}$$

where the limit is taken over the segments $R \subset \mathbb{R}^d$, contractible to x, and such that the ratio of their side-lengths is bounded.

As it is well known (see [23]), without the boundedness assumption on the ratio of the side-lengths of the segments R it could happen that the limit in

[1] Let us prove (1.8). Let $A \equiv \operatorname{ess\,sup}_{x\in E} f(x)$. If $A = +\infty$, then the domain of definition of the infimum in the right-hand side of (1.8) is empty. In this case we say that the infimum is equal to $+\infty$, so that (1.8) holds true. Let now $A < +\infty$. Then for all $\alpha' > A$ we have $|\{x \in E : \ f(x) > \alpha'\}| = 0$, because otherwise we get a contradiction with (1.6). On the other hand, if $\alpha' < A$, then (1.6) implies that there exists α such that $\alpha' < \alpha \le A$ and $|\{x \in E : \ f(x) > \alpha\}| > 0$. Since $\{x \in E : \ f(x) > \alpha\} \subset \{x \in E : \ f(x) > \alpha'\}$ we see that $|\{x \in E : \ f(x) > \alpha'\}| > 0$, i.e., any $\alpha' < A$ is out of the domain of definition of the infimum in the right-hand side of (1.8). Together with the previous observation this implies (1.8). Analogously one can prove (1.9).

the left-hand side of (1.10) does not exist on the set of points x of a positive measure. Nevertheless one can prove the following result.

Theorem 1.3 (Jessen, Marcinkiewicz, Zygmund, [23, 29]). *Let f be measurable and such that*

$$\int_{R_0} |f(y)| \ln^{d-1} (1 + |f(y)|) \, dy < \infty$$

on the segment $R_0 \subset \mathbb{R}^d$. Then for almost all $x \in R_0$

$$\lim_{\operatorname{diam} R \to 0} \frac{1}{|R|} \int_R f(y) \, dy = f(x),$$

where the limit is taken over the segments $R \subset R_0$, contractible to x.

In particular, this implies

Corollary 1.4. *Let $f \in L^p_{loc}(\mathbb{R}^d)$ for some $p > 1$. Then for almost all $x \in \mathbb{R}^d$*

$$\lim_{\operatorname{diam} R \to 0} \frac{1}{|R|} \int_R f(y) \, dy = f(x),$$

where the limit is taken over the segments $R \subset \mathbb{R}^d$, contractible to x.

Theorems 1.2 and 1.3 are the particular cases of the following theorem.

Theorem 1.5 (Zygmund [23], [81]). *Let $\mathcal{B}$ be a family of segments R from $R_0 \subset \mathbb{R}^d$, such that for every $x \in R_0$ there exists a sequence of segments $R_j(x) \in \mathcal{B}$, contractible to x. Further, assume that for every segment $R \in \mathcal{B}$ the ratios of some d_1 $(1 \le d_1 \le d)$ of its d side-lengths are bounded, while the remaining $d - d_1$ of its side-lengths are arbitrary. Then for every function f, measurable on R_0, and such that*

$$\int_{R_0} |f(y)| \ln^{d-d_1} (1 + |f(y)|) \, dy < \infty$$

at almost every point $x \in R_0$,

$$\lim_{\operatorname{diam} R \to 0} \frac{1}{|R|} \int_R f(y) \, dy = f(x),$$

where the limit is taken over the segments $R \in \mathcal{B}$, contractible to x.

Now let us consider the notion of a *dyadic cube*. First, let Q_0 be a cube in $\mathbb{R}^d$. We will call it *a dyadic cube of order zero*. Partitioning each side of the cube Q_0 in halves, we get 2^d cubes, which will be called *dyadic cubes of the first order*. In the k-th step, partitioning in halves each side of the dyadic cube of order $(k-1)$, we get 2^{dk} cubes of order k. The collection of all cubes, obtained in this way, is called *the dyadic cubes with respect to the cube Q_0*. On the whole $\mathbb{R}^d$ one can define *the dyadic cubes of order k* $(k \in \mathbb{Z})$

$$Q^{(k)}_{m_1,\dots,m_d} \equiv \{x = (x_1, \dots, x_d) :\ 2^{-k} m_i \le x_i \le 2^{-k}(m_i + 1),\ i = 1, \dots, d\},$$

where each of the numbers m_i range over the set of all integers $\mathbb{Z}$. All such cubes $Q^{(k)}_{m_1,\dots,m_d},\ m_i \in \mathbb{Z},\ i = 1, \dots, d,\ k \in \mathbb{Z}$, are called *dyadic with respect to* $\mathbb{R}^d$.

Properties of Dyadic Cubes in $\mathbb{R}^d$.

Property 1.6. *For every $k \in \mathbb{Z}$*

$$\bigcup_{m_1\in\mathbb{Z}} \cdots \bigcup_{m_d\in\mathbb{Z}} Q^{(k)}_{m_1,\dots,m_d} = \mathbb{R}^d.$$

Property 1.7. *The intersection of the interiors of any two dyadic cubes of the same order is empty.*

Property 1.8. *Every dyadic cube of order k is a union of 2^d non-intersecting dyadic cubes of order $(k+1)$, which have a common vertex.*

Property 1.9. *Any two dyadic cubes are either non-intersecting or one is contained into another.*

Property 1.10. *The collection of all dyadic cubes is countable.*

Properties 1.7 – 1.10 hold true also for the cubes, which are dyadic with respect to some fixed cube Q_0, while instead of Property 1.6 in this case we have

Property 1.11. *For any $k = 0, 1, \dots,$ the union of all dyadic cubes of order k is equal to Q_0.*

Concerning Property 1.8 let us remark, that not every union of 2^d dyadic cubes, which have a common vertex, is a dyadic cube itself. For instance, the cubes $[-1, 0]$ and $[0, 1]$ are dyadic in $\mathbb{R}$, but their union $[-1, 1]$ is not a dyadic cube.

The dyadic cubes are used, in particular, to prove the so-called *covering lemmas*. Now we will consider some of these lemmas, which will be used in what follows.

Lemma 1.12 (Bennett, De Vore, Sharpley [1]). *Let G be an open set in $\mathbb{R}^d$, $|G| < \infty$. Then there exists at most countable collection of cubes Q_j with pairwise disjoint interiors such that*

$$|G \cap Q_j| \le \frac{1}{2}|Q_j| \le |Q_j \setminus G|, \tag{1.11}$$

$$G \subset \bigcup_j Q_j, \tag{1.12}$$

$$|G| \le \sum_j |Q_j| \le 2^{d+1}|G|. \tag{1.13}$$

Proof. Let us partition the whole $\mathbb{R}^d$ into dyadic cubes of the same order. We can choose the cubes to be so big that $|Q \cap G| \leq 2^{-d-1}|Q|$ for every cube Q. This is possible since $|G| < \infty$. Among all these cubes let us select only those, which satisfy $|G \cap Q| > 0$. Let Q be one of the selected cubes. We partition it into 2^d dyadic cubes and select the cubes $Q' \subset Q$ such that $|G \cap Q'| > 0$. For each of such cubes Q' we have two possibilities.

1. $|Q' \cap G| > 2^{-d-1}|Q'|$; in this case we assign to Q' the next number and take Q' as one of the cubes of the statement of the lemma.

2. $0 < |Q' \cap G| \leq 2^{-d-1}|Q'|$; such a cube Q' will be partitioned again in the next step when we will consider the cubes of higher order.

Having done this operation with all cubes of the current order, we pass to the cubes of the next order, obtained in the second case. To each of these cubes we apply again the selection procedure described above. If there is no such cubes, the process is finished.

As the result of this procedure we obtain a family of dyadic cubes Q_j, which satisfy the condition of the first case, described above. Clearly, the interiors of these cubes are pairwise disjoint and their collection is at most countable.

Let us prove (1.11), (1.12) and (1.13). Suppose Q' is one of the cubes Q_j obtained by partition of the cube $Q \supset Q'$. Since the cube Q was not numbered in the previous step we have

$$\frac{|G \cap Q'|}{|Q'|} \leq \frac{|G \cap Q|}{|Q|} \cdot \frac{|Q|}{|Q'|} \leq 2^{-d-1} \cdot 2^d = \frac{1}{2}$$

and this proves the left inequality of (1.11). From here we obtain

$$\frac{|Q' \setminus G|}{|Q'|} = \frac{|Q'| - |G \cap Q'|}{|Q'|} = 1 - \frac{|G \cap Q'|}{|Q'|} \geq 1 - \frac{1}{2} = \frac{1}{2},$$

which is equivalent to the right inequality of (1.11).

Let us prove (1.12). Since the set G is open for any $x \in G$ all dyadic cubes, which contain x and have a big enough order, are entirely contained in G. Let Q_x be the biggest one among such cubes. Assume Q_x is of order k. The dyadic cube Q_x was obtained by partitioning of some dyadic cube $Q^{(k-1)}$ of order $k-1$. In its own turn, the cube $Q^{(k-1)}$ is a result of the partitioning of some dyadic cube $Q^{(k-2)}$ of order $k-2$, and so on. Among the cubes $Q^{(k-m)}$, $m = 1, 2, \ldots$, we select the biggest cube $Q^{(k-m_0)}$ such that $|Q^{(k-m_0)} \cap G| > 2^{-d-1}|Q^{(k-m_0)}|$. Clearly such a cube exists and $|Q^{(k-m_0)} \cap G| \geq |Q_x \cap G| = |Q_x| > 0$. Therefore $Q^{(k-m_0)}$ was obtained as a result of the partitioning of the dyadic cube $Q^{(k-m_0-1)}$ and $0 < |Q^{(k-m_0-1)} \cap G| \leq 2^{-d-1}|Q^{(k-m_0-1)}|$. This means that $Q^{(k-m_0-1)}$ is one of the cubes, which satisfy the condition of the second case, and $Q^{(k-m_0)}$ is one of the cubes Q_j. Since the point $x \in G$ was arbitrary (1.12) follows.

Finally, since every cube Q_j satisfy the condition of the first case, by (1.12),

$$|G| \le \sum_j |Q_j| \le 2^{d+1} \sum_j |Q_j \cap G| \le 2^{d+1}|G|,$$

which yields (1.13). □

Remark. From the geometrical point of view the last lemma says that any open set G can be covered by a family of non-intersecting cubes (inequality (1.12)) such that the biggest part of each of these cubes is the complement of the set G (the right inequality of (1.11)), and the sum of their measures is comparable with the measure of the set G itself (inequality (1.13)).

Recall, that the set E_1 is said to be *open with respect to the set* E_2, if there exists an open set G such that $E_1 = G \cap E_2$.

The following lemma provides the local version of Lemma 1.12.

Lemma 1.13 (Bennett, De Vore, Sharpley [1]). *Let $G \subset Q_0$ be an open set with respect to the cube $Q_0 \subset \mathbb{R}^d$, $|G| \le \frac{1}{2}|Q_0|$. Then there exists at most countable collection of cubes $Q_j \subset Q_0$ with pairwise disjoint interiors such that*

$$|G \cap Q_j| \le \frac{1}{2}|Q_j| \le |Q_j \setminus G|, \tag{1.14}$$

$$G \subset \bigcup_j Q_j, \tag{1.15}$$

$$|G| \le \sum_j |Q_j| \le 2^{d+1}|G|. \tag{1.16}$$

Proof. If $2^{-d-1}|Q_0| < |G| \le \frac{1}{2}|Q_0|$, then we take Q_0 as the only cube Q_j. Otherwise, if $|G| \le 2^{-d-1}|Q_0|$, then it is enough to repeat the proof the previous lemma, taking the cubes which are dyadic with respect to Q_0. □

The next lemma has a lot of applications in the different fields of theory of functions ([6], [70], [23], [16]).

Lemma 1.14 (Calderón, Zygmund, [6]). *Let f be a summable function on the cube $Q_0 \subset \mathbb{R}^d$, and let $\alpha \ge \frac{1}{|Q_0|}\int_{Q_0} |f(x)|\,dx$. Then there exists at most countable collection of cubes $Q_j \subset Q_0$, $j = 1, 2, \ldots$ with pairwise disjoint interiors such that*

$$\alpha < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\,dx \le 2^d\alpha,\ j = 1, 2, \ldots, \tag{1.17}$$

$$|f(x)| \le \alpha \ \ \textit{for almost every } x \in Q_0 \setminus \left(\bigcup_{j\ge 1} Q_j\right). \tag{1.18}$$

Proof. Let us partition Q_0 into 2^d cubes of order one, dyadic with respect to Q_0. Let Q' be one of these cubes. If

$$\frac{1}{|Q'|}\int_{Q'} |f(x)|\, dx > \alpha,$$

then we assign to this cube the next number j and stop to partition it. Otherwise we partition the cube Q' in the next step. Having done this selection procedure for all cubes Q' of order one, we pass to the next step.

After the k-th step we obtain a finite collection of numbered dyadic cubes of order less of equal than k, and a finite collection of dyadic cubes of order k, subject to the further partitioning in the $(k+1)$-th step. In the $(k+1)$-th step we partition every cube, subject to partitioning, into 2^d cubes of order $k+1$ and repeat the selection procedure described above.

As the result of such a process we obtain the set of cubes Q_j, which are dyadic with respect to Q_0, and such that their interiors are pairwise disjoint. Let us fix some j. If the cube Q_j is of order k, then it was obtained as a result of the partitioning of some dyadic cube $\widetilde{Q}$ of order $k-1$. Since the cube Q_j was enumerated

$$\frac{1}{|Q_j|}\int_{Q_j} |f(x)|\, dx > \alpha.$$

On the other hand, as $\widetilde{Q}$ was subject to partitioning in the k-the step,

$$\frac{1}{|Q_j|}\int_{Q_j} |f(x)|\, dx \le \frac{|\widetilde{Q}|}{|Q_j|}\frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} |f(x)|\, dx \le 2^d\alpha.$$

Therefore the cubes Q_j satisfy (1.17).

Let us prove (1.18). If $x \notin \bigcup_{j\ge 1} Q_j$, then there exists a sequence of dyadic cubes $\widetilde{Q}_k$, which are contractible to x and subject to partitioning in the k-th step, i.e. such that

$$\frac{1}{|\widetilde{Q}_k|}\int_{\widetilde{Q}_k} |f(y)|\, dy \le \alpha,\ \ k = 1, 2, \ldots$$

According to Lebesgue theorem 1.1, for almost all such x we have

$$|f(x)| = \lim_{k\to\infty} \frac{1}{|\widetilde{Q}_k|}\int_{\widetilde{Q}_k} |f(y)|\, dy,$$

and (1.18) follows. □

Remark 1.15. Sometimes the method of proof, that was used in the previous lemma, is called *the stopping times technique* (see [16]).

In the theory of differentiation of functions a very important role is played by the so-called “rising sun lemma” of F. Riesz ([65, 66, 70, 33, 32]). We give one of the several known versions of this lemma.

Lemma 1.16 (Klemes, [32]). *Let f be a summable function on $I_0 \equiv [a_0, b_0] \subset \mathbb{R}$ and let $\alpha \geq \frac{1}{|I_0|}\int_{I_0} f(x)\,dx$. Then there exists at most countable collection of pairwise disjoint intervals $I_j \subset I_0, j = 1, 2, \ldots$ such that*

$$\frac{1}{|I_j|}\int_{I_j} f(x)\,dx = \alpha,\ j = 1, 2, \ldots, \tag{1.19}$$

$$f(x) \leq \alpha \quad \textit{for almost all } x \in I_0 \setminus \left(\bigcup_{j\geq 1} I_j\right). \tag{1.20}$$

Proof. It is enough to consider just the non-trivial case $\alpha > \frac{1}{|I_0|}\int_{I_0} f(x)\,dx$. The function $F(x) \equiv \int_{a_0}^{x} f(y)\,dy - \alpha(x - a_0)$ is continuous on $[a_0, b_0]$, $F(a_0) = 0$ and $F(b_0) < 0$. Set $b = \sup\{x \in [a_0, b_0) :\ F(x) = 0\}$. Then the continuity of F implies that $a_0 \leq b < b_0$ and $F(b) = 0$. Let us denote by E the set of all $x \in (b, b_0)$ for which there exists y, $x < y < b_0$, such that $F(y) > F(x)$. Let us show that E is an open set. Indeed, if $x_0 \in E$, then for some $y_0 \in (x_0, b)$ we have $F(y_0) > F(x_0)$. Then the continuity of F implies $F(y_0) > F(x)$ also for all x close enough to x_0. This means that the point x is an inner point of E, i.e. the set E is open.

Now let us use the well-known fact stating that any open set on the real line can be presented as a at most countable collection of pairwise disjoint intervals such that their endpoints are not contained in this open set (see [60]). Let us present the open set E as a collection of such intervals $I_j \equiv (a_j, b_j)$. If $b > a_0$, we add the interval $(a_1, b_1) \equiv (a_0, b)$ to the collection of intervals $\{I_j\}$.

Let us show that the obtained collection of intervals I_j satisfies (1.19) and (1.20). First let us prove (1.20). Let $x \in I_0 \setminus \left(\bigcup_{j\geq 1} I_j\right)$. Then, by the definition of the set E, for any $y > x$ we have $F(y) \leq F(x)$, or, equivalently,

$$\frac{1}{y-x}\int_x^y f(z)\,dz \leq \alpha.$$

Since Lebesgue theorem 1.1 implies

$$f(x) = \lim_{y \to x+0} \frac{1}{y-x}\int_x^y f(z)\,dz$$

for almost all x, then (1.20) is proved.

It remains to prove (1.19). Let $I_j = (a_j, b_j)$. Then the equality $\frac{1}{|I_j|}\int_{I_j} f(x)\,dx = \alpha$ is obviously equivalent to the following one

$$F(b_j) = F(a_j). \tag{1.21}$$

If $(a_j, b_j) = (a_0, b)$, then (1.21) follows from the equalities $F(a_0) = F(b) = 0$ noticed above. Let us consider the case $b \leq a_j < b_j \leq b_0$. If $F(b_j) > F(a_j)$ and

$a_j > b$, then $a_j \in E$, which contradicts to the assumption that the endpoints of the interval (a_j, b_j) do not belong to E. If $F(b_j) > F(a_j)$ and $a_j = b$, then $0 = F(b) < F(b_j)$, and the continuity of F together with the inequality $F(b_0) < 0$ implies that there exists b', $b < b' < b_0$, such that $F(b') = 0$. This contradicts to the choice of b. Therefore the inequality $F(b_j) > F(a_j)$ is impossible. Assume $F(b_j) < F(a_j)$. Set

$$x' = \sup \left\{ x \in [a_j, b_j] : \; F(x) = \frac{1}{2} \left(F(b_j) + F(a_j) \right) \right\}.$$

Then it is clear that $x' \in (a_j, b_j) \subset E$, and hence, by virtue of the definition of E, one can find some $y > x'$ such that $F(x') < F(y)$. It is easy to see that by the definition of x' $y > b_j$. Therefore $F(b_j) < \frac{1}{2} \left(F(b_j) + F(a_j) \right) = F(x') < F(y)$, i.e., $b_j \in E$. Again we reach the contradiction to the assumption that the endpoint of the interval (a_j, b_j) does not belong to E. So, the inequality $F(b_j) < F(a_j)$ is also impossible. The only possibility which remains is given by (1.21). □

Remark 1.17. It is easy to see that the collection of intervals $\{I_j\}$, constructed in Lemma 1.16, is not unique. Indeed, let $f(x) = \chi_{[\frac{1}{3}, \frac{2}{3}]}(x)$, $x \in I_0 \equiv (0, 1)$ and $\alpha = \frac{1}{2} > \frac{1}{|I_0|} \int_{I_0} f(x)\, dx = \frac{1}{3}$. Then as the collection $\{I_j\}$ one can take any of the intervals $\left(a, a + \frac{2}{3}\right)$ with $0 \le a \le \frac{1}{3}$.

We took this proof of the "rising sun lemma" from the work of I. Klemes [32]. It is close to the original proof by F. Riesz. Now we give one more proof, based on the "stopping times technique", or, to be more precise, on its modification. This proof is particularly useful because in a certain sense it can be generalized for the multidimensional case. We will discuss it later.

Another proof of "rising sun lemma" 1.16. Let $\alpha > \frac{1}{|I_0|} \int_{I_0} f(x)\, dx$. We partition I_0 in halves and obtain two intervals $I^{(1)}$ and $I^{(2)}$. If $\frac{1}{|I^{(k)}|} \int_{I^{(k)}} f(x)\, dx < \alpha$, $k = 1, 2$, then we pass to the next step and partition both intervals $I^{(k)}$ again. Otherwise, if the mean value of f on one of these intervals (for definiteness let us take $I^{(1)} \equiv (a_0, b)$ with $b = \frac{1}{2}(a_0 + b_0)$) is greater or equal than α, then consider the function

$$F(x) = \frac{1}{x - a_0} \int_{a_0}^{x} f(y)\, dy, \; b \le x \le b_0.$$

It is easy to see that F is continuous, $F(b) \ge \alpha$ and $F(b_0) = \frac{1}{|I_0|} \int_{I_0} f(y)\, dy < \alpha$. By the Mean Value theorem there exists $b_1 \in [b, b_0)$ such that $F(b_1) = \alpha$. We assign to the interval (a_0, b_1) the next number j_0 and attribute it to the collection $\{I_j\}$ without further partitioning. Clearly, $\frac{1}{|I_{j_0}|} \int_{I_{j_0}} f(y)\, dy = F(b_1) = \alpha$, and since $\frac{1}{|I_0|} \int_{I_0} f(y)\, dy < \alpha$ for the interval $I' \equiv (a_0, b_0) \setminus (a_0, b_1]$ we have $\frac{1}{|I'|} \int_{I'} f(y)\, dy < \alpha$. We will partition the interval I' in the next step.

In each step we apply the analogous procedure to every interval, subject to further partitioning.

As the result of this process we obtain at most countable collection of pairwise disjoint intervals I_j, which satisfy (1.19). Let us prove (1.20). Let $x \in I_0 \setminus \left(\bigcup_{j\geq 1} I_j\right)$. Then there exists a sequence of intervals I'_k, obtained by partitioning in each step and containing x. Notice, that every time we were passing from one step to another, the length of each interval, subject to partitioning, was divided at least by two. Hence the intervals I'_k are contractible to x and $\frac{1}{|I'_k|}\int_{I'_k} f(y)\,dy < \alpha$. But according to Lebesgue theorem 1.1 for almost all x

$$f(x) = \lim_{k\to\infty} \frac{1}{|I'_k|}\int_{I'_k} f(y)\,dy.$$

This implies (1.20). □

Making obvious changes in the given proof of Lemma 1.16 one can obtain the following version of the "rising sun lemma".

Lemma 1.18 ([43]). *Let f be summable on $I_0 \equiv [a_0, b_0] \subset \mathbb{R}$ and let $\alpha \leq \frac{1}{|I_0|}\int_{I_0} f(x)\,dx$. Then there exists at most countable set of intervals $I_j \subset I_0,\ j = 1, 2, \ldots$ with pairwise disjoint interiors such that*

$$\frac{1}{|I_j|}\int_{I_j} f(x)\,dx = \alpha,\ \ j = 1, 2, \ldots,$$

$$f(x) \geq \alpha \text{ for almost all } x \in I_0 \setminus \left(\bigcup_{j\geq 1} I_j\right).$$

Remark 1.19. For a non-negative functions "rising sun lemma" 1.16 is more precise, than Calderón–Zygmund Lemma 1.14 in the one-dimensional case. This is why sometimes in the case $d = 1$ the "rising sum lemma" provides sharper results. We will see it below when we will consider the rearrangements of BMO-functions and also in the proof of the John–Nirenberg inequality.

In the context of the last remark the following question is natural: *Does there exist a choice of cubes $Q_j \subset Q_0$ in the Calderón–Zygmund lemma such that instead of* (1.17) *one would have*

$$\frac{1}{|Q_j|}\int_{Q_j} |f(x)|\,dx = \alpha\ ?$$

In the one-dimensional case the positive answer is provided by "rising sun lemma" 1.16. The next example shows that this is no more true in the case of higher dimension.

Example 1.20 ([45]). Let $f(x) = \chi_{[\frac{1}{2},1]\times[\frac{1}{2},1]}(x)$, where $x \equiv (x_1, x_2) \in Q_0 \equiv [0,1]^2$. Then $\frac{1}{|Q_0|}\int_{Q_0} f(x)\,dx = \frac{3}{4}$. Take $\alpha = \frac{7}{8} > \frac{3}{4}$. In order to have all points from the neighborhood of the origin to be covered by some cube $Q_1 \subset Q_0$

such that $\frac{1}{|Q_1|}\int_{Q_1} f(x)\,dx = \alpha$, the cube Q_1 necessarily should have the form $Q_1 = [0,\gamma]^2$, where $\gamma,\ \frac{1}{2} < \gamma < 1$, is to be defined from the equality

$$\frac{7}{8} = \alpha = \frac{1}{|Q_1|}\int_{Q_1} f(x)\,dx = \frac{1}{\gamma^2}\left(\gamma - \frac{1}{4}\right),$$

i.e., $\gamma = \frac{4+\sqrt{2}}{7}$. But then it is clear that for a small enough ε the points $x = (x_1, x_2) \in Q_0$ with $\gamma < x_1 < 1$ and $0 < x_2 < \varepsilon$ cannot be covered by any cube $Q_2 \subset Q_0$ in such a way that $\frac{1}{|Q_2|}\int_{Q_2} f(y)\,dy = \alpha$ and the interiors of Q_2 and Q_1 are non-intersecting. □

So, for $d \geq 2$ the analog of "rising sun lemma" 1.16 fails if we replace the one-dimensional intervals by the multidimensional cubes. The things change if instead of cubes one considers segments (multidimensional rectangles). For a function f, summable on the set $E \subset \mathbb{R}^d$ $(0 < |E| < \infty)$, we will denote by

$$f_E = \frac{1}{|E|}\int_E f(x)\,dx$$

the mean value of f on E.

Lemma 1.21 ([46, 45]). *Let $R_0 \subset \mathbb{R}^d$ be a segment. Assume $f \in L(R_0)$ and $\alpha \geq f_{R_0}$. Then there exists a family of segments $R_j \subset R_0,\ j = 1, 2, \dots$ with pairwise disjoint interiors such that $f_{R_j} = \alpha,\ j = 1, 2, \dots$. Moreover, for almost every point $x \in R_0 \setminus \left(\bigcup_{j\geq 1} R_j\right)$ there exists a sequence of contractible to x segments $J_s(x) \subset R_0,\ s = 1, 2, \dots,$ embedded one into another and such that $f_{J_s(x)} < \alpha$.*

Proof. If $f_{R_0} = \alpha$, then the statement of the lemma is trivial. Assume $f_{R_0} < \alpha$. In the first step we denote $J^{(1,1)} = R_0$. After the s-th step we obtain a collection of segments $J^{(s,k)},\ k = 1, \dots, r_s$ with $r_s \leq 2^{s-1}$ such that they satisfy the following properties

1. $f_{J^{(s,k)}} < \alpha,\ k = 1, \dots, r_s$;
2. each $J^{(s,k')}$ is contained in some $J^{(s-1,k'')}$;
3. $\left(\text{int } J^{(s,k')}\right) \bigcap \left(\text{int } J^{(s,k'')}\right) = \emptyset$ for $k' \neq k''$.

Let $J^{(s,k)} \equiv \prod_{i=1}^{d}\left[a_i^{(s,k)}, b_i^{(s,k)}\right]$ be one of such segments and denote by $l_i^{(s,k)}$ the lengths of its sides $b_i^{(s,k)} - a_i^{(s,k)}$. Choose $1 \leq i_0 \leq d$ such that $l_{i_0}^{(s,k)} = \max_{1\leq i\leq d} l_i^{(s,k)}$ and set $c = \frac{1}{2}\left(a_{i_0}^{(s,k)} + b_{i_0}^{(s,k)}\right)$, i.e. put the point c in the middle of the biggest side of the segment $J^{(s,k)}$. As the result we obtain two segments

$$R' = \left[a_1^{(s,k)}, b_1^{(s,k)}\right] \times \cdots \times \left[a_{i_0-1}^{(s,k)}, b_{i_0-1}^{(s,k)}\right] \times$$

$$\times \left[a_{i_0}^{(s,k)}, c\right] \times \left[a_{i_0+1}^{(s,k)}, b_{i_0+1}^{(s,k)}\right] \times \cdots \times \left[a_d^{(s,k)}, b_d^{(s,k)}\right]$$

and

$$R'' = \left[a_1^{(s,k)}, b_1^{(s,k)}\right] \times \cdots \times \left[a_{i_0-1}^{(s,k)}, b_{i_0-1}^{(s,k)}\right] \times$$

$$\times \left[c, b_{i_0}^{(s,k)}\right] \times \left[a_{i_0+1}^{(s,k)}, b_{i_0+1}^{(s,k)}\right] \times \cdots \times \left[a_d^{(s,k)}, b_d^{(s,k)}\right].$$

It is clear that $R' \bigcup R'' = J^{(s,k)}$ and $(\text{int } R') \bigcap (\text{int } R'') = \emptyset$. There are two mutually excluding possibilities:

1. $f_{R'} < \alpha$ and $f_{R''} < \alpha$; in this case we say that the segments $J^{(s+1,k')} \equiv R'$ and $J^{(s+1,k'')} \equiv R''$ are obtained by diminution of the i_0-the side of the segment $J^{(s,k)}$, k' and k'' being some numbers. We assign $J^{(s+1,k')}$ and $J^{(s+1,k'')}$ to the segments of the $(s+1)$-th level. This ends the partition of the segment $J^{(s,k)}$.

2. The mean value of the function f over one of the segments R' and R'' is greater of equal than α. Notice that in this case the mean value of f over another segment is smaller than α. It follows from the inequality $f_{J^{(s,k)}} < \alpha$. If $f_{R'} = \alpha$, then we set $R_j \equiv R'$ (here j is the next number), $J^{(s+1,k')} \equiv R''$, k' being some number, and at this point we stop to partition the segment $J^{(s,k)}$. Otherwise, if $f_{R'} > \alpha$, then consider the function

$$F(t) = \left[\left(t - a_{i_0}^{(s,k)}\right) \prod_{i \neq i_0} \left(b_i^{(s,k)} - a_i^{(s,k)}\right)\right]^{-1} \times$$

$$\times \int_{a_1^{(s,k)}}^{b_1^{(s,k)}} dx_1 \ldots \int_{a_{i_0-1}^{(s,k)}}^{b_{i_0-1}^{(s,k)}} dx_{i_0-1} \int_{a_{i_0}^{(s,k)}}^{t} dx_{i_0} \int_{a_{i_0+1}^{(s,k)}}^{b_{i_0+1}^{(s,k)}} dx_{i_0+1} \ldots \int_{a_d^{(s,k)}}^{b_d^{(s,k)}} f(x)\, dx_d.$$

This function is continuous on $\left[c, b_{i_0}^{(s,k)}\right]$, $F(c) = f_{R'} > \alpha$ and $F\left(b_{i_0}^{(s,k)}\right) = f_{J^{(s,k)}} < \alpha$. Thus, due to the continuity of F, there exists some δ, $0 < \delta < \frac{1}{2} l_{i_0}^{(s,k)}$ such that $F\left(b_{i_0}^{(s,k)} - \delta\right) = \alpha$. Take

$$\delta' = \sup\left\{\delta :\ F\left(b_{i_0}^{(s,k)} - \delta\right) = \alpha\right\}, \quad c' = b_{i_0}^{(s,k)} - \delta',$$

$$R_j = \left[a_1^{(s,k)}, b_1^{(s,k)}\right] \times \cdots \times \left[a_{i_0-1}^{(s,k)}, b_{i_0-1}^{(s,k)}\right] \times$$

$$\times \left[a_{i_0}^{(s,k)}, c'\right] \times \left[a_{i_0+1}^{(s,k)}, b_{i_0+1}^{(s,k)}\right] \times \cdots \times \left[a_d^{(s,k)}, b_d^{(s,k)}\right],$$

with j being the next number, and

$$J^{(s+1,k')} = \left[a_1^{(s,k)}, b_1^{(s,k)}\right] \times \cdots \times \left[a_{i_0-1}^{(s,k)}, b_{i_0-1}^{(s,k)}\right] \times$$

$$\times \left[c', b_{i_0}^{(s,k)}\right] \times \left[a_{i_0+1}^{(s,k)}, b_{i_0+1}^{(s,k)}\right] \times \cdots \times \left[a_d^{(s,k)}, b_d^{(s,k)}\right],$$

where k' is some number. Clearly, $f_{R_j} = F(c') = \alpha$, where

$$\frac{1}{2}\left(a_{i_0}^{(s,k)} + b_{i_0}^{(s,k)}\right) < c' < b_{i_0}^{(s,k)} \quad \text{and} \quad f_{J^{(s+1,k')}} < \alpha.$$

We say that the segment $J^{(s+1,k')} \subset J^{(s,k)}$ is obtained by diminution of the i_0-th side of the segment $J^{(s,k)}$.

The case $f_{R''} \geq \alpha$ can be settled analogously. This completes the analysis of the second case.

Having done the described procedure for all $k = 1, \dots, r_s$, we pass to the next $(s+1)$-th step. As the result we obtain at most countable collection of segments R_j such that $f_{R_j} = \alpha,\ j = 1, 2, \dots$, and by construction the interiors of the segments R_j are pairwise disjoint.

Let $x \in R_0 \setminus \left(\bigcup_{j\geq 1} R_j\right)$ and assume that x does not belong to any side of any segment $J^{(s,k)}$. Then for every s there exists a segment $J_s(x) \equiv J^{(s,k_s)}$, which contains x. Since $f_{J^{(s,k)}} < \alpha$ for all s and k we have $f_{J_s(x)} < \alpha$. Moreover, since passing from $J^{(s,k_s)}$ to $J^{(s+1,k_{s+1})} \subset J^{(s,k_s)}$ we decrease the biggest side of the segment $J^{(s,k_s)}$ at least by half we have the following estimate for the ratio of the diameters:

$$\left(\frac{\operatorname{diam} J^{(s+1,k_{s+1})}}{\operatorname{diam} J^{(s,k_s)}}\right)^2 = \frac{\sum_{i=1}^d \left[b_i^{(s+1,k_{s+1})} - a_i^{(s+1,k_{s+1})}\right]^2}{\sum_{i=1}^d \left[b_i^{(s,k_s)} - a_i^{(s,k_s)}\right]^2} \leq$$

$$\leq \frac{\sum_{i=1}^d \left[b_i^{(s,k_s)} - a_i^{(s,k_s)}\right]^2 - \frac{3}{4}\max_{1\leq i\leq d}\left[b_i^{(s,k_s)} - a_i^{(s,k_s)}\right]^2}{\sum_{i=1}^d \left[b_i^{(s,k_s)} - a_i^{(s,k_s)}\right]^2} \leq 1 - \frac{3}{4d}.$$

This means that $\operatorname{diam} J_s(x) \to 0$ as $s \to \infty$. Therefore the segments $J_s(x)$ are contained one into another and they are contractible to x. $\square$

In the case $d = 2$ Lemma 1.21 can be revised. More precisely, the following result is valid.

Lemma 1.22 ([45]). *Let $R_0 \subset \mathbb{R}^2$ be a segment. Assume $f \in L(R_0)$ and let $\alpha \geq f_{R_0}$. Then there exists a family of segments $R_j \subset R_0$ with pairwise disjoint interiors such that $f_{R_j} = \alpha,\ j = 1, 2, \dots$, and $f(x) \leq \alpha$ for almost every point $x \in R_0 \setminus \left(\bigcup_{j\geq 1} R_j\right)$.*

Proof. We will use the same notations as in the proof of Lemma 1.21. According to Lebesgue theorem 1.2, at almost every $x \in R_0 \setminus \left(\bigcup_{j\geq 1} R_j\right)$ it is enough to construct a sequence of segments $Q_m \equiv Q_m(x),\ m = 1, 2, \dots$, contractible to x, such that the ratio of the side-lengths of $Q_m(x)$ is bounded and $f_{Q_m(x)} \leq \alpha$.

Let $d = 2$, $x \in R_0 \setminus \left(\bigcup_{j\geq 1} R_j\right)$ and assume that x does not belong to the side of any of the segments $J^{(s,k)}$. Here R_j and $J^{(s,k)}$ are the segments, constructed in the proof of Lemma 1.21. Denote $J_s \equiv J_s(x)$ and let $l_1^{(s)}$ and $l_2^{(s)}$ be the side-lengths of the segment J_s. Then $l_1^{(s)} > 0$, $l_2^{(s)} > 0$, $s = 1, 2, \ldots,$ and $\lim_{s\to\infty} l_1^{(s)} = \lim_{s\to\infty} l_2^{(s)} = 0$. For every s consider two pairs of numbers $\left(l_1^{(s)}, l_2^{(s)}\right)$ and $\left(l_1^{(s+1)}, l_2^{(s+1)}\right)$. The smallest element of the first pair is equal to the corresponding element of the second pair, while the biggest element of the first pair is at least twice bigger than the corresponding element of the second pair. Hence there exists a sequence of numbers s_m, $m = 1, 2, \ldots$ such that

$$l_2^{(s_m+1)} < l_1^{(s_m+1)} = l_1^{(s_m)} \leq l_2^{(s_m)}.$$

We have the following three cases.

1. If $l_2^{(s_m+1)} \geq \frac{1}{2} l_1^{(s_m+1)}$, we set $Q_m = J_{s_m+1}$. Then the inequality

$$\frac{1}{2} \leq \frac{l_2^{(s_m+1)}}{l_1^{(s_m+1)}} \leq 1$$

implies that the ratio of the side-lengths of the segment Q_m does not exceed 2. Moreover, $f_{Q_m} = f_{J_{s_m+1}} < \alpha$.

2. If $l_2^{(s_m)} \leq 2 l_1^{(s_m)}$, we set $Q_m = J_{s_m}$. Then $f_{Q_m} < \alpha$, and from

$$1 \leq \frac{l_2^{(s_m)}}{l_1^{(s_m)}} \leq 2$$

we again obtain that the ratio of the side-lengths of the segment Q_m does not exceed 2.

3. It remains to consider the case

$$l_2^{(s_m+1)} < \frac{1}{2} l_1^{(s_m+1)}, \quad l_2^{(s_m)} > 2 l_1^{(s_m)}. \tag{1.22}$$

In this case the segment $J^{(s_m+1,\cdot)} \equiv J_{s_m+1}$ was obtained from $J^{(s_m,\cdot)} \equiv J_{s_m}$ by diminution of the second side of the segments $J^{(s_m,\cdot)}$ by more, than a half, i.e., we have the situation of the second case of the proof of Lemma 1.21.[2] Notice, that in this case both equalities $f_{R'} = \alpha$ and $f_{R''} = \alpha$ are impossible, because, by construction of the segments $J^{(s,k)}$, they contradict to condition (1.22).

For example, let $f_{R'} > \alpha$. Then the function F, constructed in the proof of Lemma 1.21, has the following properties on $\left[c, b_2^{(s_m,\cdot)}\right]$

$$F(c) > \alpha, \quad F\left(b_2^{(s_m,\cdot)}\right) < \alpha, \quad F\left(b_2^{(s_m,\cdot)} - l_2^{(s_m+1)}\right) = \alpha,$$

[2] In the notation $J^{(s,\cdot)}$ the point stays for a number.

and from the definition $\delta' = l_2^{(s_m+1)}$ it follows that for any l, $l_2^{(s_m+1)} < l \leq \frac{1}{2} l_2^{(s_m)}$

$$F\left(b_2^{(s_m,\cdot)} - l\right) > \alpha.$$

According to (1.22),

$$l_2^{(s_m+1)} < l_1^{(s_m)} \leq \frac{1}{2} l_2^{(s_m)},$$

so that

$$F\left(b_2^{(s_m,\cdot)} - l_1^{(s_m)}\right) > \alpha. \tag{1.23}$$

Set

$$Q_m = \left[a_1^{(s_m,\cdot)}, b_1^{(s_m,\cdot)}\right] \times \left[b_2^{(s_m,\cdot)} - l_1^{(s_m)}, b_2^{(s_m,\cdot)}\right].$$

Then Q_m is a square, $J_{s_m+1} \subset Q_m \subset J_{s_m}$, and from the inequality $f_{J_{s_m}} < \alpha$ and (1.23) it follows that $f_{Q_m} < \alpha$.

Thus the constructed sequence $Q_m,\ m = 1, 2, \ldots,$ has the required properties. □

Remark 1.23. If the dimension of the space $d = 1$, then, due to Lebesgue theorem 1.1, in Lemma 1.21 we obtain that $f(x) \leq \alpha$ almost everywhere on $R_0 \setminus (\cup_j R_j)$.

Remark 1.24. If in addition to the conditions of Lemma 1.21 we assume that $f \in L^p(R_0)$ for some $p > 1$, then from Corollary 1.4 for $d \geq 2$ it follows that $f(x) \leq \alpha$ almost everywhere on $R_0 \setminus (\cup_j R_j)$.

Lemma 1.22 is a particular case of the following statement.

Lemma 1.25. *Let $d \geq 2$. Let f be a measurable function on the segment $R_0 \subset \mathbb{R}^d$ such that*

$$\int_{R_0} |f(y)| \ln^{d-2}(1 + |f(y)|)\ dy < \infty.$$

Assume $\alpha \geq f_{R_0}$. Then there exists a family of segments $R_j \subset R_0,\ j = 1, 2, \ldots$ with pairwise disjoint interiors such that $f_{R_j} = \alpha,\ j = 1, 2, \ldots$ and $f(x) \leq \alpha$ at almost every point $x \in R_0 \setminus (\cup_j R_j)$.

The proof of this lemma is based on the application of Theorem 1.5, we will discuss it later. First let us introduce the notion of *the contraction of a segment* and prove one auxiliary lemma.

We say that the segment $R' \equiv [a', b'] \equiv \prod_{i=1}^d [a_i', b_i'] \subset \mathbb{R}^d$ is obtained by contraction of the i_0-th side of the non-degenerate segment $R \equiv [a, b] \equiv \prod_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$ with some parameter λ, $0 < \lambda \leq 1$, if $[a_i', b_i'] = [a_i, b_i]$ for $i \neq i_0$, and $\left[a_{i_0}', b_{i_0}'\right]$ is either $[a_{i_0}, a_{i_0} + \lambda(b_{i_0} - a_{i_0})]$ or $[b_{i_0} - \lambda(b_{i_0} - a_{i_0}), b_{i_0}]$.

Lemma 1.26. *Let $d \geq 2$ and let $J_s \subset \mathbb{R}^d,\ s = 1, 2, \ldots,$ be the sequence of segments such that each J_{s+1} is obtained from the biggest side of the segment J_s by contraction with λ_s, $0 < \lambda_s \leq \frac{1}{2}$. Then there exist a natural number*

$i_0,\ 2 \le i_0 \le d$, and a strictly increasing sequence of natural numbers $s_m,\ m = 1, 2, \dots$ such that for any s_m there exists a segment $Q_m \equiv \prod_{i=1}^d \left[\overline{a}_i^{(m)}, \overline{b}_i^{(m)}\right]$, obtained from the biggest side of the segment J_{s_m} by contraction with λ_m', so that

$$J_{s_m+1} \subset Q_m \subset J_{s_m}, \tag{1.24}$$

$$\lambda_{s_m} \le \lambda_m' \le \frac{1}{2}, \tag{1.25}$$

$$1 \le \frac{\overline{b}_{i_0}^{(m)} - \overline{a}_{i_0}^{(m)}}{\overline{b}_1^{(m)} - \overline{a}_1^{(m)}} \le 2. \tag{1.26}$$

Proof. Since J_{s+1} was obtained by contraction of the biggest side of the segment $J_s \equiv \prod_{i=1}^d \left[a_i^{(s)}, b_i^{(s)}\right]$ with $0 < \lambda \le \frac{1}{2}$, similarly to the proof of Lemma 1.21, we conclude that the diameters of the segments J_s tend to zero as $s \to \infty$. Taking into account that all J_s are non-degenerate, we obtain that for the infinite set $N_1 \subset \mathbb{N}$ of numbers s the side $\left[a_1^{(s)}, b_1^{(s)}\right]$ is the biggest side of the segment J_s. For $s \in N_1$ let us denote by $i_s,\ 2 \le i_s \le d$, the number such that

$$b_{i_s}^{(s)} - a_{i_s}^{(s)} = \max_{2 \le i \le d} \left(b_i^{(s)} - a_i^{(s)}\right).$$

Since the set of all i_s is finite there exists $i_0,\ 2 \le i_0 \le d$, and an infinite set $N_2 \subset N_1$ such that $i_s = i_0$ for all $s \in N_2$. Clearly, the set $\mathbb{N} \setminus N_2$ is also infinite. Hence there exists an infinite set $N_3 \subset N_2$ of numbers $s \in N_2$ such that $s + 1 \notin N_2$.

Let $s_m,\ m = 1, 2, \dots$ be the elements of the set N_3, arranged in the ascending order. For any natural number m let us denote by Q_m the segment, obtained by contraction of the biggest (i.e. the first) side of the segment J_{s_m} with

$$\lambda_m' \equiv \min\left(\frac{b_{i_0}^{(s_m)} - a_{i_0}^{(s_m)}}{b_1^{(s_m)} - a_1^{(s_m)}}, \frac{1}{2}\right),$$

so that $Q_m \supset J_{s_m+1}$. Let us show that such a segment does exist and satisfies (1.24), (1.25) and (1.26).

Since $s_m \in N_2$ and $s_m + 1 \notin N_2$

$$b_1^{(s_m)} - a_1^{(s_m)} \ge b_{i_0}^{(s_m)} - a_{i_0}^{(s_m)} \ge \max_{i \ne 1, i_0} \left(b_i^{(s_m)} - a_i^{(s_m)}\right),$$

$$b_{i_0}^{(s_m)} - a_{i_0}^{(s_m)} = b_{i_0}^{(s_m+1)} - a_{i_0}^{(s_m+1)} \ge \max_{i \ne i_0} \left(b_i^{(s_m+1)} - a_i^{(s_m+1)}\right).$$

From here, taking into account that $\lambda_{s_m} \le \frac{1}{2}$, we get

$$\lambda_{s_m} = \frac{b_1^{(s_m+1)} - a_1^{(s_m+1)}}{b_1^{(s_m)} - a_1^{(s_m)}} = \min\left(\frac{b_1^{(s_m+1)} - a_1^{(s_m+1)}}{b_1^{(s_m)} - a_1^{(s_m)}}, \frac{1}{2}\right) \le$$

$$\le \min\left(\frac{b_{i_0}^{(s_m)} - a_{i_0}^{(s_m)}}{b_1^{(s_m)} - a_1^{(s_m)}}, \frac{1}{2}\right) = \lambda'_m \le \frac{1}{2},$$

so that (1.24) and (1.25) are satisfied. Further, since

$$\left[\overline{a}_{i_0}^{(m)}, \overline{b}_{i_0}^{(m)}\right] = \left[a_{i_0}^{(s_m)}, b_{i_0}^{(s_m)}\right], \quad \overline{b}_1^{(m)} - \overline{a}_1^{(m)} = \lambda'_m \left(b_1^{(s_m)} - a_1^{(s_m)}\right)$$

we have

$$\frac{\overline{b}_{i_0}^{(m)} - \overline{a}_{i_0}^{(m)}}{\overline{b}_1^{(m)} - \overline{a}_1^{(m)}} = \frac{b_{i_0}^{(s_m)} - a_{i_0}^{(s_m)}}{\lambda'_m \left(b_1^{(s_m)} - a_1^{(s_m)}\right)} =$$

$$= \frac{b_{i_0}^{(s_m)} - a_{i_0}^{(s_m)}}{b_1^{(s_m)} - a_1^{(s_m)}} \cdot \left(\min\left(\frac{b_{i_0}^{(s_m)} - a_{i_0}^{(s_m)}}{b_1^{(s_m)} - a_1^{(s_m)}}, \frac{1}{2}\right)\right)^{-1},$$

which implies (1.26). □

Proof of Lemma 1.25. In the proof of Lemma 1.21 for almost every point $x \in E \equiv R_0 \setminus \left(\bigcup_{j\ge 1} R_j\right)$ we constructed the family of segments $J_s(x)$, contractible to x, such that each of $J_{s+1}(x)$ was obtained by contraction of the biggest side of the segment $J_s(x)$ with λ_s, $0 < \lambda_s \le \frac{1}{2}$. Hence, according to Lemma 1.26, up to a set of zero measure the whole set E can be presented as a finite collection of subsets $E_2, \dots, E_d \subset E$ such that for every $x \in E_k$ there exists a sequence of segments $Q_m(x)$, contractible to x, which satisfies (1.24), (1.25), (1.26), and such that $i_0 = k$. Moreover, taking into account the definition of the number δ', given in the proof of Lemma 1.21, from (1.24) and (1.25) we obtain that $f_{Q_m(x)} < \alpha$.

Fix some k, $2 \le k \le d$. Let us construct a collection $\mathcal{B}_k$ of segments from R_0, according to the following rule. If $x \in E_k$, then we assign to $\mathcal{B}_k$ all segments $Q_m(x)$, obtained above by application of Lemma 1.26. Otherwise, if $x \in R_0 \setminus E_k$, then we assign to $\mathcal{B}_k$ all possible cubes contained in R_0 and containing the point x. Due to (1.26), we can apply Zygmund theorem 1.5 for $d_1 = 2$ to the collection of segments $\mathcal{B}_k$. Then

$$\lim_{\mathrm{diam} Q \to 0} \frac{1}{|Q|} \int_Q f(y)\, dy = f(x)$$

for almost all $x \in R_0$. In particular, this equality holds true for almost all $x \in E_k$. On the other hand, for $x \in E_k$ we have $f_{Q(x)} < \alpha$. Thus $f(x) \le \alpha$ for almost all $x \in E_k$.

Since $k \in \{2, \dots, d\}$ was arbitrary the proof of lemma is complete. □

We conclude this section with the most complete version of the Riesz lemma for the multidimensional case. First we introduce some definitions and prove a few auxiliary results.

Let $R_0 \subset \mathbb{R}^d$. We say that the family $\mathcal{B} \equiv \{J\}$ of the segments $J \subset R_0$ has *the "dyadic" property*, , if for any two segments from $\mathcal{B}$ either the interior of their intersection is empty, or one of the segments is entirely contained into another one. The next lemma is an analog of the well-known Vitali-type lemma [70] for a family of segments with the "dyadic" property.

Lemma 1.27. *Let $R_0 \subset \mathbb{R}^d$ be a segment, and let $\mathcal{B}$ be a countable family of the segments $J \subset R_0$ with the "dyadic" property. Then there exists a subfamily $\mathcal{B}' \subset \mathcal{B}$ of segments with pairwise disjoint interiors such that*

$$\bigcup_{J\in\mathcal{B}'} J = \bigcup_{J\in\mathcal{B}} J.$$

Proof. Assume that the segments in $\mathcal{B}$ are ordered in such a way that their diameters do not increase. We throw out from $\mathcal{B}$ all segments $J_k,\ k \geq 2$ such that the interior of their intersection with J_1 is non-empty and denote the remaining family by $\mathcal{B}_1$. If $\mathcal{B}_1$ contains not more than two segments, then we terminate the process and set $\mathcal{B}' = \mathcal{B}_1$. Otherwise we pass to the next step. After the j-th step we obtain the family $\mathcal{B}_j$ of segments such that the first j of them (or $j+1$, depending on whether their total number is greater than j) have pairwise disjoint interiors. If $\mathcal{B}_j$ contains not more than $(j+1)$ segments, then we set $\mathcal{B}' = \mathcal{B}_j$ and stop the process. Otherwise we throw out of $\mathcal{B}_j$ all segments $J_k,\ k \geq j+1$, such that the interiors of their intersection with J_{j+1} are non-empty, and denote the remaining part by $\mathcal{B}_{j+1}$. Continuing the process we obtain a finite or countable collection of families $\mathcal{B}_j$. Clearly the family $\mathcal{B}' = \bigcap_j \mathcal{B}_j$ satisfies all required properties. □

Consider the segment $R_0 \subset \mathbb{R}^d$ and let $\mathcal{B}$ be a family of segments $J \subset R_0$ so that $\bigcup_{J\in\mathcal{B}} J = R_0$. Let f be a summable function on R_0. The function

$$\mathcal{M}_{\mathcal{B}} f(x) = \sup_{\mathcal{B}\ni J\ni x} \frac{1}{|J|} \int_J |f(y)|\, dy, \quad x \in R_0,$$

is called *the maximal function* generated by the family $\mathcal{B}$. Here the supremum is taken over all segments $J \in \mathcal{B}$ which contain x.

From Lemma 1.27 by the standard construction (see [70]) one can show that the maximal operator $\mathcal{M}_{\mathcal{B}}$ is of the so-called weak $(1-1)$-type. Namely, the following lemma hold true.

Lemma 1.28. *Let f be a summable function on the segment $R_0 \subset \mathbb{R}^d$, and let $\mathcal{B}$ be a countable collection of segments $J \subset R_0$ with the "dyadic" property, so that $\bigcup_{J\in\mathcal{B}} J = R_0$. Then for any $\lambda > 0$*

$$|\{x \in R_0 :\ \mathcal{M}_{\mathcal{B}} f(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{\mathcal{M}_{\mathcal{B}} f(x) > \lambda\}} |f(x)|\, dx.$$

Proof. Denote by $\mathcal{B}_1$ the family of all segments $J \in \mathcal{B}$ such that $\frac{1}{|J|}\int_J |f(x)|\, dx > \lambda$. Then, obviously,

$$\{x \in R_0 :\ \mathcal{M}_{\mathcal{B}} f(x) > \lambda\} = \bigcup_{J \in \mathcal{B}_1} J.$$

Applying Lemma 1.27 to $\mathcal{B}_1$ we obtain a family $\mathcal{B}'$, which contains segments with pairwise disjoint interiors and such that

$$\bigcup_{J \in \mathcal{B}'} J = \bigcup_{J \in \mathcal{B}_1} J.$$

Therefore

$$|\{x \in R_0 :\ \mathcal{M}_{\mathcal{B}} f(x) > \lambda\}| = \left|\bigcup_{J \in \mathcal{B}'} J\right| = \sum_{J \in \mathcal{B}'} |J| \le$$

$$\le \sum_{J \in \mathcal{B}'} \frac{1}{\lambda} \int_J |f(x)|\, dx = \frac{1}{\lambda} \int_{\bigcup_{J \in \mathcal{B}'} J} |f(x)|\, dx = \frac{1}{\lambda} \int_{\{\mathcal{M}_{\mathcal{B}} f(x) > \lambda\}} |f(x)|\, dx.$$

□

The following lemma is a version of Lebesgue theorem 1.1 for the set of segments with the “dyadic” property.

Lemma 1.29. *Let f be a summable function on the segment $R_0 \subset \mathbb{R}^d$, and let $\mathcal{B}$ be a countable collection of segments $J \subset R_0$ with the “dyadic” property, so that $\bigcup_{J \in \mathcal{B}} J = R_0$. Let E be a set of points $x \in R_0$ such that at every x there exists a sequence of parallelepipeds $J_i(x) \in \mathcal{B},\ i = 1, 2, \dots,$ containing x, whose diameters tend to zero. Then for almost all $x \in E$ there exists $D(f, x) \equiv \lim_{i \to \infty} f_{J_i(x)}$ and*

$$D(f, x) = f(x).$$

Proof. For any $x \in E$ let us denote

$$\overline{D}(f, x) = \limsup_{\operatorname{diam} J \to 0,\ J \ni x} f_J, \quad \underline{D}(f, x) = \liminf_{\operatorname{diam} J \to 0,\ J \ni x} f_J.$$

It is clear, that for any function g, continuous at $x \in E$,

$$\overline{D}(g, x) = \underline{D}(g, x) = D(g, x) = g(x).$$

Choose some $\varepsilon > 0$. Let us present f as the sum $f = g + b$, where the function g is continuous on R_0 and $\|b\|_{L(R_0)} < \varepsilon$. Then

$$\overline{D}(f,x)-\underline{D}(f,x)=\overline{D}(b,x)-\underline{D}(b,x)\leq 2\mathcal{M}_{\mathcal{B}}b(x),\quad x\in E.$$

Hence, by Lemma 1.28, for $\lambda>0$

$$\left|\left\{x\in E:\ \overline{D}(f,x)-\underline{D}(f,x)>\lambda\right\}\right|\leq\left|\left\{x\in E:\ \mathcal{M}_{\mathcal{B}}b(x)>\frac{\lambda}{2}\right\}\right|\leq$$

$$\leq\frac{2}{\lambda}\|b\|_{L(R_0)}\leq\frac{2\varepsilon}{\lambda}.$$

Since ε was arbitrary the last inequality implies

$$\left|\left\{x\in E:\ \overline{D}(f,x)-\underline{D}(f,x)>\lambda\right\}\right|=0.$$

Now, recalling that also $\lambda>0$ was arbitrary, we see that the equality $\overline{D}(f,x)=\underline{D}(f,x)$ holds true at almost every $x\in E$. This means that there exists $D(f,x)$ and $D(f,x)=\overline{D}(f,x)=\underline{D}(f,x)$.

It remains to prove that $D(f,x)=f(x)$ for almost all $x\in E$. Again, we choose some $\varepsilon>0$ and present f in the form $f=g+b$ with g being continuous on R_0 and $\|b\|_{L(R_0)}<\varepsilon$. Then, since $D(f,x)-f(x)=D(b,x)-b(x)$ at almost every $x\in E$ for any $\lambda>0$

$$|\{x\in E:\ |D(f,x)-f(x)|>\lambda\}|=|\{x\in E:\ |D(b,x)-b(x)|>\lambda\}|\leq$$

$$\leq|\{x\in E:\ |D(b,x)|+|b(x)|>\lambda\}|\leq$$

$$\leq\left|\left\{x\in E:\ |D(b,x)|>\frac{\lambda}{2}\right\}\right|+\left|\left\{x\in E:\ |b(x)|>\frac{\lambda}{2}\right\}\right|\leq$$

$$\leq\left|\left\{x\in E:\ \mathcal{M}_{\mathcal{B}}b(x)>\frac{\lambda}{2}\right\}\right|+\left|\left\{x\in E:\ |b(x)|>\frac{\lambda}{2}\right\}\right|.$$

The first term can be estimated by means of Lemma 1.28, while to the second one we apply the Chebyshev inequality. Then

$$|\{x\in E:\ |D(f,x)-f(x)|>\lambda\}|=|\{x\in E:\ |D(b,x)-b(x)|>\lambda\}|\leq$$

$$\leq\frac{2}{\lambda}\|b\|_{L(R_0)}+\frac{2}{\lambda}\|b\|_{L(R_0)}\leq\frac{4\varepsilon}{\lambda}.$$

Hence, as ε was arbitrary, we have $|\{x\in E:\ |D(f,x)-f(x)|>\lambda\}|=0$ for any $\lambda>0$, and since λ was also arbitrary it follows that $D(f,x)=f(x)$ for almost all $x\in E$. □

Now we can easily prove the analog of the Riesz lemma in the general form.

Lemma 1.30 ([47]). *Let $R_0\subset\mathbb{R}^d$ be a segment. Assume $f\in L(R_0)$ and $\alpha\geq f_{R_0}$. Then there exists a family of segments $R_j\subset R_0,\ j=1,2,\ldots,$ with pairwise disjoint interiors such that $f_{R_j}=\alpha,\ j=1,2,\ldots,$ and $f(x)\leq\alpha$ for almost every $x\in R_0\setminus\left(\bigcup_{j\geq1}R_j\right)$.*

Proof. Repeating the proof of Lemma 1.21 we obtain two families of segments: $\{R_j\}_{j\geq 1}$ and

$$\mathcal{B} \equiv \left\{ J^{(s,i)},\ i = 1, 2, \ldots, r_s,\ s = 1, 2, \ldots \right\}.$$

It is easy to see, that the family $\mathcal{B}$ satisfies the "dyadic" property. Let $E \equiv R_0 \setminus \left(\bigcup_{j\geq 1} R_j \right)$. Then the family $\mathcal{B}$ and the set E obviously verify the conditions of Lemma 1.29. According to this lemma, for almost all $x \in E$

$$f(x) = \lim_{s\to\infty} f_{J_s(x)}.$$

Since $f_{J_s(x)} < \alpha$ we see that $f(x) \leq \alpha$ for almost all $x \in E$. $\square$

We have already mentioned above that the proof of Lemma 1.16, based on the "stopping times technique", can be easily adopted for Lemma 1.18. Similarly, in the multidimensional case the proofs of Lemmas 1.21 and 1.30 can be modified in order to get the following statement.

Lemma 1.31. *Let $R_0 \subset \mathbb{R}^d$ be a segment. Assume $f \in L(R_0)$ and $\alpha \leq f_{R_0}$. Then there exists a family of segments $R_j \subset R_0,\ j = 1, 2, \ldots$ with pairwise disjoint interiors such that $f_{R_j} = \alpha,\ j = 1, 2, \ldots,$ and $f(x) \geq \alpha$ for almost every $x \in R_0 \setminus \left(\bigcup_{j\geq 1} R_j \right)$.*

2

Properties of Oscillations and the Definition of the BMO-class

2.1 Properties of Mean Oscillations

The quantity

$$\Omega(f;Q) = \frac{1}{|Q|}\int_Q |f(x) - f_Q|\ dx$$

is called *the mean oscillation* of the function $f \in L_{loc}$ on the cube $Q \subset \mathbb{R}^d$.

In this section we will study the properties of mean oscillations. First of all we note that $\Omega(f;Q) = 0$ if and only if the function f is constant a. e. on Q. Obviously, for any constant λ we have $\Omega(f+\lambda;Q) = \Omega(f;Q)$.

Property 2.1. *Let f be a summable function on Q. Then*

$$\Omega(f;Q) = \frac{2}{|Q|}\int_{\{x\in Q:\ f(x)>f_Q\}} (f(x) - f_Q)\ dx =$$

$$= \frac{2}{|Q|}\int_{\{x\in Q:\ f(x)<f_Q\}} (f_Q - f(x))\ dx. \tag{2.1}$$

Proof. Indeed, from the trivial equality

$$\int_{\{x\in Q:\ f(x)>f_Q\}} (f(x) - f_Q)\ dx + \int_{\{x\in Q:\ f(x)<f_Q\}} (f(x) - f_Q)\ dx = 0$$

it follows that

$$\int_{\{x\in Q:\ f(x)>f_Q\}} (f(x) - f_Q)\ dx = -\int_{\{x\in Q:\ f(x)<f_Q\}} (f(x) - f_Q)\ dx =$$

$$= \int_{\{x\in Q:\ f(x)<f_Q\}} (f_Q - f(x))\ dx.$$

Hence

$$\int_Q |f(x) - f_Q|\, dx =$$

$$= \int_{\{x\in Q:\ f(x)>f_Q\}} (f(x) - f_Q)\, dx + \int_{\{x\in Q:\ f(x)<f_Q\}} (f_Q - f(x))\, dx =$$

$$= 2\int_{\{x\in Q:\ f(x)>f_Q\}} (f(x) - f_Q)\, dx = 2\int_{\{x\in Q:\ f(x)<f_Q\}} (f_Q - f(x))\, dx.$$

Dividing the last equality by $|Q|$ we obtain (2.1). □

Property 2.1 will be frequently used in the further calculations. Now let us present two lemmas, which follow from Property 2.1. This lemmas will play an important role in finding the sharp estimates of equimeasurable rearrangements of functions in terms of their mean oscillations.

Lemma 2.2 ([34]). *Let the non-increasing function φ be locally summable on $[0, +\infty)$, and let $F(t) \equiv \frac{1}{t}\int_0^t \varphi(u)\, du,\ t > 0$. Then for any constant $a > 1$*

$$F\left(\frac{t}{a}\right) - F(t) \le \frac{a}{2}\frac{1}{t}\int_0^t |\varphi(u) - F(t)|\, du, \quad t > 0. \tag{2.2}$$

Proof. Let $t > 0$. If $\varphi\left(\frac{t}{a}\right) \le F(t)$, then

$$E_t \equiv \{u \in [0, t] :\ \varphi(u) > F(t)\} \subset \left[0, \frac{t}{a}\right].$$

According to Property 2.1,

$$F\left(\frac{t}{a}\right) - F(t) = \frac{a}{t}\int_0^{t/a} (\varphi(u) - F(t))\, du =$$

$$= \frac{a}{t}\int_{E_t} (\varphi(u) - F(t))\, du + \frac{a}{t}\int_{\left[0,\frac{t}{a}\right]\setminus E_t} (\varphi(u) - F(t))\, du \le$$

$$\le \frac{a}{t}\int_{E_t} (\varphi(u) - F(t))\, du = \frac{a}{2}\frac{1}{t}\int_0^t |\varphi(u) - F(t)|\, du.$$

Otherwise, if $\varphi\left(\frac{t}{a}\right) > F(t)$, then $E_t \supset \left[0, \frac{t}{a}\right]$, and, again by Property 2.1,

$$F\left(\frac{t}{a}\right) - F(t) = \frac{a}{t}\int_0^{t/a} (\varphi(u) - F(t))\, du \le \frac{a}{t}\int_{E_t} (\varphi(u) - F(t))\, du =$$

$$= \frac{a}{2}\frac{1}{t}\int_0^t |\varphi(u) - F(t)|\, du. \quad \square$$

In the same way one can prove the following analog of Lemma 2.2 for a non-decreasing function.

Lemma 2.3 ([43]). *Let the non-negative function φ be non-decreasing on $[0, +\infty)$, and let $F(t) \equiv \frac{1}{t}\int_0^t \varphi(u)\,du,\ t > 0$. Then for any constant $a > 1$*

$$F(t) - F\left(\frac{t}{a}\right) \leq \frac{a}{2}\frac{1}{t}\int_0^t |\varphi(u) - F(t)|\,du, \quad t > 0. \tag{2.3}$$

Denote

$$\Omega'(f;Q) = \inf_{c\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(x) - c|\,dx.$$

Property 2.4.

$$\Omega'(f;Q) \leq \Omega(f;Q) \leq 2\Omega'(f;Q). \tag{2.4}$$

In general the constants 1 *and* 2 *in the left and the right-hand sides are sharp.*

Proof. The left inequality in (2.4) is an obvious consequence of the definitions of $\Omega'(f;Q)$ and $\Omega(f;Q)$. On the other hand, for any constant c

$$\Omega(f;Q) = \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx = \frac{1}{|Q|}\int_Q \left|f(x) - \frac{1}{|Q|}\int_Q f(y)\,dy\right|\,dx \leq$$

$$\leq \frac{1}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|\,dy\,dx \leq \frac{2}{|Q|}\int_Q |f(x) - c|\,dx.$$

Taking the infimum over all c we obtain the right inequality of (2.4).

Further, for $f(x) = \chi_{[0,1/2]}(x) - \chi_{(1/2,1]}(x),\ x \in Q \equiv [0,1]$, we have $f_Q = 0,\ \Omega(f;Q) = 1$. If $|c| \leq 1$, then

$$\int_0^1 |f(x) - c|\,dx = \int_0^{1/2}(1 - c)\,dx + \int_{1/2}^1 (c - (-1))\,dx = 1,$$

while in the case $|c| > 1$ obviously $\int_0^1 |f(x) - c|\,dx > 1$. This means that $\Omega'(f;Q) = 1$. Therefore, the left inequality of (2.4) becomes an equality, i.e., the constant 1 of the left-hand side of (2.4) cannot be increased.

Now set $f(x) = \chi_{[0,\varepsilon]}(x)$, where $x \in Q \equiv [0,1]$ and $0 < \varepsilon < \frac{1}{2}$. Then $f_Q = \varepsilon,\ \Omega(f;Q) = 2\int_0^\varepsilon (1 - \varepsilon)\,dx = 2\varepsilon(1 - \varepsilon)$. Let us calculate $\Omega'(f;Q)$. Clearly, it is enough to consider only the constants c such that $0 \leq c \leq 1$. For such a constant we have

$$\int_0^1 |f(x) - c|\,dx = \int_0^\varepsilon (1-c)\,dx + \int_\varepsilon^1 c\,dx = \varepsilon(1-c) + c(1-\varepsilon) = \varepsilon + c(1-2\varepsilon),$$

and so $\Omega'(f;Q) = \varepsilon$. Finally,

$$\frac{\Omega(f;Q)}{\Omega'(f;Q)} = 2(1-\varepsilon) \to 2, \quad \varepsilon \to 0,$$

so that the constant 2 of the right-hand side of (2.4) cannot be decreased. □

Denote

$$\Omega''(f;Q) = \frac{1}{|Q|^2} \int_Q \int_Q |f(x) - f(y)| \, dy \, dx.$$

Property 2.5.

$$\Omega(f;Q) \leq \Omega''(f;Q) \leq 2\Omega(f;Q). \tag{2.5}$$

In general the constants 1 *and* 2 *in the left and the right-hand sides are sharp.*

Proof. The following calculation yields the left inequality of (2.5):

$$\Omega(f;Q) = \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx = \frac{1}{|Q|} \int_Q \left| f(x) - \frac{1}{|Q|} \int_Q f(y) \, dy \right| dx \leq$$

$$\leq \frac{1}{|Q|^2} \int_Q \int_Q |f(x) - f(y)| \, dy \, dx = \Omega''(f;Q).$$

The right inequality of (2.5) is also trivial:

$$\Omega''(f;Q) = \frac{1}{|Q|^2} \int_Q \int_Q |f(x) - f(y)| \, dy \, dx \leq$$

$$\leq \frac{1}{|Q|^2} \int_Q \int_Q |f(x) - f_Q| \, dy \, dx + \frac{1}{|Q|^2} \int_Q \int_Q |f(y) - f_Q| \, dy \, dx = 2\Omega(f;Q).$$

Set $f(x) = \chi_{[0,1/2]}(x),\ x \in Q \equiv [0,1]$. Then $f_Q = \frac{1}{2},\ \Omega(f;Q) = \frac{1}{2}$, and

$$\Omega''(f;Q) = \int_0^{1/2} dx \int_{1/2}^1 dy + \int_{1/2}^1 dx \int_0^{1/2} dy = \frac{1}{2}.$$

This means that the constant 1 in the left-hand side of (2.5) cannot be improved.

Further, if $0 < \varepsilon \leq \frac{1}{2}$, we set $f(x) = \chi_{[0,\varepsilon)}(x) - \chi_{(1-\varepsilon,1]}(x),\ x \in Q \equiv [0,1]$. Then $f_Q = 0,\ \Omega(f;Q) = 2\varepsilon$, and

$$\Omega''(f;Q) = \int_0^\varepsilon dx \int_\varepsilon^{1-\varepsilon} dy + \int_0^\varepsilon dx \int_{1-\varepsilon}^1 2\, dy + \int_\varepsilon^{1-\varepsilon} dx \int_0^\varepsilon dy +$$

$$+ \int_\varepsilon^{1-\varepsilon} dx \int_{1-\varepsilon}^1 dy + \int_{1-\varepsilon}^1 dx \int_0^\varepsilon 2\, dy + \int_{1-\varepsilon}^1 dx \int_\varepsilon^{1-\varepsilon} dy =$$

$$= 4\varepsilon(1-2\varepsilon) + 2\varepsilon^2 = 4\varepsilon - 6\varepsilon^2.$$

Hence
$$\frac{\Omega''(f;Q)}{\Omega(f;Q)} = 2 - 3\varepsilon \to 2, \quad \varepsilon \to 0,$$
so that the constant 2 in the right-hand side of (2.5) cannot be decreased. □

Property 2.6.
$$\Omega(|f|;Q) \le 2\Omega(f;Q), \tag{2.6}$$
and in general the constant 2 *is sharp.*

Proof. Indeed, by (2.5),
$$\Omega(|f|;Q) \le \Omega''(|f|;Q) = \frac{1}{|Q|^2}\int_Q\int_Q \big|\,|f(x)| - |f(y)|\,\big|\, dy\, dx \le$$
$$\le \frac{1}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|\, dy\, dx \le 2\Omega(f;Q).$$

On the other hand, for the function $f(x) = \chi_{[0,\varepsilon)}(x) - \chi_{(1-\varepsilon,1]}(x)$, where $x \in Q \equiv [0,1]$ and $0 < \varepsilon < \frac{1}{2}$, we have $f_Q = 0$, $\Omega(f;Q) = 2\varepsilon$, $|f|_Q = 2\varepsilon$, and
$$\Omega(|f|;Q) = 2\int_{[0,\varepsilon)\cup(1-\varepsilon,1]} (1-2\varepsilon)\, dx = 4\varepsilon(1-2\varepsilon).$$

Hence
$$\frac{\Omega(|f|;Q)}{\Omega(f;Q)} = 2(1-2\varepsilon) \to 2, \quad \varepsilon \to 0.$$
Thus the constant 2 in the right-hand side of (2.6) is sharp. □

Property 2.7. *Let* f *be an essentially bounded function on the cube* Q*. Then*
$$\Omega(f;Q) \le \frac{1}{2}\left(\operatorname*{ess\,sup}_{x\in Q} f(x) - \operatorname*{ess\,inf}_{x\in Q} f(x)\right), \tag{2.7}$$
and in general the constant $\frac{1}{2}$ *in the right-hand side is sharp.*

Proof. The inequality
$$\Omega(f;Q) \le \operatorname*{ess\,sup}_{x\in Q} f(x) - \operatorname*{ess\,inf}_{x\in Q} f(x),$$
which is more rough than (2.7), follows immediately from (2.5). Indeed, since
$$|f(x) - f(y)| \le \operatorname*{ess\,sup}_{x\in Q} f(x) - \operatorname*{ess\,inf}_{x\in Q} f(x), \quad x, y \in Q,$$
we have
$$\Omega(f;Q) \le \Omega''(f;Q) = \frac{1}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|\, dy\, dx \le$$

$$\le \operatorname*{ess\,sup}_{x\in Q} f(x) - \operatorname*{ess\,inf}_{x\in Q} f(x).$$

In order to prove (2.7) it is enough to consider the case

$$A \equiv \operatorname*{ess\,sup}_{x\in Q} f(x) = -\operatorname*{ess\,inf}_{x\in Q} f(x).$$

Otherwise it is sufficient to consider the function $f(x) - M,\ x \in Q$ with $M = \frac{1}{2}\left(\operatorname{ess\,sup}_{x\in Q} f(x) + \operatorname{ess\,inf}_{x\in Q} f(x)\right)$.

If $f_Q = 0$, then $|f(x) - f_Q| = |f(x)| \le A$ for almost all $x \in Q$. Hence

$$\Omega(f;Q) = \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx = \frac{1}{|Q|}\int_Q |f(x)|\,dx \le A,$$

and (2.7) holds true in this case.

Now let us suppose that $f_Q > 0$. Denote $E_1 = \{x \in Q:\ f(x) \ge f_Q\}$, $E_2 = Q \setminus E_1$. If $|E_1| \le |Q|/2$, then

$$\Omega(f;Q) = \frac{2}{|Q|}\int_{E_1} (f(x) - f_Q)\,dx \le \frac{2}{|Q|}\cdot A \cdot |E_1| \le A,$$

and hence (2.7) holds true. Otherwise, if $|E_1| > |Q|/2$, then $|E_2| \le |E_1|$ and we have

$$\Omega(f;Q) = \frac{1}{|Q|}\left(\int_{E_1} (f(x) - f_Q)\,dx + \int_{E_2} (f_Q - f(x))\,dx\right) =$$

$$= \frac{1}{|Q|}\left(\int_{E_1} f(x)\,dx - \int_{E_2} f(x)\,dx + f_Q\left(|E_2| - |E_1|\right)\right) \le$$

$$\le \frac{1}{|Q|}\left(\int_{E_1} f(x)\,dx - \int_{E_2} f(x)\,dx\right) \le$$

$$\le \frac{1}{|Q|}\left(\int_{\{x:\ f(x)\ge 0\}} f(x)\,dx + \int_{\{x:\ f(x)<0\}} |f(x)|\,dx\right) =$$

$$= \frac{1}{|Q|}\int_Q |f(x)|\,dx \le A.$$

Thus also in this case inequality (2.7) is satisfied.

So, we have proved (2.7) for the case $f_Q \ge 0$. The case $f_Q < 0$ can be treated analogously. This completes the proof of (2.7).

Taking $f(x) = \chi_{[0,1/2]}(x),\ x \in Q \equiv [0,1]$ as the test function it is easy to see that the constant $\frac{1}{2}$ in right-hand side of (2.7) is sharp. □

Remark 2.8. In particular, (2.7) implies

$$\Omega(f;Q) \le \|f\|_\infty. \tag{2.8}$$

Remark 2.9. We have presented the direct proof of inequality (2.8). However, it can be proved in a much easier way using the Schwartz inequality. Namely,

$$\Omega^2(f;Q) = \left\{ \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx \right\}^2 \le \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx =$$

$$= \frac{1}{|Q|} \int_Q f^2(x)\, dx - (f_Q)^2 \le \|f^2\|_\infty = \|f\|_\infty^2.$$

For $1 \le p < \infty$ the quantity

$$\Omega_p(f;Q) = \left\{ \frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right\}^{1/p}$$

is called *the mean p-oscillation* of the function $f \in L^p_{loc}$ on the cube $Q \subset \mathbb{R}^d$.

Remark 2.10. Actually, while proving inequality (2.8) in Remark 2.9, we have obtained the inequality

$$\Omega_2(f;Q) \le \|f\|_\infty,$$

which is stronger than (2.8). From here and from the Hölder inequality it follows that

$$\Omega_p(f;Q) \le \|f\|_\infty \tag{2.9}$$

for $p \le 2$. However, in general inequality (2.9) is not true for all p, $1 \le p < \infty$. Indeed, for the function $f(x) = \chi_{[0,5/8]}(x) - \chi_{(5/8,1]}(x)$, $x \in Q \equiv [0,1]$, we have $f_Q = \frac{1}{4}$,

$$\Omega_p^p(f;Q) = \int_0^1 |f(x) - f_Q|^p \, dx = \left(\frac{3}{4}\right)^p \cdot \frac{5}{8} + \left(\frac{5}{4}\right)^p \cdot \frac{3}{8},$$

so that

$$\Omega_p(f;Q) = \left\{ \left(\frac{3}{4}\right)^p \cdot \frac{5}{8} + \left(\frac{5}{4}\right)^p \cdot \frac{3}{8} \right\}^{1/p} \to \frac{5}{4} > 1 = \|f\|_\infty, \quad p \to \infty.$$

Within this context the following question is natural: *What are the exponents p which satisfy* (2.9)*?* Notice, that the inequality

$$\Omega_p(f;Q) \le 2\|f\|_\infty$$

is obviously true for all p, $1 \le p < \infty$. This is why the question, stated above, can be reformulated in the following way: *What is the minimal constant $c = c(p)$, which guarantees*

$$\Omega_p(f;Q) \le c\|f\|_\infty$$

for any function $f \in L^\infty$?

For the answer to the both questions let us denote

$$C_p = \sup_{f \in L^\infty([0,1])} \frac{\Omega_p(f;[0,1])}{\|f\|_\infty}.$$

Theorem 2.11 (Leonchik, [52]). *For $p \geq 1$*

$$C_p = 2 \sup_{0<h<1} \{h(1-h)^p + h^p(1-h)\}^{1/p}. \tag{2.10}$$

Proof. For $0 < h < 1$ let $f_h(x) = \chi_{[0,h]}(x) - \chi_{(h,1]}(x)$, $x \in [0,1]$. Then we obtain $\|f_h\|_\infty = 1$, $(f_h)_{[0,1]} = 2h - 1$,

$$\Omega_p^p(f_h;[0,1]) = 2^p \{h(1-h)^p + h^p(1-h)\}. \tag{2.11}$$

Therefore $C_p \geq B_p$, where B_p denotes the right-hand side of (2.10).

In order to prove the opposite inequality

$$C_p \leq B_p, \qquad (2.12),$$

we observe that it follows immediately from the inequality

$$\Omega_p^p(f;[0,1]) \leq 2^p \sup_{0<h<1} \{h(1-h)^p + h^p(1-h)\}, \tag{2.13}$$

where the function f is non-increasing on $[0,1]$ and $f(0) = -f(1) = 1$. Assume that f satisfies these conditions. Let h^* be such that $f(x) \geq f_{[0,1]}$ for $x \leq h^*$ and $f(x) \leq f_{[0,1]}$ for $x \geq h^*$. Set $h = \frac{1}{2}\left(1 + f_{[0,1]}\right)$. Then $0 < h < 1$ and $f_{[0,1]} = 2h - 1$. It is easy to see that there exist h_1, $0 < h_1 \leq \min(h, h^*)$, and h_2, $\max(h, h^*) \leq h_2 < 1$, such that

$$\Omega_p^p(f;[0,1]) = \int_0^{h^*} \left(f(x) - f_{[0,1]}\right)^p dx + \int_{h^*}^1 \left(f_{[0,1]} - f(x)\right)^p dx =$$

$$= \int_0^{h_1} \left(1 - f_{[0,1]}\right)^p dx + \int_{h_2}^1 \left(f_{[0,1]} + 1\right)^p dx \leq$$

$$\leq \int_0^{h} \left(1 - f_{[0,1]}\right)^p dx + \int_{h}^1 \left(f_{[0,1]} + 1\right)^p dx = \Omega_p^p(f_h;[0,1]),$$

where the function f_h was defined at the beginning of the proof. From here and from (2.11) we obtain (2.13). □

In order to study the behavior of the constants C_p, defined in Theorem 2.11, first we prove one more inequality.

Lemma 2.12 (Korneichuk, [50, Lemma 5.2.3, p. 225]). *For $0 \leq p \leq 3$*

$$h(1-h)^p + h^p(1-h) \leq 2^{-p}, \quad 0 \leq h \leq 1. \tag{2.14}$$

Proof. Notice, that in the original work [50] inequality (2.14) was given in the following equivalent form:

$$2^p\left(u^p+u\right)\leq(1+u)^{p+1},\quad u\geq 0.$$

The proof, which we are going to present now, is different from the one of [50]. Setting $h=\frac{t+1}{2}$, we transform the inequality (2.14) into

$$\left(1-t^2\right)\left[(1+t)^{p-1}+(1-t)^{p-1}\right]\leq 2,\quad -1\leq t\leq 1. \tag{2.15}$$

Let us denote the left-hand side of (2.15) by $\varphi(t)$. The function φ is even on $[-1,1]$, thus it is enough to prove (2.15) for $0\leq t\leq 1$.

Let $1\leq p\leq 2$. Denote $\psi(t)=(1+t)^{p-1}+(1-t)^{p-1}$. Then $\psi'(t)=(p-1)\left[(1+t)^{p-2}-(1-t)^{p-2}\right]\leq 0$, i.e. the function ψ does not increase on $[0,1]$. Since the function $1-t^2$ is non-increasing it follows that also the function φ is non-increasing on $[0,1]$. Therefore

$$\varphi(t)\leq\varphi(0)=2,\quad 0\leq t\leq 1.$$

The case $p=0$ is trivial. So, it remains to consider the case $p\in(0,1)\cup(2,3]$. We can write

$$\varphi(t)=\left(1-t^2\right)\left[2+\sum_{k=1}^{\infty}\frac{(p-1)\dots(p-k)}{k!}t^k+\sum_{k=1}^{\infty}\frac{(p-1)\dots(p-k)}{k!}(-t)^k\right]=$$

$$=2\left(1-t^2\right)\left[1+\sum_{s=1}^{\infty}\frac{(p-1)\dots(p-2s)}{(2s)!}t^{2s}\right]=$$

$$=2\left[1+\sum_{s=1}^{\infty}\frac{(p-1)\dots(p-2s)}{(2s)!}t^{2s}-t^2-\sum_{s=1}^{\infty}\frac{(p-1)\dots(p-2s)}{(2s)!}t^{2s+2}\right]=$$

$$=2\left\{1+\left[\frac{(p-1)(p-2)}{2!}-1\right]t^2+\right.$$

$$\left.+\sum_{s=2}^{\infty}\left[\frac{(p-1)\dots(p-2s)}{(2s)!}-\frac{(p-1)\dots(p-(2s-3))(p-(2s-2))}{(2s-2)!}\right]t^{2s}\right\}=$$

$$=2\left\{1+\left[\frac{(p-1)(p-2)}{2}-1\right]t^2+\right.$$

$$\left.+\sum_{s=2}^{\infty}\frac{(p-1)\dots(p-(2s-2))}{(2s-2)!}\left[\frac{(p-(2s-1))(p-2s)}{(2s-1)(2s)}-1\right]t^{2s}\right\}.$$

Since $p \in (0,1) \cup (2,3]$ we have

$$\frac{(p-1)(p-2)}{2} - 1 \leq 0,$$

i.e., the coefficient in the right-hand side of the last equality, which corresponds to t^2, is non-positive. Further, if $2 < p \leq 3$, the first two terms of the product $(p-1)(p-2)\dots(p-(2s-3))(p-(2s-2))$ are positive, while the other terms (we have an even number of them) are non-positive. In the case $0 < p < 1$ all terms (even number) are non-positive. So, the whole product is non-negative. Since $p \leq 4s - 1$ $(s \geq 2)$ we have $p^2 - p(4s-1) \leq 0$. This means that $\frac{(p-(2s-1))(p-2s)}{(2s-1)2s} - 1 \leq 0$. Therefore all the coefficients of the sum in the right-hand side of the last equality, which correspond to t^{2s}, are non-positive. Thus the whole sum is non-positive. Finally we have

$$\varphi(t) \leq 2, \quad 0 \leq t \leq 1$$

for $p \in (0,1) \cup (2,3]$. □

Now from Theorem 2.11 we easily derive

Corollary 2.13. *The constants C_p are monotone increasing and satisfy the following properties:*

(i) $$C_p = 1, \quad 1 \leq p \leq 3;$$

(ii) $$1 < C_p < 2, \quad 3 < p < \infty;$$

(iii) $$\lim_{p\to\infty} C_p = 2.$$

Proof. The monotonicity of C_p follows immediately from the Hölder inequality. Clearly, $C_p \geq 1$ for all $p \geq 1$. On the other hand, in order to prove (i) it is enough to use Theorem 2.11 and Lemma 2.12.

Assume $p > 3$. Denote $\varphi(h) = h(1-h)^p + h^p(1-h)$. Then $\varphi\left(\frac{1}{2}\right) = 2^{-p}$, $\varphi'\left(\frac{1}{2}\right) = 0$ and $\varphi''\left(\frac{1}{2}\right) = 2^{2-p}p(p-3) > 0$. These properties of the function φ imply that in a small enough neighborhood of the point $\frac{1}{2}$ there exists h such that $\varphi(h) > 2^{-p}$. According to Theorem 2.11, this is equivalent to the left inequality of (ii). The right inequality of (ii) is trivial since for $p > 1$ we have $\varphi(h) \leq 2h(1-h) \leq \frac{1}{2}$, so that $C_p \leq 2^{1-1/p} < 2$.

Finally, for some fixed h, $0 < h < 1$, and $p \to \infty$

$$\lim_{p\to\infty} C_p \geq \lim_{p\to\infty} \Omega\left(f_h; [0,1]\right) = 2 \lim_{p\to\infty} \left(\varphi(h)\right)^{1/p} = 2\max(h, 1-h).$$

Choosing h to be small enough, we obtain (iii). □

The next property shows the semi-linearity of mean oscillations.

Property 2.14. *For* $p \geq 1$

$$\Omega_p(f+g;Q) \leq \Omega_p(f;Q) + \Omega_p(g;Q), \tag{2.16}$$

$$\Omega_p(\lambda f;Q) = |\lambda| \Omega_p(f;Q) \tag{2.17}$$

for any constant λ.

Proof. Indeed, since $(f+g)_Q = f_Q + g_Q$ and $(\lambda f)_Q = \lambda f_Q$ by Minkowski inequality we have

$$\Omega_p(f+g;Q) = \left\{ \frac{1}{|Q|} \int_Q |f(x) + g(x) - f_Q - g_Q|^p \, dx \right\}^{1/p} \leq \Omega_p(f;Q) + \Omega_p(g;Q),$$

$$\Omega_p(\lambda f;Q) = \left\{ \frac{1}{|Q|} |\lambda f(x) - \lambda f_Q|^p \, dx \right\}^{1/p} = |\lambda| \Omega_p(f;Q). \quad \square$$

Property 2.15 (Klemes, [32]). *Let* f *be a summable monotone function on* $I_1 \equiv [\alpha_1, \beta_1]$, *and let* $I \equiv [\alpha, \beta] \subset I_1$ *be such that* $f_I = f_{I_1}$. *Then*

$$\Omega(f;I) \leq \Omega(f;I_1). \tag{2.18}$$

Proof. We can assume that f does not increase on I_1. Let us consider the non-trivial case $\Omega(f;I) > 0$. Since f is monotone there exists $\gamma \in (\alpha, \beta)$ such that $f(x) \geq f_{I_1}$ if $x \in [\alpha_1, \gamma]$ and $f(x) \leq f_{I_1}$ if $x \in [\gamma, \beta_1]$. The monotonicity of $f(x) - f_{I_1}$ implies

$$\frac{1}{\gamma - \alpha_1} \int_{\alpha_1}^{\gamma} (f(x) - f_{I_1}) \, dx \geq \frac{1}{\gamma - \alpha} \int_{\alpha}^{\gamma} (f(x) - f_I) \, dx,$$

$$\frac{1}{\beta_1 - \gamma} \int_{\gamma}^{\beta_1} (f_{I_1} - f(x)) \, dx \geq \frac{1}{\beta - \gamma} \int_{\gamma}^{\beta} (f_I - f(x)) \, dx,$$

or, equivalently,

$$\frac{\gamma - \alpha_1}{\int_{\alpha_1}^{\gamma} (f(x) - f_{I_1}) \, dx} \leq \frac{\gamma - \alpha}{\int_{\alpha}^{\gamma} (f(x) - f_I) \, dx}, \tag{2.19}$$

$$\frac{\beta_1 - \gamma}{\int_{\gamma}^{\beta_1} (f_{I_1} - f(x)) \, dx} \leq \frac{\beta - \gamma}{\int_{\gamma}^{\beta} (f_I - f(x)) \, dx}. \tag{2.20}$$

Using the equalities

$$\int_{\alpha_1}^{\gamma} (f(x) - f_{I_1})\, dx = \int_{\gamma}^{\beta_1} (f_{I_1} - f(x))\, dx, \tag{2.21}$$

$$\int_{\alpha}^{\gamma} (f(x) - f_I)\, dx = \int_{\gamma}^{\beta} (f_I - f(x))\, dx, \tag{2.22}$$

which follow from Property 2.1, and summing (2.19) and (2.20), we obtain

$$\frac{\beta_1 - \alpha_1}{\int_{\alpha_1}^{\gamma} (f(x) - f_{I_1})\, dx} \le \frac{\beta - \alpha}{\int_{\alpha}^{\gamma} (f(x) - f_I)\, dx}. \tag{2.23}$$

But, according to Property 2.1

$$\Omega(f; I_1) = \frac{2}{\beta_1 - \alpha_1} \int_{\alpha_1}^{\gamma} (f(x) - f_{I_1})\, dx,$$

$$\Omega(f; I) = \frac{2}{\beta - \alpha} \int_{\alpha}^{\gamma} (f(x) - f_I)\, dx.$$

This, together with (2.23), yields (2.18). □

If we consider the analog of inequality (2.18) for the oscillations Ω_p, $1 < p < \infty$, then the last proof fails because it is based on equalities (2.21) and (2.22) that have no analogs for $p > 1$. However, the following property is satisfied.

Property 2.16 ([40]). *Let* $f \in L^p$, $1 \le p < \infty$, *be monotone on* $I_1 \equiv [\alpha_1, \beta_1]$, *and let* $I \equiv [\alpha, \beta] \subset I_1$ *be such that* $f_I = f_{I_1}$. *Then*

$$\Omega_p(f; I) \le \Omega_p(f; I_1). \tag{2.24}$$

For the proof we will need the following two lemmas.

Lemma 2.17. *Let* $I_1 \equiv [\alpha_1, \beta_1] \supset [\alpha, \beta] \equiv I$ *and let the function* $\varphi \in L(I_1)$ *be non-increasing on* I_1, *so that*

$$\varphi_I = \varphi_{I_1} = 0. \tag{2.25}$$

Then

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} |\varphi(x)|\, dx \le 2 \frac{\frac{1}{\alpha - \alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx \cdot \frac{1}{\beta_1 - \beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx}{\frac{1}{\alpha - \alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx + \frac{1}{\beta_1 - \beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx}. \tag{2.26}$$

Proof. As in the proof of Property 2.15, let us choose $\gamma \in (\alpha, \beta)$ such that

$$\int_{\alpha}^{\gamma} \varphi(x)\, dx = -\int_{\gamma}^{\beta} \varphi(x)\, dx = \frac{1}{2} \int_{\alpha}^{\beta} |\varphi(x)|\, dx. \tag{2.27}$$

From (2.25) and from the monotonicity of φ it follows that $\varphi(x) \geq 0$ if $x \in [\alpha_1, \gamma]$, and $\varphi(x) \leq 0$ if $x \in [\gamma, \beta_1]$. Further, the monotonicity of φ implies

$$\frac{1}{\alpha - \alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx \geq \frac{1}{\gamma - \alpha} \int_{\alpha}^{\gamma} |\varphi(x)|\, dx,$$

$$\frac{1}{\beta_1 - \beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx \geq \frac{1}{\beta - \gamma} \int_{\gamma}^{\beta} |\varphi(x)|\, dx,$$

i.e.,

$$\frac{1}{\frac{1}{\alpha-\alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx} \leq \frac{\gamma - \alpha}{\int_{\alpha}^{\gamma} |\varphi(x)|\ dx},$$
$$\frac{1}{\frac{1}{\beta_1-\beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx} \leq \frac{\beta - \gamma}{\frac{1}{\beta-\gamma} \int_{\gamma}^{\beta} |\varphi(x)|\ dx}.$$

Notice that, by (2.27), the denominators of both fractions in the right-hand sides of the last inequalities are the same. Summing up we obtain

$$\frac{1}{\frac{1}{\alpha-\alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx} + \frac{1}{\frac{1}{\beta_1-\beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx} \leq \frac{\beta - \alpha}{\int_{\alpha}^{\gamma} |\varphi(x)|\ dx} = 2\frac{\beta - \alpha}{\int_{\alpha}^{\beta} |\varphi(x)|\ dx}.$$

This inequality is equivalent to (2.26) and the lemma is proved. □

We use Lemma 2.17 to prove the next inequality.

Lemma 2.18. *Assume that for some $p \geq 1$ the function $\varphi \in L^p(I_1)$ satisfies the conditions of Lemma* 2.17*. Then*

$$\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} |\varphi(x)|^p\ dx \leq \frac{\frac{1}{\alpha-\alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx \cdot \frac{1}{\beta_1-\beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx}{\frac{1}{\alpha-\alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx + \frac{1}{\beta_1-\beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx} \times$$

$$\times \left[\left(\frac{1}{\alpha - \alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx \right)^{p-1} + \left(\frac{1}{\beta_1 - \beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx \right)^{p-1} \right]. \quad (2.28)$$

Proof. From the monotonicity of φ it follows that

$$\int_{\alpha}^{\gamma} |\varphi(x)|^p\ dx \leq \left(\frac{1}{\alpha - \alpha_1} \int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx \right)^{p-1} \int_{\alpha}^{\gamma} |\varphi(x)|\ dx,$$

$$\int_{\gamma}^{\beta} |\varphi(x)|^p\, dx \leq \left(\frac{1}{\beta_1 - \beta} \int_{\beta}^{\beta_1} |\varphi(x)|\, dx \right)^{p-1} \int_{\gamma}^{\beta} |\varphi(x)|\ dx,$$

with the same γ as in the proof of Lemma 2.17. Summing up these two inequalities and using (2.27) we obtain

$$\int_\alpha^\beta |\varphi(x)|^p\, dx \le \frac{1}{2}\int_\alpha^\beta |\varphi(x)|\, dx \times$$

$$\times \left[\left(\frac{1}{\alpha-\alpha_1}\int_{\alpha_1}^{\alpha} |\varphi(x)|\, dx\right)^{p-1} + \left(\frac{1}{\beta_1-\beta}\int_{\beta}^{\beta_1} |\varphi(x)|\, dx\right)^{p-1}\right].$$

From here and from (2.26) inequality (2.28) follows. □

Proof of Property 2.16. We can assume that the function f does not increase on I_1. Set $\varphi(x) = f(x) - f_I,\ x \in I_1$. Then the function φ satisfies the conditions of Lemma 2.18. It is easy to see that in order to proof Property 2.16 it is enough to show that

$$\frac{1}{|I_1|}\int_{I_1} |\varphi(x)|^p\, dx \ge \frac{1}{|I|}\int_I |\varphi(x)|^p\, dx. \tag{2.29}$$

Denote $I' = [\alpha_1, \alpha],\ I'' = [\beta, \beta_1]$. Then the equality $\varphi_I = \varphi_{I_1} = 0$ implies

$$\int_{I'} |\varphi(x)|\, dx = \int_{I''} |\varphi(x)|\, dx. \tag{2.30}$$

Hence

$$\frac{1}{|I_1|}\int_{I_1} |\varphi(x)|^p\, dx - \frac{1}{|I|}\int_I |\varphi(x)|^p\, dx =$$

$$= \frac{1}{|I_1|}\left[|I'|\left(\frac{1}{|I'|}\int_{I'} |\varphi(x)|^p\, dx - \frac{1}{|I|}\int_I |\varphi(x)|^p\, dx\right) + \right.$$

$$\left. + |I''|\left(\frac{1}{|I''|}\int_{I''} |\varphi(x)|^p\, dx - \frac{1}{|I|}\int_I |\varphi(x)|^p\, dx\right)\right] =$$

$$= \frac{1}{|I_1|}\left[\int_{I'} |\varphi(x)|^p\, dx + \int_{I''} |\varphi(x)|^p\, dx - \frac{(|I'|+|I''|)}{|I|}\int_I |\varphi(x)|^p\, dx\right].$$

Now using the equality

$$|I''| = |I'|\frac{\frac{1}{|I'|}\int_{I'} |\varphi(x)|\, dx}{\frac{1}{|I''|}\int_{I''} |\varphi(x)|\, dx},$$

which follows from (2.30), we have

$$\frac{1}{|I_1|}\int_{I_1} |\varphi(x)|^p\, dx - \frac{1}{|I|}\int_I |\varphi(x)|^p\, dx =$$

$$= \frac{|I'|}{|I_1|}\left[\frac{1}{|I'|}\int_{I'} |\varphi(x)|^p\, dx + \frac{\frac{1}{|I'|}\int_{I'} |\varphi(x)|\, dx}{\frac{1}{|I''|}\int_{I''} |\varphi(x)|\, dx}\cdot\frac{1}{|I''|}\int_{I''} |\varphi(x)|^p\, dx - \right.$$

$$- \frac{\frac{1}{|I''|}\int_{I''}|\varphi(x)|\,dx + \frac{1}{|I'|}\int_{I'}|\varphi(x)|\,dx}{\frac{1}{|I''|}\int_{I''}|\varphi(x)|\,dx} \cdot \frac{1}{|I|}\int_I |\varphi(x)|^p\,dx\Bigg].$$

By (2.28),

$$\frac{1}{|I_1|}\int_{I_1}|\varphi(x)|^p\,dx - \frac{1}{|I|}\int_I|\varphi(x)|^p\,dx \ge$$

$$\ge \frac{|I'|}{|I_1|}\Bigg[\frac{1}{|I'|}\int_{I'}|\varphi(x)|^p\,dx + \frac{\frac{1}{|I'|}\int_{I'}|\varphi(x)|\,dx}{\frac{1}{|I''|}\int_{I''}|\varphi(x)|\,dx}\frac{1}{|I''|}\int_{I''}|\varphi(x)|^p\,dx -$$

$$-\frac{1}{|I'|}\int_{I'}|\varphi(x)|\,dx\left(\left(\frac{1}{|I'|}\int_{I'}|\varphi(x)|\,dx\right)^{p-1} + \left(\frac{1}{|I''|}\int_{I''}|\varphi(x)|\,dx\right)^{p-1}\right)\Bigg] =$$

$$= \frac{|I'|}{|I_1|}\Bigg[\frac{1}{|I'|}\int_{I'}|\varphi(x)|^p\,dx - \left(\frac{1}{|I'|}\int_{I'}|\varphi(x)|\,dx\right)^p +$$

$$+ \frac{\frac{1}{|I'|}\int_{I'}|\varphi(x)|\,dx}{\frac{1}{|I''|}\int_{I''}|\varphi(x)|\,dx}\left(\frac{1}{|I''|}\int_{I''}|\varphi(x)|^p\,dx - \left(\frac{1}{|I''|}\int_{I''}|\varphi(x)|\,dx\right)^p\right)\Bigg] \ge 0,$$

where the last inequality follows from the Hölder inequality. Thus we have proved (2.29) and Property 2.16 follows. □

Remark 2.19. The proof of inequality (2.24) that we have just presented fails if $0 < p < 1$. We do not know whether (2.24) holds true for $0 < p < 1$.

2.2 Definition of the BMO-class and Examples

We will say that the function $f \in L_{loc}$ is *of bounded mean oscillation*, if

$$\|f\|_* \equiv \sup_Q \Omega(f;Q) < \infty.$$

Here the supremum is taken over all cubes $Q \subset \mathbb{R}^d$. The class of all such functions f is denoted by BMO. This class was first defined in the work by F. John and L. Nirenberg [30] in 1961. In this paper they obtained the John–Nirenberg theorem, which plays the fundamental role for BMO. We will consider it later. Here we will study some elementary properties, implied by the properties of mean oscillations, considered in the previous section. In addition we will consider several examples that will be of use in what follows.

Denote

$$\|f\|'_* = \sup_Q \Omega'(f;Q), \quad \|f\|''_* = \sup_Q \Omega''(f;Q),$$

where, as before, the supremuma are taken over all cubes $Q \subset \mathbb{R}^d$. Then from (2.4) and (2.5) it follows immediately that

$$\|f\|'_* \le \|f\|_* \le 2\|f\|'_*,$$

$$\|f\|_* \le \|f\|''_* \le 2\|f\|_*.$$

This means that in the definition of the BMO-class one can use the mean oscillations $\Omega'(f;Q)$ and $\Omega''(f;Q)$ instead of $\Omega(f;Q)$.

Let $Q \subset \mathbb{R}^d$ be a cube. Then there exist two balls B_1 and B_2 such that $B_1 \subset Q \subset B_2$ and $c''_d|B_2| \le |Q| \le c'_d|B_1|$, where the positive constants c'_d and c''_d depend only on the dimension d of the space. We have

$$\Omega(f;Q) \le \frac{1}{|Q|^2}\int_Q\int_Q |f(x)-f(y)|\,dy\,dx \le \frac{2}{|Q|}\int_Q |f(x)-f_{B_2}|\,dx \le$$

$$\le \frac{2}{c''_d}\frac{1}{|B_2|}\int_{B_2}|f(x)-f_{B_2}|\,dx = \frac{2}{c''_d}\Omega(f;B_2),$$

$$\Omega(f;B_1) \le \frac{1}{|B_1|^2}\int_{B_1}\int_{B_1}|f(x)-f(y)|\,dy\,dx \le \frac{2}{|B_1|}\int_{B_1}|f(x)-f_Q|\,dx \le$$

$$\le 2c'_d\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx = 2c'_d\Omega(f;Q).$$

From these two inequalities we see that the mean oscillations in the definition of the BMO-class can be calculated over all possible balls $B \subset \mathbb{R}^d$.

For any fixed cube $Q_0 \subset \mathbb{R}^d$ we will denote by $BMO \equiv BMO(Q_0)$ the class of functions f such that

$$\|f\|_* = \sup_{Q\subset Q_0}\Omega(f;Q) < \infty$$

where the supremum is taken over all cubes $Q \subset Q_0$. Notice that in order to show that the function f belongs to $BMO(Q_0)$ it is enough to assume the boundedness of the oscillations only on the small enough cubes. Indeed, if we fix some δ, $0<\delta<1$, then for the cube $Q \subset Q_0$ with $|Q| \ge \delta|Q_0|$

$$\Omega(f;Q) \le \Omega''(f;Q) \le \left(\frac{|Q_0|}{|Q|}\right)^2 \frac{1}{|Q_0|^2}\int_{Q_0}\int_{Q_0}|f(x)-f(y)|\,dy\,dx \le$$

$$\le \frac{2}{\delta^2}\frac{1}{|Q_0|}\int_{Q_0}|f(x)|\,dx < \infty.$$

Obviously, $BMO(Q_0) \supset BMO(\mathbb{R}^d)$ for any cube $Q_0 \subset \mathbb{R}^d$, and if $\|f\|_* \le c$ in each $BMO(Q_0)$ with the constant c being independent on the cube $Q_0 \subset \mathbb{R}^d$, then $\|f\|_* \le c$ in $BMO(\mathbb{R}^d)$.

It is clear that the equality $\|f\|_* = 0$ means that the function f is constant a. e. It is also obvious that for any constant c

$$\|f + c\|_* = \|f\|_*.$$

Further, Property 2.14 implies, that for any functions $f,\ g$ and a number λ

$$\|f + g\|_* \le \|f\|_* + \|g\|_*,$$

$$\|\lambda f\|_* = |\lambda| \cdot \|f\|_*,$$

i.e., $\| \cdot \|_*$ is a norm of the space of functions of bounded mean oscillation, factorized by the set of all constant functions.

From Property 2.6 of the mean oscillations it follows immediately that if f belongs to BMO, then $|f| \in BMO$ and

$$\|\,|f|\,\|_* \le 2\|f\|_*.$$

The opposite it not true. We will see it later (see Example 2.21).

By virtue of (2.8), every essentially bounded function f belongs to BMO and

$$\|f\|_* \le \|f\|_\infty.$$

However, BMO contains also unbounded functions. A typical example of an unbounded BMO-function is the logarithmic function.

Example 2.20. Assume $f(x) = \ln \frac{1}{|x|},\ x \in \mathbb{R}^d$. Let us show that $f \in BMO(\mathbb{R}^d)$. As we have already noticed above, it is enough to estimate the oscillations over all possible balls $B \subset \mathbb{R}^d$.

Fix some ball $B \equiv B(x_0, r) \subset \mathbb{R}^d$. If $r \le \frac{|x_0|}{2}$, then for any $x,\ y \in B$ we obviously have $\frac{1}{3} \le \frac{|y|}{|x|} \le 3$, and hence

$$\Omega(f; B) \le \frac{1}{|B|^2} \int_B \int_B |f(x) - f(y)|\, dy\, dx = \frac{1}{|B|^2} \int_B \int_B \left| \ln \frac{|y|}{|x|} \right| dy\, dx \le \ln 3.$$

Now let us assume that $r > \frac{|x_0|}{2}$. Denote $R = 3r$. Then $B_1 \equiv B_1(0, R) \supset B$ and

$$\Omega(f; B) \le 2\Omega'(f; B) = 2 \inf_c \frac{1}{|B|} \int_B |f(x) - c|\, dx \le$$

$$\le \frac{2}{|B|} \int_B \left| \ln \frac{1}{|x|} - \ln \frac{1}{R} \right| dx = 2 \frac{|B_1|}{|B|} \frac{1}{|B_1|} \int_{B_1} \ln \frac{R}{|x|}\, dx =$$

$$= 2 \cdot 3^d \frac{1}{e_d R^d} \int_{B_1} \ln \frac{R}{|x|}\, dx.$$

Here e_d is the volume of the unit ball in $\mathbb{R}^d$ ($e_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$, where Γ is the Euler gamma-function (see [12, p. 393])).

Since the integral

$$\frac{1}{R^d} \int_{B_1} \ln \frac{R}{|x|}\, dx = 2 \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^1 \rho^{d-1} \ln \frac{1}{\rho}\, d\rho$$

does not depend on the radius R of the ball the oscillations over all possible balls $B \subset \mathbb{R}^d$ can be dominated by some constant, which depends only on the dimension d of the space. This means that $f \in BMO$. We remark that, at least to our knowledge, even in the one dimensional case the problem of calculation of $\|f\|_*$ is still open.

In the particular case $d = 1$ the function $f(x) = \ln \frac{1}{|x|}$ for $-\infty < x < \infty$ belongs to the BMO-class. Since as we have already mentioned above $BMO([-1,1]) \supset BMO(\mathbb{R})$ it follows that $f \in BMO([-1,1])$. □

Example 2.21. Let us show that the function $g(x) = \operatorname{sign}(x) \cdot \ln \frac{1}{|x|}$ does not belong to $BMO([-1,1])$. Indeed, for $0 < h < 1$ and $I \equiv [-h,h]$ we have $g_I = 0$ and

$$\Omega(g;I) = \frac{1}{2h} \int_{-h}^{h} \left| \ln \frac{1}{|x|} \right| dx = \frac{1}{h} \int_0^h \ln \frac{1}{x}\, dx = 1 + \ln \frac{1}{h} \to \infty, \quad h \to 0.$$

This example shows that if the absolute value of a function belongs to the BMO-class, this does not imply that the function itself is a BMO-function. □

For $1 \le p < \infty$ we will denote by BMO_p the class of all functions $f \in L^p_{loc}$ such that

$$\|f\|_{*,p} \equiv \sup_Q \Omega_p(f;Q) < \infty.$$

The Hölder inequality implies that $BMO_p \subset BMO_q$ for $1 \le q < p < \infty$. Later we will show that all the classes BMO_p, $1 \le p < \infty$, coincide (see Remark 3.19).

Properties 2.15 and 2.16 essentially simplify the calculation of $\|f\|_{*,p}$ if f is a monotone function.

Lemma 2.22. *Let $1 \le p < \infty$, and let $f \in L^p_{loc}(\mathbb{R}_+)$ be non-increasing on $\mathbb{R}_+ \equiv [0,+\infty)$. Then*

$$\|f\|_{*,p} = \sup_{\beta > 0} \Omega_p(f;[0,\beta]).$$

Proof. Let $I \equiv [a,b] \subset \mathbb{R}_+$. If $f_I = f(b)$, then obviously $\Omega_p(f;I) = 0$. Otherwise, if $f_I > f(b)$, then using the continuity of the integral with respect to the upper limit we can find $\beta \ge b$ such that

$$\frac{1}{\beta}\int_0^\beta f(x)\,dx = f_I.$$

Since $[0,\beta] \supset I$ and $f_{[0,\beta]} = f_I$ according to Property 2.16 we have

$$\Omega_p(f;I) \le \Omega_p(f;[0,\beta]),$$

provided f is monotone. Obviously this implies the statement of the lemma. □

For $\mathbb{R}$ the following analog of Lemma 2.22 is valid.

Lemma 2.23. *Let $1 \le p < \infty$, and assume that $f \in L^p_{loc}(\mathbb{R})$ is non-decreasing on $(-\infty,0)$ and non-increasing on $(0,+\infty)$. Then*

$$\|f\|_{*,p} = \sup_{\alpha<0<\beta} \Omega_p(f;[\alpha,\beta]).$$

Proof. It is enough to show that for any interval $I' \subset (-\infty,0)\cup(0,+\infty)$ and for any $\varepsilon > 0$ there exists a interval $I \equiv [\alpha,\beta]$ such that $\alpha < 0 < \beta$ and

$$\Omega_p(f;I) > \Omega_p(f;I') - \varepsilon.$$

For instance, let $I' \subset (0,+\infty)$. Similarly to the proof of Lemma 2.22, for $\Omega_p(f;I') > 0$ we find $\beta > 0$ such that $I' \subset [0,\beta]$ and $f_{[0,\beta]} = f_{I'}$. Then, since f is monotone on $(0,+\infty)$, by Property 2.16,

$$\Omega_p(f;[0,\beta]) \ge \Omega_p(f;I').$$

Further, since the absolute continuity of the integral implies the continuity of the function $\varphi(\tau) \equiv \Omega_p(f;[\tau,\beta])$, $\tau < \beta$, one can find some $\alpha < 0$ such that

$$\Omega_p(f;[\alpha,\beta]) > \Omega_p(f;I') - \varepsilon.$$

The case $I' \subset (-\infty,0)$ can be treated analogously. □

Example 2.24. Assume $f(x) = \ln\frac{1}{x}$, $x \in \mathbb{R}_+$. Let us calculate $\|f\|_*$. By Lemma 2.22, in order to compute $\|f\|_*$ it is enough to take the supremum only over the intervals of the form $[0,\beta]$, $\beta > 0$.

For $\beta > 0$

$$f_{[0,\beta]} = \frac{1}{\beta}\int_0^\beta \ln\frac{1}{x}\,dx = 1 + \ln\frac{1}{\beta},$$

$$\Omega(f;[0,\beta]) = \frac{1}{\beta}\int_0^\beta \left|\ln\frac{1}{x} - \left(1+\ln\frac{1}{\beta}\right)\right| dx =$$

$$= \frac{1}{\beta}\int_0^\beta \left|\ln\frac{\beta}{\mathrm{e}x}\right| dx = \int_0^1 \left|\ln\frac{1}{\mathrm{e}x}\right| dx.$$

By Property 2.1,

$$\Omega(f;[0,\beta]) = 2\int_0^{1/\mathrm{e}} \left(\ln\frac{1}{x} - 1\right)\,dx = \frac{2}{\mathrm{e}}.$$

Finally, for the function $f(x) = \ln\frac{1}{x},\ 0 < x < \infty$,

$$\|f\|_* = \frac{2}{\mathrm{e}}.$$

It is easy to see that this equality still holds true if we consider the norm $\|f\|_*$ in the space $BMO([0,\beta_0])$ for any $\beta_0 > 0$. Recall that, as we have already mentioned above, we do not know the norm $\|g\|_*$ for the function $g(x) = \ln\frac{1}{|x|},\ -\infty < x < \infty$. However, it is easy to show that

$$\frac{2}{\mathrm{e}} < \|g\|_* \le \frac{4}{\mathrm{e}},$$

(see Example 2.29). □

Concerning Example 2.24, it is interesting to remark that $BMO(\mathbb{R})$ does not contain monotone unbounded functions. Namely,

Proposition 2.25. *If the function $f \in BMO(\mathbb{R})$ is monotone, then it is bounded.*

Proof. Let us assume the opposite, for example, assume that the function f is non-decreasing and unbounded from above. We can assume that $f(0) = 0$. Fix some $B > \lim_{x\to 0+} f(x) \ge 0$. Using the continuity of the integral with respect to the upper limit, we can find some $x_0 > 0$ such that $f_{[0,x_0]} = B$. Further, let us find $c \in [0, x_0]$ such that $f(x) \ge B$ if $x \ge c$ and $f(x) \le B$ if $x \le c$. Obviously, for every $b > x_0$ there exists $a \equiv a(b) \le 0$ such that $f_{[a,b]} = B$. Clearly, $a(b) \to -\infty$ as $b \to +\infty$. So, we can choose b to be big enough to provide the inequalities

$$\frac{1}{b-c}\int_c^b (f(x) - B)\,dx > \frac{B}{2}, \qquad \frac{|a|}{c-a} > \frac{1}{2}.$$

Then, by Property 2.1,

$$\Omega(f;[a,b]) = 2\frac{b-c}{b-a}\frac{1}{b-c}\int_c^b (f(x) - B)\,dx > B\frac{b-c}{b-a},$$

$$\Omega(f;[a,b]) = \frac{2}{b-a}\int_a^c (B - f(x))\,dx \ge 2\frac{c-a}{b-a}\frac{1}{c-a}\int_a^0 B\,dx =$$

$$= 2\frac{c-a}{b-a}\frac{|a|}{c-a}B > \frac{c-a}{b-a}B.$$

Summing up these two inequalities, we get

$$\Omega(f;[a,b]) \geq \frac{1}{2}B.$$

Since we can choose B arbitrarily big $f \notin BMO(\mathbb{R})$. □

The next statement provides the rule for the calculation of $\|f\|_*$ for any monotone function f on $\mathbb{R}$.

Proposition 2.26. *If f is a monotone function on $\mathbb{R}$, then*

$$\|f\|_* = \frac{1}{2}\left|\lim_{x\to+\infty} f(x) - \lim_{x\to-\infty} f(x)\right|. \tag{2.31}$$

Proof. If the right-hand side of (2.31) (we will denote it by B) is infinite, then, by Proposition 2.25, $\|f\|_* = \infty$. Assume that $B < \infty$. It is enough to consider the case when f is non-decreasing and satisfy

$$\lim_{x\to+\infty} f(x) = -\lim_{x\to-\infty} f(x) = B > 0.$$

Let us choose some b_0 such that $f(x) \geq 0$ if $x \geq b_0$ and $f(x) \leq 0$ if $x \leq b_0$. Further, for a big enough $b > b_0$ there exists $a \equiv a(b) < b_0$ such that $f_{[a,b]} = 0$. Moreover, $a(b) \to -\infty$ as $b \to +\infty$. Fix some $\varepsilon > 0$ and choose $b > b_0$ so big, that

$$\frac{1}{b-b_0}\int_{b_0}^{b} f(x)\,dx > B - \varepsilon, \qquad \frac{1}{b_0-a}\int_{a}^{b_0} f(x)\,dx < -B + \varepsilon.$$

Then we have $f_{[a,b]} = 0$ and

$$\Omega(f;[a,b]) = \frac{1}{b-a}\int_a^b |f(x)|\,dx \geq \frac{b-b_0}{b-a}\frac{1}{b-b_0}\int_{b_0}^{b} f(x)\,dx-$$

$$-\frac{b_0-a}{b-a}\frac{1}{b_0-a}\int_a^{b_0} f(x)\,dx \geq \frac{b-b_0}{b-a}(B-\varepsilon) + \frac{b_0-a}{b-a}(B-\varepsilon) = B - \varepsilon.$$

Then $\|f\|_* \geq B$, provided ε is arbitrary. On the other hand, the inequality $\|f\|_* \leq B$ follows immediately from Property 2.7. □

Example 2.27. Again, let $f(x) = \ln\frac{1}{x},\ 0 < x < \infty$. Let us show that if p is natural, then

$$\|f\|_{*,p} = \left\{\frac{p!}{\mathrm{e}}\left[1 + (-1)^{p-1}\left(1 - \mathrm{e}\sum_{k=0}^{p}\frac{(-1)^k}{k!}\right)\right]\right\}^{1/p}. \tag{2.32}$$

Indeed, as in Example 2.24, using Lemma 2.22 it is easy to see that

$$\|f\|_{*,p}^p = \Omega_p^p(f;[0,1]) = \int_0^1 \left|\ln\frac{1}{x} - 1\right|^p dx =$$

$$= \int_0^{1/\mathrm{e}} \ln^p \frac{1}{\mathrm{e}x}\, dx + \int_{1/\mathrm{e}}^1 \ln^p(\mathrm{e}x)\, dx = \frac{1}{\mathrm{e}}\left[\int_0^\infty x^p \mathrm{e}^{-x}\, dx + \int_0^1 x^p \mathrm{e}^x\, dx\right].$$

The first integral in the brackets in the right-hand side is equal to $\Gamma(p+1) = p!$ with Γ being the Euler gamma-function. In order to compute the second integral, we denote it by $J(p)$ and perform the integration by parts. Then

$$J(p) = \mathrm{e} - pJ(p-1), \quad p \geq 2, \tag{2.33}$$

and

$$J(1) = \int_0^1 x\mathrm{e}^x\, dx = 1.$$

So, to conclude the proof of (2.32) it remains to show that

$$J(p) = (-1)^{p-1} p! \left(1 - \mathrm{e}\sum_{k=0}^{p} \frac{(-1)^k}{k!}\right). \tag{2.34}$$

We prove this equality by induction. Formula (2.34) is true for $p = 1$. By (2.33),

$$J(p) = \mathrm{e} - pJ(p-1) = \mathrm{e} - p\left[(-1)^p (p-1)! \left(1 - \mathrm{e}\sum_{k=0}^{p-1} \frac{(-1)^k}{k!}\right)\right] =$$

$$= (-1)^{p-1} p! + \mathrm{e}\left[1 + (-1)^p p! \sum_{k=0}^{p-1} \frac{(-1)^k}{k!}\right] =$$

$$= (-1)^{p-1} p! + (-1)^{p-1}\mathrm{e}\left[(-1)^{p-1} - p!\sum_{k=0}^{p-1} \frac{(-1)^k}{k!}\right] =$$

$$= (-1)^{p-1} p! + (-1)^{p-1}\mathrm{e}p!\left[-\sum_{k=0}^{p-1} \frac{(-1)^k}{k!} - \frac{(-1)^p}{p!}\right] =$$

$$= (-1)^{p-1} p! \left(1 - \mathrm{e}\sum_{k=0}^{p} \frac{(-1)^k}{k!}\right).$$

This completes the proof of equality (2.32). □

Example 2.28. Let us show that for any α, $0 < \alpha < 1$, the function $f(x) = x^{-\alpha}$, $0 < x \leq 1$, does not belong to $BMO([0,1])$. As we will see later, this fact can be easily derived from the John–Nirenberg theorem. Here we prove it using just the definition and the elementary properties of oscillations.

For $0 < h < 1$ we have $f_{[0,h]} = \frac{1}{h}\int_0^h x^{-\alpha}\,dx = \frac{h^{-\alpha}}{1-\alpha}$ and

$$\Omega(f;[0,h]) = \frac{1}{h}\int_0^h \left|f(x) - f_{[0,h]}\right|\,dx = \frac{2}{h}\int_0^{h(1-\alpha)^{1/\alpha}} \left(x^{-\alpha} - \frac{h^{-\alpha}}{1-\alpha}\right)\,dx =$$

$$= \frac{2}{h}\frac{1}{1-\alpha}h^{1-\alpha}\left[(1-\alpha)^{(1-\alpha)/\alpha} - (1-\alpha)^{1/\alpha}\right] =$$

$$= 2\alpha(1-\alpha)^{1/\alpha-2}h^{-\alpha} \to \infty, \quad h \to 0.$$

Therefore, $f \notin BMO([0,1])$. □

Example 2.29. Let us show that if the function f is even on $\mathbb{R}$, then

$$\|f\|_* \le 2 \sup_{I\subset[0,+\infty)} \Omega(f;I). \tag{2.35}$$

Assume that zero is an inner point of the interval $J \subset \mathbb{R}$. Denote $I' = [0,+\infty) \cap J$, $I'' = (-\infty,0] \cap J$. Let $|I'| \ge |I''|$ and denote $J' = [-|I'|, |I'|]$. Then $J' \supset J$, $|I'| \le |J| \le |J'| = 2|I'|$, and since f is even we have $f_{J'} = f_{I'}$. If $f_{J'} \le f_J$, then, by Property 2.1,

$$\Omega(f;J) = \frac{2}{|J|}\int_{\{x\in J:\ f(x)>f_J\}} (f(x) - f_J)\,dx \le$$

$$\le \frac{2}{|J|}\int_{\{x\in J:\ f(x)>f_{J'}\}} (f(x) - f_{J'})\,dx \le$$

$$\le \frac{2}{|I'|}\int_{\{x\in J':\ f(x)>f_{J'}\}} (f(x) - f_{J'})\,dx =$$

$$= \frac{4}{|J'|}\int_{\{x\in J':\ f(x)>f_{J'}\}} (f(x) - f_{J'})\,dx.$$

The expression in the right-hand side is equal to $2\Omega(f;I')$, provided f is even. Hence

$$\Omega(f;J) \le 2\Omega(f;I').$$

Otherwise, if $f_{J'} > f_J$, then, again by virtue of Property 2.1,

$$\Omega(f;J) = \frac{2}{|J|}\int_{\{x\in J:\ f(x)<f_J\}} (f_J - f(x))\,dx \le$$

$$\le \frac{2}{|J|}\int_{\{x\in J:\ f(x)<f_{J'}\}} (f_{J'} - f(x))\,dx \le$$

$$\le \frac{2}{|I'|}\int_{\{x\in J':\ f(x)<f_{J'}\}} (f_{J'} - f(x))\,dx =$$

$$= \frac{4}{|J'|} \int_{\{x\in J':\ f(x)<f_{J'}\}} (f_{J'} - f(x))\ dx = 2\Omega(f;I').$$

In the case $|I''| > |I'|$ in the same manner we obtain

$$\Omega(f;J) \le 2\Omega(f;I'') = 2\Omega(f;[0,|I''|]).$$

These inequalities obviously imply (2.35). □

Example 2.30. Let $\gamma > 0$. Then the function

$$f(x) = \ln(1+|x|)\chi_{[0,+\infty)}(x) + \ln(1+\gamma|x|)\chi_{(-\infty,0)}(x), \quad x \in \mathbb{R},$$

is contained in $BMO(\mathbb{R})$.

Denote $g(x) = \ln x,\ x \in [0,+\infty)$. In Example 2.24 we showed that $g \in BMO([0,+\infty))$ and $\|g\|_* = \frac{2}{\mathrm{e}}$. Now we will use this fact. Let us prove that $\|f\|_* \le C_\gamma$, where the constant C_γ depends only on γ.

If $[a,b] \subset [0,+\infty)$, then

$$\Omega(f;[a,b]) = \Omega(g;[a+1,b+1]) \le \frac{2}{\mathrm{e}}.$$

Analogously, if $[a,b] \subset (-\infty,0]$, then

$$\Omega(f;[a,b]) = \frac{1}{b-a}\int_a^b \left|\ln(1-\gamma x) - \frac{1}{b-a}\int_a^b \ln(1-\gamma y)\,dy\right| dx =$$

$$= \frac{1}{(1-\gamma a)-(1-\gamma b)}\int_{1-\gamma b}^{1-\gamma a}\left|\ln t - \frac{1}{(1-\gamma a)-(1-\gamma b)}\int_{1-\gamma b}^{1-\gamma a}\ln u\,du\right| dt =$$

$$= \Omega(g;[1-\gamma b, 1-\gamma a]) \le \frac{2}{\mathrm{e}}.$$

It remains to consider the case $a < 0 < b$. If $b \le \gamma|a|$, we set $b_1 = \gamma|a|$ and obtain $I_1 \equiv [a,b_1] \supset [a,b] \equiv I$,

$$|I_1| = b_1 - a = (\gamma+1)|a| \le (\gamma+1)(b+|a|) = (\gamma+1)|I|.$$

Otherwise, if $b > \gamma|a|$, we set $a_1 = -\frac{b}{\gamma}$ and then $I_1 \equiv [a_1,b] \supset [a,b]$ and

$$|I_1| = b - a_1 = \left(1+\frac{1}{\gamma}\right)b \le \left(1+\frac{1}{\gamma}\right)(b+|a|) = \left(1+\frac{1}{\gamma}\right)|I|.$$

Setting $c_\gamma = 1+\gamma+\frac{1}{\gamma}$, in both cases we obtain the segment $I_1 \supset I$ such that $|I_1| \le c_\gamma|I|$. Therefore

$$\Omega(f;I) \le \Omega''(f;I) = \frac{1}{|I|^2}\int_I\int_I |f(x)-f(y)|\,dx\,dy \le$$

$$\leq \frac{|I_1|^2}{|I|^2}\frac{1}{|I_1|^2}\int_{I_1}\int_{I_1}|f(x)-f(y)|\,dx\,dy \leq c_\gamma^2 \Omega''(f;I_1) \leq 2c_\gamma^2\Omega(f;I_1).$$

Notice, that the interval I_1 has the form $[-b,\gamma b]$ with $b>0$. So, it remains to estimate the oscillations of the function f over the intervals $I \equiv [-b,\gamma b]$. We have

$$f_I = \frac{1}{(\gamma+1)b}\int_{-b}^{\gamma b} f(x)\,dx =$$

$$= \frac{1}{(\gamma+1)b}\int_{-b}^{0}\ln(1-\gamma x)\,dx + \frac{1}{(\gamma+1)b}\int_{0}^{\gamma b}\ln(1+x)\,dx =$$

$$= \frac{1}{\gamma(\gamma+1)b}\int_{1}^{1+\gamma b}\ln t\,dt + \frac{1}{(\gamma+1)b}\int_{1}^{1+\gamma b}\ln t\,dt =$$

$$= \frac{1}{\gamma+1}g_{[1,1+\gamma b]} + \frac{\gamma}{\gamma+1}g_{[1,1+\gamma b]} = g_{[1,1+\gamma b]},$$

$$f_{[0,\gamma b]} = \frac{1}{\gamma b}\int_{0}^{\gamma b}\ln(1+x)\,dx = \frac{1}{\gamma b}\int_{1}^{1+\gamma b}\ln t\,dt = g_{[1,1+\gamma b]},$$

so that $f_{[-b,0]} = f_{[0,\gamma b]} = f_I = g_{[1,1+\gamma b]}$. From here, by Property 2.1,

$$\Omega(f;I) = \frac{2}{|I|}\int_{\{x\in I:\ f(x)>f_I\}}(f(x)-f_I)\ dx =$$

$$= \frac{2}{|I|}\int_{\{x\in[-b,0]:\ f(x)>f_{[-b,0]}\}}\left(f(x)-f_{[-b,0]}\right)\ dx +$$

$$+\frac{2}{|I|}\int_{\{x\in[0,\gamma b]:\ f(x)>f_{[0,\gamma b]}\}}\left(f(x)-f_{[0,\gamma b]}\right)\ dx =$$

$$= \frac{b}{|I|}\Omega(f;[-b,0]) + \frac{\gamma b}{|I|}\Omega(f;[0,\gamma b]).$$

But, as we have already shown above,

$$\Omega(f;[-b,0]) \leq \frac{2}{\mathrm{e}},\quad \Omega(f;[0,\gamma b]) \leq \frac{2}{\mathrm{e}},$$

so that

$$\Omega(f;I) \leq \frac{2}{\mathrm{e}}\left(\frac{b}{|I|}+\frac{\gamma b}{|I|}\right) = \frac{2}{\mathrm{e}}.$$

Finally,

$$\|f\|_* \leq \frac{4}{\mathrm{e}}c_\gamma^2 \equiv C_\gamma,$$

and this completes the analysis of this example. □

Example 2.31. If $f \in BMO([a_0, b_0])$ with $-\infty \le a_0 < b_0 \le +\infty$, then for any numbers $\alpha \ne 0$ and β the function $g(x) = f(\alpha x + \beta)$ belongs to $BMO([a_1, b_1])$ and $\|f\|_* = \|g\|_*$. Here we use the notations $a_1 = \min\left(\frac{a_0-\beta}{\alpha}, \frac{b_0-\beta}{\alpha}\right)$, $b_1 = \max\left(\frac{a_0-\beta}{\alpha}, \frac{b_0-\beta}{\alpha}\right)$. In other words, *the linear change of variable does not change the BMO-norm of functions.*

Indeed, it follows immediately from the following obvious equality:

$$\Omega(f; [a, b]) = \Omega(g; [a', b']),$$

where $[a, b] \subset [a_0, b_0]$, $a' = \min\left(\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}\right)$, $b' = \max\left(\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}\right)$, $[a', b'] \subset [a_1, b_1]$. $\square$

Until now we have considered the oscillations of functions on the cubes $Q \subset \mathbb{R}^d$ and the BMO-class, related to them. It is easy to see that the properties of oscillations, considered in the previous section, hold true if we replace the cubes Q by the multidimensional segments R. We will denote by BMO^R the class of all functions $f \in L_{loc}(\mathbb{R}^d)$ such that

$$\|f\|_{*,R} \equiv \sup_R \Omega(f; R) < \infty.$$

Here the supremum in taken over all possible multidimensional segments $R \subset \mathbb{R}^d$. Analogously one can define the class $BMO^R(R_0)$ with respect to a fixed segment $R_0 \subset \mathbb{R}^d$. The classes BMO^R are called *the anisotropic BMO-classes.*

If $d = 1$, then obviously $BMO^R = BMO$. Further, $BMO \supset BMO^R$ since a cube is a particular case of a segment. The next example shows that if $d \ge 2$, the classes BMO and BMO^R do not coincide.

Example 2.32 of a function $f \in BMO$, which does not belong to BMO^R.

Let us consider the case $d = 2$. Set

$$f(x) = \sum_{k=1}^{\infty} \chi_{[0,2^{-k+1}]\times\left[0,\frac{1}{k}\right]}(x), \quad x \equiv (x_1, x_2) \in [0, 1]^2 \equiv Q_0.$$

As it was noticed before (see p. 40), in order to prove that f belongs to $BMO(Q_0)$ it is enough to prove that the mean oscillations of f are bounded with respect to all cubes $Q \subset Q_0$ such that $l(Q) \le 2^{-10}$. So, we will consider only such cubes.

Let $Q \equiv [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subset Q_0$.

If $\alpha_1 > \frac{1}{2}\beta_1$, then it is easy to see that there exists an integer k such that $2^{-k-1} \le \alpha_1 < \beta_1 \le 2^{-k+1}$. If $k \ge 8$, then for $m = 0, 1, \ldots, k$

$$\beta_2 - \alpha_2 = l(Q) = \beta_1 - \alpha_1 \le 3 \cdot 2^{-k-1} \le \frac{1}{(k+1)(k+2)} \le \frac{1}{m+1} - \frac{1}{m+2},$$

and if $k < 8$, then

$$\beta_2 - \alpha_2 = l(Q) \le 2^{-10} \le \frac{1}{(k+1)(k+2)} \le \frac{1}{m+1} - \frac{1}{m+2}, \quad m = 0, 1, \dots, k.$$

Therefore for any two points $x, y \in Q$ we have the inequality $|f(x) - f(y)| \le 1$, so that in this case (see Property 2.5)

$$\Omega(f; Q) \le 1.$$

It remains to consider the case $\alpha_1 \le \frac{1}{2}\beta_1$. Choose an integer k such that $2^{-k} \le \beta_1 \le 2^{-k+1}$. Then

$$l(Q) = \beta_1 - \alpha_1 \ge \frac{1}{2}\beta_1 \ge 2^{-k-1} = \frac{1}{4}2^{-k+1}.$$

Let us denote by Q' the cube such that its projections on the Ox_1 and Ox_2-axes are equal to $[0, 2^{-k+1}]$ and $[\alpha_2', \beta_2'] \supset [\alpha_2, \beta_2]$ respectively, and $\beta_2' - \alpha_2' = 2^{-k+1}$. Then $Q' \supset Q,\ l(Q') \le 4l(Q)$, so that

$$\Omega(f; Q) \le \frac{1}{|Q|^2} \int_Q \int_Q |f(x) - f(y)|\, dx\, dy \le$$

$$\le \frac{|Q'|^2}{|Q|^2} \frac{1}{|Q'|^2} \int_{Q'} \int_{Q'} |f(x) - f(y)|\, dx\, dy \le 4^4 \cdot 2\Omega(f; Q').$$

So, it remains to check the boundedness of the mean oscillations of the function f over all possible cubes $Q \subset Q_0$ of the form

$$Q = \left[0, 2^{-k+1}\right] \times \left[\alpha, \alpha + 2^{-k+1}\right], \quad 0 \le \alpha \le 1 - 2^{-k+1},\ k \ge 8. \tag{2.36}$$

Let the cube Q be of the form (2.36). If $\alpha \ge \frac{1}{k+1}$, then from the inequality

$$l(Q) = 2^{-k+1} \le \frac{1}{k(k+1)} \le \frac{1}{m} - \frac{1}{m+1}, \quad m = 1, 2, \dots, k,$$

it follows that for any two points $x, y \in Q$ we have $|f(x) - f(y)| \le 1$. That is, again we obtain

$$\Omega(f; Q) \le 1.$$

If $\alpha < \frac{1}{k+1}$, then the inequality

$$\alpha + l(Q) \le \frac{1}{k+1} + 2^{-k+1} \le \frac{1}{k}$$

imply that

$$\inf_{x \in Q} f(x) = k.$$

Therefore (see Property 2.4)

$$\Omega(f; Q) \le 2 \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(x) - c|\, dx \le \frac{2}{|Q|} \int_Q \left(f(x) - \inf_{y \in Q} f(y) \right) dy =$$

$$= 2 \cdot \frac{1}{2^{-k+1}} \frac{1}{2^{-k+1}} \int_0^{2^{-k+1}} dx_1 \int_\alpha^{\alpha+2^{-k+1}} (f(x_1,x_2) - k)\ dx_2 =$$

$$= 2 \cdot \frac{1}{2^{-k+1}} \frac{1}{2^{-k+1}} \sum_{s=k}^{\infty} \int_{2^{-s}}^{2^{-s+1}} dx_1 \int_\alpha^{\alpha+2^{-k+1}} (f(x_1,x_2) - k)\ dx_2 \leq$$

$$\leq 2 \cdot \frac{1}{2^{-k+1}} \frac{1}{2^{-k+1}} \sum_{s=k}^{\infty} \int_{2^{-s}}^{2^{-s+1}} dx_1 \int_\alpha^{\alpha+2^{-k+1}} (s-k)\, dx_2 =$$

$$= 2 \cdot \frac{1}{2^{-k+1}} \sum_{s=k}^{\infty} (s-k) 2^{-s} = \sum_{s=0}^{\infty} s 2^{-s} < \infty.$$

So, we have proved that $f \in BMO(Q_0)$.

Now let us show that $f \notin BMO^R(Q_0)$. For $k \geq 100$ denote $R_k = \left[0, 2^{-k+1}\right] \times [0,1]$. Then

$$f_{R_k} = \frac{1}{2^{-k+1}} \left\{ \sum_{s=1}^{k-1} s \left(\frac{1}{s} - \frac{1}{s+1} \right) \cdot 2^{-k+1} + \right.$$

$$\left. + \sum_{s=k}^{\infty} s \left(\frac{1}{s} \cdot 2^{-s+1} - \frac{1}{s+1} \cdot 2^{-s} \right) \right\} =$$

$$= \sum_{s=1}^{k} \frac{1}{s} + \sum_{s=1}^{\infty} \frac{1}{s+k} \cdot 2^{-s}.$$

Notice, that

$$f_{R_k} \geq \sum_{s=1}^{k} \frac{1}{s} \geq \ln(k+1) \geq [\ln(k+1)] \equiv L_k,$$

where $[\cdot]$ denotes the integer part function. From here we obtain (see Property 2.1)

$$\Omega(f; R_k) = \frac{2}{|R_k|} \int_{\left\{x \in R_k:\ f(x) < f_{R_k}\right\}} (f_{R_k} - f(x))\ dx \geq$$

$$\geq \frac{2}{|R_k|} \int_{\{x \in R_k:\ f(x) < L_k\}} (L_k - f(x))\ dx \geq$$

$$\geq 2^k \sum_{s=1}^{L_k} \int_{\frac{1}{s+1}}^{\frac{1}{s}} dx_2 \int_0^{2^{-k+1}} (L_k - f(x_1,x_2))\ dx_1 =$$

$$= 2\sum_{s=1}^{L_k}(L_k - s)\left(\frac{1}{s} - \frac{1}{s+1}\right) = 2L_k\left(1 - \frac{1}{L_k+1}\right) - 2\sum_{s=1}^{L_k}\frac{1}{s+1} \geq$$

$$\geq 2L_k - 2 - 2\ln(L_k+1) \;\to\; \infty, \quad k \to \infty.$$

This concludes the analysis of the example. □

In the paper of R. Coifman and R. Rochberg [9] there was defined *the* $BLO(\mathbb{R}^d)$*-class of functions of bounded lower oscillation*. This class consists of all locally summable functions f which are locally essentially bounded from below and such that

$$\|f\|_{BLO} \equiv \sup_Q L(f;Q) < \infty.$$

Here the supremum is taken over all cubes $Q \subset \mathbb{R}^d$, and the quantity

$$L(f;Q) \equiv \frac{1}{|Q|}\int_Q f(x)\,dx - \operatorname*{ess\,inf}_{x\in Q} f(x)$$

is called *the lower oscillation* of the function f on the cube Q. Analogously one can define the class $BLO(Q_0)$ for a fixed cube $Q_0 \subset \mathbb{R}^d$. It is easy to see that, unlikely BMO, the BLO-class is not a linear space.

From the trivial inequality

$$\Omega(f;Q) \leq 2\Omega'(f;Q) = 2\inf_c \frac{1}{|Q|}\int_Q |f(x) - c|\,dx \leq$$

$$\leq 2\frac{1}{|Q|}\int_Q \left(f(x) - \operatorname*{ess\,inf}_{y\in Q} f(y)\right)dx = 2L(f;Q), \quad Q \subset \mathbb{R}^d, \qquad (2.37)$$

it follows immediately that $BLO \subset BMO$ and $\|f\|_* \leq 2\|f\|_{BLO}$. The inverse inclusion $BLO \supset BMO$ is not true just because the BMO-class contains locally essentially unbounded from below functions. But even for the non-negative functions the BMO-class is substantially larger, than BLO. Indeed, in Example 2.20 we showed that the function $f(x) = \ln(1+|x|),\ x \in \mathbb{R}$ belongs to BMO, while

$$L(f;[0,a]) = \frac{1}{a}\int_0^a \ln(1+x)\,dx \to +\infty, \quad \text{as } a \to +\infty,$$

so that $f \notin BLO(\mathbb{R})$. Let us give one more example, which shows that also in the case of the finite interval $I_0 \subset \mathbb{R}$ the class of essentially bounded from below functions of bounded mean oscillation is larger than $BLO(I_0)$.

Example 2.33. Let $I_0 = \left[0, \frac{1}{2}\right]$. Let us show that the non-negative function

$$f(x) = \sum_{k=1}^{\infty} \ln\Big(2^k - \left(2^k - 1\right)2^{k+2}\left|x - 3\cdot 2^{-k-2}\right|\Big)\chi_{(2^{-k-1},2^{-k}]}(x), \quad x \in I_0,$$

does not belong to $BLO(I_0)$, but $f \in BMO(I_0)$.

Indeed, denote $I_k = \left[2^{-k-1}, 2^{-k}\right],\ k = 1, 2, \dots$ Then $\operatorname{ess\,inf}_{x\in I_k} f(x) = 0$ and

$$L(f; I_k) = f_{I_k} = 2^{k+1} \int_{2^{-k-1}}^{2^{-k}} f(x)\,dx =$$

$$= 2^{k+1} \int_{2^{-k-1}}^{2^{-k}} \ln\left(2^k - \left(2^k - 1\right) 2^{k+2} \left|x - 3 \cdot 2^{-k-2}\right|\right)\,dx =$$

$$= 2^{k+1} \int_{-2^{-k-2}}^{2^{-k-2}} \ln\left(2^k - \left(2^k - 1\right) 2^{k+2} |t|\right)\,dt =$$

$$= 2^{k+2} \int_0^{2^{-k-2}} \ln\left(2^k - \left(2^k - 1\right) 2^{k+2} t\right)\,dt = \frac{1}{2^k - 1} \int_1^{2^k} \ln v\,dv =$$

$$= \frac{1}{2^k - 1}\left(2^k k \ln 2 - 2^k + 1\right) = k \ln 2 - 1 + \frac{k \ln 2}{2^k - 1} \ge k \ln 2 - 1. \tag{2.38}$$

This means that $f \notin BLO(I_0)$.

Now let us show that $f \in BMO(I_0)$. Let $I \equiv [a, b] \subset I_0$, and let the integer k be such that $2^{-k-1} < b \le 2^{-k}$. We have to distinguish between two cases.

1. If $a \le \frac{3}{4} b$, then $b - a \ge \frac{1}{4} b \ge 2^{-k-3}$, i.e., $I'_k \equiv \left[0, 2^{-k}\right] \supset [a, b]$ and $|I'_k| \le 8|I|$. Hence

$$\Omega(f; I) \le \Omega''(f; I) = \frac{1}{|I|^2} \int_I \int_I |f(x) - f(y)|\,dx\,dy \le$$

$$\le \frac{|I'_k|^2}{|I|^2} \frac{1}{|I'_k|^2} \int_{I'_k} \int_{I'_k} |f(x) - f(y)|\,dx\,dy \le 64\Omega''\left(f; I'_k\right) \le 128\Omega\left(f; I'_k\right).$$

Let us estimate $\Omega\left(f; I'_k\right)$. By (2.38) and the properties of oscillations,

$$f_{I'_k} = 2^k \int_0^{2^{-k}} f(x)\,dx = 2^k \sum_{j=k}^{\infty} \int_{2^{-j-1}}^{2^{-j}} f(x)\,dx =$$

$$= 2^k \sum_{j=k}^{\infty} 2^{-j-1} f_{I_j} = 2^k \sum_{j=k}^{\infty} 2^{-j-1} \left(j \ln 2 - 1 + \frac{j \ln 2}{2^j - 1}\right) =$$

$$= \sum_{s=0}^{\infty} 2^{-s-1} \left((s + k) \ln 2 - 1 + \frac{(s + k) \ln 2}{2^{s+k} - 1}\right) =$$

$$= k\ln 2 + \ln 2\sum_{s=0}^{\infty} s2^{-s-1} - 1 + \sum_{s=0}^{\infty} 2^{-s-1}\frac{(s+k)\ln 2}{2^{s+k}-1} \geq$$

$$\geq k\ln 2 + \ln 2\sum_{s=0}^{\infty} s2^{-s-1} - 1 = k\ln 2 + \ln 2 - 1,$$

$$\Omega\left(f; I'_k\right) = 2\cdot 2^k \int_{\left\{x\in I'_k:\ f(x)>f_{I'_k}\right\}} \left(f(x) - f_{I'_k}\right) dx \leq$$

$$\leq 2^{k+1}\int_{\left\{x\in I'_k:\ f(x)>k\ln 2+\ln 2-1\right\}} (f(x) - (k\ln 2 + \ln 2 - 1))\ dx =$$

$$= 2^{k+1}\sum_{j=k}^{\infty}\int_{\{x\in I_j:\ f(x)>k\ln 2+\ln 2-1\}} (f(x) - (k\ln 2 + \ln 2 - 1))\ dx \leq$$

$$\leq 2^{k+1}\sum_{j=k}^{\infty} 2^{-j-1}\left(\max_{x\in I_j} f(x) - k\ln 2 - \ln 2 + 1\right) =$$

$$= 2^{k+1}\sum_{j=k}^{\infty} 2^{-j-1}\left(j\ln 2 - k\ln 2 - \ln 2 + 1\right) =$$

$$= \sum_{j=k}^{\infty} 2^{-(j-k)}\left((j-k)\ln 2 - \ln 2 + 1\right) = \sum_{s=0}^{\infty} 2^{-s}(s\ln 2 - \ln 2 + 1) =$$

$$= \ln 2\sum_{s=0}^{\infty} s2^{-s} - 2\ln 2 + 2 = 2. \qquad (2.39)$$

2. If $\frac{3}{4}b < a < b$, then $[a,b] \subset \left[3\cdot 2^{-k-3}, 2^{-k}\right]$. If $2^{-k-1} \notin (a,b)$ and $3\cdot 2^{-k-2} \notin (a,b)$, then from Example 2.31 we see that $\Omega(f;[a,b])$ does not exceed the BMO-norm of the function $g(x) = \ln\frac{1}{x},\ x > 0$, i.e.,

$$\Omega(f;[a,b]) \leq \frac{2}{\mathrm{e}}. \qquad (2.40)$$

If $3{\cdot}2^{-k-2} \in (a,b)$, then $[a,b] \subset \left[2^{-k-1}, 2^{-k}\right]$ and, according to Examples 2.31, 2.29 and 2.24,

$$\Omega(f;[a,b]) \leq \frac{4}{\mathrm{e}}. \qquad (2.41)$$

If $2^{-k-1} \in (a,b)$, then $[a,b] \subset \left[3\cdot 2^{-k-3}, 3\cdot 2^{-k-2}\right]$. Making the change of variables $\tau = 2^{k+2}\left(2^k - 1\right)\left(x - 2^{-k-1}\right)$ and taking into account Example 2.31, we obtain that $\Omega(f;[a,b])$ does not exceed the BMO-norm of the function

$$\varphi(\tau)=\ln(1+|\tau|)\chi_{[0,+\infty)}(\tau)+\ln\left(1+\frac{2\left(2^{k+1}-1\right)}{2^k-1}|\tau|\right)\chi_{(-\infty,0)}(\tau),\qquad \tau\in\mathbb{R}.$$

We can apply the result of Example 2.30 to the function φ, taking $\gamma=2\left(2^{k+1}-1\right)/\left(2^k-1\right)$. Since $2\le\gamma\le 6$ we see that

$$C_\gamma=\frac{4}{\mathrm{e}}c_\gamma^2=\frac{4}{\mathrm{e}}\left(1+\gamma+\frac{1}{\gamma}\right)^2<128.$$

Thus

$$\Omega(f;[a,b])\le 128. \tag{2.42}$$

From (2.39) – (2.42) it follows that $f\in BMO(I_0)$. □

Now let us consider more in detail the case, when the function f is non-increasing on $\mathbb{R}_+$. Clearly f is locally essentially bounded from below. In this case $\|f\|_{BLO}$, together with $\|f\|_*$, is equal to the supremum of the oscillations over all possible intervals, having the left end at zero. Moreover, we have

Lemma 2.34 ([41]). *Let the locally summable function f be non-increasing on $\mathbb{R}_+$. Then*

$$\|f\|_{BLO}=\sup_{b>0}L(f;[a,b])=\sup_{b>0}\left(\frac{1}{b}\int_0^b f(x)\,dx-f(b)\right). \tag{2.43}$$

Proof. Let $[a,b]\subset\mathbb{R}_+$. Since f is non-increasing

$$L(f;[a,b])=\frac{1}{b-a}\int_a^b f(x)\,dx-\operatorname*{ess\,inf}_{x\in[a,b]}f(x)\le$$

$$\le\frac{1}{b}\int_0^b f(x)\,dx-\operatorname*{ess\,inf}_{x\in[0,b]}f(x)=L(f;[0,b]).$$

This implies the first equality of (2.43).

In order to prove the second equality let us remark, that the inequality $f(b)\le\operatorname{ess\,inf}_{x\in[a,b]}f(x)$ implies

$$L(f;[a,b])\le\frac{1}{b}\int_0^b f(x)\,dx-f(b),$$

so that

$$\sup_{b>0}L(f;[a,b])\le\sup_{b>0}\left(\frac{1}{b}\int_0^b f(x)\,dx-f(b)\right). \tag{2.44}$$

On the other hand, by the continuity of the integral with respect to the upper limit, for $b>0$

$$L(f;[0,\beta]) = \frac{1}{\beta}\int_0^\beta f(x)\,dx - \operatorname*{ess\,inf}_{x\in[0,\beta]} f(x) \geq$$

$$\geq \frac{1}{\beta}\int_0^\beta f(x)\,dx - f(b) \;\to\; \frac{1}{b}\int_0^b f(x)\,dx - f(b) \quad \text{as} \quad \beta \to b+0.$$

Then

$$\sup_{\beta>0} L(f;[0,\beta]) \geq \frac{1}{b}\int_0^b f(x)\,dx - f(b),$$

so that

$$\sup_{\beta>0} L(f;[0,\beta]) \geq \sup_{b>0}\left(\frac{1}{b}\int_0^b f(x)\,dx - f(b)\right). \tag{2.45}$$

Estimates (2.44) and (2.45) yield the second equality of (2.43). □

Lemma 2.35 ([41]). *Let f be a summable non-increasing function on $I \equiv [a,b]$, and let $s \in (a,b)$ be such that $f(x) \geq f_I$ if $a \leq x \leq s$, and $f(x) \leq f_I$ if $s \leq x \leq b$. Then*

$$\sup_{\gamma\in[a,b]} \int_a^\gamma (f(x) - f_I)\;dx = \int_a^s (f(x) - f_I)\;dx = \frac{b-a}{2}\Omega(f;[a,b]) =$$

$$= \int_s^b (f_I - f(x))\;dx = \sup_{\gamma\in[a,b]} \int_\gamma^b (f_I - f(x))\;dx. \tag{2.46}$$

Proof. The second and the third equalities of this chain follow from Property 2.1. The first and the last equalities of (2.46) can be proved analogously. We will prove only the first one. It is enough to show, that for any $\gamma \in [a,b]$

$$\int_0^\gamma (f(x) - f_I)\;dx \leq \int_a^s (f(x) - f_I)\;dx. \tag{2.47}$$

If $\gamma < s$, then (2.47) follows from the fact that $f(x) \geq f_I$ for $\gamma \leq x \leq s$. Otherwise, if $\gamma > s$, then (2.47) follows from the inequality $f(x) \leq f_I$ for $s \leq x \leq \gamma$. □

The next theorem provides the exact relations between the BMO and BLO norms of a non-increasing function. In particular, it shows that for the non-increasing on $\mathbb{R}_+$ functions the classes BMO and BLO coincide.

Theorem 2.36 ([41]). *Let f be a non-increasing function on $\mathbb{R}_+$. Then*

$$\frac{1}{2}\|f\|_{BLO} \leq \|f\|_* \leq \frac{2}{\mathrm{e}}\|f\|_{BLO}. \tag{2.48}$$

Moreover, in general the constants $\frac{1}{2}$ and $\frac{2}{\mathrm{e}}$ in the left and right-hand sides are sharp.

Proof. For $b > 0$ the monotonicity of f and Lemma 2.35 imply

$$\frac{1}{b}\int_0^b f(x)\,dx - f(b) \le \frac{1}{b}\int_0^b f(x)\,dx - \frac{1}{b}\int_b^{2b} f(x)\,dx =$$

$$= 2\left[\frac{1}{2b}\int_0^b \big(f(x) - f_{[0,2b]}\big)\,dx + \frac{1}{2b}\int_b^{2b} \big(f_{[0,2b]} - f(x)\big)\,dx\right] \le 2\Omega(f;[0,2b]).$$

Now the left inequality of (2.48) follows from Lemma 2.34.

For the function $f(x) = \chi_{[0,1]}(x),\ x \ge 0$ we have $\|f\|_* = \frac{1}{2},\ \|f\|_{BLO} = 1$, so that the constant $\frac{1}{2}$ in the left-hand side of (2.48) cannot be increased.

Now let us prove the right inequality of (2.48). By Lemma 2.34,

$$A \equiv \|f\|_{BLO} = \sup_{y>0}\left(\frac{1}{y}\int_0^y f(x)\,dx - f(y)\right).$$

Let us rewrite this inequality as follows:

$$\frac{1}{y}\int_0^y f(x)\,dx \le A + f(y), \quad y > 0.$$

Dividing by y and making the integration from ε to s with $0 < \varepsilon < s$, we obtain

$$\int_\varepsilon^s \int_0^y f(x)\,dx\,\frac{dy}{y^2} \le A\int_\varepsilon^s \frac{dy}{y} + \int_\varepsilon^s f(y)\,\frac{dy}{y} = A\ln\frac{s}{\varepsilon} + \int_\varepsilon^s f(y)\,\frac{dy}{y}. \quad (2.49)$$

Changing the order of integration in the left-hand side of (2.49), we see that the left-hand side of (2.49) is equal to

$$\int_0^\varepsilon f(x)\int_\varepsilon^s \frac{dy}{y^2}\,dx + \int_\varepsilon^s f(x)\int_x^s \frac{dy}{y^2}\,dx =$$

$$= \frac{1}{\varepsilon}\int_0^\varepsilon f(x)\,dx - \frac{1}{s}\int_0^s f(x)\,dx + \int_\varepsilon^s f(x)\,\frac{dx}{x}.$$

From here and from (2.49)

$$\frac{1}{\varepsilon}\int_0^\varepsilon f(x)\,dx - \frac{1}{s}\int_0^s f(x)\,dx \le A\ln\frac{s}{\varepsilon}, \quad 0 < \varepsilon < s.$$

Hence

$$\frac{1}{s}\int_0^s f(x)\,dx - \frac{1}{s-\varepsilon}\int_\varepsilon^s f(x)\,dx =$$

$$= \frac{1}{s}\int_0^s f(x)\,dx - \frac{1}{s-\varepsilon}\left(\int_0^s f(x)\,dx - \int_0^\varepsilon f(x)\,dx\right) =$$

$$= \frac{\varepsilon}{s-\varepsilon}\frac{1}{\varepsilon}\int_0^{\varepsilon} f(x)\,dx - \frac{\varepsilon}{s-\varepsilon}\frac{1}{s}\int_0^{s} f(x)\,dx \le A\frac{\varepsilon}{s-\varepsilon}\ln\frac{s}{\varepsilon},$$

or, equivalently,

$$\frac{1}{s-\varepsilon}\int_{\varepsilon}^{s} \big(f_{[0,s]} - f(x)\big)\,dx \le A\frac{\varepsilon}{s-\varepsilon}\ln\frac{s}{\varepsilon},$$

i.e.

$$\frac{1}{s}\int_{\varepsilon}^{s} \big(f_{[0,s]} - f(x)\big)\,dx \le A\frac{\varepsilon}{s}\ln\frac{s}{\varepsilon}. \tag{2.50}$$

Since $\frac{\ln t}{t} \le \frac{1}{\mathrm{e}},\ t \ge 1$ for $0 < \varepsilon < s$ inequality (2.50) implies

$$\frac{1}{s}\int_{\varepsilon}^{s} \big(f_{[0,s]} - f(x)\big)\,dx \le \frac{A}{\mathrm{e}}. \tag{2.51}$$

On the other hand, according to Lemma 2.35,

$$\sup_{\{\varepsilon:\ 0<\varepsilon\le s\}} \frac{1}{s}\int_{\varepsilon}^{s} \big(f_{[0,s]} - f(x)\big)\,dx = \frac{1}{2}\Omega(f;[0,s]),$$

so that (2.51) implies

$$\Omega(f;[0,s]) \le \frac{2}{\mathrm{e}}A. \tag{2.52}$$

It remains to notice, that, as it follows from Property 2.15, if f is a non-increasing function on $\mathbb{R}_+$, then

$$\|f\|_* = \sup_{s>0}\Omega(f;[0,s]).$$

Hence, taking the supremum over all $s > 0$ in (2.52), we obtain the right inequality of (2.48).

Finally, as we saw in Example 2.24, for the function $f(x) = \ln\frac{1}{x},\ x > 0$ we have $\|f\|_* = \frac{2}{\mathrm{e}}$. On the other hand, according to Lemma 2.34,

$$\|f\|_{BLO} = \sup_{b>0}\left(\frac{1}{b}\int_0^b \ln\frac{1}{x}\,dx - \ln\frac{1}{b}\right) = 1.$$

So, the constant $\frac{2}{\mathrm{e}}$ in the right-hand side of (2.48) cannot be decreased. □

3

Estimates of Rearrangements and the John–Nirenberg Theorem

3.1 Estimates of Rearrangements of the BMO-functions

The aim of the present section is to show that the non-increasing rearrangement f^* of a BMO-function f is also a BMO-function. First we will consider the case of the function f defined on the whole $\mathbb{R}^d$. As it was mentioned above, in addition we have to assume that $f^*(t)$ is defined for all $t > 0$.

Theorem 3.1 (Bennett, De Vore, Sharpley, [1]). *Let* $f \in BMO(\mathbb{R}^d)$. *Then*

$$f^{**}(t) - f^*(t) \le 2^{d+4}\|f\|_*, \quad 0 < t < \infty. \tag{3.1}$$

Proof. Since $\|\,|f|\,\|_* \le 2\|f\|_*$ it is enough to prove the inequality

$$f^{**}(t) - f^*(t) \le 2^{d+3}\|f\|_* \tag{3.2}$$

for a non-negative function f.

Fix $t > 0$ and denote $E = \{x \in \mathbb{R}^d : \ f(x) > f^*(t)\}$. Then $|E| \le t$. Let us construct an open set $G \supset E$ such that $|G| \le 2t$. Applying Lemma 1.12 to the set G we obtain a collection of cubes Q_j with pairwise disjoint interiors, which satisfy properties (1.11), (1.12) and (1.13) of the lemma. Then

$$t\left(f^{**}(t) - f^*(t)\right) = \int_0^t \left(f^*(u) - f^*(t)\right)\,du =$$

$$= \int_0^{|E|} \left(f^*(u) - f^*(t)\right)\,du + \int_{|E|}^t \left(f^*(u) - f^*(t)\right)\,du =$$

$$= \int_0^{|E|} \left(f^*(u) - f^*(t)\right)\,du = \int_E \left(f(x) - f^*(t)\right)\,dx =$$

$$= \sum_j \int_{E \cap Q_j} (f(x) - f^*(t)) \, dx =$$

$$= \sum_j \int_{E \cap Q_j} \left(f(x) - f_{Q_j}\right) \, dx + \sum_j \left(f_{Q_j} - f^*(t)\right) |E \cap Q_j| \leq$$

$$\leq \sum_j \int_{E \cap Q_j} \left|f(x) - f_{Q_j}\right| \, dx + {\sum_j}' \left(f_{Q_j} - f^*(t)\right) |E \cap Q_j| , \tag{3.3}$$

where ${\sum_j}'$ denotes the sum over all numbers j such that $f_{Q_j} > f^*(t)$. We have

$${\sum_j}' \left(f_{Q_j} - f^*(t)\right) |E \cap Q_j| \leq {\sum_j}' \left(f_{Q_j} - f^*(t)\right) |G \cap Q_j| \leq$$

$$\leq {\sum_j}' \left(f_{Q_j} - f^*(t)\right) |Q_j \setminus G| = {\sum_j}' \int_{Q_j \setminus G} \left(f_{Q_j} - f^*(t)\right) \, dx \leq$$

$$\leq {\sum_j}' \int_{Q_j \setminus G} \left(f_{Q_j} - f(x)\right) \, dx \leq {\sum_j}' \int_{Q_j} \left|f(x) - f_{Q_j}\right| \, dx \leq$$

$$\leq \sum_j \int_{Q_j} \left|f(x) - f_{Q_j}\right| \, dx.$$

Then (3.3) becomes

$$t \left(f^{**}(t) - f^*(t)\right) \leq 2 \sum_j \int_{Q_j} \left|f(x) - f_{Q_j}\right| \, dx \leq$$

$$\leq 2\|f\|_* \sum_j |Q_j| \leq 2^{d+2} \|f\|_* \cdot |G| \leq 2^{d+3} \|f\|_* \cdot t,$$

which is exactly (3.2). □

In particular, from this lemma it follows that the rearrangement operator is bounded in BMO.

Theorem 3.2 (Garsia, Rodemich ($d = 1$)**, [17]; Bennett, De Vore, Sharpley** ($d \geq 1$)**, [1]).** *Let $f \in BMO(\mathbb{R}^d)$. Then $f^* \in BMO([0, \infty))$ and*

$$\|f^*\|_* \leq c\|f\|_*,$$

where the constant c depends only on the dimension d of the space (one can take $c = 2^{d+5}$).

Proof. Since f^* is a non-increasing function on $[0,\infty)$

$$\|f^*\|_* = \sup_{t>0} \Omega(f^*;[0,t]).$$

But by the properties of oscillations

$$\Omega(f^*;[0,t]) \le 2\Omega'(f^*;[0,t]) = 2\inf_c \frac{1}{t}\int_0^t |f^*(u)-c|\,du \le$$

$$\le \frac{2}{t}\int_0^t (f^*(u)-f^*(t))\,du = 2\left(f^{**}(t)-f^*(t)\right),$$

and the result follows from the previous theorem. □

Now let us consider the case $f \in BMO(Q_0)$ for a fixed cube $Q_0 \subset \mathbb{R}^d$. In this case the presented proof of inequality (3.1) is valid only for t such that $0 < t \le \frac{1}{4}|Q_0|$ because it is based on the application of Lemma 1.13, which requires $|G| \le \frac{1}{2}|Q_0|$. Therefore the following theorem is valid.

Theorem 3.3 (Bennett, De Vore, Sharpley, [1]). *Let $f \in BMO(Q_0)$. Then*

$$f^{**}(t) - f^*(t) \le 2^{d+4}\|f\|_*, \quad 0 < t \le \frac{1}{4}|Q_0|. \tag{3.4}$$

Let us show that (3.4) fails as $t \to |Q_0|$ even if the coefficient in its right-hand side is arbitrarily big. Indeed, for $0 < h < 1$ set $f(x) = \ln\frac{1-x}{h}$, $x \in Q_0 \equiv [0,1-h]$. Since f does not increase on $[0,1-h]$ it follows that $f^*(t) = f(t)$, $0 < t \le 1-h = |Q_0|$, and $f^*(1-h) = 0$. It is easy to see that $\|f\|_*$ does not exceed the BMO-norm of the function $\ln\frac{1}{x}$, $0 < x < \infty$, so that $\|f\|_* \le \frac{2}{\mathrm{e}}$. Thus it remains to show that $f^{**}(1-h) \to \infty$ as $h \to 0$. But this is indeed true, because

$$f^{**}(1-h) = \frac{1}{1-h}\int_0^{1-h} f^*(u)\,du = \frac{1}{1-h}\int_0^{1-h} \ln\frac{1-u}{h}\,du =$$

$$= \frac{h}{1-h}\int_h^1 \ln\frac{1}{z}\,\frac{dz}{z^2} = \frac{1}{1-h}\ln\frac{1}{h} - 1 \;\to\; \infty, \quad h \to 0.$$

We see that (3.4) fails for $t = |Q_0|$, and hence also for t close to $|Q_0|$. However, the analog of Theorem 3.2 for $BMO(Q_0)$ is true.

Theorem 3.4 (Garsia, Rodemich ($d=1$), [17]; Bennett, De Vore, Sharpley ($d \ge 1$), [1]). *Let $f \in BMO(Q_0)$. Then $f^* \in BMO([0,|Q_0|])$ and*

$$\left\|(f - f_{Q_0})^*\right\|_* \le c\|f\|_*,$$

where the constant c depends only on the dimension d of the space.

Proof. We can assume that $f_{Q_0} = 0$. Fix the interval $[\alpha, \beta] \subset [0, |Q_0|]$. If $f^*_{[\alpha,\beta]} \le |f|_{Q_0}$, then

$$\Omega(f^*; [\alpha, \beta]) \le \frac{1}{\beta - \alpha} \int_\alpha^\beta \left| f^*(u) - f^*_{[\alpha,\beta]} \right| du \le$$

$$\le 2 f^*_{[\alpha,\beta]} \le 2|f|_{Q_0} = \frac{2}{|Q_0|} \int_{Q_0} |f(x)|\, dx \le 2\|f\|_*.$$

In remains to consider the non-trivial case $f^*_{[\alpha,\beta]} > |f|_{Q_0}$. Choose β_0, $\beta \le \beta_0 \le |Q_0|$, such that $f^*_{[\alpha,\beta]} = f^*_{[0,\beta_0]}$. If $\beta_0 \ge \frac{1}{4}|Q_0|$, then

$$\Omega(f^*; [\alpha, \beta]) \le \Omega(f^*; [0, \beta_0]) = \frac{2}{\beta_0} \int_{\{u:\ f^*(u) > f^{**}(\beta_0)\}} (f^*(u) - f^{**}(\beta_0))\, du \le$$

$$\le \frac{2}{\beta_0} \int_{\{u:\ f^*(u) > f^{**}(\beta_0)\}} (f^*(u) - |f|_{Q_0})\, du \le$$

$$\le \frac{2}{\beta_0} \int_{\{u:\ f^*(u) > |f|_{Q_0}\}} (f^*(u) - |f|_{Q_0})\, du =$$

$$= \frac{2}{\beta_0} \int_{\{x \in Q_0:\ |f(x)| > |f|_{Q_0}\}} (|f(x)| - |f|_{Q_0})\, dx =$$

$$= \frac{|Q_0|}{\beta_0} \Omega(|f|; Q_0) \le 4 \cdot 2 \cdot \Omega(f; Q_0) \le 8\|f\|_*.$$

Otherwise, if $\beta_0 \le \frac{1}{4}|Q_0|$, then, by Theorem 3.3,

$$\Omega(f^*; [\alpha, \beta]) \le \Omega(f^*; [0, \beta_0]) \le 2\Omega'(f^*; [0, \beta_0]) \le$$

$$\le \frac{2}{\beta_0} \int_0^{\beta_0} (f^*(u) - f^*(\beta_0))\, du = 2\left(f^{**}(\beta_0) - f^*(\beta_0)\right) \le 2^{d+5}\|f\|_*.$$

Since the interval $[\alpha, \beta] \subset [0, |Q_0|]$ was arbitrary the theorem is proved. □

The estimates of the non-increasing rearrangement, which were obtained in Theorems 3.2 and 3.4, are based on the applications of Theorems 3.1 and 3.3 respectively, while for the proofs of Theorems 3.1 and 3.3 we used Lemma 1.12. Now we are going to consider another method of getting estimates for the BMO-norm of the non-increasing rearrangement, based on the application of "rising sun lemma" 1.16. For this we will use the non-increasing equimeasurable rearrangement f_d.

Theorem 3.5 (Klemes, [32]). *Let $f \in BMO([a_0, b_0])$. Then*

$$\|f_d\|_* \le \|f\|_*.$$

Proof. Fix the segment $J \subset [0, b_0 - a_0]$ and denote $\alpha = \frac{1}{|J|}\int_J f_d(u)\,du$. First let us consider the case $\alpha \geq \frac{1}{b_0-a_0}\int_{a_0}^{b_0} f(x)\,dx$. Applying "rising sun lemma" 1.16 we construct the pairwise disjoint intervals $I_j \subset [a_0, b_0]$, $j = 1, 2, \dots,$ such that $f_{I_j} = \alpha$ and $f(x) \leq \alpha$ at almost every $x \in [a_0, b_0] \setminus E$ with $E = \bigcup_{j\geq 1} I_j$. If we prove the inequality

$$\frac{1}{|J|}\int_J |f_d(t) - \alpha|\,dt \leq \frac{1}{|E|}\int_E |f(x) - \alpha|\,dx, \tag{3.5}$$

then it will remain to use the fact, that

$$f_E = \frac{1}{|E|}\int_E f(x)\,dx = \frac{1}{\sum_j |I_j|}\sum_j \int_{I_j} f(x)\,dx = \alpha, \tag{3.6}$$

and

$$\frac{1}{|E|}\int_E |f(x) - \alpha|\,dx = \frac{1}{|E|}\sum_j \int_{I_j} |f(x) - \alpha|\,dx =$$

$$= \frac{1}{|E|}\sum_j |I_j| \frac{1}{|I_j|}\int_{I_j} \left|f(x) - f_{I_j}\right|\,dx = \frac{1}{|E|}\sum_j |I_j| \Omega(f; I_j) \leq \|f\|_*.$$

In order to prove (3.5), choose the maximal $t \in (0, b_0 - a_0]$ such that $J \subset [0, t]$ and $\frac{1}{t}\int_0^t f_d(u)\,du = \alpha$. The existence of such a t is guaranteed by the condition

$$\frac{1}{b_0 - a_0}\int_0^{b_0 - a_0} f_d(u)\,du = \frac{1}{b_0 - a_0}\int_{a_0}^{b_0} f(x)\,dx \leq \alpha = \frac{1}{|J|}\int_J f_d(u)\,du.$$

Using the monotonicity of the function f_d and applying Property 2.15, we obtain

$$\frac{1}{|J|}\int_J |f_d(u) - \alpha|\,du \leq \frac{1}{t}\int_0^t |f_d(u) - \alpha|\,du.$$

Now for the proof of (3.5) it is enough to show that

$$\frac{1}{t}\int_0^t |f_d(u) - \alpha|\,du \leq \frac{1}{|E|}\int_E |f(x) - \alpha|\,dx. \tag{3.7}$$

But (3.7) is a consequence of the following two relations

$$t \geq |E|, \tag{3.8}$$

$$\int_0^t |f_d(u) - \alpha|\,du = \int_E |f(x) - \alpha|\,dx. \tag{3.9}$$

Concerning (3.8), notice that by the definition of the non-increasing rearrangement we have

$$\int_0^{|E|} f_d(u)\,du \geq \int_E f(x)\,dx,$$

so that, by (3.6),

$$\frac{1}{|E|}\int_0^{|E|} f_d(u)\,du \geq \frac{1}{|E|}\int_E f(x)\,dx = \alpha = \frac{1}{t}\int_0^t f_d(u)\,du.$$

From here and from the monotonicity of f_d inequality (3.8) follows. In order to prove (3.9) let us use the fact that $f(x) \leq \alpha$ almost everywhere on $[a_0, b_0] \setminus E$ (see (1.20)). Then, by (3.6) and the properties of mean oscillations,

$$\int_0^t |f_d(u) - \alpha|\;du = 2\int_{\{u\in[0,t]:\ f_d(u)>\alpha\}} (f_d(u) - \alpha)\;du =$$

$$= 2\int_{\{x\in[a_0,b_0]:\ f(x)>\alpha\}} (f(x) - \alpha)\,dx = 2\int_{\{x\in E:\ f(x)>\alpha\}} (f(x) - \alpha)\,dx =$$

$$= \int_E |f(x) - \alpha|\,dx.$$

This concludes the proof of inequality (3.7).

In the case $\alpha < \frac{1}{b_0-a_0}\int_{a_0}^{b_0} f(x)\,dx$ it is enough to apply the previous arguments to the function $-f$ and notice that the equality $(-f)_d(t) = -f_d(b_0 - a_0 - t)$ holds true for all $t \in [0, b_0 - a_0]$ except the set of measure zero of the points of discontinuity of the function $(-f)_d$. In this case again we have

$$\frac{1}{|J|}\int_J |f_d(u) - \alpha|\;du \leq \|f\|_*,$$

and this complete the proof of the theorem. □

Remark 3.6. As it was already noticed, if the function f is non-negative on $[a_0, b_0]$, then $f^* = f_d$. So, in this case Theorem 3.5 leads to the inequality

$$\|f^*\|_* \leq \|f\|_*, \tag{3.10}$$

which is sharp in the sense of constants. In this sense the estimate (3.10) for $d = 1$ is better than the one provided by Theorem 3.2.

If we drop the assumption that f is non-negative, then

$$f^* = |f|_d.$$

If in addition we take into account (Property 2.6) that

$$\Omega(|f|; I) \leq 2\Omega(f; I), \quad I \subset [a_0, b_0], \tag{3.11}$$

then, applying Theorem 3.5, we obtain

$$\|f^*\|_* = \|\ |f|_d\ \|_* \leq \|\ |f|\ \|_* \leq 2\|f\|_*. \tag{3.12}$$

However, the last inequality is not sharp despite of the fact that the constant 2 in (3.11) cannot be decreased (Property 2.6). Actually there holds true the following theorem.

Theorem 3.7 ([34]). *Let* $f \in BMO([a_0, b_0])$. *Then*

$$\| \, |f| \, \|_* \leq \|f\|_*. \tag{3.13}$$

Proof. Fix the interval $I \subset [a_0, b_0]$ and denote by $g = f|I$ the restriction of f to I. Obviously then

$$\Omega(|f|; I) = \Omega(|g|; I) = \Omega(|g_d|; [0, |I|]).$$

But in view of Theorem 3.5, for any interval $J \subset [0, |I|]$ we have

$$\Omega(g_d; J) \leq \sup_{K \subset I} \Omega(g; K) = \sup_{K \subset I} \Omega(f; K) \leq \|f\|_*.$$

Hence in order to prove the theorem it is enough to prove the inequality

$$\Omega(|g_d|; [0, |I|]) \leq \sup_{J \subset [0, |I|]} \Omega(g_d; J). \tag{3.14}$$

Without loss of generality we can assume that $|I| = 1$. Denote $K = [0, 1], \ h = g_d, \ \beta = |h|_K, \ \gamma = h_K$. Then (3.14) becomes

$$\int_K | \, |h(t)| - \beta| \, dt \leq \sup_{J \subset K} \frac{1}{|J|} \int_J |h(t) - h_J| \, dt. \tag{3.15}$$

The proof of the theorem splits into the following three cases:

1. $\lim_{t \to 1-0} h(t) \geq -\beta$; obviously in this case $\lim_{t \to 0+} h(t) > \beta$;

2. $\lim_{t \to 0+} \leq \beta$; obviously in this case $\lim_{t \to 1-0} h(t) < -\beta$;

3. $\lim_{t \to 1-0} h(t) < -\beta$ and $\lim_{t \to 0+} h(t) > \beta$.

In the first case, by properties of mean oscillations,

$$\int_K | \, |h(t)| - \beta| \, dt = 2 \int_{\{t \in K: \ |h(t)| > \beta\}} (|h(t)| - \beta) \, dt =$$

$$= 2 \int_{\{t \in K: \ h(t) > \beta\}} (h(t) - \beta) \, dt \leq 2 \int_{\{t \in K: \ h(t) > \gamma\}} (h(t) - \gamma) \, dt =$$

$$= \int_K |h(t) - \gamma| \, dt = \int_K |h(t) - h_K| \, dt.$$

Similarly, in the second case we have

$$\int_K | \, |h(t)| - \beta| \, dt = 2 \int_{\{t \in K: \ |h(t)| > \beta\}} (|h(t)| - \beta) \, dt =$$

$$= 2\int_{\{t\in K:\ h(t)<-\beta\}} (-h(t)-\beta)\,dt \le 2\int_{\{t\in K:\ h(t)<\gamma\}} (-h(t)-(-\gamma))\,dt =$$

$$= \int_K |h(t)-\gamma|\,dt = \int_K |h(t)-h_K|\,dt.$$

For the third case let us consider the function $\varphi(\tau) = \frac{1}{\tau}\int_0^\tau |h(t)|\,dt$. This function is continuous on $(0,1]$, $\lim_{\tau\to 0+}\varphi(\tau) > \beta$, $\varphi(1) = \beta$, and for $\varepsilon > 0$ small enough

$$\varphi(1-\varepsilon) = \frac{1}{1-\varepsilon}\int_0^{1-\varepsilon} |h(t)|\,dt = \frac{1}{1-\varepsilon}\int_0^1 |h(t)|\,dt - \frac{\varepsilon}{1-\varepsilon}\frac{1}{\varepsilon}\int_{1-\varepsilon}^1 |h(t)|\,dt < \beta.$$

From the properties of the function φ it follows that there exists $\tau_0 \in (0,1)$ such that $\varphi(\tau_0) = \beta$. Denote $K_1 = [0,\tau_0],\ K_2 = [\tau_0, 1]$. Then

$$|h|_{K_1} = \varphi(\tau_0) = \beta,$$

$$|h|_{K_2} = \frac{1}{1-\tau_0}\int_{\tau_0}^1 |h(t)|\,dt = \frac{1}{1-\tau_0}\left(\int_0^1 |h(t)|\,dt - \int_0^{\tau_0} |h(t)|\,dt\right) =$$

$$= \frac{1}{1-\tau_0}(\beta - \tau_0\beta) = \beta,$$

$$\int_K \big|\,|h(t)|-\beta\big|\,dt = \tau_0\frac{1}{|K_1|}\int_{K_1} \big|\,|h(t)|-\beta\big|\,dt + (1-\tau_0)\frac{1}{|K_2|}\int_{K_2} \big|\,|h(t)|-\beta\big|\,dt \le$$

$$\le \max_{i=1,2}\frac{1}{|K_i|}\int_{K_i} \big|\,|h(t)|-\beta\big|\,dt.$$

If we show that

$$\int_{K_i} \big|\,|h(t)|-\beta\big|\,dt \le \int_{K_i} |h(t)-h_{K_i}|\,dt, \quad i = 1,2, \tag{3.16}$$

then

$$\int_K \big|\,|h(t)|-\beta\big|\,dt \le \max_{i=1,2}\frac{1}{|K_i|}\int_{K_i} \big|\,|h(t)|-\beta\big|\,dt \le$$

$$\le \max_{i=1,2}\frac{1}{|K_i|}\int_{K_i} |h(t)-h_{K_i}|\,dt \le \sup_{J\subset K}\frac{1}{|J|}\int_J |h(t)-h_J|\,dt,$$

i.e., (3.15). So, it remains to prove (3.16).

For $i = 1$

$$\int_{K_1} |\,|h(t)| - \beta|\, dt = 2\int_{\{t\in K_1:\ |h(t)|>\beta\}} (|h(t)| - \beta)\, dt \le$$

$$\le 2\int_{\{t\in K_1:\ h(t)>h_{K_1}\}} \left(h(t) - h_{K_1}\right)\, dt = \int_{K_1} |h(t) - h_{K_1}|\, dt.$$

Similarly, for $i = 2$

$$\int_{K_2} |\,|h(t)| - \beta|\, dt = 2\int_{\{t\in K_2:\ |h(t)|>\beta\}} (|h(t)| - \beta)\, dt =$$

$$= 2\int_{\{t\in K_2:\ h(t)<-\beta\}} (-h(t) - \beta)\, dt \le$$

$$\le 2\int_{\{t\in K_2:\ h(t)<h_{K_2}\}} \left(-h(t) + h_{K_2}\right)\, dt = \int_{K_2} |h(t) - h_{K_2}|\, dt.$$

This proves (3.16) and completes the proof of (3.15). □

Remark 3.8. We have proved (3.13) in the one-dimensional case. If the dimension of the space $d \ge 2$, then Property 2.6 immediately implies that

$$\|\,|f|\,\|_* \le 2\|f\|_*.$$

For $d \ge 2$ we do not know the minimal constant c (which possibly depends on d) for the inequality

$$\|\,|f|\,\|_* \le c\|f\|_*.$$

By means of Theorem 3.7 one can improve the last inequality in (3.12) and obtain the following

Corollary 3.9 ([34]). *Let $f \in BMO([a_0, b_0])$. Then*

$$\|f^*\|_* \le \|f\|_*. \tag{3.17}$$

Remark 3.10. For $d \ge 2$ we do not know the minimal constant c in the inequality

$$\|f^*\|_* \le c\|f\|_*.$$

Now let us consider the estimates of the BMO-norm of the non-increasing equimeasurable rearrangement of a BMO^R-function. Recall that BMO^R differs from BMO if $d \ge 2$ because the oscillations must be calculated over all possible rectangles, not only the cubes. First of all we prove the multidimensional analog of Theorem 3.5.

Theorem 3.11 ([45]). *Let $f \in BMO^R(R_0)$, where $R_0 \subset \mathbb{R}^d$ is a multidimensional segment. Then*

$$\|f_d\|_* \le \|f\|_{*,R}. \tag{3.18}$$

Proof. Essentially we will repeat the proof of Theorem 3.5. Fix the interval $J \subset [0, |R_0|]$ and denote $\alpha = \frac{1}{|J|} \int_J f_d(u)\, du$. Let $\alpha \geq \frac{1}{|R_0|} \int_{R_0} f(x)\, dx$. Applying the multidimensional analog of the Riesz "rising sun lemma" (Lemma 1.30) [1], we construct the pairwise disjoint segments $I_j \subset R_0$, $j = 1, 2, \ldots$ such that $f_{I_j} = \alpha$, $j = 1, 2, \ldots$, and $f(x) \leq \alpha$ for almost all $x \in R_0 \setminus E$ with $E = \bigcup_{j \geq 1} I_j$. If we prove the inequality

$$\frac{1}{|J|} \int_J |f_d(t) - \alpha|\, dt \leq \frac{1}{|E|} \int_E |f(x) - \alpha|\, dx, \tag{3.19}$$

then in order to complete the proof it will remain to use the relations

$$f_E = \frac{1}{|E|} \int_E f(x)\, dx = \frac{1}{\sum_j |I_j|} \sum_j \int_{I_j} f(x)\, dx = \alpha, \tag{3.20}$$

$$\frac{1}{|E|} \int_E |f(x) - \alpha|\, dx = \frac{1}{|E|} \sum_j \int_{I_j} |f(x) - \alpha|\, dx =$$

$$= \frac{1}{|E|} \sum_j |I_j| \frac{1}{|I_j|} \int_{I_j} \left| f(x) - f_{I_j} \right|\, dx = \frac{1}{|E|} \sum_j |I_j| \Omega(f; I_j) \leq \|f\|_{*,R}.$$

For the proof of (3.19) let us choose the maximal $t \in (0, |R_0|]$ such that $J \subset [0, t]$ and $\frac{1}{t} \int_0^t f_d(u)\, du = \alpha$. The existence of such a t follows from the condition

$$\frac{1}{|R_0|} \int_0^{|R_0|} f_d(u)\, du = \frac{1}{|R_0|} \int_{R_0} f(x)\, dx \leq \alpha = \frac{1}{|J|} \int_J f_d(u)\, du.$$

Using the monotonicity of the function f_d and applying Property 2.15, we obtain

$$\frac{1}{|J|} \int_J |f_d(u) - \alpha|\, du \leq \frac{1}{t} \int_0^t |f_d(u) - \alpha|\, du.$$

So, in order to prove (3.19) it is enough to show that

$$\frac{1}{t} \int_0^t |f_d(u) - \alpha|\, du \leq \frac{1}{|E|} \int_E |f(x) - \alpha|\, dx. \tag{3.21}$$

In its own turn the inequality (3.21) is a consequence of the following two statements:

[1] We could also use Lemma 1.21 and Remark 1.24. But in order to use Remark 1.24 one should prove that $f \in L^p(R_0)$ for some $p > 1$. Indeed, from the John–Nirenberg inequality, which will be proved in the next section, it follows that $f \in L^p(R_0)$ for every $p < \infty$. This is not a vicious circle, because we do not need Theorem 3.11 to prove the John–Nirenberg inequality (for $d = 2$ it is enough to use Lemma 1.22).

$$t \geq |E|, \tag{3.22}$$

$$\int_0^t |f_d(u) - \alpha| \, du = \int_E |f(x) - \alpha| \, dx. \tag{3.23}$$

For the proof of (3.22) notice, that by the definition of the non-increasing rearrangement

$$\int_0^{|E|} f_d(u) \, du \geq \int_E f(x) \, dx,$$

so that, by (3.20),

$$\frac{1}{|E|} \int_0^{|E|} f_d(u) \, du \geq \frac{1}{|E|} \int_E f(x) \, dx = \alpha = \frac{1}{t} \int_0^t f_d(u) \, du.$$

Taking into account the monotonicity of f_d, from here we obtain (3.22).

For the proof of (3.23) we will use the fact that $f(x) \leq \alpha$ almost everywhere on $R_0 \setminus E$. Then, applying (3.20) and the properties of mean oscillations, we get

$$\int_0^t |f_d(u) - \alpha| \, du = 2 \int_{\{u \in [0,t]:\ f_d(u) > \alpha\}} (f_d(u) - \alpha) \, du =$$

$$= 2 \int_{\{x \in R_0:\ f(x) > \alpha\}} (f(x) - \alpha) \, dx = 2 \int_{\{x \in E:\ f(x) > \alpha\}} (f(x) - \alpha) \, dx =$$

$$= \int_E |f(x) - \alpha| \, dx.$$

This concludes the proof of (3.21).

In the case $\alpha < \frac{1}{|R_0|} \int_{R_0} f(x) \, dx$ it is enough to apply the preceding arguments to the function $-f$ and to note that for all $t \in [0, |R_0|]$, except the set of zero measure of the points of discontinuity of $(-f)_d$, we have $(-f)_d(t) = -f_d(|R_0| - t)$. In addition, in this case

$$\frac{1}{|J|} \int_J |f_d(u) - \alpha| \, du \leq \|f\|_{*,R},$$

and this completes the proof of the theorem. □

The next theorem is the multidimensional analog of Theorem 3.7 (it is interesting to compare it with Remark 3.10).

Theorem 3.12 ([45]). *Let $f \in BMO^R(R_0)$, where $R_0 \subset \mathbb{R}^d$ is a multidimensional segment. Then*

$$\| \, |f| \, \|_{*,R} \leq \|f\|_{*,R}.$$

Proof. Fix the segment $I \subset R_0$ and denote by $g = f|I$ the restriction of the function f to I. Obviously then

$$\Omega(|f|; I) = \Omega(|g|; I) = \Omega(|g_d|; [0, |I|]).$$

On the other hand, according to Theorem 3.11, for every interval $J \subset [0, |I|]$

$$\Omega(g_d; J) \le \sup_{K \subset I} \Omega(g; K) = \sup_{K \subset I} \Omega(f; K) \le \|f\|_{*,R}.$$

Hence, in order to prove the theorem it is enough to prove the inequality

$$\Omega(|g_d|; [0, |I|]) \le \sup_{J \subset [0,|I|]} \Omega(g_d; J). \tag{3.24}$$

But we have already obtained (3.24) while proving Theorem 3.7 (inequality (3.14)). Indeed, formulas (3.14) and (3.24) express the relation between the oscillation of the function g_d and its absolute value independently on the dimension of the space. □

Now we can easily get the multidimensional analog of inequality (3.17).

Theorem 3.13 ([45]). *Let $f \in BMO^R(R_0)$, where $R_0 \subset \mathbb{R}^d$ is a multidimensional segment. Then*

$$\|f^*\|_* \le \|f\|_{*,R}.$$

Proof. Using the trivial equality $f^* = |f|_d$ and applying Theorems 3.11 and 3.12 we obtain

$$\|f^*\|_* = \|\ |f|_d\ \|_* \le \|\ |f|\ \|_{*,R} \le \|f\|_{*,R}. \quad \square$$

3.2 The John–Nirenberg Inequality

We have already mentioned (see p. 41), that the logarithmic function is a typical representative of the BMO-class. This means that the distribution function of the BMO-function decreases exponentially.

Theorem 3.14 (John, Nirenberg, [30]). *There exist constants b and B (possibly depending on the dimension d of the space) such that for any function $f \in BMO(\mathbb{R}^d)$ and any cube $Q_0 \subset \mathbb{R}^d$*

$$|\{x \in Q_o : \ |f(x) - f_{Q_0}| > \lambda\}| \le B \cdot |Q_0| \cdot \exp\left(-\frac{b\lambda}{\|f\|_*}\right), \quad \lambda > 0. \tag{3.25}$$

Proof. Since inequality (3.25) is homogeneous with respect to the multiplication of the function f by a constant we can assume that $\|f\|_* = 1$. Let us apply the Calderón–Zygmund lemma (Lemma 1.14) with $\alpha = \frac{3}{2}$ to the function $|f - f_{Q_0}|$. As the result we obtain a collection of cubes $\left\{Q_j^{(1)}\right\}_{j\geq 1}$ with pairwise disjoint interiors and verifying the following properties:

$$\frac{3}{2} < \frac{1}{\left|Q_j^{(1)}\right|} \int_{Q_j^{(1)}} |f(x) - f_{Q_0}|\, dx \leq 2^d \cdot \frac{3}{2}, \quad j = 1, 2, \dots, \tag{3.26}$$

$$|f(x) - f_{Q_0}| \leq \frac{3}{2} \quad \text{for a.e. } x \in Q_0 \setminus \left(\bigcup_{j\geq 1} Q_j^{(1)} \right).$$

The left inequality of (3.26) implies

$$\sum_{j\geq 1} \left|Q_j^{(1)}\right| \leq \frac{1}{3/2} \sum_{j\geq 1} \int_{Q_j^{(1)}} |f(x) - f_{Q_0}|\, dx \leq$$

$$\leq \frac{2}{3} \int_{Q_0} |f(x) - f_{Q_0}|\, dx \leq \frac{2}{3}|Q_0| \cdot \|f\|_* = \frac{2}{3}|Q_0|,$$

while from the right inequality we have

$$\left|f_{Q_j^{(1)}} - f_{Q_0}\right| = \left| \frac{1}{\left|Q_j^{(1)}\right|} \int_{Q_j^{(1)}} (f(x) - f_{Q_0})\, dx \right| \leq$$

$$\leq \frac{1}{\left|Q_j^{(1)}\right|} \int_{Q_j^{(1)}} |f(x) - f_{Q_0}|\, dx \leq 2^d \cdot \frac{3}{2}, \quad j = 1, 2, \dots$$

To every cube $Q_j^{(1)}$ we apply again Calderón–Zygmund lemma 1.14.

In the k-th step, applying Calderón–Zygmund lemma 1.14 with $\alpha = \frac{3}{2}$ to the function $\left|f - f_{Q_j^{(k-1)}}\right|$ on every cube $Q_j^{(k-1)}$, $j = 1, 2 \dots$, we obtain a family of cubes $Q_{i,j}^{(k)} \subset Q_j^{(k-1)}$, $i = 1, 2, \dots$, with pairwise disjoint interiors, and such that

$$\frac{3}{2} < \frac{1}{\left|Q_{i,j}^{(k)}\right|} \int_{Q_{i,j}^{(k)}} \left|f(x) - f_{Q_j^{(k-1)}}\right| dx \leq 2^d \cdot \frac{3}{2}, \tag{3.27}$$

$$\left|f(x) - f_{Q_j^{(k-1)}}\right| \leq \frac{3}{2} \quad \text{for a.e.} \quad x \in Q_j^{(k-1)} \setminus \left(\bigcup_{i\geq 1} Q_{i,j}^{(k)} \right). \tag{3.28}$$

Inequality (3.27) implies

$$\sum_{i\geq 1}\left|Q_{i,j}^{(k)}\right| \leq \frac{2}{3}\sum_{i\geq 1}\int_{Q_{i,j}^{(k)}}\left|f(x)-f_{Q_j^{(k-1)}}\right|\,dx \leq$$

$$\leq \frac{2}{3}\int_{Q_j^{(k-1)}}\left|f(x)-f_{Q_j^{(k-1)}}\right|\,dx \leq \frac{2}{3}\left|Q_j^{(k-1)}\right|\cdot\|f\|_* = \frac{2}{3}\left|Q_j^{(k-1)}\right|, \quad (3.29)$$

$$\left|f_{Q_{i,j}^{(k)}} - f_{Q_j^{(k-1)}}\right| \leq \frac{1}{\left|Q_{i,j}^{(k)}\right|}\int_{Q_{i,j}^{(k)}}\left|f(x)-f_{Q_j^{(k-1)}}\right|\,dx \leq 2^d\cdot\frac{3}{2}. \quad (3.30)$$

Numbering all cubes $Q_{i,j}^{(k)},\ i,j=1,2,\dots$ we get the collection $\left\{Q_j^{(k)}\right\}_{j\geq 1}$. In addition, by (3.29),

$$\sum_{j\geq 1}\left|Q_j^{(k)}\right| \leq \frac{2}{3}\sum_{j\geq 1}\left|Q_j^{(k-1)}\right| \leq \dots \leq \left(\frac{2}{3}\right)^{k-1}\sum_{j\geq 1}\left|Q_j^{(1)}\right| \leq \left(\frac{2}{3}\right)^{k}|Q_0|, \quad (3.31)$$

while from (3.28) and (3.30) it follows that

$$|f(x)-f_{Q_0}| \leq \left|f(x)-f_{Q_{j_{k-1}}^{(k-1)}}\right| + \left|f_{Q_{j_{k-1}}^{(k-1)}} - f_{Q_{j_{k-2}}^{(k-2)}}\right| + \dots + \left|f_{Q_{j_1}^{(1)}} - f_{Q_0}\right| \leq$$

$$\leq \frac{3}{2}+2^d(k-1)\frac{3}{2} \leq k\cdot 2^d\cdot\frac{3}{2} \quad \text{for a.e. } x\in\left(\cup_{j\geq 1}Q_j^{(k-1)}\right)\setminus\left(\cup_{j\geq 1}Q_j^{(k)}\right), \quad (3.32)$$

where $Q_{j_{i+1}}^{(i+1)}\subset Q_{j_i}^{(i)},\ i=1,\dots,k-2$. Then we pass to the next, $(k+1)$-th step.

Take an arbitrary number $\lambda>0$. If $k\cdot 2^d\cdot\frac{3}{2}<\lambda\leq(k+1)\cdot 2^d\cdot\frac{3}{2}$ for some $k\in\mathbb{N}$, then, by (3.31) and (3.32),

$$|\{x\in Q_0:\ |f(x)-f_{Q_0}|>\lambda\}| \leq \left|\left\{x\in Q_0:\ |f(x)-f_{Q_0}|>k\cdot 2^d\cdot\frac{3}{2}\right\}\right| \leq$$

$$\leq \sum_{j\geq 1}\left|Q_j^{(k)}\right| \leq \left(\frac{2}{3}\right)^k\cdot|Q_0| = |Q_0|\exp\left(-k\ln\frac{3}{2}\right) \leq$$

$$\leq |Q_0|\exp\left(\left(1-\frac{\lambda}{2^d\cdot\frac{3}{2}}\right)\ln\frac{3}{2}\right) = \frac{3}{2}|Q_0|\exp(-b\lambda),$$

where $b=\frac{2}{3}\cdot\ln\frac{3}{2}\cdot 2^{-d}$. Otherwise, if $\lambda\leq 2^d\cdot\frac{3}{2}$, then

$$|\{x\in Q_o:\ |f(x)-f_{Q_0}|>\lambda\}| \leq |Q_0|\exp(-b\lambda)\cdot\exp\left(b\cdot 2^d\frac{3}{2}\right) \equiv$$

$$\equiv |Q_0|B_1\exp(-b\lambda),$$

where $B_1=\exp\left(b\cdot 2^d\cdot\frac{3}{2}\right)$. Setting $B=B_1+\frac{3}{2}$, we obtain (3.25). □

Remark 3.15. In terms of equimeasurable rearrangements inequality (3.25) can be rewritten in the following form:

$$(f - f_{Q_0})^* (t) \le \frac{\|f\|_*}{b} \ln \frac{B|Q_0|}{t}, \quad 0 < t \le |Q_0|. \tag{3.33}$$

So, if $f \in BMO$, then its equimeasurable rearrangement do not grow faster than the logarithmic function as the argument tends to zero.

Remark 3.16. In a certain sense the John–Nirenberg theorem is invertible. Namely, if f is a locally summable on $\mathbb{R}^d$ function such that for any cube $Q_0 \subset \mathbb{R}^d$

$$|\{x \in Q_0 : \ |f(x) - f_{Q_0}| > \lambda\}| \le B|Q_0| \cdot \exp(-b\lambda), \quad \lambda > 0, \tag{3.34}$$

where the constants B and b do not depend on Q_0, then $f \in BMO(\mathbb{R}^d)$. Indeed, let us rewrite (3.34) in the form

$$(f - f_{Q_0})^* (t) \le \frac{1}{b} \ln \frac{B|Q_0|}{t}, \quad 0 < t \le |Q_0|. \tag{3.35}$$

Then

$$\frac{1}{|Q_0|} \int_{Q_0} |f(x) - f_{Q_0}| \, dx = \frac{1}{|Q_0|} \int_0^{|Q_0|} (f - f_{Q_0})^* (t) \, dt \le$$

$$\le \frac{1}{b} \frac{1}{|Q_0|} \int_0^{|Q_0|} \ln \frac{B|Q_0|}{t} \, dt = \frac{1}{b} \int_0^1 \ln \frac{B}{u} \, du = \frac{1}{b}(1 + \ln B).$$

Taking the supremum over all cubes $Q_0 \subset \mathbb{R}^d$, we obtain $\|f\|_* \le \frac{1}{b}(1 + \ln B)$.

Remark 3.17. Now let $f \in BMO(Q_0)$ for some fixed cube $Q_0 \subset \mathbb{R}^d$. Obviously then the proof of inequality (3.25) holds true, and so do (3.33). However, (3.34), as well as its equivalent form (3.35), does not imply $f \in BMO(Q_0)$. One can easily construct the corresponding example, we omit this point here.

The John–Nirenberg theorem implies

Corollary 3.18. *If $f \in BMO(\mathbb{R}^d)$, then $f \in L^p_{loc}$ for any $p < \infty$.*

Proof. It is enough to prove that $f - f_{Q_0} \in L^p(Q_0)$ for any cube $Q_0 \subset \mathbb{R}^d$. The John–Nirenberg inequality in the form (3.33) yields

$$\int_{Q_0} |f(x) - f_{Q_0}|^p \, dx = \int_0^{|Q_0|} \left[(f - f_{Q_0})^* (t)\right]^p \, dt \le$$

$$\le \left(\frac{\|f\|_*}{b}\right)^p \int_0^{|Q_0|} \ln^p \frac{B|Q_0|}{t} \, dt =$$

$$= \left(\frac{\|f\|_*}{b}\right)^p |Q_0| \cdot B \int_0^{1/B} \ln^p \frac{1}{u} \, du < \infty. \quad \square \tag{3.36}$$

Remark 3.19. Inequality (3.36) can be rewritten as follows:

$$\Omega_p(f; Q_0) \le c_{p,d}\|f\|_*, \quad Q_0 \subset \mathbb{R}^d,$$

where the constant $c_{p,d}$ depends only on p and d. Hence

$$\|f\|_{*,p} \le c_{p,d}\|f\|_*.$$

As we have already mentioned, the inequality $\|f\|_* \le \|f\|_{*,p}$ for $1 < p < \infty$ is a direct consequence of the Hölder inequality. Therefore all the classes $BMO_p(\mathbb{R}^d)$ coincide for all p, $1 \le p < \infty$. Analogously, for any fixed cube $Q_0 \subset \mathbb{R}^d$ all the classes $BMO_p(Q_0)$ coincide.

The John–Nirenberg inequality in the form (3.33) can be easily derived from the estimate of the rearrangement, provided by Theorem 3.4. Indeed, let $Q_0 \subset \mathbb{R}^d$, $f \in BMO(Q_0)$ and $f_{Q_0} = 0$. Then

$$f^{**}\left(\frac{t}{2}\right) - f^{**}(t) = \frac{2}{t}\int_0^{t/2} (f^*(u) - f^{**}(t))\, du \le$$

$$\le \frac{2}{t}\int_0^{t} |f^*(u) - f^{**}(t)|\, du \le 2\|f^*\|_*, \quad 0 < t \le |Q_0|.$$

According to Theorem 3.4,

$$f^{**}\left(\frac{t}{2}\right) - f^{**}(t) \le 2c\|f\|_*, \quad 0 < t \le |Q_0|, \tag{3.37}$$

where the constant c depends only on the dimension d of the space.

Fix some $t \in (0, |Q_0|]$ and choose n such that $2^{-n-1}|Q_0| < t \le 2^{-n}|Q_0|$. Applying (3.37) we obtain

$$f^{**}(t) \le f^{**}\left(2^{-n-1}|Q_0|\right) \le f^{**}\left(2^{-n}|Q_0|\right) + 2c\|f\|_* \le$$

$$\le f^{**}\left(2^{-n+1}|Q_0|\right) + 2\cdot(2c\|f\|_*) \le \cdots \le f^{**}\left(|Q_0|\right) + (n+1)(2c\|f\|_*).$$

Taking into account that $f_{Q_0} = 0$ implies

$$f^{**}\left(|Q_0|\right) = \frac{1}{|Q_0|}\int_0^{|Q_0|} f^*(u)\, du = \frac{1}{|Q_0|}\int_{Q_0} |f(x)|\, dx \le \|f\|_*,$$

we get

$$f^*(t) \le f^{**}(t) \le 2c(n+2)\|f\|_* \le 2c\left(\frac{\ln\frac{|Q_0|}{t}}{\ln 2} + 2\right)\|f\|_* =$$

$$= \frac{2c}{\ln 2}\|f\|_* \ln\frac{4|Q_0|}{t}, \quad 0 < t \le |Q_0|,$$

which for $f_{Q_0} = 0$ yields (3.33) with $B = 4$, $b = \frac{\ln 2}{2c}$.

Notice, that the assumption $f_{Q_0} = 0$ is not restrictive, and one could obtain (3.33) without this additional condition. □

3.2.1 One-Dimensional Case

For the proof of the John–Nirenberg inequality we were using the arguments, based on the estimates of the equimeasurable rearrangements of functions. These arguments were used in the original work [1]. One can improve this result and get the exact exponent in the John–Nirenberg inequality (3.25) for the one-dimensional case. Indeed, Lemma 2.2 has the following

Corollary 3.20. *Let* $f \in BMO\,([a_0, b_0])$ *with* $[a_0, b_0] \subset \mathbb{R}$ *and* $f_{[a_0,b_0]} = 0$. *Then for any* $a > 1$

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \le \frac{a}{2}\|f\|_*, \quad 0 < t \le b_0 - a_0. \tag{3.38}$$

Proof. Taking $\varphi = f^*$ in Lemma 2.2, we have $F = f^{**}$. Then, by (2.2),

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \le \frac{a}{2}\|f^*\|_*, \quad 0 < t \le b_0 - a_0.$$

This, together with inequality (3.17) (see Corollary 3.9), implies (3.38). □

Using (3.38) it is easy to derive the John–Nirenberg inequality with the exact exponent in the one-dimensional case.

Theorem 3.21 ([34]). *Let* $f \in BMO\,([a_0, b_0])$. *Then*

$$\left(f - f_{[a_0,b_0]}\right)^*(t) \le \frac{\|f\|_*}{2/\mathrm{e}} \ln \frac{B\,(b_0 - a_0)}{t}, \quad 0 < t \le b_0 - a_0, \tag{3.39}$$

with $B = \exp\left(1 + \frac{2}{\mathrm{e}}\right)$. *Moreover, in general the denominator* $2/\mathrm{e}$ *in the fraction, preceding to the logarithm, cannot be increased.*

Proof. Without loss of generality, we can assume that $f_{[a_0,b_0]} = 0$. Let $a > 1$ (we will choose this constant later). Summing up the inequalities

$$f^{**}\left(\frac{b_0 - a_0}{a^i}\right) - f^{**}\left(\frac{b_0 - a_0}{a^{i-1}}\right) \le \frac{a}{2}\|f\|_*, \quad i = 1, \ldots, k+1,$$

which follow from (3.38), we get

$$f^{**}\left(\frac{b_0 - a_0}{a^{k+1}}\right) \le (k+1)\frac{a}{2}\|f\|_* + f^{**}\,(b_0 - a_0)\,. \tag{3.40}$$

Since $f_{[a_0,b_0]} = 0$ we have

$$f^{**}\,(b_0 - a_0) = \frac{1}{b_0 - a_0}\int_0^{b_0 - a_0} f^*(u)\,du = \frac{1}{b_0 - a_0}\int_{a_0}^{b_0} |f(x)|\,dx =$$

$$= \Omega\,(f;[a_0,b_0]) \le \|f\|_*,$$

so that

$$f^{**}\left(\frac{b_0-a_0}{a^{k+1}}\right) \le \left((k+1)\frac{a}{2}+1\right)\|f\|_*, \quad k=0,1,\ldots \tag{3.41}$$

Choose some $t \in (0, b_0-a_0]$ and k such that

$$\frac{b_0-a_0}{a^{k+1}} < t \le \frac{b_0-a_0}{a^k}.$$

Then $k \le \frac{1}{\ln a}\ln\frac{b_0-a_0}{t}$ and from (3.41) we obtain

$$f^*(t) \le f^{**}(t) \le f^{**}\left(\frac{b_0-a_0}{a^{k+1}}\right) \le \left((k+1)\frac{a}{2}+1\right)\|f\|_* \le$$

$$\le \left(\left(\frac{1}{\ln a}\cdot\ln\frac{b_0-a_0}{t}+1\right)\frac{a}{2}+1\right)\|f\|_* =$$

$$= \left(\frac{1}{2}\frac{a}{\ln a}\cdot\ln\frac{b_0-a_0}{t}+\frac{a}{2}+1\right)\|f\|_*. \tag{3.42}$$

The function $\frac{a}{\ln a}$ for $a>1$ takes its minimal value at $a=\mathrm{e}$. Substituting $a=\mathrm{e}$ in (3.42) for $0<t\le b_0-a_0$ we have

$$f^*(t) \le \left(\frac{\mathrm{e}}{2}\ln\frac{b_0-a_0}{t}+\frac{\mathrm{e}}{2}+1\right)\|f\|_* = \frac{\|f\|_*}{2/\mathrm{e}}\ln\frac{\exp\left(1+\frac{2}{\mathrm{e}}\right)(b_0-a_0)}{t},$$

and (3.39) follows.

It remains to show that the constant $2/\mathrm{e}$ in the denominator in the right-hand side of (3.39) cannot be increased. Indeed, for the function $f(x) = \ln\frac{1}{x}-1,\ 0\le x\le 1$, we have $f_{[0,1]}=0$. Moreover, as it was already shown (see Example 2.24), $\|f\|_* = \frac{2}{\mathrm{e}}$. Hence (3.39) becomes

$$f^*(t) \le \ln\frac{1}{t}+1+\frac{2}{\mathrm{e}}, \quad 0<t\le 1.$$

On the other hand, it is easy to see that $f^*(t) = \ln\frac{1}{\mathrm{e}t},\ 0<t\le \mathrm{e}^{-2}$. Therefore the coefficient of the logarithm in the right-hand side of (3.39) cannot be decreased. □

In terms of the distribution function, inequality (3.39) can be rewritten in the following way.

Corollary 3.22 ([34]). *Let $f \in BMO\,([a_0,b_0])$. Then*

$$\left|\left\{x\in[a_0,b_0]:\ \left|f(x)-f_{[a_0,b_0]}\right|>\lambda\right\}\right| \le B\,(b_0-a_0)\exp\left(-\frac{2/\mathrm{e}}{\|f\|_*}\lambda\right), \quad \lambda>0, \tag{3.43}$$

where $B=\exp\left(1+\frac{2}{\mathrm{e}}\right)$, and in general the constant $2/\mathrm{e}$ in the exponent cannot be increased.

3.2.2 Anisotropic Case

To our knowledge in the multidimensional case the problem of finding the upper bound of those b that provide John–Nirenberg inequality (3.25) is still open. If instead of $\|f\|_*$ in (3.25) we consider $\|f\|_{*,R}$, then the maximal value of the constant b in the John–Nirenberg inequality is equal to $\frac{2}{\mathrm{e}}$ as in the one-dimensional case. Namely, we have

Theorem 3.23 ([45]). *Let $f \in BMO^R(\mathbb{R}^d)$. Then for any segment $R_0 \subset \mathbb{R}^d$*

$$|\{x \in R_0 : \ |f(x) - f_{R_0}| > \lambda\}| \le B \cdot |R_0| \cdot \exp\left(-\frac{2/\mathrm{e}}{\|f\|_{*,R}}\lambda\right), \quad \lambda > 0, \quad (3.44)$$

where $B = \exp\left(1 + \frac{2}{\mathrm{e}}\right)$, and in general the constant $\frac{2}{\mathrm{e}}$ in the exponent cannot be increased.

Proof. Without loss of generality we can assume that $f_{R_0} = 0$. Then rewriting inequality (3.44) in terms of equimeasurable rearrangements we have

$$f^*(t) \le \frac{\|f\|_{*,R}}{2/\mathrm{e}} \ln \frac{B|R_0|}{t}, \quad 0 < t \le |R_0|. \quad (3.45)$$

Essentially in order to prove (3.45) we have to repeat the same arguments as in the proof of Theorem 3.21. Indeed, setting $\varphi = f^*$ in Lemma 2.2, we have

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \le \frac{a}{2}\|f^*\|_*, \quad 0 < t \le |R_0|,$$

which together with Theorem 3.13 leads to the following multidimensional analog of inequality (3.38):

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \le \frac{a}{2}\|f\|_{*,R}, \quad 0 < t \le |R_0|$$

for an arbitrary $a > 1$. Now it remains to repeat completely the proof of Theorem 3.21, taking R_0 instead of $[a_0, b_0]$, and $\|f\|_{*,R}$ instead of $\|f\|_*$.

The fact that the denominator $\frac{2}{\mathrm{e}}$ in the right-hand side of (3.45) cannot be increased can be easily checked on the following function:

$$f(x_1, \dots, x_d) = \ln \frac{1}{x_1} - 1, \quad x \equiv (x_1, \dots, x_d) \in R_0 \equiv [0, 1]^d. \quad \square$$

4

The BMO-estimates of the Hardy-type Transforms

4.1 Estimates of Oscillations of the Hardy Transform

The operator $\mathcal{P}$ defined via the formula

$$\mathcal{P}f(t) = \frac{1}{t}\int_0^t f(u)\,du, \quad t \neq 0$$

is called the Hardy transform (operator) of the function $f \in L_{loc}(\mathbb{R})$. This operator has plenty of applications. We have seen some of them in Section 1.1. Namely, it easy to see that $f^{**}(t) = \mathcal{P}f^*(t),\ t \in \mathbb{R}_+$. The *Hardy inequality* [26]:

$$\int_0^\infty |\,\mathcal{P}f(x)|^p\,dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p\,dx$$

provides the boundedness of the Hardy operator in $L^p(\mathbb{R}_+)$. Here the constant $\left(\frac{p}{p-1}\right)^p$ is sharp. For $p=\infty$ the analogous inequality $\|\,\mathcal{P}f\,\|_\infty \le \|f\|_\infty$ is trivial. For $p=1$ the analog of the Hardy inequality is false, in order to see this it is enough to consider the function

$$f_0(x) = \frac{1}{x\ln^2\frac{1}{x}}\chi_{\left(0,\frac{1}{\mathrm{e}}\right)}(x), \quad x \in \mathbb{R}_+.$$

Indeed, $f_0 \in L(\mathbb{R}_+)$ and for $0<x<\frac{1}{\mathrm{e}}$ we have $\mathcal{P}f_0(x) = \left(x\ln\frac{1}{x}\right)^{-1}$, so that $\mathcal{P}f_0 \notin L_{loc}(\mathbb{R}_+)$. It is easy to see that in general *the reverse Hardy inequality*

$$\int_0^\infty |\,\mathcal{P}f(x)|^p\,dx \ge c_p \int_0^\infty |f(x)|^p\,dx \tag{4.1}$$

fails for an arbitrary constant $c_p > 0$. However, if the non-negative function f is non-increasing on $\mathbb{R}_+$, then (4.1) is true for $c_p = \frac{p}{p-1}$ and this value cannot be increased (see [53, 64, 58]). If $p=\infty$ and f is a non-positive non-increasing function on $\mathbb{R}_+$, then obviously $\|\,\mathcal{P}f\,\|_\infty = \|f\|_\infty = \lim_{x\to0+} f(x)$.

In this section we will study the behavior of the Hardy operator in the spaces BMO, BMO_p and BLO. The boundedness of $\mathcal{P}$ in BMO in different cases was proved by several authors in [73, 74, 71, 24, 19, 20, 80]. In particular, in [80] it was proved the following result.

Theorem 4.1 (Jie Xiao, [80]). *The operator $\mathcal{P}$ is bounded in $BMO(\mathbb{R}_+)$ and*

$$\| \mathcal{P}f \|_* \leq \|f\|_*.$$

On the other hand, if f is a positive non-increasing function on $\mathbb{R}_+$, then

$$\| \mathcal{P}f \|_* \geq \frac{1}{17}\|f\|_*.$$

Here we will prove some more general facts. Let us start with the direct estimate of the Hardy transform.

Theorem 4.2 ([40]). *Let $1 \leq p < \infty$. Then if f belongs to $BMO_p(\mathbb{R})$, then $\mathcal{P}f \in BMO_p(\mathbb{R})$ and*

$$\| \mathcal{P}f \|_{*,p} \leq \|f\|_{*,p}. \tag{4.2}$$

Moreover, in general the constant 1 in the right-hand side of (4.2) is sharp.

Proof. As in [80], we will use the equality

$$\mathcal{P}f(t) = \int_0^1 f(tu)\,du, \quad t \in \mathbb{R} \setminus \{0\}. \tag{4.3}$$

Fix the interval $[a,b] \equiv I \subset \mathbb{R}$. By the Fubini theorem,

$$(\mathcal{P}f)_I = \frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt = \int_0^1 \frac{1}{|I|}\int_I f(tu)\,dt\,du.$$

Denote $uI \equiv [ua, ub]$. Applying again the Fubini theorem and the Hölder inequality, we have

$$\Omega_p^p(\mathcal{P}f; I) = \frac{1}{|I|}\int_I \left| \int_0^1 f(\tau u)\,du - \int_0^1 \frac{1}{|I|}\int_I f(tu)\,dt\,du \right|^p d\tau \leq$$

$$\leq \int_0^1 \frac{1}{|I|}\int_I \left| f(\tau u) - \frac{1}{|I|}\int_I f(tu)\,dt \right|^p d\tau\,du =$$

$$= \int_0^1 \frac{1}{|uI|}\int_{uI} \left| f(v) - \frac{1}{uI}\int_{uI} f(\xi)\,d\xi \right|^p dv\,du = \int_0^1 \Omega_p^p(f; uI)\,du \leq \|f\|_{*,p}^p,$$

and (4.2) follows.

For the function $f(x) = \ln \frac{1}{|x|},\ x \in \mathbb{R}$, we have $\mathcal{P}f(x) = 1 + \ln \frac{1}{|x|}$. Hence for this choice of f inequality (4.2) becomes an equality, so that the constant in the right-hand side of (4.2) cannot be smaller than 1. $\square$

If in the proof of Theorem 4.2 we choose $I \subset \mathbb{R}_+$, then $uI \subset \mathbb{R}_+$ for $u > 0$. Hence, repeating the proof of Theorem 4.2, we obtain the following statement.

Theorem 4.3 ([40]). *Let $1 \le p < \infty$. If $f \in BMO_p(\mathbb{R}_+)$, then $\mathcal{P}f \in BMO_p(\mathbb{R}_+)$ and*

$$\| \mathcal{P}f \|_{*,p} \le \|f\|_{*,p}.$$

Moreover, the constant 1 in the right-hand side is sharp.

Similarly one can obtain the estimates for the "norm" of the Hardy transform in BLO.

Theorem 4.4 ([41]). *Let $f \in BLO(\mathbb{R})$. Then $\mathcal{P}f \in BLO(\mathbb{R})$,*

$$\| \mathcal{P}f \|_{BLO} \le \|f\|_{BLO}, \tag{4.4}$$

and the constant 1 in the right-hand side is sharp.

Theorem 4.5 ([41]). *Let $f \in BLO(\mathbb{R}_+)$. Then $\mathcal{P}f \in BLO(\mathbb{R}_+)$,*

$$\| \mathcal{P}f \|_{BLO} \le \|f\|_{BLO},$$

and the constant 1 in the right-hand side is sharp.

As in the case of Theorems 4.2 and 4.3, the proofs of both Theorems 4.4 and 4.5 are similar. Here we give just one of them.

Proof of Theorem 4.4. Let $I \subset \mathbb{R}$ and $x \in I,\ x \ne 0$. By (4.3),

$$\frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt - \mathcal{P}f(x) = \int_0^1 \frac{1}{|I|}\int_I f(tu)\,du - \int_0^1 f(xu)\,du \le$$

$$\le \int_0^1 \left[\frac{1}{|I|}\int_I f(tu)\,dt - f(xu)\right] du = \int_0^1 \left[\frac{1}{|uI|}\int_{uI} f(v)\,dv - f(xu)\right] du.$$

Since $x \in I$ implies $ux \in uI$ for $u > 0$ we have

$$\frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt - \mathcal{P}f(x) \le \int_0^1 \left[\frac{1}{|uI|}\int_{uI} f(v)\,dv - \operatorname*{ess\,inf}_{y\in uI} f(y)\right] du =$$

$$= \int_0^1 L(f;uI)\,du \le \|f\|_{BLO}.$$

Hence, using the equality

$$L(\mathcal{P}f;I) = \frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt - \operatorname*{ess\,inf}_{x\in I} \mathcal{P}f(x) = \operatorname*{ess\,sup}_{x\in I}\left[\frac{1}{|I|}\int_I \mathcal{P}f(t)\,dt - \mathcal{P}f(x)\right],$$

and taking the essential supremum over all $x \in I,\ x \neq 0$, we obtain

$$L(\mathcal{P}f; I) \leq \|f\|_{BLO}, \quad I \subset \mathbb{R}.$$

The same arguments as in the proof of Theorem 4.2 show that the constant 1 in the right-hand side of (4.4) is sharp. □

Now let us consider the lower bounds for the norm of the Hardy transform. It is easy to see that, similarly to (4.1), the inequality

$$\| \mathcal{P}f \|_{*} \geq c\|f\|_{*} \tag{4.5}$$

in general fails for arbitrary f and $c > 0$. But if we consider the functions f that are non-increasing and non-negative on $\mathbb{R}_{+}$, then, according to Theorem 4.1, inequality (4.5) holds true for $c = \frac{1}{17}$. In what follows we will derive (4.5) with the value of c grater than $\frac{1}{17}$ and find its upper bound (see Corollary 4.16).

Theorem 4.6 ([40]). *Let $1 \leq p < \infty$, and assume that the function f is non-decreasing on $(-\infty, 0)$ and non-increasing on $(0, +\infty)$. Then $f \in BMO_p(\mathbb{R})$ if and only if $\mathcal{P}f \in BMO_p(\mathbb{R})$ and*

$$\| \mathcal{P}f \|_{*,p} \leq \|f\|_{*,p} \leq \frac{2}{3-\sqrt{7}} \| \mathcal{P}f \|_{*,p}. \tag{4.6}$$

Proof. The left inequality of (4.6) is contained in Theorem 4.2. Moreover, it is true even if f is not monotone. Now let us prove the right inequality.

Let $\lambda > 1$ (we will choose it later). Consider the function

$$g(t) = \frac{1}{\lambda}\, \mathcal{P}f(t) + \frac{\lambda - 1}{\lambda} f(t), \quad t \in \mathbb{R} \setminus \{0\}.$$

Then $f(t) = \frac{\lambda}{\lambda-1} g(t) - \frac{1}{\lambda-1}\, \mathcal{P}f(t)$, so that by Minkowski inequality

$$\|f\|_{*,p} \leq \frac{\lambda}{\lambda - 1} \|g\|_{*,p} + \frac{1}{\lambda - 1} \| \mathcal{P}f \|_{*,p}. \tag{4.7}$$

Let us estimate $\|g\|_{*,p}$. The monotonicity of f on $(-\infty, 0)$ and $(0, +\infty)$ implies

$$\mathcal{P}f(\lambda t) = \frac{1}{\lambda}\frac{1}{t} \int_0^t f(u)\, du + \frac{\lambda - 1}{\lambda}\, \frac{1}{\lambda t - t} \int_t^{\lambda t} f(u)\, du \leq \frac{1}{\lambda}\, \mathcal{P}f(t) + \frac{\lambda - 1}{\lambda} f(t)$$

for $t \neq 0$. Hence, using again the monotonicity of f, we obtain

$$\mathcal{P}f(\lambda t) \leq g(t) \leq \mathcal{P}f(t), \quad t \neq 0. \tag{4.8}$$

Assume $I \equiv [\alpha, \beta],\ \alpha < 0 < \beta$. By (4.8),

$$\frac{1}{|I|}\int_I \mathcal{P}f(\lambda t)\,dt = (\mathcal{P}f)_{\lambda I} \le g_I \le (\mathcal{P}f)_I.$$

The last inequality, together with (4.8), imply

$$g(t) - g_I \le \mathcal{P}f(t) - (\mathcal{P}f)_{\lambda I}, \quad t \ne 0, \tag{4.9}$$

$$g_I - g(t) \le (\mathcal{P}f)_I - \mathcal{P}f(\lambda t), \quad t \ne 0. \tag{4.10}$$

Denote $E_+ \equiv \{t \in I : \; g(t) \ge g_I\}$, $E_- \equiv \{t \in I : \; g(t) < g_I\}$. Multiplying (4.9) and (4.10) by $\chi_{E_+}(t)$ and $\chi_{E_-}(t)$ respectively and summing up the obtained inequalities for $t \ne 0$ we get

$$|g(t) - g_I| \le (\mathcal{P}f(t) - (\mathcal{P}f)_{\lambda I})\,\chi_{E_+}(t) + ((\mathcal{P}f)_I - \mathcal{P}f(\lambda t))\,\chi_{E_-}(t) =$$

$$= (\mathcal{P}f(t) - (\mathcal{P}f)_I)\,\chi_{E_+}(t) + ((\mathcal{P}f)_{\lambda I} - \mathcal{P}f(\lambda t))\,\chi_{E_-}(t) + ((\mathcal{P}f)_I - (\mathcal{P}f)_{\lambda I}).$$

Using Lemma 2.35 and the inclusion $\lambda I \supset I$, one can find the following estimate for the last term in the right hand side:

$$(\mathcal{P}f)_I - (\mathcal{P}f)_{\lambda I} = \frac{1}{|I|}\int_I (\mathcal{P}f(t) - (\mathcal{P}f)_{\lambda I})\;dt \le$$

$$\le \lambda \frac{1}{|\lambda I|}\int_{\{t\in\lambda I:\; \mathcal{P}f(t) > (\mathcal{P}f)_{\lambda I}\}} (\mathcal{P}f(t) - (\mathcal{P}f)_{\lambda I})\;dt = \frac{\lambda}{2}\Omega(\mathcal{P}f;\lambda I) \le \frac{\lambda}{2}\|\,\mathcal{P}f\,\|_*.$$

Thus, by Minkowski inequality,

$$\Omega_p(g;I) \le \left\{\frac{1}{|I|}\int_{E_+} |\mathcal{P}f(t) - (\mathcal{P}f)_I|^p\;dt\right\}^{\frac{1}{p}} +$$

$$+ \left\{\frac{1}{|I|}\int_{E_-} |(\mathcal{P}f)_{\lambda I} - \mathcal{P}f(\lambda t)|^p\;dt\right\}^{\frac{1}{p}} + \frac{\lambda}{2}\|\,\mathcal{P}f\,\|_* \le$$

$$\le \Omega_p(\mathcal{P}f;I) + \Omega_p(\mathcal{P}f;\lambda I) + \frac{\lambda}{2}\|\,\mathcal{P}f\,\|_* \le \left(2 + \frac{\lambda}{2}\right)\|\,\mathcal{P}f\,\|_{*,p}.$$

Notice that both functions g and f are non-decreasing on $(-\infty,0)$ and non-increasing on $(0,+\infty)$. Since I is an arbitrary segment, which contains zero, the last inequality together with Lemma 2.23 imply

$$\|g\|_{*,p} \le \left(2 + \frac{\lambda}{2}\right)\|\,\mathcal{P}f\,\|_{*,p}.$$

Substituting this bound in (4.7), we obtain

$$\|f\|_{*,p} \le \frac{1}{\lambda - 1}\left[\lambda\left(2 + \frac{\lambda}{2}\right) + 1\right]\|\,\mathcal{P}f\,\|_{*,p}. \tag{4.11}$$

It remains to choose the constant $\lambda > 1$ which provides the minimal value to the function $\psi(\lambda) \equiv \frac{1}{\lambda-1}\left[\lambda\left(2+\frac{\lambda}{2}\right)+1\right]$. An easy calculation shows that

$$\min_{\lambda>1}\psi(\lambda) = \psi(1+\sqrt{7}) = \frac{2}{3-\sqrt{7}}.$$

Therefore, (4.11) implies the right inequality of (4.6). □

If in the proof of Theorem 4.6 instead of Lemma 2.23 we apply Lemma 2.22, then we get the following statement.

Theorem 4.7 ([40]). *Let $1 \le p < \infty$ and let $f \in L^p_{loc}(\mathbb{R}_+)$ be non-increasing on $\mathbb{R}_+$. Then $f \in BMO_p(\mathbb{R}_+)$ if and only if $\mathcal{P}f \in BMO_p(\mathbb{R}_+)$, and*

$$\|\,\mathcal{P}f\,\|_{*,p} \le \|f\|_{*,p} \le \frac{2}{3-\sqrt{7}}\|\,\mathcal{P}f\,\|_{*,p}. \tag{4.12}$$

In the particular case $p = 1$ the right inequality of (4.12) becomes

$$\|\,\mathcal{P}f\,\|_* \ge \frac{3-\sqrt{7}}{2}\|f\|_*, \quad f \in BMO(\mathbb{R}_+),\ f \text{ do not increase.} \tag{4.13}$$

Since $\frac{3-\sqrt{7}}{2} > \frac{1}{17}$ the new inequality is stronger than the inequality in the second part of Theorem 4.1.

The next theorem is an analog of Theorem 4.7 for the BLO-"norm".

Theorem 4.8 ([41]). *Let $f \in L_{loc}(\mathbb{R}_+)$ be non-increasing on $\mathbb{R}_+$. Then*

$$\frac{1}{\mathrm{e}}\|f\|_{BLO} \le \|\,\mathcal{P}f\,\|_{BLO} \le \|f\|_{BLO}, \tag{4.14}$$

and in general the constants $\frac{1}{\mathrm{e}}$ and 1 in the left and right-hand sides are sharp.

Proof. The left inequality of (4.14) was already proved in Theorem 4.5. Let us show that the constant $\frac{1}{\mathrm{e}}$ in the left-hand side of (4.14) cannot be increased. For this consider the function $f_0(x) = \chi_{[0,1)}(x),\ x \in \mathbb{R}_+$. By Lemma 2.34,

$$\mathcal{P}f_0(x) = \min\left(1, \frac{1}{x}\right), \quad \|f_0\|_{BLO} = \sup_{x>0}\left[\mathcal{P}f_0(x) - f_0(x)\right] = 1,$$

and for $x > 1$

$$L\left(\mathcal{P}f_0; [0,x]\right) = \frac{1}{x}\int_0^x \mathcal{P}f_0(t)\,dt - \mathcal{P}f_0(x) = \frac{1}{x}(1+\ln x) - \frac{1}{x} = \frac{\ln x}{x}.$$

Hence

$$\|\,\mathcal{P}f_0\,\|_{BLO} = \sup_{x>1}\frac{\ln x}{x} = \frac{1}{\mathrm{e}} = L\left(\mathcal{P}f_0; [0,\mathrm{e}]\right). \tag{4.15}$$

Therefore the constant $\frac{1}{\mathrm{e}}$ in the left-hand side of (4.14) cannot be increased.

It remains to prove the left inequality of (4.14) for an arbitrary f. By virtue of Lemma 2.34, it is enough to show that for any $x > 0$ there exists $y > 0$ such that

$$L(\mathcal{P}f;[0,y]) \geq \frac{1}{\mathrm{e}} L(f;[0,x]). \tag{4.16}$$

Without loss of generality we can assume that

$$x = 1, \quad f(1) = 0, \quad \int_0^1 f(t)\,dt = 1, \quad L(f;[0,1]) = 1. \tag{4.17}$$

As before, set $f_0(x) = \chi_{[0,1)}(x),\ x \in \mathbb{R}_+$. Let us show that

$$\mathcal{P}(f - f_0)(t) \geq (f - f_0)(t), \quad t > 0. \tag{4.18}$$

Indeed, if $0 < t \leq 1$, then (4.18) follows from the monotonicity of $f - f_0$. Otherwise, if $t > 1$, then $f(t) \leq 0$. Taking into account assumptions (4.17), from the monotonicity of f we obtain

$$\frac{1}{t}\int_0^t [f(u) - f_0(u)]\,du = \frac{1}{t}\int_0^1 [f(u) - 1]\,du + \frac{1}{t}\int_1^t f(u)\,du = \frac{1}{t}\int_1^t f(u)\,du =$$

$$= \left(1 - \frac{1}{t}\right)\frac{1}{t-1}\int_1^t f(u)\,du \geq \left(1 - \frac{1}{t}\right) f(t) \geq f(t) = (f - f_0)(t).$$

Now, by (4.18) and (4.15),

$$\frac{1}{\mathrm{e}}\int_0^{\mathrm{e}} \mathcal{P}f(t)\,dt - \mathcal{P}f(\mathrm{e}) - \frac{1}{\mathrm{e}} =$$

$$= \frac{1}{\mathrm{e}}\left[\int_0^{\mathrm{e}} \mathcal{P}f(t)\,dt - \int_0^{\mathrm{e}} \mathcal{P}f_0(t)\,dt\right] - \mathcal{P}f(\mathrm{e}) + \mathcal{P}f_0(\mathrm{e}) =$$

$$= \frac{1}{\mathrm{e}}\int_0^{\mathrm{e}} [\mathcal{P}f(t) - f(t) - \mathcal{P}f_0(t) + f_0(t)]\,dt =$$

$$= \frac{1}{\mathrm{e}}\int_0^{\mathrm{e}} [\mathcal{P}(f - f_0)(t) - (f - f_0)(t)]\,dt \geq 0.$$

Then

$$L(\mathcal{P}f;[0,\mathrm{e}]) \geq \frac{1}{\mathrm{e}}.$$

So, inequality (4.16) is proved and (4.14) follows. □

Let us come back to the estimate given by (4.13). One can improve this estimate using Theorem 4.8.

Corollary 4.9. *Let f be a non-increasing function on $\mathbb{R}_+$. Then*

$$\|\,\mathcal{P}f\,\|_* \geq \frac{1}{4}\|f\|_*. \tag{4.19}$$

Proof. Applying successively Theorems 2.36, 4.8 and again Theorem 2.36, we have

$$\| \mathcal{P}f \|_* \geq \frac{1}{2} \| \mathcal{P}f \|_{BLO} \geq \frac{1}{2}\frac{1}{\mathrm{e}} \|f\|_{BLO} \geq \frac{1}{2}\frac{1}{\mathrm{e}}\frac{\mathrm{e}}{2} \|f\|_* = \frac{1}{4}\|f\|_*. \quad \square$$

Let us show that inequality (4.19) can be also improved. For this we will need some auxiliary statements.

Lemma 4.10. *The equation*

$$\psi(x) \equiv \ln \frac{x}{1+\ln x} - \frac{\ln x}{1+\ln x} = 0 \tag{4.20}$$

has a unique root γ_1 *on* $(1, +\infty)$.

Proof. One can easily check that $\psi(1) = 0$,

$$\lim_{x\to+\infty} \psi(x) = +\infty, \quad \psi'(x) = \frac{\ln^2 x + \ln x - 1}{x(1+\ln x)^2},$$

$\psi'(1) = -1$, and the equation $\psi'(x) = 0$ has a unique root on the interval $(1, +\infty)$. Since the function ψ is differentiable on $[1, +\infty)$ the listed properties imply the statement of the lemma. $\square$

Let $\gamma_1 > 1$ be the root of equation (4.20). In what follows we will use the following constants:

$$\beta_0 = \frac{1+\ln\gamma_1}{\gamma_1}, \quad \gamma_0 = \frac{1}{\beta_0}, \quad \alpha_0 = \frac{4}{\gamma_1}\ln\gamma_0. \tag{4.21}$$

The approximate values of these constants are

$$\alpha_0 \approx 0.52, \quad \beta_0 \approx 0.546, \quad \gamma_0 \approx 1.83, \quad \gamma_1 \approx 4.65.$$

The next lemma explains the original meaning of equation (4.20) and of constants $\alpha_0,\ \beta_0,\ \gamma_0$, defined by (4.21).

Lemma 4.11. *For the function* $f_0(x) = \chi_{[0,1)}(x),\ x \in \mathbb{R}_+$,

$$\|f_0\|_* = \Omega(f_0; [0,2]) = \frac{1}{2}, \tag{4.22}$$

$$\| \mathcal{P}f_0 \|_* = \Omega\left(\mathcal{P}f_0; [0, \gamma_1]\right) = \frac{1}{2}\alpha_0, \tag{4.23}$$

where γ_1 *is a root of equation* (4.20), *and* α_0 *is defined by* (4.21).

Proof. According to Property 2.7, $\Omega(f_0; I) \leq \frac{1}{2}$ for any interval $I \subset \mathbb{R}_+$ and $\Omega(f_0; [0,2]) = \frac{1}{2}$. Thus equality (4.22) follows.

In order to prove (4.23) we use the equality $\mathcal{P}f_0(x) = \min\left(1, \frac{1}{x}\right),\ x \in \mathbb{R}_+$. Then for $x > 1$ and $I \equiv [0, x]$

$$(\mathcal{P}f_0)_I = \frac{1}{x}\int_0^1 \mathcal{P}f_0(t)\,dt + \frac{1}{x}\int_1^x \mathcal{P}f_0(t)\,dt = \frac{1+\ln x}{x}.$$

Set $x_0 = \frac{x}{1+\ln x}$. Then $\mathcal{P}f_0(t) \geq (\mathcal{P}f_0)_I$ if $t \leq x_0$, and $\mathcal{P}f_0(t) \leq (\mathcal{P}f_0)_I$ if $t \geq x_0$. Hence, by Property 2.1,

$$\Omega\left(\mathcal{P}f_0; I\right) = \frac{2}{x}\int_{x_0}^x \left[(\mathcal{P}f_0)_I - \mathcal{P}f_0(t)\right]\,dt =$$

$$= \frac{2}{x}\left[\frac{1+\ln x}{x}\left(x - \frac{x}{1+\ln x}\right) - \ln(1+\ln x)\right] = \frac{2}{x}[\ln x - \ln(1+\ln x)] \equiv \varphi(x).$$

We have

$$\varphi'(x) = \frac{2}{x^2}\left[\frac{\ln x}{1+\ln x} - \ln\frac{x}{1+\ln x}\right] = \frac{2}{x^2}\psi(x),$$

where the function ψ was defined in Lemma 4.10. Applying Lemmas 4.10 and 2.34 it is easy to see that

$$\|\,\mathcal{P}f_0\,\|_* = \sup_{x>1}\Omega\left(\mathcal{P}f_0;[0,x]\right) = \max_{x>1}\varphi(x) = \varphi(\gamma_1) = \Omega\left(\mathcal{P}f_0;[0,\gamma_1]\right) = \frac{1}{2}\alpha_0,$$

which proves (4.23). □

Remark 4.12. The following formula explains the meaning of the constants β_0 and γ_0, defined by (4.21),

$$(\mathcal{P}f_0)_{[0,\gamma_1]} = \mathcal{P}f_0\left(\gamma_0\right) = \beta_0.$$

Remark 4.13. Let us denote the maximal value of the constant c in (4.5) by

$$c_* = \sup\left\{c:\ \frac{\|\,\mathcal{P}f\,\|_*}{\|f\|_*} \geq c \quad \forall\, f\downarrow \quad \text{on } \mathbb{R}_+,\ f \in BMO,\ f \neq Const\right\} =$$

$$= \inf\left\{\frac{\|\,\mathcal{P}f\,\|_*}{\|f\|_*}:\ f \in BMO,\ f\downarrow \quad \text{on } \mathbb{R}_+,\ f \neq Const\right\}. \qquad (4.24)$$

Lemma 4.11 implies that $c_* \leq \alpha_0$. On the other hand, according to (4.19), we have that $c_* \geq \frac{1}{4}$.

The next theorem allows to improve the lower bound for c_*. We hope that this result could be of interest also outside of the present context.

Theorem 4.14 ([41]). *Let f be a non-increasing function on $\mathbb{R}_+$. Then*

$$\| \mathcal{P}f \|_* \geq \frac{\alpha_0}{2} \|f\|_{BLO}, \tag{4.25}$$

where α_0 *is defined by* (4.21)*, and in general the constant* $\frac{\alpha_0}{2}$ *in the right-hand side of* (4.25) *is sharp.*

In order to prove Theorem 4.14 we need the following statement.

Lemma 4.15. *Let* $g \in L_{loc}(\mathbb{R}_+)$ *be such that* $g(x) \geq g(y),\ 0 \leq x \leq y \leq 1,$ *and*

$$\int_0^1 g(t)\,dt \geq g(x) \geq g(y), \quad 1 \leq x \leq y. \tag{4.26}$$

Then the function $\mathcal{P}g$ *does not increase on* $\mathbb{R}_+$*.*

Proof. The monotonicity of $\mathcal{P}g$ on $[0,1]$ follows immediately from the monotonicity of the function g on $[0,1]$. If we prove that

$$\mathcal{P}g(x) \geq \mathcal{P}g(y), \quad 1 < x < y, \tag{4.27}$$

then, due to the continuity of $\mathcal{P}g$, we immediately obtain the statement of the lemma. For $1 < x < y$

$$\mathcal{P}g(y) - \mathcal{P}g(x) = \left(1 - \frac{x}{y}\right)\left\{\frac{1}{x}\left[\frac{1}{x-1}\int_1^x g(t)\,dt - \int_0^1 g(t)\,dt\right] + \right.$$

$$\left. + \left[\frac{1}{y-x}\int_x^y g(t)\,dt - \frac{1}{x-1}\int_1^x g(t)\,dt\right]\right\}.$$

Now it is easy to see that (4.27) follows from (4.26). $\square$

Proof of Theorem 4.14. According to Lemmas 2.22 and 2.34, it is enough to show that for any $\gamma > 0$ there exists $\gamma' > 0$ such that

$$\Omega\left(\mathcal{P}f; [0,\gamma']\right) \geq \frac{\alpha_0}{2}\left[\mathcal{P}f(\gamma) - f(\gamma)\right]. \tag{4.28}$$

We can assume $\gamma = 1,\ f(1) = 0,\ \mathcal{P}f(1) = 1$. Set $f_0(x) = \chi_{[0,1)}(x),\ x \in \mathbb{R}_+$. Then the function $g \equiv f - f_0$ obviously satisfies the conditions of Lemma 4.15. By this lemma, the function $\mathcal{P}g$ does not increase on $\mathbb{R}_+$. Let us denote $h(x) = g(x) - \mathcal{P}g(\gamma_0)$, with γ_0 defined by (4.21). Then $\mathcal{P}h$ is also non-increasing on $\mathbb{R}_+$ and $\mathcal{P}h(\gamma_0) = 0$. Therefore

$$0 \leq \int_0^{\gamma_0} \mathcal{P}h(t)\,dt = \int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f_0(t)\right]\,dt - \mathcal{P}f(\gamma_0) + \mathcal{P}f_0(\gamma_0),$$

or, equivalently,

$$\frac{1}{\gamma_0}\int_0^{\gamma_0} \left[\mathcal{P}f_0(t) - \mathcal{P}f_0(\gamma_0)\right]\,dt \leq \frac{1}{\gamma_0}\int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f(\gamma_0)\right]\,dt. \tag{4.29}$$

Now choose γ' such that

$$\frac{1}{\gamma'}\int_0^{\gamma'} \mathcal{P}f(t)\,dt = \mathcal{P}f(\gamma_0).$$

Clearly, $\gamma' > \gamma_0$. Comparing γ' with γ_1, which was defined in Lemma 4.10 as a root of equation (4.20), we see that the following two situations are possible:

1. $\gamma' \le \gamma_1$; in this case, by Property 2.1, (4.29) implies

$$\Omega\left(\mathcal{P}f;[0,\gamma']\right) = \frac{2}{\gamma'}\int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f(\gamma_0)\right]\,dt \ge$$

$$\ge \frac{2}{\gamma_1}\int_0^{\gamma_0} \left[\mathcal{P}f_0(t) - \mathcal{P}f_0(\gamma_0)\right]\,dt = \Omega\left(\mathcal{P}f_0;[0,\gamma_1]\right) = \frac{\alpha_0}{2},$$

so that (4.28) holds true.

2. If $\gamma' > \gamma_1$, then the listed above properties of function $\mathcal{P}h$ imply

$$0 \ge \int_{\gamma_0}^{\gamma_1} \mathcal{P}h(t)\,dt = \int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f(t) - \mathcal{P}f_0(t)\right]\,dt - \mathcal{P}f(\gamma_0) + \mathcal{P}f_0(\gamma_0),$$

i.e.,

$$\frac{1}{\gamma_1-\gamma_0}\int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f(\gamma_0) - \mathcal{P}f(t)\right]\,dt \ge \frac{1}{\gamma_1-\gamma_0}\int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f_0(\gamma_0) - \mathcal{P}f_0(t)\right]\,dt.$$

But since the function $\mathcal{P}f(\gamma_0) - \mathcal{P}f(t),\ t > \gamma_0$, is non-decreasing and $\gamma' > \gamma_1$

$$\frac{1}{\gamma'-\gamma_0}\int_{\gamma_0}^{\gamma'} \left[\mathcal{P}f(\gamma_0) - \mathcal{P}f(t)\right]\,dt \ge \frac{1}{\gamma_1-\gamma_0}\int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f_0(\gamma_0) - \mathcal{P}f_0(t)\right]\,dt. \tag{4.30}$$

One can rewrite inequalities (4.29) and (4.30) in the following form

$$\frac{\gamma_0}{\int_0^{\gamma_0} \left[\mathcal{P}f_0(t) - \mathcal{P}f_0(\gamma_0\right]\,dt} \ge \frac{\gamma_0}{\int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f(\gamma_0)\right]\,dt},$$

$$\frac{\gamma_1-\gamma_0}{\int_{\gamma_0}^{\gamma_1} \left[\mathcal{P}f_0(\gamma_0) - \mathcal{P}f_0(t)\right]\,dt} \ge \frac{\gamma'-\gamma_0}{\int_{\gamma_0}^{\gamma'} \left[\mathcal{P}f(\gamma_0) - \mathcal{P}f(t)\right]\,dt}.$$

Notice that, by Property 2.1, the denominators of the fractions in the right and left-hand sides are the same. Summing up, we obtain

$$\frac{1}{\gamma_1}\int_0^{\gamma_0} \left[\mathcal{P}f_0(t) - \mathcal{P}f_0(\gamma_0)\right]\,dt \le \frac{1}{\gamma'}\int_0^{\gamma_0} \left[\mathcal{P}f(t) - \mathcal{P}f(\gamma_0)\right]\,dt.$$

Then, by Property 2.1,

$$\Omega\left(\mathcal{P}f;[0,\gamma']\right) \ge \Omega\left(\mathcal{P}f_0;[0,\gamma_1]\right) = \frac{\alpha_0}{2}$$

(see also the proof of Lemma 4.11). Therefore, in this case (4.28) holds true, too.

It remains to show that the constant $\frac{\alpha_0}{2}$ in the right-hand side of (4.25) cannot be increased. But, according to Lemma 4.11, for the function $f_0(x) = \chi_{[0,1)}(x)$, $x \in \mathbb{R}_+$, we have $\| \mathcal{P} f_0 \|_* = \frac{\alpha_0}{2}$, and hence $\|f_0\|_{BLO}$ is obviously equal to 1. □

From Theorem 4.14 we immediately get

Corollary 4.16 ([41]).

$$\frac{e\alpha_0}{4} \le c_* \le \alpha_0 \tag{4.31}$$

where c_* is the constant defined by (4.24).

Proof. As we have already mentioned, the right inequality of (4.31) follows from Lemma 4.11. On the other hand, if f is an arbitrary non-increasing function on $\mathbb{R}_+$, then by Theorems 4.14 and 2.36

$$\| \mathcal{P} f \|_* \ge \frac{\alpha_0}{2} \|f\|_{BLO} \ge \frac{\alpha_0}{2} \frac{e}{2} \|f\|_*,$$

which implies the left inequality of (4.31). □

Remark 4.17. We do not know the value of c_*, defined by equality (4.24).

4.2 Estimates of the Oscillations of the Conjugate Hardy Transform and the Calderón Transform

In this section we consider the non-negative summable functions f on $\mathbb{R}_+$ such that the integral $\int_1^{+\infty} f(x)\, \frac{dx}{x}$ converges. The following formulas define *the conjugate Hardy operator* $\mathcal{P}^*$ and *the Calderón operator* $\mathcal{S}$ respectively (see [51, 3]):

$$\mathcal{P}^* f(t) = \int_t^{+\infty} f(x)\, \frac{dx}{x}, \quad t > 0,$$

$$\mathcal{S} f(t) = \frac{1}{t} \int_0^t f(x)\, dx + \int_t^{+\infty} f(x)\, \frac{dx}{x} = \mathcal{P} f(t) + \mathcal{P}^* f(t), \quad t > 0.$$

The operators $\mathcal{P}^*$ and $\mathcal{S}$, together with the operator $\mathcal{P}$, are often used in various fields of mathematics.

Example 4.18. Let $f_0(x) = \ln \frac{1}{x} \chi_{[0,1)}(x)$, $x \in \mathbb{R}_+$. Then according to Example 2.24, $f_0 \in BMO(\mathbb{R}_+)$. In the same time, $\mathcal{P}^* f_0(x) = \frac{1}{2} \ln^2 x \chi_{[0,1)}(x)$, and so $\mathcal{P}^* f_0 \notin BMO(\mathbb{R}_+)$. Indeed, the assumption $\mathcal{P}^* f_0 \in BMO$ contradicts to John–Nirenberg inequality (3.33). Similarly, it can be shown that $\mathcal{S} f_0 \notin BMO(\mathbb{R}_+)$. □

So, unlikely the operator $\mathcal{P}$, the operators $\mathcal{P}^*$ and $\mathcal{S}$ do not act from BMO into BMO. However, it is easy to see that $\mathcal{P}^* f \in BMO(\mathbb{R}_+)$ and $\mathcal{S} f \in BMO(\mathbb{R}_+)$ for $f \in L^\infty(\mathbb{R}_+)$. Moreover, the following theorem holds true.

Theorem 4.19 ([39]). *Let f be a non-negative locally summable function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\,\frac{dx}{x} < \infty$. Then*

$$\|\,\mathcal{P}^* f\,\|_{BLO} = \|\,\mathcal{P} f\,\|_\infty, \tag{4.32}$$

$$\|\,\mathcal{S} f\,\|_{BLO} = \|\mathcal{P}(\mathcal{P} f)\|_\infty. \tag{4.33}$$

Proof. Since $f(x) \geq 0$ for $x \in \mathbb{R}_+$ it is clear that the function $\mathcal{P}^* f$ does not increase on $\mathbb{R}_+$. Further, for $0 < t < s$

$$\mathcal{S} f(t) - \mathcal{S} f(s) = \left(\frac{1}{t} - \frac{1}{s}\right) \int_0^t f(x)\,dx + \int_t^s f(x) \left(\frac{1}{x} - \frac{1}{s}\right) dx \geq 0,$$

so that also $\mathcal{S} f$ does not increase on $\mathbb{R}_+$. Hence, by Lemma 2.34,

$$\|\,\mathcal{P}^* f\,\|_{BLO} = \sup_{t>0} \left(\frac{1}{t} \int_0^t \mathcal{P}^* f(u)\,du - \mathcal{P}^* f(t)\right) =$$

$$= \sup_{t>0} \left(\frac{1}{t} \int_0^t \int_u^{+\infty} f(x)\,\frac{dx}{x}\,du - \int_t^{+\infty} f(x) \frac{dx}{x}\right) = \sup_{t>0} \frac{1}{t} \int_0^t \int_u^t f(x)\,\frac{dx}{x}\,du =$$

$$= \sup_{t>0} \frac{1}{t} \int_0^t f(x)\,dx = \sup_{t>0} \mathcal{P} f(t) = \|\,\mathcal{P} f\,\|_\infty.$$

Similarly,

$$\|\,\mathcal{S} f\,\|_{BLO} = \sup_{t>0} \left(\frac{1}{t} \int_0^t \mathcal{S} f(u)\,du - \mathcal{S} f(t)\right) =$$

$$= \sup_{t>0} \left(\frac{1}{t} \int_0^t \frac{1}{u} \int_0^u f(x)\,dx\,du - \frac{1}{t} \int_0^t f(x)\,dx + \frac{1}{t} \int_0^t \int_u^t f(x)\,\frac{dx}{x}\,du\right) =$$

$$= \sup_{t>0} \frac{1}{t} \int_0^t \frac{1}{u} \int_0^u f(x)\,dx\,du = \sup_{t>0} \mathcal{P}(\mathcal{P} f)(t) = \|\mathcal{P}(\mathcal{P} f)\|_\infty.$$

□

Theorem 4.20 ([39]). *Let f be a non-negative locally summable function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\,\frac{dx}{x} < \infty$. Then*

$$\frac{1}{2}\|\,\mathcal{P} f\,\|_\infty \leq \|\,\mathcal{P}^* f\,\|_* \leq \frac{2}{\mathrm{e}}\|\,\mathcal{P} f\,\|_\infty, \tag{4.34}$$

$$\frac{1}{2}\|\mathcal{P}(\mathcal{P} f)\|_\infty \leq \|\,\mathcal{S} f\,\|_* \leq \frac{2}{\mathrm{e}}\|\mathcal{P}(\mathcal{P} f)\|_\infty. \tag{4.35}$$

Proof. As it was shown in the proof of Theorem 4.19, both functions $\mathcal{P}^* f$ and $\mathcal{S}f$ do not increase on $\mathbb{R}_+$. Then, by virtue of (4.32) and (4.33), Theorem 2.36 applied to these functions immediately yields (4.34) and (4.35) respectively.

Let us show that the constant $\frac{1}{2}$ in the left-hand side of (4.34) cannot be increased. For $0 < \varepsilon < 1$ let $f_\varepsilon(x) = \frac{1}{\varepsilon}\chi_{[1-\varepsilon,1]}(x)$ $x \in \mathbb{R}_+$. Then $\| \mathcal{P} f_\varepsilon \|_\infty = 1,\ \mathcal{P}^* f_\varepsilon(t) = \frac{1}{\varepsilon} \min\left(\ln \frac{1}{1-\varepsilon}, \ln \frac{1}{t}\right) \chi_{[0,1]}(t),\ t \in \mathbb{R}_+$. Hence, by Property 2.7,

$$\| \mathcal{P}^* f_\varepsilon \|_* \le \frac{1}{2}\frac{1}{\varepsilon} \ln \frac{1}{1-\varepsilon} \le \frac{1}{2}\frac{1}{1-\varepsilon} \to \frac{1}{2}, \quad \varepsilon \to 0+ .$$

Therefore the constant $\frac{1}{2}$ in the left-hand side of (4.34) is sharp.

For $\varepsilon = 1$ we have $f_1(x) = \chi_{[0,1]}(x),\ x \in \mathbb{R}_+,\ \| \mathcal{P} f_1 \|_\infty = 1,\ \mathcal{P}^* f_1(t) = \ln \frac{1}{t}\chi_{[0,1]}(t),\ t \in \mathbb{R}_+$. As it was shown in Example 2.28, this implies

$$\| \mathcal{P}^* f_1 \|_* \ge \Omega\left(\mathcal{P}^* f_1; [0,1]\right) = \frac{2}{\mathrm{e}},$$

so that the constant $\frac{2}{\mathrm{e}}$ in the right-hand side of (4.34) is sharp, too.

It remains to show that the constant $\frac{2}{\mathrm{e}}$ in the right-hand side of (4.35) cannot be decreased. For the function $f_1(x) = \chi_{[0,1]}(x),\ x \in \mathbb{R}_+$, we have

$$\|\mathcal{P}(\mathcal{P} f_1)\|_\infty = 1, \quad \mathcal{S} f_1(t) = \left(1 + \ln \frac{1}{t}\right) \chi_{[0,1]}(t) + \frac{1}{t}\chi_{(1,+\infty)}(t), \quad t \in \mathbb{R}_+.$$

So, it is enough to show that for the function $g \equiv \mathcal{S} f_1$

$$\|g\|_* = \frac{2}{\mathrm{e}}. \tag{4.36}$$

Since g is non-increasing on $\mathbb{R}_+$ according to Lemma 2.22 the last relation follows from the equality

$$\sup_{t>0} \Omega(g; [0,t]) = \frac{2}{\mathrm{e}}. \tag{4.37}$$

But for $0 < t \le 1$

$$\Omega(g; [0,t]) = \frac{2}{t} \int_0^{t/\mathrm{e}} \left(\ln \frac{1}{u} - \ln \frac{\mathrm{e}}{t}\right) du = \frac{2}{\mathrm{e}},$$

so that in order to prove (4.37) it remains to show that

$$\Omega(g; [0,t]) \le \frac{2}{\mathrm{e}}, \quad 1 < t < \infty. \tag{4.38}$$

Let $t_0,\ t_0 > \mathrm{e}$, be the root of the equation $\ln x = x - 2$. We have to consider the following two cases.

1. If $1 < t \le t_0$, then $g_{[0,t]} \ge 1$. Denote $h(u) = 1 + \ln\frac{1}{u}$, $u \in \mathbb{R}_+$. Since $h(u) \le \frac{1}{u}$, $u \ge 1$, there exists t_1, $1 \le t_1 < t$ such that $g_{[0,t]} = \frac{1}{t_1}\int_0^{t_1}\left(1 + \ln\frac{1}{u}\right)du = h_{[0,t_1]}$. Now, by Property 2.1,

$$\Omega(g;[0,t]) = \frac{2}{t}\int_{\left\{u:\ 1+\ln\frac{1}{u} > g_{[0,t]}\right\}}\left(1 + \ln\frac{1}{u} - g_{[0,t]}\right)du \le$$

$$\le \frac{2}{t_1}\int_{\left\{u:\ 1+\ln\frac{1}{u} \ge g_{[0,t]}\right\}}\left(1 + \ln\frac{1}{u} - h_{[0,t_1]}\right)du = \Omega(h;[0,t_1]) = \frac{2}{\mathrm{e}},$$

so that (4.38) holds true in this case.

2. Let $t > t_0$, i.e., $g_{[0,t]} = \frac{1}{t}(2 + \ln t) < 1$. In this case, applying Property 2.1, we obtain

$$\Omega(g;[0,t]) = \frac{2}{t}\int_0^{t_2}\left(g(u) - g_{[0,t]}\right)du = 2\frac{1 + \ln t - \ln(2 + \ln t)}{t},$$

where t_2 is to be defined from the condition $\frac{1}{t_2} = g_{[0,t]}$, i.e. $t_2 = \frac{t}{2+\ln t}$. Denote $\psi(t) = \frac{1+\ln t-\ln(2+\ln t)}{t}$, $t \ge t_0$. Since $t_0 > \mathrm{e}$ we have $\psi(t_0) = \frac{1}{t_0} < \frac{1}{\mathrm{e}}$. It is easy to see that $\psi'(t) \le 0$ for $t \ge t_0$. Hence $\psi(t) \le \frac{1}{\mathrm{e}}$ for $t \ge t_0$, and in this case (4.38) holds true as well. $\square$

Remark 4.21. We do not know whether the constant $\frac{1}{2}$ in the left-hand side of (4.35) is sharp.

Remark 4.22. Clearly, $\|\mathcal{P}f\|_\infty \le \|f\|_\infty$, though the condition $\|f\|_\infty < \infty$ is not necessary for the boundedness of $\mathcal{P}f$. On the other hand, is f is non-negative on $\mathbb{R}_+$, then obviously $u\mathcal{P}f(u) \ge t\mathcal{P}f(t)$, $u \ge t > 0$, so that

$$\mathcal{P}(\mathcal{P}f)(2t) \ge \frac{1}{2t}\int_t^{2t}\mathcal{P}f(u)\,du \ge \frac{\ln 2}{2}\,\mathcal{P}f(t), \quad t > 0.$$

This means that the conditions $\mathcal{P}f \in L^\infty$ and $\mathcal{P}(\mathcal{P}f) \in L^\infty$ are equivalent. In other words, Theorems 4.19 and 4.20 show, that for a non-negative on $\mathbb{R}_+$ function f the boundedness of $\mathcal{P}f$ (and not the essential boundedness of f) is the necessary and sufficient condition for $\mathcal{P}^*f$ and $\mathcal{S}f$ to belong to BLO and BMO.

The next theorem provides the lower bound of the BMO-norms of $\mathcal{P}^*f$ and $\mathcal{S}f$, which reflect the behavior of the function f in the neighborhood of zero.

Theorem 4.23 ([39]). *Let f be a non-negative locally summable function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\,\frac{dx}{x} < \infty$. Then*

$$\|\mathcal{P}^*f\|_* \ge \frac{2}{\mathrm{e}}\lim_{t\to 0+}\operatorname*{ess\,inf}_{u\in(0,t)} f(u), \tag{4.39}$$

$$\|\mathcal{S}f\|_* \geq \frac{2}{\mathrm{e}} \lim_{t\to 0+} \operatorname*{ess\,inf}_{u\in(0,t)} f(u), \tag{4.40}$$

and in general the constants $\frac{2}{\mathrm{e}}$ *in the right-hand sides of* (4.39) *and* (4.40) *are sharp.*

Proof. Let us denote $A = \lim_{t\to 0+} \operatorname{ess\,inf}_{u\in(0,t)} f(u)$. If $A = 0$, then (4.39) and (4.40) are trivial. Let $A > 0$. Fix a, $0 < a < A$, and choose $\varepsilon > 0$ such that $f(u) > a$ for almost all $u \in (0, \varepsilon)$. Then for $t < \varepsilon$

$$\mathcal{P}^* f(t) \geq \int_t^\varepsilon f(u)\,\frac{du}{u} \geq a \ln\frac{\varepsilon}{t}. \tag{4.41}$$

Now let us use the John–Nirenberg inequality with exact exponent (Theorem 3.21). Assuming $\mathcal{P}^* f \in BMO$ and taking into account the monotonicity of $\mathcal{P}^* f$, one can rewrite the John-Nirenberg inequality (3.39) in the following way

$$\mathcal{P}^* f(t) \leq (\mathcal{P}^* f)_{[0,1]} + \frac{\mathrm{e}}{2}\|\mathcal{P}^* f\|_* \left(\ln\frac{1}{t} + \ln B\right). \tag{4.42}$$

Here the constant $B = \exp\left(1 + \frac{2}{\mathrm{e}}\right)$ is taken from Theorem 3.21, and $t > 0$ is small. Comparing (4.41) and (4.42), we have

$$a \ln\frac{\varepsilon}{t} \leq (\mathcal{P}^* f)_{[0,t]} + \frac{\mathrm{e}}{2}\|\mathcal{P}^* f\|_* \left(\ln\frac{1}{t} + \ln B\right)$$

for $t > 0$ small enough. This immediately implies $\|\mathcal{P}^* f\|_* \geq a$. As a was an arbitrary number smaller than A, inequality (4.39) is proved.

The same arguments lead the following inequality

$$a \ln\frac{\varepsilon}{t} \leq (\mathcal{S}f)_{[0,t]} + \frac{\mathrm{e}}{2}\|\mathcal{S}f\|_* \left(\ln\frac{1}{t} + \ln B\right),$$

with $a < A$ being an arbitrary number. This inequality implies (4.40).

It remains to prove that the constant $\frac{2}{\mathrm{e}}$ in the right-hand sides of (4.39) and (4.40) cannot be increased. In the proof of Theorem 4.20 we showed that for the function $f_1(x) = \chi_{[0,1]}(x)$, $x \in \mathbb{R}_+$, one has $\|\mathcal{S}f_1\|_* = \frac{2}{\mathrm{e}}$. Hence for the function f_1 the inequality (4.40) becomes an equality, so that the constant $\frac{2}{\mathrm{e}}$ in (4.40) is sharp. In order to proof that $\frac{2}{\mathrm{e}}$ in (4.39) is also sharp, obviously it is enough to show that

$$\|\mathcal{P}^* f_1\|_* = \frac{2}{\mathrm{e}}. \tag{4.43}$$

Denote $g(t) \equiv \mathcal{P}^* f_1(t) = \ln\frac{1}{t}\chi_{[0,1]}(t)$, $t \in \mathbb{R}_+$. If $0 < t \leq 1$, then it is easy to see that $\Omega(g; [0,t]) = \frac{2}{\mathrm{e}}$. Otherwise, if $t > 1$, then for the function $h(x) = \ln\frac{1}{x}$, $x \in \mathbb{R}_+$, there exists t_1, $1 < t_1 \leq t$, such that $g_{[0,t]} = h_{[0,t_1]}$. Therefore, by Property 2.1,

$$\Omega(g; [0,t]) = \frac{2}{t}\int_{\left\{u:\ g(u) > g_{[0,t]}\right\}} \left(g(u) - g_{[0,t]}\right) du \leq$$

$$\leq \frac{2}{t_1} \int_{\{u:\ h(u) > h_{[0,t_1]}\}} \big(h(u) - h_{[0,t_1]}\big)\, du = \Omega(h; [0, t_1]) = \frac{2}{\mathrm{e}}.$$

So, we have proved (4.43), and this completes the proof of the theorem. □

Remark 4.24. We cannot substitute the ess inf in the left-hand sides of (4.39) and (4.40) by ess sup. Indeed, for the function

$$f(x) = \sum_{k=0}^{\infty} 2^{k+1} \chi_{[2^{-k} - 2^{-2k-2}, 2^{-k}]}(x), \quad x \in \mathbb{R}_+,$$

we obviously have $\lim_{t \to 0+} \operatorname{ess\,sup}_{u \in (0,t)} f(u) = +\infty$. Let us show that $\| \mathcal{P} f \|_\infty \leq 2$. Indeed, if $x > 1$, then

$$\mathcal{P} f(x) \leq \int_0^1 f(t)\, dt = \sum_{k=0}^{\infty} 2^{k+1} \cdot 2^{-2k-2} = \sum_{k=0}^{\infty} 2^{-k-1} = 1.$$

If $0 < x \leq 1$, we can find an integer n such that $2^{-n-1} < x \leq 2^{-n}$. Then

$$\mathcal{P} f(x) \leq \frac{1}{2^{-n-1}} \int_0^{2^{-n}} f(t)\, dt = 2^{n+1} \sum_{k=n}^{\infty} 2^{k+1} \cdot 2^{-2k-2} = 2.$$

Hence

$$\|\mathcal{P}(\mathcal{P} f)\|_\infty \leq \| \mathcal{P} f \|_\infty \leq 2,$$

and, according to Theorem 4.20,

$$\| \mathcal{P}^* f \|_* \leq \frac{4}{\mathrm{e}}, \quad \| \mathcal{S} f \|_* \leq \frac{4}{\mathrm{e}}.$$

This shows that (4.39) and (4.40) fail if we substitute ess inf by ess sup.

Now let f be a non-negative non-increasing function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x) \frac{dx}{x} < \infty$. Clearly, in this case

$$\|f\|_\infty = \| \mathcal{P} f \|_\infty = \lim_{t \to 0+} f(t).$$

Thus Theorems 4.20 and 4.23 immediately lead to the following results.

Corollary 4.25 ([39]). *If f is a non-negative non-increasing function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x) \frac{dx}{x} < \infty$, then*

$$\| \mathcal{P}^* f \|_* = \| \mathcal{S} f \|_* = \frac{2}{\mathrm{e}} \|f\|_\infty. \tag{4.44}$$

Corollary 4.26 ([39]). *If f is a non-negative non-increasing function on $\mathbb{R}_+$ such that $\int_1^{+\infty} f(x)\,\frac{dx}{x} < \infty$, then*

$$\| \mathcal{P}^* f \|_* \geq \alpha_0 \|f\|_*, \tag{4.45}$$

$$\| \mathcal{S} f \|_* \geq \alpha_0 \|f\|_*, \tag{4.46}$$

where the constant α_0 is defined by (4.21).

Proof. Applying successively (4.44), the inequality $\| \mathcal{P} f \|_* \leq \frac{1}{2} \| \mathcal{P} f \|_\infty$ (which follows from Property 2.7), and (4.31), we obtain

$$\| \mathcal{P}^* f \|_* = \frac{2}{\mathrm{e}} \| \mathcal{P} f \|_\infty \geq \frac{4}{\mathrm{e}} \| \mathcal{P} f \|_* \geq \frac{4}{\mathrm{e}} c_* \|f\|_* \geq \alpha_0 \|f\|_*,$$

where the constant c_* is defined by (4.24).

Analogously one can prove (4.46). □

Remark 4.27. Without the monotonicity assumption on f the inequalities (4.45) and (4.46) fail even if the constants α_0 in their right-hand sides are arbitrarily small. It can be easily seen from the following example. Take $f_0(x) = \chi_{[1-\varepsilon,1]}(x),\ x \in \mathbb{R}_+$, with $0 < \varepsilon < 1$. Then $\|f_0\|_* = \frac{1}{2}$ and

$$\max \left(\| \mathcal{P}^* f_0 \|_*, \| \mathcal{S} f_0 \|_* \right) \leq \| \mathcal{S} f_0 \|_\infty \leq \varepsilon + \ln \frac{1}{1-\varepsilon} \to 0, \quad \varepsilon \to 0+ .$$

On the other hand, the boundedness condition in Corollary 4.26 can be neglected. Indeed, if f is unbounded, then, by (4.44), the left-hand sides of (4.45) and (4.46) are infinite.

Remark 4.28. Equality (4.44) implies that it is impossible to get the upper bounds of $\| \mathcal{P}^* f \|_*$ and $\| \mathcal{S} f \|_*$ in terms of $\|f\|_*$ even for the monotone bounded function f. Indeed, if such upper bounds exist, equality (4.44) would imply $\|f\|_\infty \leq c\|f\|_*$ with some constant $c > 0$, which is wrong. In order to see this it is enough to consider the function

$$f_N(x) = \frac{1}{N} \min \left(N, \ln \frac{1}{x} \right) \chi_{[0,1)}(x), \quad x \in \mathbb{R}_+.$$

We have $\|f_N\|_\infty = 1$ and one can easily check that $\|f_N\|_* \to 0$ as $N \to \infty$.

5

The Gurov–Reshetnyak Class of Functions

5.1 Embedding in the Gehring Class

The BMO-class is closely related to the class of functions, which was studied first by Gurov and Reshetnyak in [21, 22]. This class could be also defined in terms of mean oscillations of functions in the following way.

Let $Q_0 \subset \mathbb{R}^d$ be a fixed cube. We will say that the non-negative function $f \in L(Q_0)$ satisfies the *Gurov–Reshetnyak condition*, if

$$\Omega(f; Q) \leq \varepsilon f_Q, \quad Q \subset Q_0, \tag{5.1}$$

where the constant ε does not depend on the cube Q. The class of all such functions f is called *the Gurov–Reshetnyak class.* We will denote it by $GR \equiv GR(\varepsilon) \equiv GR(\varepsilon, Q_0)$. Often inequality (5.1) is called *the Gurov–Reshetnyak inequality.*

Remark 5.1. Obviously, for $\varepsilon = 2$ inequality (5.1) holds, but if $\varepsilon < 2$ in general it is no more true. Indeed, for the function $f_N(x) = N\chi_{[0,\frac{1}{N}]}(x),\ x \in [0,1] \equiv Q_0$, we have $(f_N)_{Q_0} = 1,\ \Omega(f_N; Q_0) = 2\left(1 - \frac{1}{N}\right)$, so that

$$\frac{\Omega(f_N; Q_0)}{(f_N)_{Q_0}} \to 2, \quad N \to \infty.$$

Hence condition (5.1) is substantial only for $0 < \varepsilon < 2$.

Remark 5.2. For any $\varepsilon < 2$ if the function f satisfies condition (5.1), then it is either positive almost everywhere or equivalent to zero. This fact is a consequence of the following statement.

Proposition 5.3. *If $f \not\sim 0$ is a non-negative locally summable function such that $|\{x : \ f(x) = 0\}| > 0$, then*

$$\sup_Q \frac{\Omega(f; Q)}{f_Q} = 2. \tag{5.2}$$

Proof. Denote $A = \{x : f(x) = 0\}$. Since by virtue of Lebesgue theorem 1.1 almost every point of the set A is its density point for any $\delta > 0$ there exists a cube Q such that $f_Q > 0$ and

$$|E| < \frac{\delta}{2}|Q|,$$

for $E \equiv Q \setminus A$. Then the equality

$$\int_{\{x\in E:\, f(x)\le f_Q\}} f(x)dx \le f_Q|\{x \in E : f(x) \le f_Q\}| =$$

$$= \frac{|\{x \in E : f(x) \le f_Q\}|}{|Q|} \int_E f(x)dx$$

implies

$$\frac{|\{x \in E : f(x) > f_Q\}|}{|Q|} \int_E f(x)dx + \int_{\{x\in E:\, f(x)\le f_Q\}} f(x)dx \le$$

$$\le \frac{1}{|Q|} \left(|\{x \in E : f(x) > f_Q\}| + |\{x \in E : f(x) \le f_Q\}|\right) \int_E f(x)dx =$$

$$= \frac{|E|}{|Q|} \int_E f(x)dx < \frac{\delta}{2} \int_E f(x)dx.$$

Thus

$$\frac{|Q|}{2}(\Omega(f;Q) - (2-\delta)f_Q) =$$

$$= \int_{\{x\in E:\, f(x)>f_Q\}} (f(x) - f_Q)dx - \frac{2-\delta}{2} \int_E f(x)dx =$$

$$= \int_{\{x\in E:\, f(x)>f_Q\}} f(x)dx - f_Q|\{x \in E : f(x) > f_Q\}| -$$

$$- \int_E f(x)dx + \frac{\delta}{2} \int_E f(x)dx \ge$$

$$\ge \int_{\{x\in E:\, f(x)\le f_Q\}} f(x)dx - \frac{|\{x \in E : f(x) > f_Q\}|}{|Q|} \int_E f(x)dx +$$

$$+ \frac{|\{x \in E : f(x) > f_Q\}|}{|Q|} \int_E f(x)dx - \int_{\{x\in E:\, f(x)\le f_Q\}} f(x)dx = 0,$$

i.e.,

$$\frac{\Omega(f;Q)}{f_Q} \ge 2 - \delta.$$

Since $\delta > 0$ is arbitrary equality (5.2) follows. □

The fundamental property of the GR-class, which stipulates numerous applications, is described by the following theorem.

Theorem 5.4 (Gurov, Reshetnyak, [22]). *There exists a number* $\varepsilon_0 \equiv \varepsilon_0(d)$, $0 < \varepsilon_0 \le 2$ *such that for any* ε, $0 < \varepsilon < \varepsilon_0$, *one can find* $p_0 \equiv p_0(\varepsilon, d) > 1$ *such that if the function* f *satisfies condition* (5.1), *then* $f \in L^p(Q_0)$ *for any* $p < p_0$.

Various proofs, generalizations and refinements of this theorem were found by a number of different authors ([27, 4, 15, 76, 13, 14] and others). In what follows we will derive this theorem as a corollary of a more general result (see Corollary 5.8).

Let us introduce the following quantity

$$\nu(f;\sigma) \equiv \sup_{l(Q)\le\sigma} \frac{\Omega(f;Q)}{f_Q}, \quad 0 < \sigma \le l\,(Q_0)\,.$$

The supremum here is taken over all cubes $Q \subset Q_0$ such that their side-lengths $l(Q)$ are less or equal than σ. In addition, if $f_Q = 0$ for some cube $Q \subset Q_0$, then $\Omega(f;Q) = 0$, provided f is non-negative on Q_0. In this case we assume that $\frac{\Omega(f;Q)}{f_Q} = 0$. According to Remark 5.1, for any $f \in L(Q_0)$ we have $\nu(f;\sigma) \le 2$ for $0 < \sigma \le l\,(Q_0)$. Some properties of the function f in terms of $\nu(f;\sigma)$ were studied in [13, 14].

Theorem 5.5 ([37]). *Let* $Q_0 \subset \mathbb{R}^d$ *be a cube , and let the function* $f \in L(Q_0)$ *be non-negative. Then*

$$\frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\, du \le 3 \cdot 2^d \nu\left(f; 2t^{1/d}\right) f^{**}(t), \quad 0 < t \le 2^{-d}\,|Q_0|\,. \tag{5.3}$$

For the proof of this theorem we will need the following refinement of Calderón–Zygmund lemma 1.14.

Lemma 5.6 (Calderón, Zygmund, [70]). *Let* f *be a non-negative function, summable on the cube* $Q_0 \subset \mathbb{R}^d$, *and let* $\alpha \ge \frac{1}{|Q_0|}\int_{Q_0} f(x)\, dx$. *Then there exist cubes* $Q_j \subset Q_j' \subset Q_0$, $j = 1, 2, \ldots$, *with pairwise disjoint interiors such that* $\left|Q_j'\right| = 2^d\,|Q_j|$,

$$f_{Q_j'} \le \alpha < f_{Q_j} \le 2^d \alpha, \tag{5.4}$$

and

$$f(x) \le \alpha \ \ \text{for almost all} \ \ x \in Q_0 \setminus \left(\bigcup_{j\ge 1} Q_j\right). \tag{5.5}$$

Proof. Essentially the proof of this lemma repeats the proof of Lemma 1.14. We only have to notice that as the cube Q_j' it is enough to take the cube, whose partition results the dyadic cube Q_j, $j = 1, 2, \ldots$. Clearly, in this case the left inequality of (5.4) holds true. The other statements of the lemma follow from Lemma 1.14. □

Proof of Theorem 5.5. Let us fix some t, $0 < t \leq 2^{-d}|Q_0|$. Applying Lemma 5.6 with $\alpha = f^{**}(t)$, we obtain the cubes Q_j and Q'_j, which satisfy the properties stated by this lemma.

Denote $E = \cup_{j\geq 1} Q_j$. Using Property 2.1, together with the definition of the rearrangement f^* and (5.5), we obtain

$$\int_0^t |f^*(u) - f^{**}(t)|\, du = 2 \int_{\{u:\ f^*(u)>\alpha\}} (f^*(u) - \alpha)\, du =$$

$$= 2 \int_{\{x\in Q_0:\ f(x)>\alpha\}} (f(x) - \alpha)\, dx = 2 \int_{\{x\in Q_0:\ f(x)>\alpha\}\cap E} (f(x) - \alpha)\, dx.$$

Since the interiors of the cubes Q_j are pairwise disjoint

$$\int_0^t |f^*(u) - f^{**}(t)|\, du = 2 \sum_{j\geq 1} \int_{\{x\in Q_0:\ f(x)>\alpha\}\cap Q_j} (f(x) - \alpha)\, dx =$$

$$= 2 \sum_{j\geq 1} \int_{\{x\in Q_j:\ f(x)>\alpha\}} (f(x) - \alpha)\, dx =$$

$$= 2 \sum_{j\geq 1} \int_{\{x\in Q_j:\ f(x)>\alpha\}} \left(f(x) - f_{Q_j}\right) dx +$$

$$+ 2 \sum_{j\geq 1} \left(f_{Q_j} - \alpha\right) |\{x \in Q_j :\ f(x) > \alpha\}| =$$

$$= 2 \sum_{j\geq 1} \int_{\left\{x\in Q_j:\ f(x)>f_{Q_j}\right\}} \left(f(x) - f_{Q_j}\right) dx +$$

$$+ 2 \sum_{j\geq 1} \int_{\left\{x\in Q_j:\ \alpha<f(x)\leq f_{Q_j}\right\}} \left(f(x) - f_{Q_j}\right) dx +$$

$$+ 2 \sum_{j\geq 1} \left(f_{Q_j} - \alpha\right) |\{x \in Q_j :\ f(x) > \alpha\}| \equiv S_1 + S_2 + S_3. \tag{5.6}$$

In order the estimate $S_i,\ i = 1, 2, 3$, let us notice that, by (5.4),

$$\frac{1}{|E|} \int_E f(x)\, dx = \frac{1}{|E|} \sum_{j\geq 1} \int_{Q_j} f(x)\, dx =$$

$$= \frac{1}{|E|} \sum_{j\geq 1} |Q_j|\, f_{Q_j} \geq \frac{\alpha}{|E|} \sum_{j\geq 1} |Q_j| = \alpha.$$

Therefore, by the definition of the rearrangement f^*,

$$\frac{1}{t}\int_0^t f^*(u)\,du = f^{**}(t) = \alpha \le \frac{1}{|E|}\int_E f(x)\,dx \le \frac{1}{|E|}\int_0^{|E|} f^*(u)\,du.$$

From here it follows that

$$|E| \le t, \tag{5.7}$$

provided f^* is monotone, so that $|Q_j| \le t$ and $\left|Q'_j\right| \le 2^d t,\ j = 1, 2, \ldots$. Therefore, by Property 2.1,

$$S_1 = 2\sum_{j\ge1}\int_{\{x\in Q_j:\ f(x)>f_{Q_j}\}}\left(f(x) - f_{Q_j}\right)\,dx = \sum_{j\ge1}\int_{Q_j}\left|f(x) - f_{Q_j}\right|\,dx =$$

$$= \sum_{j\ge1}\frac{\Omega\left(f;Q_j\right)}{f_{Q_j}}\int_{Q_j} f(x)\,dx \le \nu\left(f;t^{1/d}\right)\sum_{j\ge1}\int_{Q_j} f(x)\,dx \le$$

$$\le \nu\left(f;t^{1/d}\right)\sum_{j\ge1}\int_{Q'_j} f(x)\,dx, \tag{5.8}$$

provided $\nu(f;\sigma)$ is monotone. Further, by (5.4),

$$S_3 = 2\sum_{j\ge1}\left(f_{Q_j} - \alpha\right)|\{x\in Q_j:\ f(x) > \alpha\}| \le 2\sum_{j\ge1}\left(f_{Q_j} - f_{Q'_j}\right)|Q_j| \le$$

$$\le 2\sum_{j\ge1}\int_{Q_j}\left|f(x) - f_{Q'_j}\right|\,dx \le 2\sum_{j\ge1}\int_{Q'_j}\left|f(x) - f_{Q'_j}\right|\,dx =$$

$$= 2\sum_{j\ge1}\frac{\Omega\left(f;Q'_j\right)}{f_{Q'_j}}\int_{Q'_j} f(x)\,dx \le 2\nu\left(f;2t^{1/d}\right)\sum_{j\ge1}\int_{Q'_j} f(x)\,dx. \tag{5.9}$$

Taking into account that $S_2 \le 0$, from (5.6), (5.8), (5.9) and from the monotonicity of $\nu(f;\sigma)$ we obtain

$$\int_0^t |f^*(u) - f^{**}(t)|\,du \le 3\nu\left(f;2t^{1/d}\right)\sum_{j\ge1}\int_{Q'_j} f(x)\,dx.$$

Now the application of (5.4) and (5.7) leads to the inequality

$$\int_0^t |f^*(u) - f^{**}(t)|\,du \le 3\nu\left(f;2t^{1/d}\right)\cdot\alpha\sum_{j\ge1}\left|Q'_j\right| =$$

$$= 3\alpha\cdot 2^d\nu\left(f;2t^{1/d}\right)\sum_{j\ge1}|Q_j| = 3\alpha\cdot 2^d\nu\left(f;2t^{1/d}\right)|E| \le 3\alpha\cdot 2^d\nu\left(f;2t^{1/d}\right)\cdot t.$$

Since $\alpha = f^{**}(t)$ the last inequality is equivalent to (5.3). □

Inequality (5.3) leads to the following estimate of the rearrangement of the function f in terms of $\nu(f;\sigma)$.

Theorem 5.7 ([37]). *There exist constants $c_1 \equiv c_1(d)$ and $c_2 \equiv c_2(d)$ such that for any cube $Q_0 \subset \mathbb{R}^d$ and for any non-negative function $f \in L(Q_0)$*

$$f^{**}(t) \le c_1 f_{Q_0} \cdot \exp\left(c_2 \int_{t^{1/d}}^{l(Q_0)} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}\right), \quad 0 < t \le |Q_0|\,. \tag{5.10}$$

Proof. Applying Lemma 2.2 to the function $\varphi = f^*$ with $a = 2$ and using Theorem 5.5, for $0 < t \le 2^{-d}|Q_0|$ we get

$$f^{**}\left(\frac{t}{2}\right) - f^{**}(t) \le \frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\,du \le 3\cdot 2^d \nu\left(f; 2t^{1/d}\right) f^{**}(t),$$

or, equivalently,

$$f^{**}\left(\frac{t}{2}\right) \le \left(1 + 3\cdot 2^d \nu\left(f; 2t^{1/d}\right)\right) f^{**}(t), \quad 0 < t \le 2^{-d}|Q_0|\,.$$

The recurrent application of the last inequality yields

$$f^{**}\left(2^{-d-s}|Q_0|\right) \le f^{**}\left(2^{-d}|Q_0|\right)\cdot \prod_{i=1}^{s}\left(1 + 3\cdot 2^d \nu\left(f; \left(2^{-s+i}|Q_0|\right)^{1/d}\right)\right) =$$

$$= f^{**}\left(2^{-d}|Q_0|\right)\cdot \exp\left(\sum_{i=0}^{s-1} \ln\left(1 + 3\cdot 2^d \nu\left(f; 2^{-i/d} l\,(Q_0)\right)\right)\right) \le$$

$$\le f^{**}\left(2^{-d}|Q_0|\right)\cdot \exp\left(3\cdot 2^d \sum_{i=0}^{s-1} \nu\left(f; 2^{-i/d} l\,(Q_0)\right)\right), \quad s = 1, 2, \ldots$$

On the other hand,

$$\nu\left(f; 2^{-i/d} l\,(Q_0)\right) \le \frac{2^{1/d}}{2^{1/d} - 1}\int_{2^{-i/d}l(Q_0)}^{2^{-(i-1)/d}l(Q_0)} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}, \quad i = 1, \ldots, s-1,$$

provided $\nu(f;\sigma)$ is monotone. Therefore, taking into account that $\nu(f; l\,(Q_0)) \le 2$, we get

$$f^{**}\left(2^{-d-s}|Q_0|\right) \le \exp\left(3\cdot 2^{d+1}\right) f^{**}\left(2^{-d}|Q_0|\right) \times$$

$$\times \exp\left(3\cdot 2^d \cdot \frac{2^{1/d}}{2^{1/d} - 1}\sum_{i=1}^{s-1} \int_{2^{-i/d}l(Q_0)}^{2^{-(i-1)/d}l(Q_0)} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}\right) \le$$

$$\leq 2^d \exp\left(3 \cdot 2^{d+1}\right) f_{Q_0} \exp\left(3 \cdot 2^d \cdot \frac{2^{1/d}}{2^{1/d}-1} \int_{2^{-(s-1)/d} l(Q_0)}^{l(Q_0)} \nu(f;\sigma) \frac{d\sigma}{\sigma}\right).$$

Fix some $t \in (0, |Q_0|]$ and choose $s \in \mathbb{N}$ such that $2^{-s}|Q_0| < t \leq 2^{-s+1}|Q_0|$. Then, since f^{**} is monotone the last inequality implies

$$f^{**}(t) \leq f^{**}\left(2^{-s}|Q_0|\right) \leq f^{**}\left(2^{-d-s}|Q_0|\right) \leq$$

$$\leq 2^d \exp\left(3 \cdot 2^{d+1}\right) f_{Q_0} \exp\left(3 \cdot 2^d \cdot \frac{2^{1/d}}{2^{1/d}-1} \int_{2^{(-s+1)/d} l(Q_0)}^{l(Q_0)} \nu(f;\sigma) \frac{d\sigma}{\sigma}\right) \leq$$

$$\leq c_1 \cdot f_{Q_0} \exp\left(c_2 \int_{t^{1/d}}^{l(Q_0)} \nu(f;\sigma) \frac{d\sigma}{\sigma}\right),$$

where $c_1 = 2^d \exp\left(3 \cdot 2^{d+1}\right)$, $c_2 = 3 \cdot 2^d \cdot \frac{2^{1/d}}{2^{1/d}-1}$. □

Corollary 5.8 (Gurov-Reshetnyak theorem 5.4). *For any ε such that*

$$0 < \varepsilon < \varepsilon_0(d) \equiv \frac{d}{c_2} = \frac{d\left(2^{1/d}-1\right)}{3 \cdot 2^d \cdot 2^{1/d}},$$

there exists $p_0 \equiv p_0(\varepsilon, d) > 1$ such that if the function f satisfies condition (5.1), then $f \in L^p(Q_0)$ for any $p < p_0$.

Proof. Condition (5.1) is equivalent to the inequality $\nu(f;\sigma) \leq \varepsilon$ for $0 < \sigma \leq l(Q_0)$. Thus (5.10) implies

$$f^{**}(t) \leq c_1 \cdot f_{Q_0} \cdot \exp\left(\frac{c_2}{d} \cdot \varepsilon \cdot \ln \frac{|Q_0|}{t}\right) = c_1 \cdot f_{Q_0} \left(\frac{|Q_0|}{t}\right)^{\frac{\varepsilon}{\varepsilon_0(d)}}, \quad 0 < t \leq |Q_0|. \tag{5.11}$$

Since $\varepsilon < \varepsilon_0(d)$ we have

$$p_0(\varepsilon, d) \equiv \frac{\varepsilon_0(d)}{\varepsilon} = \frac{1}{\varepsilon} \cdot \frac{d\left(2^{1/d}-1\right)}{3 \cdot 2^d \cdot 2^{1/d}} > 1. \tag{5.12}$$

If $p < p_0(\varepsilon, d)$, then, by (5.11),

$$\left(f^{**}(t)\right)^p \leq c_1^p \left(f_{Q_0}\right)^p \left(\frac{|Q_0|}{t}\right)^{\frac{\varepsilon}{\varepsilon_0(d)} \cdot p}, \quad 0 < t \leq |Q_0|.$$

As $p < p_0(\varepsilon, d) = \frac{\varepsilon_0(d)}{\varepsilon}$, we have

$$\|f\|_p^p = \int_0^{|Q_0|} \left(f^*(t)\right)^p dt \leq \int_0^{|Q_0|} \left(f^{**}(t)\right)^p dt \leq c_1^p \frac{\varepsilon_0(d)}{\varepsilon_0(d) - p\varepsilon} |Q_0| \left(f_{Q_0}\right)^p. \; \square \tag{5.13}$$

Remark 5.9. Fix the cube $Q_0 \subset \mathbb{R}^d$ and choose some $Q \subset Q_0$. Then (5.1) implies that $f \in GR(\varepsilon, Q)$. Hence, applying Corollary 5.8 to the cube Q, one can rewrite inequality (5.13) in the following way:

$$\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p} \le c_3 \frac{1}{|Q|}\int_Q f(x)\,dx, \quad Q \subset Q_0, \tag{5.14}$$

where $1 < p < p_0(\varepsilon, d)$, $c_3 = c_3(\varepsilon, p, d)$, and c_3 does not depend on $Q \subset Q_0$. Inequality (5.14) is called *the reverse Hölder inequality*, or *the Gehring inequality* [18].

Corollary 5.10. *If the function f is non-negative on the cube $Q_0 \subset \mathbb{R}^d$ and satisfies Gurov–Reshetnyak condition* (5.1) *for some $\varepsilon < \varepsilon_0(d)$, then it also satisfies Gehring inequality* (5.14) *for all $p < p_0(\varepsilon, d)$.*

Remark 5.11. In Gurov–Reshetnyak theorem 5.4 we have found that $p_0(\varepsilon, d) = \frac{\varepsilon_0(d)}{\varepsilon} \to \infty$ as $\varepsilon \to 0+$. The estimate $p_0(\varepsilon, d) = \underline{O}\left(\frac{1}{\varepsilon}\right)$, $\varepsilon \to 0+$, was first obtained in [4, 76]. As it was noticed in [4], it turns out that this limiting behavior cannot be improved. In what follows we will find the maximal possible value of $p_0(\varepsilon, 1)$ (see Corollary 5.35).

Remark 5.12. The proof of inequality (5.10) is based on the application of Lemma 2.2 with $a = 2$. Generally speaking, the parameter $a > 1$ in Lemma 2.2 could be chosen in the "better way" in order to minimize the value of the constant c_2 in the exponent in right-hand side of (5.10). This could allow to slightly increase the values of $\varepsilon_0(d)$ and $p_0(\varepsilon, d)$, obtained in Corollary 5.8. However, this method does not lead to the desired result (i.e. $\varepsilon_0(d) = 2$ and maximal possible $p_0(\varepsilon, d)$), because in order to prove (5.9) we have used estimate (5.3), which is overstated in the sense of constants.

From Theorem 5.7 one can derive the following

Corollary 5.13 (Franciosi, [13]). *If the function $f \in L(Q_0)$ is non-negative on the cube $Q_0 \subset \mathbb{R}^d$ and $\nu(f;\sigma) \to 0$ as $\sigma \to 0+$, then $f \in L^p(Q_0)$ for any $p < \infty$.*

Proof. Take some $p < \infty$ and choose t_0, $0 < t_0 \le |Q_0|$, such that $\nu(f;\sigma) \le \frac{d}{2c_2 p}$ for $0 < \sigma \le t_0^{1/d}$. Then, by (5.10), for $0 < t \le t_0$

$$f^{**}(t) \le c_1 \cdot f_{Q_0} \cdot \exp\left(c_2 \int_{t_0^{1/d}}^{|Q_0|} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}\right) \cdot \left(\frac{t_0}{t}\right)^{\frac{1}{2p}}.$$

Obviously this implies that the function $(f^{**})^p$ is summable on $[0, |Q_0|]$, and hence $f \in L^p(Q_0)$. □

It is clear that inequality (5.10) provides the sufficient conditions for f to belong to the various Orlicz spaces in terms of the order of $\nu(f;\sigma)$ as $\sigma \to 0+$. If $\nu(f;\sigma)$ is such that

$$\int_0^{l(Q_0)} \nu(f;\sigma)\,\frac{d\sigma}{\sigma} < \infty, \tag{5.15}$$

then (5.10) immediately implies the following result.

Corollary 5.14. *If the function $f \in L(Q_0)$ is non-negative on the cube $Q_0 \subset \mathbb{R}^d$ and satisfies* (5.15), *then $f \in L^\infty(Q_0)$.*

Corollary 5.14 can be sharpen. In order to show this, we will need one result due to Spanne [69] (see also [7, 57]). Denote

$$\nu_1(f;\delta) = \sup_{Q \subset Q_0,\ l(Q) \le \delta} \Omega(f;Q), \quad 0 < \delta \le l(Q_0).$$

Theorem 5.15 (Spanne, [69]). *If $f \in L(Q_0)$ is such that*

$$\int_0^{l(Q_0)} \nu_1(f;\delta)\,\frac{d\delta}{\delta} < \infty, \tag{5.16}$$

then to any cube $Q \subset Q_0$ with $l(Q) \le \frac{1}{2} l(Q_0)$

$$\operatorname*{ess\,sup}_{x \in Q} |(f - f_Q)(x)| \le c_d \int_0^{2l(Q)} \nu_1(f;\delta)\,\frac{d\delta}{\delta}, \tag{5.17}$$

where the constant c_d depends only on the dimension d of the space.

Proof. Fix the cube $Q \subset Q_0$ such that $l(Q) \le l(Q_0)/2$. In order to prove (5.17) without loss of generality we can assume $f_Q = 0$. According to Lebesgue theorem 1.1, we have $f(x) = \lim_{j\to\infty} f_{Q_j}$ for almost every $x \in Q$, where Q_j is a sequence of dyadic (with respect to Q) cubes of order $j = 1, 2, \dots$, contractible to x. Thus, for the proof of (5.17) it is enough to show that

$$\left|f_{Q_j}\right| \le c_d \int_0^{2l(Q)} \nu_1(f;\delta)\,\frac{d\delta}{\delta}, \quad j = 1, 2, \dots \tag{5.18}$$

Let us prove (5.18). For $i \ge 1$

$$\left|f_{Q_{i+1}} - f_{Q_i}\right| \le \frac{1}{|Q_{i+1}|} \int_{Q_{i+1}} |f(x) - f_{Q_i}|\,dx \le$$

$$\le 2^d \frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}|\,dx \le 2^d \nu_1\left(f; l(Q_i)\right),$$

while the condition $f_Q = 0$ implies

$$|f_{Q_1}| \le 2^d \nu_1(f; l(Q)).$$

Therefore

$$\left|f_{Q_j}\right| \le |f_{Q_1}| + \sum_{i=1}^{j-1} \left|f_{Q_{i+1}} - f_{Q_i}\right| \le 2^d \left(\nu_1(f;l(Q)) + \sum_{i=1}^{\infty} \nu_1\left(f;l(Q_i)\right)\right) \le$$

$$\le 2^d \left(\frac{1}{l(Q)} \int_{l(Q)}^{2l(Q)} \nu_1(f;\delta)\, d\delta + \frac{1}{l(Q)-l(Q_1)} \int_{l(Q_1)}^{l(Q)} \nu_1(f;\delta)\, d\delta + \right.$$

$$\left. + \sum_{i=2}^{\infty} \frac{1}{l(Q_{i-1})-l(Q_i)} \int_{l(Q_i)}^{l(Q_{i-1})} \nu_1(f;\delta)\, d\delta \right) =$$

$$= 2^d \left(\frac{1}{l(Q)} \int_{l(Q)}^{2l(Q)} \nu_1(f;\delta)\, d\delta + \frac{1}{l(Q_1)} \int_{l(Q_1)}^{l(Q)} \nu_1(f;\delta)\, d\delta + \right.$$

$$\left. + \sum_{i=2}^{\infty} \frac{1}{l(Q_i)} \int_{l(Q_i)}^{l(Q_{i-1})} \nu_1(f;\delta)\, d\delta \right) \le$$

$$\le 2^{d+1} \left(\int_{l(Q)}^{2l(Q)} \nu_1(f;\delta)\, \frac{d\delta}{\delta} + \int_{l(Q_1)}^{l(Q)} \nu_1(f;\delta)\, \frac{d\delta}{\delta} + \sum_{i=2}^{\infty} \int_{l(Q_i)}^{l(Q_{i-1})} \nu_1(f;\delta)\, \frac{d\delta}{\delta} \right) =$$

$$= 2^{d+1} \int_0^{2l(Q)} \nu_1(f;\delta)\, \frac{d\delta}{\delta}. \quad \square$$

From Spanne's theorem 5.15 and condition (5.16) it follows immediately, that the function f is equivalent to some function g, which is continuous on Q_0 and such that its *modulus of continuity* $\omega(g;\sigma)$ satisfies the condition

$$\omega(g;\sigma) \equiv \sup_{x,y \in Q_0,\ |x-y| \le \sigma} |g(x)-g(y)| \le c_d \int_0^{2\sigma} \nu_1(f;\delta)\, \frac{d\delta}{\delta}, \quad 0 < \sigma \le \frac{1}{2} l(Q_0). \tag{5.19}$$

Indeed, by Lebesgue theorem 1.1, the function $g(x) \equiv \lim_{l(Q)\to 0} f_Q$ for $x \in Q_0$ is equivalent to f. If $x, y \in Q_0$ and $|x-y| \le \sigma$, then we choose the cube $Q \subset Q_0$, which contains x, y, and such that $l(Q) \le \sigma$. Then

$$|g(x)-g(y)| \le |g(x)-f_Q| + |g(y)-f_Q| \le 2 \operatorname*{ess\,sup}_{x \in Q} |(f-f_Q)(x)|$$

and Theorem 5.15 imply (5.19).

Now let us suppose that f is non-negative on the cube Q_0 and satisfies (5.15). Then (5.10) implies

$$\|f\|_\infty \le K \equiv c_1 f_{Q_0} \exp\left(c_2 \int_0^{l(Q_0)} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}\right).$$

Hence for any cube $Q \subset Q_0$

$$f_Q \le \|f\|_\infty \le K.$$

Therefore

$$\nu(f;\delta) = \sup_{l(Q)\le\delta} \frac{\Omega(f;Q)}{f_Q} \ge \frac{1}{K} \sup_{l(Q)\le\delta} \Omega(f;Q) = \frac{1}{K}\nu_1\left(f;\delta\right), \quad 0<\delta\le l\left(Q_0\right). \tag{5.20}$$

So condition (5.15) implies

$$\int_0^{l(Q_0)} \nu_1(f;\delta)\,\frac{d\delta}{\delta} \le K \int_0^{|Q_0|} \nu(f;\delta)\,\frac{d\delta}{\delta} < \infty,$$

i.e., condition (5.16) is also satisfied. Now, applying Spanne theorem 5.15, from (5.20) we obtain the following statement.

Theorem 5.16. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying condition* (5.15)*. Then for any cube $Q \subset Q_0$ with $l(Q) \le \frac{1}{2} l\left(Q_0\right)$*

$$\operatorname{ess\,sup}_{x\in Q} \left|\left(f - f_Q\right)(x)\right| \le c_d K \int_0^{2l(Q)} \nu(f;\delta)\,\frac{d\delta}{\delta}.$$

In the same way as the estimate of the modulus of continuity follows from Spanne theorem 5.15, the last Theorem implies

Corollary 5.17 ([37]). *If the function f is non-negative on the cube $Q_0 \subset \mathbb{R}^d$ and satisfies condition* (5.15)*, then f is equivalent to the function g, which is continuous on Q_0 and*

$$\omega(g;\delta) = \underline{O}\left(\int_0^{\delta} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}\right), \quad \delta \to 0.$$

Remark 5.18. Inequality (5.20) was obtained under assumption (5.15). This assumption is necessary to guarantee that $\nu(f;\delta)$ dominates $\nu_1\left(f;\delta\right)$. Indeed, let us consider the function $f_0(x) = \ln\frac{1}{x}$, $0 < x \le \beta_0$, where $\beta_0 > 0$ is small enough. Then, as it was shown in Example 2.24, $\nu_1(f_0;\delta) = \Omega(f;[0,\delta]) = \frac{2}{\mathrm{e}}$, and

$$\nu(f_0;\delta) \le \frac{1}{\ln\frac{1}{\beta_0}}\Omega(f_0;[0,\delta]) = \frac{2/\mathrm{e}}{\ln\frac{1}{\beta_0}}, \quad 0<\delta\le\beta_0.$$

So, for the function f_0 inequality (5.20) fails for any K, which does not depend on β_0.

Let us come back to Gurov–Reshetnyak theorem 5.4. As we remarked in Corollary 5.8, Gurov–Reshetnyak condition (5.1) for $0 < \varepsilon < \varepsilon_0(d)$ implies that f is summable for some $p > 1$. On the other hand, condition (5.1) is non-trivial for any $\varepsilon < 2$. In this context the following question is natural. *For which $\varepsilon < 2$ one can increase the exponent of summability of the function f, using condition* (5.1)*?* The next theorem provides the answer to this question.

Theorem 5.19 (Coifman, Fefferman, [8]). *Let $f \in L(Q_0)$ be a non-negative summable function on the cube $Q_0 \subset \mathbb{R}^d$. If*

$$|\{x \in Q : \ f(x) > \sigma \cdot f_Q\}| > \theta \cdot |Q|, \quad Q \subset Q_0, \tag{5.21}$$

where the constants $0 < \sigma, \theta < 1$ do not depends on Q, then there exists $r \equiv r(\sigma, \theta, d) > 0$ such that $f \in L^{1+r}(Q_0)$ and

$$\left\{ \frac{1}{|Q_0|} \int_{Q_0} f^{1+r}(x)\, dx \right\}^{\frac{1}{1+r}} \le c \frac{1}{|Q_0|} \int_{Q_0} f(x)\, dx \tag{5.22}$$

with $c = c(\sigma, \theta, d, r)$.

Proof. First we prove the inequality

$$\int_{\{x \in Q_0 : \ f(x) > \alpha\}} f(x)\, dx \le c' \cdot \alpha \, |\{x \in Q_0 : \ f(x) > \sigma \cdot \alpha\}| \, , \tag{5.23}$$

where $\alpha \ge f_{Q_0}$ and the constant c' depends only on θ and d. Let us fix some $\alpha \ge f_{Q_0}$ and apply Caldrón–Zygmund lemma 1.14. Then we obtain a collection of cubes $Q_j \subset Q_0, \ j = 1, 2, \ldots$, with pairwise disjoint interiors such that

$$\alpha < \frac{1}{|Q_j|} \int_{Q_j} f(x)\, dx \le 2^d \alpha,$$

and $f(x) \le \alpha$ for almost all $x \in Q_0 \setminus \left(\bigcup_{j \ge 1} Q_j \right)$. From here, using (5.21), we have

$$\int_{\{x \in Q_0 : \ f(x) > \alpha\}} f(x)\, dx \le \sum_{j \ge 1} \int_{Q_j} f(x)\, dx \le$$

$$\le 2^d \alpha \sum_{j \ge 1} |Q_j| \le \frac{2^d \alpha}{\theta} \sum_{j \ge 1} \left| \{x \in Q_j : \ f(x) > \sigma \cdot f_{Q_j}\} \right| \le$$

$$\le \frac{2^d}{\theta} \cdot \alpha \sum_{j \ge 1} |\{x \in Q_j : \ f(x) > \sigma \cdot \alpha\}| \le c' \cdot \alpha \cdot |\{x \in Q_0 : \ f(x) > \sigma \cdot \alpha\}| \, ,$$

where $c' = 2^d / \theta$, and this proves (5.23).

Now, multiplying (5.23) by α^{r-1} and integrating, we find that

$$\int_{f_{Q_0}}^{\infty} \alpha^{r-1} \left(\int_{\{x \in Q_0 : \ f(x) > \alpha\}} f(x)\, dx \right) d\alpha \le$$

$$\leq c' \int_0^\infty \alpha^r \left|\{x \in Q_0 : \ f(x) > \sigma \cdot \alpha\}\right| \, d\alpha =$$

$$= \frac{c'}{\sigma^{1+r}} \int_0^\infty \tau^r \left|\{x \in Q_0 : \ f(x) > \tau\}\right| \, d\tau = c'' \int_{Q_0} f^{1+r}(x) \, dx, \qquad (5.24)$$

where $c'' = c' \cdot \sigma^{-1-r}/(1+r)$. On the other hand, using the Fubini theorem one can get the following estimate for the left hand-side of the last inequality

$$\int_{f_{Q_0}}^\infty \alpha^{r-1} \left(\int_{\{x \in Q_0 : \ f(x) > \alpha\}} f(x) \, dx \right) d\alpha =$$

$$= \int_{\left\{x \in Q_0 : \ f(x) > f_{Q_0}\right\}} f(x) \left(\int_{f_{Q_0}}^{f(x)} \alpha^{r-1} \, d\alpha \right) dx =$$

$$= \frac{1}{r} \int_{\left\{x \in Q_0 : \ f(x) > f_{Q_0}\right\}} f(x) \left(f^r(x) - (f_{Q_0})^r\right) \, dx \geq$$

$$\geq \frac{1}{r} \int_{Q_0} f(x) \left(f^r(x) - (f_{Q_0})^r\right) \, dx = \frac{1}{r} \int_{Q_0} f^{1+r}(x) \, dx - \frac{|Q_0|}{r} \left(f_{Q_0}\right)^{1+r}.$$

From here and (5.24) it follows that

$$\left(\frac{1}{r} - c'' \right) \frac{1}{|Q_0|} \int_{Q_0} f^{1+r}(x) \, dx \leq \frac{1}{r} \left(f_{Q_0}\right)^{1+r}.$$

Choosing $r > 0$ so small that $\frac{1}{r} - c'' > 0$, we obtain (5.22). $\square$

Remark 5.20. The presented proof of Theorem 5.19 needs a small refinement. Indeed, in (5.24) we a priori assumed that $f \in L^{1+r}(Q_0)$. This vicious circle can be avoided in the following way.

For $N > f_{Q_0}$ let us consider *the cut-off function* $[f]_N(x) = \min(N, f(x)) \leq f(x)$, $x \in Q_0$. If $f_{Q_0} \leq \alpha < N$, then

$$\{x \in Q_0 : \ f(x) > \alpha\} = \{x \in Q_0 : \ [f]_N(x) > \alpha\}$$

and

$$\{x \in Q_0 : \ f(x) > \sigma \cdot \alpha\} = \{x \in Q_0 : \ [f]_N(x) > \sigma \cdot \alpha\},$$

so that, by (5.23),

$$\int_{\{x \in Q_0 : \ [f]_N(x) > \alpha\}} [f]_N(x) \, dx \leq c' \cdot \alpha \left|\{x \in Q_0 : \ [f]_N(x) > \sigma \cdot \alpha\}\right|. \qquad (5.25)$$

Otherwise, if $\alpha \geq N$, then (5.25) is trivial because the domain of the definition of the integral in the left inequality is empty. So, (5.23) implies (5.25) for all $\alpha \geq f_{Q_0}$ and $N > f_{Q_0}$. It is also clear that $[f]_N \in L^{1+r}(Q_0)$ for any $r > 0$.

Now we repeat the proof of Theorem 5.19. In the proof of (5.24), substituting f by $[f]_N$, we have

$$\left\{\frac{1}{|Q_0|}\int_{Q_0}[f]_N^{1+r}(x)\,dx\right\}^{\frac{1}{1+r}} \le c\frac{1}{|Q_0|}\int_{Q_0}[f]_N(x)\,dx, \quad N > f_{Q_0}.$$

Finally, since $\frac{1}{|Q_0|}\int_{Q_0}[f]_N(x)\,dx \le \frac{1}{|Q_0|}\int_{Q_0} f(x)\,dx$ in order to complete the proof of (5.22) it is enough to send $N \to \infty$ and use the Levi theorem.

Remark 5.21. Condition (5.21) is one of the equivalent forms of the so-called A_∞*–Muckenhoupt condition* (see [8, 72]).

We see that condition (5.21) allows to increase the exponent of summability of the function f. Therefore, in order to proof Gurov–Reshetnyak theorem 5.4 it is enough to show that Gurov–Reshetnyak condition (5.1) implies (5.21). Actually it turn out that conditions (5.1) and (5.21)are equivalent. This fact is the content of the next theorem.

Theorem 5.22 ([42]). *Let $f \in L(Q_0)$ be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$. Then*

(*i*) *if for some ε, $0 < \varepsilon < 2$, the function f satisfies Gurov-Reshetnyak condition* (5.1)*, then there exist σ and θ, $0 < \sigma, \theta < 1$, which depend only on ε and such that* (5.21) *holds true;*

(*ii*) *if for some σ and θ, $0 < \sigma, \theta < 1$, the function f satisfies* (5.21)*, then*

$$\Omega(f; Q) \le 2(1 - \sigma\theta) f_Q, \quad Q \subset Q_0.$$

Proof. To prove (*i*) let us choose a number λ such that $\varepsilon < \lambda < 2$. Fix some cube $Q \subset Q_0$ and consider the set $E = \left\{x \in Q : \; f(x) > \frac{\lambda-\varepsilon}{\lambda} \cdot f_Q\right\}$. Since $\frac{\varepsilon}{\lambda} \cdot f_Q \le f_Q - f(x)$ for all $x \in Q \setminus E$ we have

$$\frac{\varepsilon}{\lambda} \cdot f_Q \le \inf_{x \in Q\setminus E} (f_Q - f(x)) \le \frac{1}{|Q \setminus E|}\int_{Q\setminus E} (f_Q - f(x))\,dx.$$

On the other hand, $\frac{\lambda-\varepsilon}{\lambda} < 1$, so that

$$Q \setminus E = \left\{x \in Q : \; f(x) \le \frac{\lambda-\varepsilon}{\lambda} \cdot f_Q\right\} \subset \{x \in Q : \; f(x) < f_Q\}.$$

Now applying Property 2.1 and condition (5.1) to the last inequality, we get

$$\frac{\varepsilon}{\lambda} \cdot f_Q \le \frac{1}{|Q \setminus E|}\int_{\{x\in Q:\; f(x)<f_Q\}} (f_Q - f(x))\,dx =$$

$$= \frac{1}{2} \cdot \frac{|Q|}{|Q \setminus E|} \cdot \Omega(f;Q) \le \frac{\varepsilon}{2} \cdot \frac{|Q|}{|Q \setminus E|} \cdot f_Q.$$

Hence $|Q \setminus E| \le \frac{\lambda}{2} \cdot |Q|$, i.e. $|E| \ge \left(1 - \frac{\lambda}{2}\right) \cdot |Q|$. This inequality coincides with (5.21) for $\sigma = \frac{\lambda - \varepsilon}{\lambda}$ and $\theta = 1 - \frac{\lambda}{2}$.

Let us prove (ii). Fix some cube $Q \subset Q_0$. By Property 2.1 and condition (5.21),

$$\Omega(f;Q) = \frac{2}{|Q|} \int_{\{x \in Q:\ f(x) \le f_Q\}} (f_Q - f(x))\, dx =$$

$$= \frac{2}{|Q|} \int_{\{x \in Q:\ \sigma f_Q < f(x) \le f_Q\}} (f_Q - f(x))\, dx +$$

$$+ \frac{2}{|Q|} \int_{\{x \in Q:\ f(x) \le \sigma f_Q\}} (f_Q - f(x))\, dx \le$$

$$\le \frac{2}{|Q|}(1-\sigma)\, |\{x \in Q:\ f(x) > \sigma f_Q\}| + \frac{2}{|Q|} f_Q\, |\{x \in Q:\ f(x) \le \sigma f_Q\}| =$$

$$= \frac{2}{|Q|} f_Q \Big[(1-\sigma)\big(|Q| - |\{x \in Q:\ f(x) \le \sigma f_Q\}|\big) + |\{x \in Q:\ f(x) \le \sigma f_Q\}|\Big] =$$

$$= \frac{2}{|Q|} f_Q \Big[|Q| - \sigma|Q| - |\{x \in Q:\ f(x) \le \sigma f_Q\}| + \sigma\, |\{x \in Q:\ f(x) \le \sigma f_Q\}| +$$

$$+ |\{x \in Q:\ f(x) \le \sigma f_Q\}|\Big] \le$$

$$\le \frac{2}{|Q|} f_Q \Big[|Q|(1-\sigma) + \sigma(1-\theta)|Q|\Big] = 2(1 - \sigma\theta) f_Q. \quad \square$$

As we have already mentioned above, the next result is the immediate consequence of Theorems 5.19 and 5.22.

Corollary 5.23 ([42]). *Let* $0 < \varepsilon < 2$ *and let* f *be a non-negative function on the cube* $Q_0 \subset \mathbb{R}^d$*, satisfying Gurov–Reshetnyak condition* (5.1)*. Then there exists* $r > 0$*, which depends only on* ε *and* d*, such that* (5.22) *holds true with* c*, depending only on* ε*,* d *and* r*.*

Condition (5.1) of Corollary 5.23 can be replaced by the weaker condition

$$\Omega(f;Q) \le \varepsilon \cdot f_Q, \quad Q \subset Q_0,\ l(Q) \le \delta \cdot l(Q_0), \tag{5.26}$$

where $\delta \in (0,1]$ is fixed. In other words, in the Gurov–Reshetnyak theorem it is sufficient to require that the condition $\Omega(f;Q) \le \varepsilon \cdot f_Q$ is verified on small enough cubes. Indeed, we have

Corollary 5.24. *Let* $0 < \varepsilon < 2$ *and let* f *be a non-negative function on the cube* $Q_0 \subset \mathbb{R}^d$*, satisfying* (5.26) *for some* $\delta \in (0,1]$*. Then there exists* $r > 0$*, which depends only on* ε *and* d*, such that* $f \in L^{1+r}(Q_0)$ *and*

$$\left\{ \frac{1}{|Q_0|} \int_{Q_0} f^{1+r}(x)\,dx \right\}^{\frac{1}{1+r}} \le c_1 \frac{1}{|Q_0|} \int_{Q_0} f(x)\,dx. \tag{5.27}$$

Here the constant c_1 depends only on ε, d, r and δ.

Proof. Let us partition the cube Q_0 into $N = \left(\left[\frac{1}{\delta}\right] + 1\right)^d$ cubes Q_j with pairwise disjoint interiors, dividing each side of the cube Q_0 into $\left[\frac{1}{\delta}\right] + 1$ equal parts. Then $l(Q_j) \le \delta \cdot l(Q_0),\ j = 1, \ldots, N$, and by virtue of (5.26) to each cube Q_j we can apply Corollary 5.23. Then

$$\left\{ \frac{1}{|Q_j|} \int_{Q_j} f^{1+r}(x)\,dx \right\}^{\frac{1}{1+r}} \le c \frac{1}{|Q_j|} \int_{Q_j} f(x)\,dx, \quad j = 1, \ldots, N, \tag{5.28}$$

where $r = r(\varepsilon, d) > 0$ and $c = c(\varepsilon, d, r)$. Since $Q_0 = \bigcup_{j=1}^N Q_j$ inequality (5.28) implies $f \in L^{1+r}(Q_0)$. Further, as $|Q_j| = \frac{|Q_0|}{N},\ j = 1, \ldots, N$, (5.28) yields

$$\frac{1}{|Q_0|} \int_{Q_0} f^{1+r}(x)\,dx = \frac{1}{N} \sum_{j=1}^N \frac{1}{|Q_j|} \int_{Q_j} f^{1+r}(x)\,dx \le$$

$$\le c^{1+r} \frac{1}{N} \sum_{j=1}^N \left(\frac{1}{|Q_j|} \int_{Q_j} f(x)\,dx \right)^{1+r} \le c^{1+r} \left(\max_{1 \le j \le N} \frac{1}{|Q_j|} \int_{Q_j} f(x)\,dx \right)^{1+r} \le$$

$$\le c^{1+r} N^{1+r} |Q_0|^{-1-r} \left(\sum_{j=1}^N \int_{Q_j} f(x)\,dx \right)^{1+r} =$$

$$= c^{1+r} N^{1+r} \left(\frac{1}{|Q_0|} \int_{Q_0} f(x)\,dx \right)^{1+r}.$$

This implies (5.27) with $c_1 = cN \le c\left(\frac{1}{\delta} + 1\right)^d$. □

Theorem 5.22 establish the equivalence of Gurov–Reshetnyak condition (5.1) and A_∞–Muckenhoupt condition (5.21). On the other hand, in [8] it was shown that Muckenhoupt condition (5.21) is equivalent to Gehring inequality (5.22) for some $r > 0$. Thus Gurov–Reshetnyak condition (5.1) and Gehring condition (5.22) are also equivalent. Now we will give another proof of the equivalence of conditions (5.1), (5.21) and (5.22). Namely, besides Theorems 5.19 and 5.22, one has to apply the following statement.

Theorem 5.25 ([44]). *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying the Gehring condition*

$$\left\{ \frac{1}{|Q|} \int_Q f^p(x)\,dx \right\}^{1/p} \le B \cdot \frac{1}{|Q|} \int_Q f(x)\,dx, \quad Q \subset Q_0, \tag{5.29}$$

for some $p,\ B > 1$. Then there exists $\varepsilon,\ 0 < \varepsilon < 2$, which depends only on p and B, such that Gurov–Reshetnyak inequality (5.1) holds true.

Proof. Let us fix an arbitrary cube $Q \subset Q_0$ and denote $E = \{x \in Q : \ f(x) \geq f_Q\}$. We can assume that $f_Q > 0$. By the Hölder inequality,

$$\frac{\Omega(f;Q)}{f_Q} = \frac{1}{f_Q}\frac{2}{|Q|}\int_E (f(x) - f_Q)\ dx = 2\frac{|E|}{|Q|}\frac{1}{f_Q}\frac{1}{|E|}\int_E f(x)\,dx - 2\frac{|E|}{|Q|} \leq$$

$$\leq 2\frac{|E|}{|Q|}\frac{1}{f_Q}\left\{\frac{1}{|E|}\int_E f^p(x)\,dx\right\}^{1/p} - 2\frac{|E|}{|Q|} \leq$$

$$\leq 2\left(\frac{|E|}{|Q|}\right)^{1-1/p}\frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p} - 2\frac{|E|}{|Q|}.$$

On the other hand, by condition (5.29),

$$(f_Q)^{-1}\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p} \leq B,$$

so that

$$\frac{\Omega(f;Q)}{f_Q} \leq 2\left(B\left(\frac{|E|}{|Q|}\right)^{1-1/p} - \frac{|E|}{|Q|}\right). \tag{5.30}$$

Let us consider the function $\varphi(\lambda) = B\cdot\lambda^{1-1/p} - \lambda,\ \lambda > 0$. The analysis of the derivative shows, that φ is increasing on $(0,\lambda_0)$ and decreasing on $(\lambda_0,+\infty)$, where $\lambda_0 = (B(p-1)/p)^p$. We also notice that from the trivial inequality

$$\frac{\Omega(f;Q)}{f_Q} = 2\frac{|Q\setminus E|}{|Q|}\frac{1}{|Q\setminus E|}\int_{Q\setminus E}\left(1 - \frac{f(x)}{f_Q}\right)\,dx \leq 2\frac{|Q\setminus E|}{|Q|}$$

it follows that

$$\frac{|E|}{|Q|} = 1 - \frac{|Q\setminus E|}{|Q|} \leq 1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}. \tag{5.31}$$

First let us consider the case

$$B < \left(\frac{p}{p-1}\right)^{(p-1)/p}. \tag{5.32}$$

Then

$$\lambda_0 < \left(\left(\frac{p}{p-1}\right)^{(p-1)/p}\frac{p-1}{p}\right)^p = \frac{p-1}{p} < 1.$$

Assume that

$$\frac{\Omega(f;Q)}{f_Q} \geq 2\,(1-\lambda_0)\,. \tag{5.33}$$

Then $\lambda_0 \geq 1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}$, and (5.30) and (5.31) imply

$$\frac{\Omega(f;Q)}{f_Q} \le 2\varphi\left(\frac{|E|}{|Q|}\right) \le 2\varphi\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right) =$$

$$= 2\left(B\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} - \left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)\right) =$$

$$= 2B\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} - 2 + \frac{\Omega(f;Q)}{f_Q},$$

provided φ is monotone on $(0, \lambda_0)$. Thus

$$B\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} \ge 1,$$

or, equivalently,

$$\frac{\Omega(f;Q)}{f_Q} \le 2\left(1 - B^{-p/(p-1)}\right). \tag{5.34}$$

Comparing the last inequality with (5.33), we find

$$1 - \lambda_0 \le 1 - B^{-p/(p-1)}. \tag{5.35}$$

But $\lambda_0 = (B(p-1)/p)^p$, so that (5.35) is equivalent to

$$B \ge \left(\frac{p}{p-1}\right)^{(p-1)/p},$$

which contradicts (5.32). Therefore, condition (5.32) excludes (5.33) and implies

$$\frac{\Omega(f;Q)}{f_Q} < 2\left(1 - \lambda_0\right), \tag{5.36}$$

and so $\lambda_0 < 1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}$. Taking into account that λ_0 is a point of maximum of φ, from (5.30) we obtain

$$\frac{\Omega(f;Q)}{f_Q} \le 2\varphi\left(\frac{|E|}{|Q|}\right) \le 2\varphi\left(\lambda_0\right) = 2\varphi\left(\left(B\frac{p-1}{p}\right)^p\right) =$$

$$= 2\left[B\left(\left(B\frac{p-1}{p}\right)^p\right)^{1-1/p} - \left(B\frac{p-1}{p}\right)^p\right] = 2B^p\frac{(p-1)^{p-1}}{p^p} =$$

$$= \frac{2}{p-1}\lambda_0 < \frac{2}{p-1}\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right) = \frac{2}{p-1} - \frac{1}{p-1}\frac{\Omega(f;Q)}{f_Q}.$$

This implies

$$\left(1+\frac{1}{p-1}\right)\frac{\Omega(f;Q)}{f_Q}\le\frac{2}{p-1},$$

i.e.,

$$\frac{\Omega(f;Q)}{f_Q}\le\frac{2}{p}. \tag{5.37}$$

Notice, that condition (5.32) is equivalent to the following one

$$\frac{2}{p}<2\left(1-\lambda_0\right).$$

This means that bound (5.37) is stronger than (5.36).

It remains to consider the case

$$B\ge\left(\frac{p}{p-1}\right)^{(p-1)/p}. \tag{5.38}$$

If we suppose that condition (5.33) is satisfied, then, as before, we come to inequality (5.34). Otherwise we obtain the opposite to (5.33) inequality (5.36). Notice also that (5.38) implies

$$1-\lambda_0\le\frac{1}{p}\le 1-B^{-p/(p-1)}.$$

So, we conclude, that among estimates (5.34), (5.36) and (5.37), the estimate provided by (5.34) is the best one that can be achieved in the case (5.38).

Finally, setting

$$\varepsilon\equiv\varepsilon(B,p)=\begin{cases}\frac{2}{p}, & \text{if } B<(p/(p-1))^{(p-1)/p},\\ 2\left(1-B^{-p/(p-1)}\right), & \text{if } B\ge(p/(p-1))^{(p-1)/p},\end{cases}$$

we get (5.1). □

For $1<p<\infty$ and $B>1$ we denote by $G_p(B)$ the class of all functions f, which are non-negative on the cube $Q_0\subset\mathbb{R}^d$ and satisfy the Gehring inequality

$$\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p}\le B\frac{1}{|Q|}\int_Q f(x)\,dx,\quad Q\subset Q_0.$$

The set $G_p\equiv\bigcup_{B>1}G_p(B)$ is called *the Gehring class* .

Remark 5.26. In the proof of Theorem 5.25 we have found that $\varepsilon(B,p)\to 2-0$ as $B\to\infty$ for any fixed $p>1$. This fact is essential. Indeed, let us consider the function $f_b(x)=b\chi_{(-\infty,0)}(x)+\chi_{[0,+\infty)}(x),\ x\in\mathbb{R}$ for $b>1$. An easy computation shows that $f_b\in GR\left(\varepsilon_0\right)$ with the minimal possible value $\varepsilon_0\equiv\varepsilon_0(b)=2\frac{\sqrt{b}-1}{\sqrt{b}+1}\to 2-0$ as $b\to\infty$. In addition, $f_b\in G_p$ for any $p>1$.

This means that the Gurov–Reshetnyak class $GR(\varepsilon)$ does not contain the Gehring class G_p for all $\varepsilon < 2$. Moreover, this example shows that for $\varepsilon < 2$ the set $\bigcap_{p>1} G_p$ does not belong to any $GR(\varepsilon)$. On the other hand, if $p > 1$ and $B \to 1$, then the value $\varepsilon(B,p) = \frac{2}{p}$, obtained in the proof of Theorem 5.25, does not tend to zero. In this sense the constant $\varepsilon(B,p)$ from Theorem 5.25 is overestimated. Indeed, as it was remarked in [4], for $p \geq 2$ and $1 < B < \sqrt{5}$ the condition $f \in G_p(B)$ implies $f \in G_2(B)$, so that by Hölder inequality

$$\Omega^2(f;Q) \leq \frac{1}{|Q|}\int_Q |f(x) - f_Q|^2 \, dx = \frac{1}{|Q|}\int_Q f^2(x)\,dx - \left(\frac{1}{|Q|}\int_Q f(x)\,dx\right)^2 \leq$$

$$\leq \left(B^2 - 1\right)\left(\frac{1}{|Q|}\int_Q f(x)\,dx\right)^2.$$

So, if $G_p(B) \subset GR(\varepsilon)$ for $p \geq 2$, then one can assure that $\varepsilon \to 0$ as $B \to 1$.

Remark 5.27. Let us fix $B > 1$. Then $\varepsilon(B,p) \to 2\frac{B-1}{B}$ as $p \to \infty$. In other words, we have

$$\bigcap_{1<p<\infty} G_p(B) \subset \bigcap_{\varepsilon_1<\varepsilon<2} GR(\varepsilon), \tag{5.39}$$

where $\varepsilon_1 = 2\frac{B-1}{B} > 0$. We do not know the minimal value of $\varepsilon_1(B)$ (possibly depending on d), which guarantees (5.39). Notice, that (5.39) fails for $\varepsilon_1 = 0$. Moreover, for the function f_b, defined in Remark 5.26, if $b = B$ then we have $f_B \in G_p(B)$ for any $p > 1$. On the other hand, from Remark 5.26 we know that $f_B \notin GR(\varepsilon)$ for any $\varepsilon < \varepsilon_0(B) = 2\frac{\sqrt{B}-1}{\sqrt{B}+1}$. Hence (5.39) fails if $\varepsilon_1 < 2\frac{\sqrt{B}-1}{\sqrt{B}+1}$, and moreover, it is easy to see that this fact is valid in the space of any dimension $d \geq 1$. Therefore, if $\varepsilon_1(B)$ is the minimal value such that (5.39) holds, then

$$2\frac{\sqrt{B}-1}{\sqrt{B}+1} \leq \varepsilon_1(B) \leq 2\frac{B-1}{B}.$$

Let us consider the other limit case $p \to 1+0$. For some fixed $B > 1$ we have $\varepsilon(B,p) \to 2-0$, i.e.,

$$\bigcup_{1<p<\infty} G_p(B) \subset \bigcup_{0<\varepsilon<2} GR(\varepsilon). \tag{5.40}$$

The same example of the function f_b, defined in Remark 5.26, shows that the constant 2 in the right-hand side of (5.40) is sharp. Indeed, it is easy to see that $f_b \in G_p\left(B_{p,b}\right)$ for the fixed value $b > 1$, where

$$B_{p,b} = \frac{(p-1)^{(p-1)/p}}{p}\,\frac{b^p - 1}{(b-1)^{1/p}}\,\frac{1}{\left(b^p - b\right)^{(p-1)/p}} \to 1+0 \text{ as } p \to 1 \tag{5.41}$$

is the minimal possible value. Fix some $B > 1$, $\varepsilon_1 < 2$ and choose $b > 1$ so big, that $\varepsilon_0(b) = 2\frac{\sqrt{b}-1}{\sqrt{b}+1} > \varepsilon_1$. Then, due to (5.41), for this value of b there exists $p > 1$ such that $B_{p,b} < B$ and $f_b \in \bigcup_{1<p<\infty} G_p(B)$. Obviously, $f_b \notin \bigcup_{0<\varepsilon<\varepsilon_1} GR(\varepsilon)$.

Another proof of Theorem 5.25. Without loss of generality we can assume

$$B \geq \frac{p}{p-1}. \tag{5.42}$$

Fix an arbitrary cube $Q \subset Q_0$ and denote $E = \{x \in Q : \ f(x) \geq f_Q\}$. Then the properties of the oscillations together with the Hölder inequality imply

$$\frac{\Omega(f;Q)}{f_Q} = \frac{1}{f_Q}\frac{2}{|Q|}\int_E (f(x) - f_Q)\, dx \leq$$

$$\leq 2\frac{|E|}{|Q|}\frac{1}{f_Q}\left\{\frac{1}{|E|}\int_E f^p(x)\, dx\right\}^{1/p} - 2\frac{|E|}{|Q|} \leq$$

$$\leq 2\left(\frac{|E|}{|Q|}\right)^{1-1/p}\frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_Q f^p(x)\, dx\right\}^{1/p} - 2\frac{|E|}{|Q|}.$$

Applying (5.29) to the integral in the right-hand side we have

$$\frac{\Omega(f;Q)}{f_Q} \leq 2\left[B\left(\frac{|E|}{|Q|}\right)^{1-1/p} - \frac{|E|}{|Q|}\right]. \tag{5.43}$$

Further, from the inequality

$$\frac{\Omega(f;Q)}{f_Q} = 2\frac{|Q\setminus E|}{|Q|}\frac{1}{|Q\setminus E|}\int_{Q\setminus E}\left(1 - \frac{f(x)}{f_Q}\right) dx \leq 2\frac{|Q\setminus E|}{|Q|}$$

it follows that

$$\frac{|E|}{|Q|} = 1 - \frac{|Q\setminus E|}{|Q|} \leq 1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}. \tag{5.44}$$

It is easy to see, that the function $\varphi(\lambda) = B\lambda^{1-1/p} - \lambda,\ \lambda > 0$ increases on $(0,\lambda_0)$, $\lambda_0 = \left(B\frac{p-1}{p}\right)^p$. Notice that due to (5.42), $\lambda_0 > 1$. Since the right-hand side of (5.44) is less or equal than 1 inequalities (5.43) and (5.44) yield

$$\frac{\Omega(f;Q)}{f_Q} \leq 2\varphi\left(\frac{|E|}{|Q|}\right) \leq 2\varphi\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right) =$$

$$= 2\left(B\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} - \left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)\right) =$$

$$= 2B\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} - 2 + \frac{\Omega(f;Q)}{f_Q}.$$

This inequality implies

$$B\left(1 - \frac{1}{2}\frac{\Omega(f;Q)}{f_Q}\right)^{1-1/p} \geq 1,$$

or, equivalently,

$$\frac{\Omega(f;Q)}{f_Q} \leq 2\left(1 - B^{-p/(p-1)}\right).$$

Therefore, the function f satisfies inequality (5.1) for

$$\varepsilon = 2\left(1 - \left[\max\left(B, \frac{p}{p-1}\right)\right]^{-p/(p-1)}\right) < 2. \quad \square$$

We conclude this section by one interesting property of functions that satisfy the Gehring condition.

Theorem 5.28 (Iwaniec, [28]). *Let $f \in L^p\left(\mathbb{R}^d\right)$, $p > 1$ be a non-negative function, satisfying the Gehring condition*

$$\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{\frac{1}{p}} \leq B\frac{1}{|Q|}\int_Q f(x)\,dx, \quad Q \subset \mathbb{R}^d, \tag{5.45}$$

where the constant $B > 1$ does not depend on the cube Q. Then f is equivalent to zero on $\mathbb{R}^d$.

Proof. Let us assume the contrary. Without loss of generality assume that

$$c \equiv \int_{Q_0} f^p(x)\,dx > 0$$

on the cube $Q_0 \equiv [-1,1]^d$. Then, by condition (5.45), for any cube $Q \supset Q_0$ centered in the origin we have

$$B\frac{1}{|Q|}\int_Q f(x)\,dx \geq \left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{\frac{1}{p}} \geq$$

$$\geq \left\{\frac{1}{|Q|}\int_{Q_0} f^p(x)\,dx\right\}^{\frac{1}{p}} = |Q|^{-\frac{1}{p}}c^{\frac{1}{p}}.$$

Thus

$$\int_Q f(x)\,dx \geq c_1|Q|^{1-\frac{1}{p}}, \tag{5.46}$$

where $c_1 = B^{-1} c^{\frac{1}{p}} > 0$. Let us show that (5.46) contradicts the condition $f \in L^p(\mathbb{R}^d)$. Since the cube Q_0 is fixed we can construct by induction a sequence of cubes $Q_k \supset Q_{k-1},\ k = 1, 2, \ldots$ such that

$$|Q_k| \geq \left(\frac{2}{c_1} \int_{Q_{k-1}} f(x)\, dx \right)^{\frac{p}{p-1}} \tag{5.47}$$

and

$$\frac{|Q_k|}{|Q_k \setminus Q_{k-1}|} \leq \frac{3}{2}. \tag{5.48}$$

By (5.46),

$$c_1 |Q_k|^{-\frac{1}{p}} \leq \frac{1}{|Q_k|} \int_{Q_k} f(x)\, dx =$$

$$= \frac{1}{|Q_k|} \left(\int_{Q_k \setminus Q_{k-1}} f(x)\, dx + \int_{Q_{k-1}} f(x)\, dx \right) =$$

$$= \frac{|Q_k \setminus Q_{k-1}|}{|Q_k|} \frac{1}{|Q_k \setminus Q_{k-1}|} \int_{Q_k \setminus Q_{k-1}} f(x)\, dx + \frac{1}{|Q_k|} \int_{Q_{k-1}} f(x)\, dx.$$

Now, using (5.47) and (5.48), we obtain

$$\frac{1}{|Q_k \setminus Q_{k-1}|} \int_{Q_k \setminus Q_{k-1}} f(x)\, dx \geq$$

$$\geq \frac{|Q_k|}{|Q_k \setminus Q_{k-1}|} \left(c_1 |Q_k|^{-\frac{1}{p}} - \frac{1}{|Q_k|} \int_{Q_{k-1}} f(x)\, dx \right) \geq$$

$$\geq \frac{|Q_k|}{|Q_k \setminus Q_{k-1}|} \left(c_1 |Q_k|^{-\frac{1}{p}} - \frac{c_1}{2} |Q_k|^{-\frac{1}{p}} \right) \geq \frac{c_1}{2} |Q_k|^{-\frac{1}{p}} \geq$$

$$\geq \frac{c_1}{2} \left(\frac{3}{2} \right)^{-\frac{1}{p}} |Q_k \setminus Q_{k-1}|^{-\frac{1}{p}} \equiv c_2 |Q_k \setminus Q_{k-1}|^{-\frac{1}{p}}.$$

Therefore, by the Hölder inequality,

$$\int_{\mathbb{R}^d} f^p(x)\, dx = \int_{Q_0} f^p(x)\, dx + \sum_{k=1}^{\infty} \int_{Q_k \setminus Q_{k-1}} f^p(x)\, dx =$$

$$= \int_{Q_0} f^p(x)\, dx + \sum_{k=1}^{\infty} |Q_k \setminus Q_{k-1}| \frac{1}{|Q_k \setminus Q_{k-1}|} \int_{Q_k \setminus Q_{k-1}} f^p(x)\, dx \geq$$

$$\geq \int_{Q_0} f^p(x)\,dx + \sum_{k=1}^{\infty} |Q_k \setminus Q_{k-1}| \left(\frac{1}{|Q_k \setminus Q_{k-1}|} \int_{Q_k \setminus Q_{k-1}} f(x)\,dx \right)^p \geq$$

$$\geq \int_{Q_0} f^p(x)\,dx + \sum_{k=1}^{\infty} |Q_k \setminus Q_{k-1}|\, c_2^p \, |Q_k \setminus Q_{k-1}|^{-1} = \infty,$$

and this completes the proof. □

5.1.1 One-Dimensional Case

Let us consider more in detail the case $d = 1$. If in the proof of Theorem 5.5 instead of Calderón–Zygmund lemma 5.6 we use "rising sun lemma" 1.16, which is sharper in the one-dimensional case, then we obtain the following statement.

Theorem 5.29 ([38]). *Let f be a non-negative function, summable on $I_0 \subset \mathbb{R}$. Then*

$$\frac{1}{t} \int_0^t |f^*(u) - f^{**}(t)|\; du \leq \nu(f;t) f^{**}(t), \quad 0 < t \leq |I_0|. \tag{5.49}$$

Proof. Essentially we will follow the proof of Theorem 5.5. Moreover, in the present case the calculations are even simpler.

Let us fix some t, $0 < t \leq |I_0|$, and apply Lemma 1.16 with $\alpha = f^{**}(t)$. As the result we obtain a family of pairwise disjoint intervals $I_j \subset I_0$, $j = 1, 2, \ldots,$ such that

$$\frac{1}{|I_j|} \int_{I_j} f(x)\,dx = \alpha, \tag{5.50}$$

$$f(x) \leq \alpha \quad \text{for almost all} \quad x \in I_0 \setminus E, \tag{5.51}$$

where $E = \cup_{j \geq 1} I_j$. Using Property 2.1, the definition of the rearrangement f^*, formulas (5.50), (5.51) and the monotonicity of $\nu(f;\sigma)$, we obtain

$$\int_0^t |f^*(u) - f^{**}(t)|\; du = 2 \int_{\{u:\; f^*(u) > \alpha\}} (f^*(u) - \alpha)\; du =$$

$$= 2 \int_{\{x \in I_0:\; f(x) > \alpha\}} (f(x) - \alpha)\; dx = 2 \int_{\{x \in I_0:\; f(x) > \alpha\} \cap E} (f(x) - \alpha)\; dx =$$

$$= 2 \sum_{j \geq 1} \int_{\{x \in I_j:\; f(x) > \alpha\}} (f(x) - \alpha)\; dx =$$

$$= 2 \sum_{j \geq 1} \int_{\{x \in I_j:\; f(x) > \alpha\}} \left(f(x) - f_{I_j}\right)\; dx =$$

$$= \sum_{j\geq 1} |I_j|\, \Omega\,(f; I_j) \leq \sum_{j\geq 1} \nu\,(f; |I_j|)\, |I_j|\, f_{I_j} \leq$$

$$\leq \nu(f; |E|) \sum_{j\geq 1} |I_j|\, f_{I_j} = \alpha \cdot |E| \cdot \nu(f; |E|). \tag{5.52}$$

On the other hand, (5.50) and the properties of the rearrangement imply

$$\frac{1}{t}\int_0^t f^*(u)\,du = f^{**}(t) = \alpha = \frac{1}{|E|}\int_E f(x)\,dx \leq \frac{1}{|E|}\int_0^{|E|} f^*(u)\,du,$$

so that $|E| \leq t$. Therefore the monotonicity of $\nu(f;\sigma)$ and (5.52) yield

$$\int_0^t |f^*(u) - f^{**}(t)|\;du \leq \alpha \cdot t \cdot \nu(f;t),$$

i.e. (5.49). □

We will proceed with the further detalization of the case $d = 1$ in the following two directions.

1). Refinement of Theorem 5.7 (i.e. sharpening of the constants in inequality (5.10)).

2). Refinement of Gurov–Reshetnyak Theorem 5.4.

The next theorem is the refined analog of Theorem 5.7 for the case $d = 1$.

Theorem 5.30 ([38]). *Let f be a non-negative function, summable on $I_0 \subset \mathbb{R}$. Then*

$$f^{**}(t) \leq c \cdot f_{I_0} \cdot \exp\left(\frac{\mathrm{e}}{2}\int_t^{|I_0|} \nu(f;\sigma)\,\frac{d\sigma}{\sigma}\right), \quad 0 < t \leq |I_0|\,, \tag{5.53}$$

where $c = \exp(1+\mathrm{e})$*, and in general the coefficient* $\mathrm{e}/2$ *is sharp.*

Proof. Let $a > 1$ (we will find the optimal value of this constant later). Applying Lemma 2.2 to the function $\varphi = f^*$ and using Theorem 5.29, for $0 < t \leq |I_0|$ we have

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \leq \frac{a}{2}\frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\;du \leq \frac{a}{2}\nu(f;t)f^{**}(t),$$

or, equivalently,

$$f^{**}\left(\frac{t}{a}\right) \leq \left(1 + \frac{a}{2}\nu(f;t)\right) f^{**}(t), \quad 0 < t \leq |I_0|\,. \tag{5.54}$$

Let us fix some $t \in \left(0, \frac{|I_0|}{a}\right]$ and denote $s = \left[\ln^{-1} a \cdot \ln\frac{|I_0|}{t}\right]$ (here the square brackets denote the integer part function). By (5.54),

$$f^{**}\left(\frac{t}{a}\right) \le f^{**}\left(\frac{|I_0|}{a}\right) \prod_{k=0}^{s} \left(1 + \frac{a}{2}\nu\left(f; a^k t\right)\right) \le$$

$$\le a \cdot f_{I_0} \prod_{k=0}^{s} \exp\left(\frac{a}{2}\nu\left(f; a^k\right)\right) = a \cdot f_{I_0} \exp\left(\frac{a}{2}\sum_{k=0}^{s} \nu\left(f; a^k t\right)\right). \tag{5.55}$$

On the other hand,

$$\nu\left(f; a^k t\right) \cdot \ln a \le \int_{a^k t}^{a^{k+1} t} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}, \quad k = 0, 1, \ldots, s-1,$$

provided $\nu(f;\sigma)$ is monotone. Hence, taking into account the inequality $\nu(f;|I_0|) \le 2$, from (5.55) we obtain

$$f^{**}\left(\frac{t}{a}\right) \le a \cdot f_{I_0} \exp\left(\frac{1}{2}\frac{a}{\ln a}\left(\sum_{k=0}^{s-1}\int_{a^k t}^{a^{k+1} t} \nu(f;\sigma)\, \frac{d\sigma}{\sigma} + \nu\left(f; |I_0|\right)\right)\right) \le$$

$$\le a \cdot \exp\left(\frac{a}{\ln a}\right) \cdot f_{I_0} \exp\left(\frac{1}{2}\frac{a}{\ln a}\int_t^{|I_0|} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}\right).$$

Since the function $\psi(a) \equiv a/\ln a$ for $a > 1$ achieves its minimal value at $a = \mathrm{e}$ we have

$$f^{**}\left(\frac{t}{\mathrm{e}}\right) \le c \cdot f_{I_0} \cdot \exp\left(\frac{\mathrm{e}}{2}\int_t^{|I_0|} \nu(f;\sigma)\, \frac{d\sigma}{\sigma}\right), \tag{5.56}$$

where $c = \exp(1 + \mathrm{e})$. This, together with the monotonicity of f^{**}, implies (5.53) for $0 < t \le \frac{|I_0|}{\mathrm{e}}$. If $t \in \left(\frac{|I_0|}{\mathrm{e}}, |I_0|\right]$, then (5.53) follows from (5.56) because $f^{**}(t) \le f^{**}\left(\frac{|I_0|}{\mathrm{e}}\right)$.

It remains to show that the coefficient $2/\mathrm{e}$ in the right-hand side of (5.53) cannot be decreased. For this let us consider the function $f_0(x) = \ln\frac{1}{x}$, $0 < x \le \beta_0$, where the constant $\beta_0 > 0$ is sufficiently small, we will define it later in Proposition 5.31. In addition, we will show there that

$$\nu(f_0;\sigma) = \frac{2/\mathrm{e}}{1 + \ln\frac{1}{\sigma}}, \quad 0 < \sigma \le \beta_0.$$

Thus if we put some constant $a < \frac{\mathrm{e}}{2}$ in the exponent in (5.53), then the right-hand side becomes

$$c \cdot (f_0)_{[0,\beta_0]} \cdot \exp\left(a\int_t^{\beta_0} \nu\left(f_0;\sigma\right)\, \frac{d\sigma}{\sigma}\right) =$$

$$= c\left(1 + \ln\frac{1}{\beta_0}\right)\exp\left(\frac{2a}{\mathrm{e}}\int_t^{\beta_0} \frac{d\sigma}{\sigma\left(1 + \ln\frac{1}{\sigma}\right)}\right) =$$

$$= c\left(1+\ln\frac{1}{\beta_0}\right)^{1-2a/\mathrm{e}}\cdot\left(1+\ln\frac{1}{t}\right)^{2a/\mathrm{e}} = \overline{o}\left(\ln\frac{1}{t}\right)^{2a/\mathrm{e}}, \quad t\to 0.$$

On the other hand, $f_0^{**}(t) = 1+\ln\frac{1}{t}$, $0<t\le\beta_0$. Comparing this equality with the previous one, we see that for $a<\frac{\mathrm{e}}{2}$ inequality (5.53) fails. □

Proposition 5.31. *For the function $f(x)=\ln\frac{1}{x}$, $0<x\le\beta_0$, where β_0 is a positive constant,*

$$\nu(f;\sigma) = \frac{2/\mathrm{e}}{1+\ln\frac{1}{\sigma}}, \quad 0<\sigma\le\beta_0.$$

Proof. Let us suppose that $\beta_0=\mathrm{e}^{-M}$, where the number $M>1$ is to be defined later. Further, let $0<\sigma\le\beta_0$ and $a\ge 0$ be such that $a+\sigma\le\beta_0$. Denote $I=[a,a+\sigma]$. Then

$$f_I = \frac{1}{\sigma}\int_a^{a+\sigma}\ln\frac{1}{x}\,dx = 1+\ln\frac{1}{a+\sigma}-\frac{a}{\sigma}\ln\left(1+\frac{\sigma}{a}\right).$$

Let $x_0=\mathrm{e}^{-f_I}$. Then $\ln\frac{1}{x_0}=f_I$ and

$$\Omega(f;I) = \frac{1}{|I|}\int_I |f(x)-f_I|\,dx = \frac{2}{\sigma}\int_a^{x_0}\left(\ln\frac{1}{x}-f_I\right)dx =$$

$$= 2\left(1+\frac{\sigma}{a}\right)\left[\exp\left(\frac{\ln(1+\sigma/a)}{\sigma/a}-1\right)-\frac{\ln(1+\sigma/a)}{\sigma/a}\right].$$

In order to proof the proposition, it is enough to show that

$$\frac{\Omega(f;I)}{f_I} \le \frac{2/\mathrm{e}}{1+\ln 1/\sigma} \equiv \frac{\Omega(f;[0,\sigma])}{f_{[0,\sigma]}}. \tag{5.57}$$

Let us denote

$$\alpha=\frac{\sigma}{a}, \quad \beta=\frac{1}{1+\ln 1/\sigma}.$$

Then the conditions $0<\sigma\le\beta_0\equiv\mathrm{e}^{-M}$ and $a+\sigma\le\beta_0=\mathrm{e}^{-M}$ become

$$0<\alpha<+\infty,$$

$$\beta \le \frac{1}{1+M+\ln\left(1+\frac{1}{\alpha}\right)} < \frac{1}{1+M}, \tag{5.58}$$

while inequality (5.57) can be rewritten in the following way

$$\frac{\mathrm{e}\left(1+\frac{1}{\alpha}\right)\left[\frac{\ln(1+\alpha)}{\alpha}-\exp\left(\frac{\ln(1+\alpha)}{\alpha}-1\right)\right]+1}{\frac{\ln(1+\alpha)}{\alpha}+\ln\left(1+\frac{1}{\alpha}\right)} \geq \beta. \tag{5.59}$$

Assume $0 < \alpha < 1$. Then, by virtue of (5.58), inequality (5.59) can be derived from the inequality

$$\frac{\mathrm{e}\left(1+\frac{1}{\alpha}\right)\left[\frac{\ln(1+\alpha)}{\alpha}-\exp\left(\frac{\ln(1+\alpha)}{\alpha}-1\right)\right]+1}{\frac{\ln(1+\alpha)}{\alpha}+\ln\left(1+\frac{1}{\alpha}\right)} \geq \frac{1}{1+M+\ln\left(1+\frac{1}{\alpha}\right)},$$

which is equivalent to the following one

$$\mathrm{e}\left(1+\frac{1}{\alpha}\right)\left[\frac{1}{\exp\left(1-\frac{\ln(1+\alpha)}{\alpha}\right)}-\frac{\ln(1+\alpha)}{\alpha}\right] \leq \frac{1+M-\frac{\ln(1+\alpha)}{\alpha}}{1+M+\ln\left(1+\frac{1}{\alpha}\right)}. \tag{5.60}$$

Let us prove (5.60). In order to find the upper bound for the left-hand side of (5.60), set $t \equiv 1-\frac{\ln(1+\alpha)}{\alpha}$. Notice that $0 < t < 1-\ln 2$. Moreover,

$$\frac{1}{\exp\left(1-\frac{\ln(1+\alpha)}{\alpha}\right)}-\frac{\ln(1+\alpha)}{\alpha} = \mathrm{e}^{-t}+t-1 \leq \frac{1}{2}t^2.$$

Hence we have the following bound for the left-hand side of (5.60) (we denote it by L)

$$L \leq \frac{\mathrm{e}}{2}\frac{\alpha+1}{\alpha}\left[1-\frac{\ln(1+\alpha)}{\alpha}\right]^2 = \frac{\mathrm{e}}{2}\frac{\alpha+1}{\alpha^3}\left[\alpha-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\alpha^k}{k}\right]^2 =$$

$$= \frac{\mathrm{e}}{2}\frac{\alpha+1}{\alpha^3}\left[\sum_{k=2}^{\infty}(-1)^k\frac{\alpha^k}{k}\right]^2 = \frac{\mathrm{e}}{2}(\alpha+1)\alpha\left[\sum_{k=2}^{\infty}(-1)^k\frac{\alpha^{k-2}}{k}\right]^2.$$

Since

$$\left|\sum_{k=2}^{\infty}(-1)^k\frac{\alpha^{k-2}}{k}\right| \leq \frac{1}{2}, \quad 0 < \alpha < 1$$

we see that

$$L \leq \frac{\mathrm{e}}{8}\alpha(\alpha+1) \leq \frac{\mathrm{e}\alpha}{4}.$$

The right-hand side of (5.60) can be estimate as follows

$$\frac{1+M-\frac{\ln(1+\alpha)}{\alpha}}{1+M+\ln\left(1+\frac{1}{\alpha}\right)} \geq \frac{M}{1+M+\ln\left(1+\frac{1}{\alpha}\right)}.$$

Therefore, (5.60) follows from the inequality

$$\frac{\mathrm{e}\alpha}{4} \leq \frac{M}{1 + M + \ln\left(1 + \frac{1}{\alpha}\right)},$$

or, which is the same, from

$$\ln\left(1 + \frac{1}{\alpha}\right) \leq \frac{1}{\alpha}\frac{4M}{\mathrm{e}} - M - 1. \tag{5.61}$$

Inequality (5.61) is valid for $\alpha = 1$ whenever $M \geq M_1$, where $M_1 \equiv (\ln 2 + 1) \,/\, \left(\frac{4}{\mathrm{e}} - 1\right)$. Otherwise, if $0 < \alpha < 1$, then (5.61) follows from the fact that the function

$$\varphi(\alpha) = \frac{1}{\alpha}\frac{4M}{\mathrm{e}} - M - 1 - \ln\left(1 + \frac{1}{\alpha}\right)$$

is decreasing on $(0,1)$. This can be easily checked by calculation of the derivative of φ.

In order to prove (5.59) in the case $\alpha \geq 1$ we rewrite it in the following form

$$\alpha\left(\exp\left(\frac{\ln(1+\alpha)}{\alpha}\right) - 1\right) \leq$$

$$\leq \ln(1+\alpha)\left[\mathrm{e}\left(1 + \frac{1}{\alpha}\right) - \beta\right] - \alpha\beta\ln\left(1 + \frac{1}{\alpha}\right) - \exp\left(\frac{\ln(1+\alpha)}{\alpha}\right). \tag{5.62}$$

Let us estimate the last two terms of the right-hand side. Denote $t = \frac{\ln(1+\alpha)}{\alpha}$. Then $0 < t \leq \ln 2$. Using the inequality $\mathrm{e}^t - 1 \leq \frac{t}{\ln 2}$, $0 < t \leq \ln 2$, we have

$$\mathrm{e}^t \leq 1 + \frac{t}{\ln 2} \iff -\exp\left(\frac{\ln(1+\alpha)}{\alpha}\right) \geq -1 - \frac{1}{\ln 2}\frac{\ln(1+\alpha)}{\alpha},$$

$$\alpha\ln\left(1 + \frac{1}{\alpha}\right) \leq 1 \iff -\beta\alpha\ln\left(1 + \frac{1}{\alpha}\right) \geq -\beta.$$

Therefore the right-hand side of (5.62) admits the following lower bound

$$\ln(1+\alpha)\left[\mathrm{e}\left(1 + \frac{1}{\alpha}\right) - \beta\right] - \alpha\beta\ln\left(1 + \frac{1}{\alpha}\right) - \exp\left(\frac{\ln(1+\alpha)}{\alpha}\right) \geq$$

$$\geq \ln(1+\alpha)\left[\mathrm{e}\left(1 + \frac{1}{\alpha}\right) - \beta\right] - \beta - 1 - \frac{1}{\ln 2}\frac{\ln(1+\alpha)}{\alpha}.$$

To estimate the left-hand side of (5.62) observe that

$$\mathrm{e}^t \leq 1 + \frac{t}{\ln 2}, \ 0 < t \leq \ln 2.$$

Hence

$$\alpha\left(\exp\left(\frac{\ln(1+\alpha)}{\alpha}\right)-1\right)\le\alpha\frac{1}{\ln 2}\frac{\ln(1+\alpha)}{\alpha}=\frac{\ln(1+\alpha)}{\alpha},\quad \alpha\ge 1.$$

So, in order to prove (5.62) it is enough to show that

$$\frac{\ln(1+\alpha)}{\ln 2}\le\ln(1+\alpha)\left[\mathrm{e}\left(1+\frac{1}{\alpha}\right)-\beta\right]-\beta-1-\frac{1}{\ln 2}\frac{\ln(1+\alpha)}{\alpha},$$

or, equivalently,

$$\ln(1+\alpha)\ge\frac{1+\beta}{\left(\mathrm{e}-\frac{1}{\ln 2}\right)\left(1+\frac{1}{\alpha}\right)-\beta}. \tag{5.63}$$

The proof of (5.63) splits into the following two cases.

1. $1\le\alpha\le 2$; in this case $\frac{3}{2}\le 1+\frac{1}{\alpha}\le 2,\ \ln(1+\alpha)\ge\ln 2$. The inequality

$$\ln 2>\frac{1}{\frac{3}{2}\left(\mathrm{e}-\frac{1}{\ln 2}\right)}$$

implies that for the sufficiently small β $\left(\beta\le\frac{1}{M_2+1}\right)$

$$\ln(1+\alpha)\ge\ln 2>\frac{1+\beta}{\frac{3}{2}\left(\mathrm{e}-\frac{1}{\ln 2}\right)-\beta}\ge\frac{1+\beta}{\left(\mathrm{e}-\frac{1}{\ln 2}\right)\left(1+\frac{1}{\alpha}\right)-\beta},$$

and so (5.63) follows.

2. $\alpha\ge 2$; in this case $1+\frac{1}{\alpha}\ge 1,\ \ln(1+\alpha)\ge\ln 3$. Since

$$\ln 3>\frac{1}{\mathrm{e}-\frac{1}{\ln 2}}$$

we have that for any sufficiently small β $\left(\beta\le\frac{1}{M_3+1}\right)$

$$\ln(1+\alpha)\ge\ln 3>\frac{1+\beta}{\mathrm{e}-\frac{1}{\ln 2}-\beta}\ge\frac{1+\beta}{\left(\mathrm{e}-\frac{1}{\ln 2}\right)\left(1+\frac{1}{\alpha}\right)-\beta},$$

and (5.63) follows in this case as well.

Setting $M=\max(M_1,M_2,M_3)$, we obtain (5.57). □

Let us come back to Gurov–Reshetnyak theorem 5.4. Theorem 5.30 has the following immediate corollary.

Corollary 5.32 ([38]). *Let f be a non-negative function on $I_0\subset\mathbb{R}$, satisfying condition* (5.1) *for some $\varepsilon<2/\mathrm{e}$. Then $f\in L^p(I_0)$ for any $p<p_0'$, where $p_0'\equiv p_0'(\varepsilon)=\frac{2}{\mathrm{e}}\cdot\frac{1}{\varepsilon}$.*

Remark 5.33. We have already mentioned (see Remark 5.11) that in Gurov-Reshetnyak theorem 5.4 the limiting exponent of summability of the function, satisfying Gurov–Reshetnyak condition (5.1), is equal to $p_0(\varepsilon,d)=\frac{\varepsilon_0(1)}{\varepsilon}$ (see (5.12)). Moreover, $p_0(\varepsilon,d)=\underline{O}\left(\frac{1}{\varepsilon}\right)$ as $\varepsilon\to 0+$, and this limiting behavior cannot be improved. In the case $d = 1$ Corollary 5.32 provides the bigger limiting exponent of summability: $p_0'(\varepsilon) = \frac{2}{\mathrm{e}} \cdot \frac{1}{\varepsilon} > p_0(\varepsilon, 1)$. The value $p_0'(\varepsilon)$ is the maximal possible also in the sense of equivalence. In what follows we will derive this fact as a corollary of Theorem 5.34. Moreover, Corollary 5.32 states that Gurov–Reshetnyak condition (5.1) assures the possibility to increase the exponent of summability of f only for $\varepsilon < \frac{2}{\mathrm{e}}$, and it leaves open the problem in the case $\frac{2}{\mathrm{e}} \leq \varepsilon < 2$, though the possibility of a certain increment of the exponent of summability for any $\varepsilon < 2$ is provided by Corollary 5.23.

Theorem 5.34 ([34]). *Let $\varepsilon,\ 0 < \varepsilon < 2$ be fixed, and let $p_0'' \equiv p_0''(\varepsilon) > 1$ be a root of the equation*

$$\frac{p^p}{(p-1)^{p-1}} = \frac{2}{\varepsilon}. \tag{5.64}$$

Then

(*i*) *if f is a non-negative function on $I_0 \subset \mathbb{R}$, satisfying Gurov–Reshetnyak condition* (5.1) *with the given ε, then*

$$f^{**}(t) \leq c \cdot f^{**}\left(|I_0|\right) \cdot \left(\frac{t}{|I_0|}\right)^{-1/p_0''}, \quad 0 < t \leq |I_0|, \tag{5.65}$$

where the constant c depends only on ε;

(*ii*) *there exists a function $f_0 \in L([0,1])$, satisfying* (5.1) *such that*

$$t^{1/p_0''} \cdot f_0^{**}(t) \geq c > 0, \quad 0 < t \leq 1. \tag{5.66}$$

Proof. Let us denote

$$\varphi(p) = \frac{p^p}{(p-1)^{p-1}}, \quad p > 1.$$

It is easy to see, that $\varphi'(p) > 0,\ \lim_{p\to 1+0} \varphi(p) = 1,\ \lim_{p\to+\infty} \varphi(p) = +\infty$, i.e. φ is continuous and increasing between 1 and $+\infty$. Hence for any $\varepsilon,\ 0 < \varepsilon < 2$, the equation (5.64) has a unique root $p_0'' = p_0''(\varepsilon) > 1$.

Let us prove (*i*). As condition (5.1) means that $\nu(f;t) \leq \varepsilon,\ 0 < t \leq |I_0|$, hence, by Theorem 5.29,

$$\frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\, du \leq \varepsilon \cdot f^{**}(t), \quad 0 < t \leq |I_0|.$$

Assume $a > 1$. According to Lemma 2.2,

$$f^{**}\left(\frac{t}{a}\right) - f^{**}(t) \leq \frac{a}{2} \cdot \frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\, du \leq \frac{a}{2} \cdot \varepsilon \cdot f^{**}(t),$$

so that

$$f^{**}\left(\frac{t}{a}\right) \le \left(\frac{a}{2}\cdot\varepsilon+1\right) f^{**}(t), \quad 0<t\le |I_0|. \tag{5.67}$$

Set

$$a=\left(\frac{p_0''}{p_0''-1}\right)^{p_0''} > 1.$$

Then, by (5.64), we have $\left(\frac{a}{2}\cdot\varepsilon+1\right)^{p_0''}=a$ and hence (5.67) takes the form

$$f^{**}\left(\frac{t}{a}\right) \le a^{1/p_0''}\cdot f^{**}(t), \quad 0<t\le |I_0|.$$

The successive application of this inequality leads to the following one

$$f^{**}\left(a^{-j}\,|I_0|\right) \le a^{j/p_0''}\cdot f^{**}\left(|I_0|\right), \quad j=1,2,\ldots \tag{5.68}$$

If $t,\ 0<t\le |I_0|$ is given, then we can choose j such that $a^{-j}|I_0|<t\le a^{-j+1}|I_0|$. Then (5.68) yields

$$f^{**}(t)\le f^{**}\left(a^{-j}\,|I_0|\right) \le \left(a^j\right)^{1/p_0''}\cdot f^{**}\left(|I_0|\right)\le$$

$$\le\left(a\frac{|I_0|}{t}\right)^{1/p_0''}\cdot f^{**}\left(|I_0|\right)=c\cdot f^{**}\left(|I_0|\right)\cdot\left(\frac{t}{|I_0|}\right)^{-1/p_0''}$$

with $c=a^{1/p_0''}=\frac{p_0''}{p_0''-1}$. Clearly, c depends only on ε.

In order to prove (ii), denote $q=p_0''$ and set $f_0(x)=x^{-1/q}+B$ with $B=\frac{q}{q-1}$. Then for $0<t\le 1$

$$t^{1/q}\cdot f_0^{**}(t)=t^{1/q}\cdot\frac{1}{t}\int_0^t f_0(x)\,dx=t^{1/q}\cdot\left(\frac{t^{-1/q}}{1-\frac{1}{q}}+B\right)\ge\frac{q}{q-1}\equiv c,$$

so that (5.66) holds true. It remains to show that the function f_0 satisfies condition (5.1).

Denote $g(x)=x^{-1/q},\ x>0$. Let $I\subset[0,1]$. If $(f_0)_I\le(f_0)_{[0,1]}$, then we choose $h,\ 0<h\le 1$, such that $(f_0)_I=(f_0)_{[0,h]}$. According to Property 2.15, the monotonicity of f_0 on $[0,h]$ implies $\Omega(f_0;I)\le\Omega(f_0;[0,h])$. Further, $\Omega(f_0;[0,h])=\Omega(g;[0,h])$. Since, as it was shown in Example 2.28,

$$\Omega(g;[0,h])=2h^{-1/q}\frac{(q-1)^{q-2}}{q^{q-1}}$$

we have

$$\frac{\Omega\left(f_0;I\right)}{(f_0)_I}\le\frac{\Omega(g;[0,h])}{(f_0)_{[0,h]}}=\frac{2h^{-1/q}\cdot\frac{(q-1)^{q-2}}{q^{q-1}}}{h^{-1/q}\cdot\frac{q}{q-1}+B}\le 2\cdot\frac{(q-1)^{q-1}}{q^q}=\varepsilon,$$

where the last equality follows from (5.64). In the case $(f_0)_I > (f_0)_{[0,1]}$ one can find $h > 1$ such that $(f_0)_I = (f_0)_{[0,h]}$. Then the same arguments lead to the inequality

$$\Omega\,(f_0; I) = \Omega(g; I) \le \Omega(g; [0,h]) = 2h^{-1/q} \cdot \frac{(q-1)^{q-2}}{q^{q-1}} \le$$

$$\le 2 \cdot \frac{(q-1)^{q-1}}{q^q} \cdot \frac{q}{q-1} = \varepsilon \cdot B \le \varepsilon \cdot (f_0)_I\,.$$

Therefore, (5.1) is true for all intervals $I \subset [0,1]$. □

Using the same arguments as in the proof of Proposition 5.9, from part (i) of Theorem 5.34 we immediately obtain the following *one-dimensional Gurov–Reshetnyak theorem* 5.4 *with exact limiting exponent of summability.*

Corollary 5.35 ([34]). *Let f be a non-negative function on $I_0 \subset \mathbb{R}$, satisfying the condition*

$$\Omega(f; I) \le \varepsilon \cdot f_I, \quad I \subset I_0,$$

for some $\varepsilon < 2$. Then f satisfies the Gehring inequality

$$\left\{\frac{1}{|I|}\int_I f^p(x)\,dx\right\}^{1/p} \le c \cdot \frac{1}{|I|}\int_I f(x)\,dx, \quad I \subset I_0,$$

for any $p < p_0''$, where $p_0'' = p_0''(\varepsilon) > 1$ is a root of equation (5.64)*, and the constant $c \equiv c\,(\varepsilon, p)$ depends only on ε and p (for example, one can take $c = \frac{(p_0'')^{1+1/p}}{(p_0''-1)(p_0''-p)^{1/p}}$).*

On the other hand, part (ii) of Theorem 5.34 shows, that in general the limiting exponent p_0'' in Corollary 5.35 cannot be increased.

Remark 5.36. It is easy to see that the root $p_0''(\varepsilon)$ of equation (5.64) satisfies the following relations:

$$p_0''(\varepsilon) > \frac{2}{\mathrm{e}} \cdot \frac{1}{\varepsilon} \quad \text{and} \quad p_0''(\varepsilon) \sim \frac{2}{\mathrm{e}} \cdot \frac{1}{\varepsilon}, \quad \varepsilon \to 0+\,.$$

Therefore Corollary 5.35 revises Corollary 5.32. On the other hand, since the value $p_0''(\varepsilon)$ is the maximal possible in Corollary 5.35 it follows that, as we mentioned in Remark 5.33, the exponent of summability $p_0'(\varepsilon) = \frac{2}{\mathrm{e}} \cdot \frac{1}{\varepsilon}$, obtained in Corollary 5.32, is equivalent to the maximal exponent $p_0''(\varepsilon)$ as $\varepsilon \to 0+$.

5.1.2 Anisotropic Case

Let us come back to the multidimensional case. By analogy with BMO^R-class, let us define *the anisotropic Gurov–Reshetnyak class* $GR^R \equiv GR^R(\varepsilon) \equiv GR^R(\varepsilon, R_0)$ as a class of all functions f that are non-negative on the segment $R_0 \subset \mathbb{R}^d$ and satisfy the Gurov–Reshetnyak condition

$$\Omega(f;R) \leq \varepsilon \cdot f_R, \quad R \subset R_0, \tag{5.69}$$

where the constant ε, $0 < \varepsilon < 2$, does not depend on R. Clearly, $GR^R(\varepsilon) \subset GR(\varepsilon)$ for any $\varepsilon \in (0,2)$. The following example shows that in general the opposite inclusion is not true.

Example 5.37. For a given $\varepsilon \in (0,2)$ let us construct a function from the Gurov–Reshetnyak class $GR(\varepsilon)$, which does not belong $GR^R(\varepsilon_1)$ for any $\varepsilon_1 < 2$.

As in Example 2.32, we set

$$f(x) = \sum_{k=1}^{\infty} \chi_{[0,2^{-k+1}]\times[0,\frac{1}{k}]}(x), \quad x \equiv (x_1,x_2) \in [0,1]^2 \equiv Q_0.$$

In Example 2.32 it was already shown that $f \in BMO\,(Q_0)$. Therefore there exists B such that $\Omega(f;Q) \leq B$ for any cube $Q \subset Q_0$. Choose some $\varepsilon \in (0,2)$ and set $g(x) = f(x) + \frac{B}{\varepsilon}$, $x \in Q_0$. Then

$$\frac{\Omega(g;Q)}{g_Q} = \frac{\Omega(f;Q)}{f_Q + \frac{B}{\varepsilon}} \leq \frac{B}{f_Q + \frac{B}{\varepsilon}} \leq \varepsilon, \quad Q \subset Q_0,$$

so that the function g satisfies Gurov–Reshetnyak condition (5.1), i.e. $g \in GR\,(\varepsilon, Q_0)$.

On the other hand, let us show that $g \notin GR^R(\varepsilon_1)$ for any $\varepsilon_1 < 2$. For this we will use the following inequalities obtained in Example 2.32 ($k \geq 100$):

$$L_k \equiv [\ln(k+1)] \leq \ln(k+1) \leq f_{R_k} = \sum_{s=1}^{k} \frac{1}{s} + \sum_{s=1}^{\infty} \frac{1}{s+k} \cdot 2^{-s} \leq 2 + \ln k,$$

$$\Omega\,(f;R_k) \geq 2L_k - 2 - 2\ln\,(L_k+1)\,.$$

Here $R_k = \left[0, 2^{-k+1}\right] \times [0,1]$. Thus

$$\frac{\Omega\,(g;R_k)}{g_{R_k}} = \frac{\Omega\,(f;R_k)}{f_{R_k} + \frac{B}{\varepsilon}} \geq \frac{2L_k - 2 - 2\ln\,(L_k+1)}{2 + \ln k + \frac{B}{\varepsilon}} \sim \frac{2\ln k - 2\ln\ln k}{\ln k} \ \to \ 2$$

as $k \to \infty$. Hence the function g does not belong to the Gurov–Reshetnyak class $GR^R(\varepsilon_1)$ for any $\varepsilon_1 < 2$. □

Let $R_0 \subset \mathbb{R}^d$ be a segment, and let f be a non-negative function on R_0. Define

$$\nu^R(f;\sigma) = \sup_{|R|\leq\sigma} \frac{\Omega(f;R)}{f_R}, \quad 0 < \sigma \leq |R_0|,$$

where the supremum is taken over all segments $R \subset R_0$ of measure smaller than σ. As before, if $f_R = 0$, then we assume $\frac{\Omega(f;R)}{f_R} = 0$.

Theorem 5.38 ([43]). *Let f be a non-negative function, summable on the segment $R_0 \subset \mathbb{R}^d$. Then*

$$\frac{1}{t}\int_0^t |f^*(u) - f^{**}(t)|\, du \le \nu^R(f;t)\cdot f^{**}(t), \quad 0 < t \le |R_0|. \tag{5.70}$$

Proof. We will follow the same way as in the proof of Theorem 5.29. Fix $t,\ 0 < t \le |R_0|$, and apply Lemma 1.30 with $\alpha = f^{**}(t)$. Then we obtain at most countable family of segments $R_j \subset R_0,\ j = 1, 2, \dots$ such that

$$\frac{1}{|R_j|}\int_{R_j} f(x)\, dx = \alpha, \tag{5.71}$$

$$f(x) \le \alpha \quad \text{for almost all} \quad x \in R_0 \setminus E, \tag{5.72}$$

where $E = \cup_{j\ge 1} R_j$. Hence

$$\int_0^t |f^*(u) - f^{**}(t)|\, du = 2\int_{\{u:\ f^*(u)>\alpha\}} (f^*(u) - \alpha)\, du =$$

$$= 2\int_{\{x\in R_0:\ f(x)>\alpha\}} (f(x) - \alpha)\, dx = 2\int_{\{x\in R_0:\ f(x)>\alpha\}\cap E} (f(x) - \alpha)\, dx =$$

$$= 2\sum_{j\ge 1}\int_{\{x\in R_j:\ f(x)>\alpha\}} (f(x) - \alpha)\, dx =$$

$$= 2\sum_{j\ge 1}\int_{\{x\in R_j:\ f(x)>\alpha\}} \left(f(x) - f_{R_j}\right)\, dx =$$

$$= \sum_{j\ge 1} |R_j|\, \Omega\,(f;R_j) \le \sum_{j\ge 1} \nu^R\,(f;|R_j|)\,|R_j|\cdot f_{R_j} \le$$

$$\le \nu^R\,(f;|E|)\sum_{j\ge 1} |R_j|\cdot f_{R_j} = \alpha\cdot |E|\cdot \nu^R(f;|E|).$$

But (5.71) implies

$$\frac{1}{t}\int_0^t f^*(u)\, du = f^{**}(t) = \alpha = \frac{1}{|E|}\int_E f(x)\, dx \le \frac{1}{|E|}\int_0^{|E|} f^*(u)\, du, \tag{5.73}$$

so that $|E| \le t$. Therefore, using the monotonicity of $\nu^R(f;\sigma)$, from (5.73) we get

$$\int_0^t |f^*(u) - f^{**}(t)|\, du \le \alpha\cdot t\cdot \nu^R(f;t). \quad \square$$

Theorem 5.39 ([43]). *Let* $\varepsilon,\ 0 < \varepsilon < 2$ *be given, and assume that* $p_0'' \equiv p_0''(\varepsilon) > 1$ *is a root of equation* (5.64). *Then* *(i) if f is non-negative on the segment $R_0 \subset \mathbb{R}^d$ and satisfies Gurov–Reshetnyak condition* (5.69) *with the given ε, then*

$$f^{**}(t) \le c \cdot f^{**}(|R_0|) \cdot \left(\frac{t}{|R_0|} \right)^{-1/p_0''}, \quad 0 < t \le |R_0|, \tag{5.74}$$

where the constant c depends only on ε;

(ii) there exists a function $f_0 \in L\left([0,1]^d\right)$, satisfying (5.69) *such that*

$$t^{1/p_0''} \cdot f_0^{**}(t) \ge c > 0, \quad 0 < t \le 1.$$

Proof. For the prof of (i) we will use Theorem 5.38. Condition (5.69) implies that $\nu^R(f;t) \le \varepsilon,\ 0 < t \le |R_0|$, and so (5.70) becomes

$$\frac{1}{t} \int_0^t |f^*(u) - f^{**}(t)|\, du \le \varepsilon \cdot f^{**}(t), \quad 0 < t \le |R_0|.$$

Further, as in the proof of Theorem 5.34, for $a = \left(p_0''/\left(p_0''-1\right)\right)^{p_0''} > 1$

$$f^{**}\left(\frac{t}{a}\right) \le \left(\frac{a}{2} \cdot \varepsilon + 1\right) f^{**}(t), \quad 0 < t \le |R_0|.$$

Now the same arguments as in the proof of Theorem 5.34 lead to (5.74).

Part (ii) can be proved in the same way as part (ii) of Theorem 5.34. Indeed, it is enough to consider the function

$$f_0\left(x_1, \ldots, x_d\right) = x_1^{-1/p_0''} + \frac{p_0''}{p_0''-1}, \quad \left(x_1, \ldots, x_d\right) \in [0,1]^d. \quad \square$$

Let $d \ge 1$. Fix some segment $R_0 \subset \mathbb{R}^d$. The application of Theorem 5.39 to an arbitrary segment $R \subset R_0$ immediately leads to the following *multidimensional analog of the Gurov–Reshetnyak theorem with exact limiting exponent of summability.*

Corollary 5.40. *Let f be a non-negative function on the segment $R_0 \subset \mathbb{R}^d$ such that*

$$\Omega(f;R) \le \varepsilon \cdot f_R, \quad R \subset R_0,$$

for some $\varepsilon < 2$. Then

(i) f satisfies the Gehring inequality

$$\left\{ \frac{1}{|R|} \int_R f^p(x)\, dx \right\}^{1/p} \le c \cdot \frac{1}{|R|} \int_R f(x)\, dx, \quad R \subset R_0,$$

for any $p < p_0''$, where $p_0''(\varepsilon) > 1$ is a root of equation (5.64) *and the constant $c = c(\varepsilon, p)$ depends only on ε and p;*

(ii) the value of p_0'' in (i) cannot be increased.

5.2 Embedding in the Muckenhoupt Class

By analogy with the Gehring class, let us consider the class $A_q \equiv A_q(C)$ of the non-negative functions f, satisfying *the reverse Hölder inequality* with the negative exponent

$$\frac{1}{|Q|}\int_Q f(x)\,dx \left\{\frac{1}{|Q|}\int_Q f^{-1/(q-1)}(x)\,dx\right\}^{q-1} \le C, \quad Q \subset Q_0. \tag{5.75}$$

Here the cube $Q_0 \subset \mathbb{R}^d$ is fixed, and the constants $q,\ C > 1$ do not depend on the cube $Q \subset Q_0$. Condition (5.75) is called *the A_q–Muckenhoupt condition* ([59]). The classes of Muckenhoupt functions are closely related to Gehring classes. Namely, every Gehring class is contained in some Muckenhoupt class and vice versa (see [8, 72]). In Section 5.1 we saw that Gehring condition (5.14) is equivalent to Gurov–Reshetnyak condition (5.1). Therefore Muckenhoupt condition (5.75) is also equivalent to Gurov–Reshetnyak condition (5.1). Here we will give the direct proof of this fact. First we prove that every Muckenhoupt class is contained in some Gurov–Reshetnyak class.

Theorem 5.41 ([43]). *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Muckenhoupt condition* (5.75) *for some $q, C > 1$. Then f belongs to the Gurov–Reshetnyak class $GR(\varepsilon)$ with $\varepsilon = 2\left(1-(qC)^{-1}\right),\ 0 < \varepsilon < 2$.*

Proof. Fix some cube $Q \subset Q_0$. Due to condition (5.75), for $0 < u \le 1$ we have

$$f_Q \le C\left\{\frac{1}{|Q|}\int_Q f^{-1/(q-1)}(x)\,dx\right\}^{-(q-1)} =$$

$$= C\left\{\frac{1}{|Q|}\int_0^{|Q|} (f\chi_Q)_*^{-1/(q-1)}(t)\,dt\right\}^{-(q-1)} \le$$

$$\le C\left\{\frac{1}{|Q|}\int_0^{u|Q|} (f\chi_Q)_*^{-1/(q-1)}(t)\,dt\right\}^{-(q-1)} \le C\,(f\chi_Q)_*\,(u|Q|)\cdot u^{-(q-1)}.$$

Thus,

$$(f\chi_Q)_*\,(t) \ge \frac{1}{C} f_Q \left(\frac{t}{|Q|}\right)^{q-1}, \quad 0 < t \le |Q|.$$

Therefore, according to Property 2.1,

$$\Omega(f;Q) = \frac{2}{|Q|}\int_{\{x\in Q:\ f(x)<f_Q\}} (f_Q - f(x))\,dx =$$

$$= \frac{2}{|Q|}\int_{\left\{t\in(0,|Q|):\ (f\chi_Q)_*(t)<f_Q\right\}} \left(f_Q - (f\chi_Q)_*\,(t)\right)\,dt \le$$

$$\leq \frac{2}{|Q|} f_Q \int_0^{|Q|} \left(1 - \frac{1}{C}\left(\frac{t}{|Q|}\right)^{q-1}\right) dt = 2\left(1 - \frac{1}{qC}\right) f_Q.$$

This means that $f \in GR(\varepsilon)$, where $\varepsilon = 2\left(1 - (qC)^{-1}\right) < 2$. □

Now let us prove that every Gehring class is contained in some Muckenhoupt class. This fact, joined to Theorem 5.41 and the results of Section 5.1, will complete the proof of the equivalence of the Gurov–Reshetnyak and Muckenhoupt classes.

Theorem 5.42 (Coifman, Fefferman, [8]). *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying the Gehring condition*

$$\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p} \leq B \cdot \frac{1}{|Q|}\int_Q f(x)\,dx, \quad Q \subset Q_0 \tag{5.76}$$

for some $p,\ B > 1$. Then there exist $q,\ C > 1$, which depend only on $p,\ B$ and d such that Muckenhoupt inequality (5.75) *holds true.*

In [8] this theorem was obtain as a consequence of a series of propositions. Here we reconstruct the original proof from [8]. First we need some auxiliary lemmas (see [8]).

Lemma 5.43. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Gehring condition* (5.76). *Then for every $\theta,\ 0 < \theta < 1$, there exists $\sigma,\ 0 < \sigma < 1$ such that for any cube $Q \subset Q_0$ and for any measurable subset $E \subset Q$ the condition $\frac{|E|}{|Q|} \geq 1 - \sigma$ implies*

$$\frac{\int_E f(x)\,dx}{\int_Q f(x)\,dx} \geq 1 - \theta. \tag{5.77}$$

Proof. Let $\theta,\ 0 < \theta < 1$ and set $\sigma = \left(\frac{\theta}{B}\right)^{p/(p-1)}$, $0 < \sigma < 1$. Let $E_1 \subset Q$ be a measurable set such that $\frac{|E_1|}{|Q|} < \sigma$. Then, by the Hölder inequality, from (5.76) we obtain

$$\frac{1}{|Q|}\int_{E_1} f(x)\,dx = \frac{1}{|Q|}\int_Q f(x)\chi_{E_1}(x)\,dx \leq$$

$$\leq \left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p} \left\{\frac{1}{|Q|}\int_Q \chi_{E_1}^{p/(p-1)}(x)\,dx\right\}^{(p-1)/p} \leq$$

$$\leq B\frac{1}{|Q|}\int_Q f(x)\,dx \left(\frac{|E_1|}{|Q|}\right)^{(p-1)/p} \leq \theta\frac{1}{|Q|}\int_Q f(x)\,dx.$$

So,

$$\frac{|E_1|}{|Q|} < \left(\frac{\theta}{B}\right)^{p/(p-1)} \quad \Longrightarrow \quad \frac{\int_{E_1} f(x)\,dx}{\int_Q f(x)\,dx} \leq \theta. \tag{5.78}$$

Now consider the measurable set $E \subset Q,\ |E| \geq \left(1 - \left(\frac{\theta}{B}\right)^{p/(p-1)}\right)|Q|$. Denote $E_1 = Q \setminus E$. Then

$$\frac{|E_1|}{|Q|} = 1 - \frac{|E|}{|Q|} \leq \left(\frac{\theta}{B}\right)^{p/(p-1)},$$

so that, by virtue of (5.78),

$$\frac{\int_E f(x)\,dx}{\int_Q f(x)\,dx} = 1 - \frac{\int_{E_1} f(x)\,dx}{\int_Q f(x)\,dx} \geq 1 - \theta. \quad \square$$

Lemma 5.44. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Gehring condition (5.76). Then for any $\theta,\ 0 < \theta < 1$, and for any cube $Q \subset Q_0$*

$$\int_Q f(x)\,dx \leq \frac{1}{1-\theta} \int_{\left\{x \in Q:\ f(x) \leq \frac{B^{p/(p-1)}}{\theta^{1/(p-1)}} f_Q\right\}} f(x)\,dx. \tag{5.79}$$

Proof. For σ, defined in the previous lemma, set

$$\beta = \frac{\sigma^{1/p}}{B} = \frac{\left(\frac{\theta}{B}\right)^{1/(p-1)}}{B} = \frac{\theta^{1/(p-1)}}{B^{p/(p-1)}}$$

and denote

$$E' = \left\{x \in Q:\ f(x) > \frac{f_Q}{\beta}\right\}.$$

Then condition (5.76) implies

$$\frac{1}{\beta}\left(\frac{|E'|}{|Q|}\right)^{1/p} = \frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_{E'}\left(\beta\frac{1}{f_Q}\right)^{-p} dx\right\}^{1/p} \leq$$

$$\leq \frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_{E'}\left(\frac{1}{f(x)}\right)^{-p} dx\right\}^{1/p} \leq \frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_Q\left(\frac{1}{f(x)}\right)^{-p} dx\right\}^{1/p} =$$

$$= \frac{1}{f_Q}\left\{\frac{1}{|Q|}\int_Q f^p(x)\,dx\right\}^{1/p} \leq B,$$

i.e.,

$$\frac{|E'|}{|Q|} \leq (\beta B)^p = \sigma.$$

Denote $E = \left\{x \in Q:\ f(x) \leq \frac{f_Q}{\beta}\right\}$. Then $\frac{|E|}{|Q|} = 1 - \frac{|E'|}{|Q|} \geq 1 - \sigma$ and from (5.77) we obtain

$$\frac{\int_E f(x)\,dx}{\int_Q f(x)\,dx} \geq 1-\theta,$$

which is equivalent to (5.79). $\square$

Lemma 5.45. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Gehring condition* (5.76)*. Then for any θ, $0<\theta<1$, and for any cube $Q_1 \subset Q \subset Q_0$ such that $|Q_1| = t|Q|$, $0<t<1$,*

$$\int_{Q_1} f(x)\,dx \geq (1-\theta)^{\left(\ln\frac{1}{t}\right)/\left(\ln\frac{1}{1-\sigma}\right)+1} \int_Q f(x)\,dx, \tag{5.80}$$

where σ is defined in Lemma 5.43.

Proof. Consider the cubes $Q_1 \subset Q \subset Q_0$, $|Q_1| = t|Q|$, $0<t<1$. Let us construct the cubes $Q_1 \subset Q_2 \subset \cdots \subset Q_k \subset Q$ such that $|Q_i| \geq (1-\sigma)\,|Q_{i+1}|$, $i = 1, \ldots, k-1$, $|Q_k| \geq (1-\sigma)|Q|$. Then

$$|Q| \leq \frac{1}{1-\sigma}\,|Q_k| \leq \left(\frac{1}{1-\sigma}\right)^2 |Q_{k-1}| \leq \cdots \leq \left(\frac{1}{1-\sigma}\right)^k |Q_1|,$$

where k is chosen in such a way that $(1-\sigma)^{k+1} < t \leq (1-\sigma)^k$, i.e.

$$k \leq \frac{\ln t}{\ln(1-\sigma)} < k+1,$$

$$k = \left[\frac{\ln\frac{1}{t}}{\ln\frac{1}{1-\sigma}}\right] + 1.$$

Then, by Lemma 5.43,

$$\int_{Q_1} f(x)\,dx \geq (1-\theta)\int_{Q_2} f(x)\,dx \geq \cdots \geq (1-\theta)^k \int_Q f(x)\,dx,$$

i.e.,

$$\int_{Q_1} f(x)\,dx \geq (1-\theta)^{\left(\ln\frac{1}{t}\right)/\left(\ln\frac{1}{1-\sigma}\right)+1} \int_Q f(x)\,dx. \quad \square$$

The next lemma is similar to Calderón–Zygmund lemma 1.14. But unlikely the Calderón–Zygmund lemma, the Gehring condition in this case is essential.

Lemma 5.46. *Let f be a non-negative function on the cube $Q_0 \subset \mathbb{R}^d$, satisfying Gehring condition* (5.76)*. Then for any cube $Q \subset Q_0$ and any $\lambda > \frac{1}{f_Q}$ there exists a collection of cubes $Q_j \subset Q$, $j = 1, 2, \ldots$, with pairwise disjoint interiors such that*

$$\delta\frac{1}{\lambda} \leq \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \leq \frac{1}{\lambda}, \tag{5.81}$$

$$f(x) \geq \frac{1}{\lambda} \quad \text{for almost all} \quad x \in Q_0 \setminus \left(\bigcup_{j\geq 1} Q_j\right), \tag{5.82}$$

where $\delta = \delta(B, \theta, d) > 0$.

Proof. Fix some cube $Q \subset Q_0$. Let us partition Q into 2^d congruent cubes, dividing in halves each side of the cube Q. Assume that Q' is one of the obtained cubes. If $f_{Q'} > \frac{1}{\lambda}$, then we partition the cube Q' again in the next step. Otherwise, if $f_{Q'} \le \frac{1}{\lambda}$, then we assign to Q' the next number j. Then, taking into account the equality $\frac{|Q'|}{|Q|} = 2^{-d}$ and (5.80), we have

$$\frac{1}{|Q'|}\int_{Q'} f(x)\,dx = 2^d \frac{1}{|Q|}\int_{Q'} f(x)\,dx \ge$$

$$\ge 2^d\,(1-\theta)^{\left(\ln 2^d\right)/\left(\ln \frac{1}{1-\sigma}\right)+1}\,\frac{1}{|Q|}\int_Q f(x)\,dx > \delta\frac{1}{\lambda},$$

where $\delta = 2^d\,(1-\theta)^{\left(\ln 2^d\right)/\left(\ln \frac{1}{1-\sigma}\right)+1}$ and σ is defined in Lemma 5.43. This means that the left inequality of (5.81) holds true. Sorting out all cubes Q' in this way, we pass to the next step.

As the result of the described process we obtain a collection of cubes Q_j with pairwise disjoint interiors, which satisfy (5.81). Let $x \in Q \setminus \left(\bigcup_{j\ge1} Q_j\right)$. Then one can choose a sequence of cubes $\overline{Q}_i$, contractible to x such that $f_{\overline{Q}_i} > \frac{1}{\lambda}$. Then (5.82) follows from Lebesgue theorem 1.1. □

Proof of Theorem 5.42. Fix an arbitrary θ, $0 < \theta < 1$. Let $Q \subset Q_0$, and assume that δ is as defined in Lemma 5.46 and $\lambda > \frac{1}{f_Q}$. Now we apply successively condition (5.82), the left inequality of (5.81), condition (5.79) and the right inequality of (5.81). Then

$$\left|\left\{x \in Q:\ f(x) < \frac{1}{\lambda}\right\}\right| \le \sum_{j\ge1}|Q_j| \le \frac{1}{\delta}\lambda\sum_{j\ge1}\int_{Q_j} f(x)\,dx \le$$

$$\le \frac{1}{\delta}\frac{1}{1-\theta}\lambda\sum_{j\ge1}\int_{\left\{x\in Q_j:\ f(x)<\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}f_{Q_j}\right\}} f(x)\,dx \le$$

$$\le \frac{1}{\delta}\frac{1}{1-\theta}\lambda\sum_{j\ge1}\int_{\left\{x\in Q_j:\ f(x)<\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\frac{1}{\lambda}\right\}} f(x)\,dx \le$$

$$\le \frac{1}{\delta}\frac{1}{1-\theta}\lambda\int_{\left\{x\in Q:\ f(x)<\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\frac{1}{\lambda}\right\}} f(x)\,dx, \tag{5.83}$$

provided the interiors of the cubes Q_j, obtained in Lemma 5.46, are pairwise disjoint. Now let $0 < \varepsilon < 1$ (we will choose it later). Then (5.83) yields

$$\int_{1/f_Q}^{\infty}\lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)} > \lambda\right\}\right|\,d\lambda \le$$

$$\le \frac{1}{\delta}\frac{1}{1-\theta}\int_{1/f_Q}^{\infty}\lambda^{\varepsilon}\int_{\left\{x\in Q:\ f(x)<\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\frac{1}{\lambda}\right\}} f(x)\,dx\,d\lambda \le$$

$$\leq \frac{1}{\delta}\frac{1}{1-\theta}\int_0^\infty \lambda^\varepsilon \int_{\left\{x\in Q:\ \frac{1}{f(x)} > \frac{\theta^{1/(p-1)}}{B^{p/(p-1)}}\lambda\right\}} f(x)\,dx\,d\lambda =$$

$$= \frac{1}{\delta}\frac{1}{1-\theta}\left(\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\right)^{1+\varepsilon}\int_0^\infty u^\varepsilon \int_{\left\{x\in Q:\ \frac{1}{f(x)} > u\right\}} f(x)\,dx\,du. \tag{5.84}$$

Applying the Fubini theorem to the integral in the right-hand side, we get

$$\int_0^\infty u^\varepsilon \int_{\left\{x\in Q:\ \frac{1}{f(x)} > u\right\}} f(x)\,dx\,du = \int_Q f(x)\int_0^{1/f(x)} u^\varepsilon\,du\,dx =$$

$$= \frac{1}{1+\varepsilon}\int_Q f(x)\left(\frac{1}{f(x)}\right)^{1+\varepsilon}dx = \frac{1}{1+\varepsilon}\int_Q f^{-\varepsilon}(x)\,dx.$$

Therefore, (5.84) implies

$$\int_{1/f_Q}^\infty \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)} > \lambda\right\}\right| d\lambda \leq$$

$$\leq \frac{1}{\delta}\frac{1}{1-\theta}\left(\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}}\right)^{1+\varepsilon}\frac{1}{1+\varepsilon}\int_Q f^{-\varepsilon}(x)\,dx. \tag{5.85}$$

Transforming the left-hand side of (5.85) we get

$$\int_{1/f_Q}^\infty \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)} > \lambda\right\}\right| d\lambda =$$

$$= \int_0^\infty \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)} > \lambda\right\}\right| d\lambda -$$

$$- \int_0^{1/f_Q} \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)} > \lambda\right\}\right| d\lambda =$$

$$= \frac{1}{\varepsilon}\int_Q f^{-\varepsilon}(x)\,dx - \int_0^{1/f_Q} \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)} > \lambda\right\}\right| d\lambda. \tag{5.86}$$

The last integral can be estimated as follows:

$$\int_0^{1/f_Q} \lambda^{\varepsilon-1}\left|\left\{x\in Q:\ \frac{1}{f(x)} > \lambda\right\}\right| d\lambda \leq$$

$$\leq |Q|\int_0^{1/f_Q} \lambda^{\varepsilon-1}\,d\lambda = \frac{|Q|}{\varepsilon}\left(\frac{1}{f_Q}\right)^\varepsilon = \frac{|Q|}{\varepsilon}\,(f_Q)^{-\varepsilon},$$

so that (5.86) becomes

$$\int_{1/f_Q}^{\infty} \lambda^{\varepsilon-1} \left| \left\{ x \in Q : \frac{1}{f(x)} > \lambda \right\} \right| d\lambda \geq \frac{1}{\varepsilon} \int_Q f^{-\varepsilon}(x)\, dx - \frac{|Q|}{\varepsilon} (f_Q)^{-\varepsilon}.$$

Substitution of this estimate into (5.85) gives

$$\left(\frac{1}{\varepsilon} - \frac{1}{\delta} \frac{1}{1-\theta} \left(\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}} \right)^{1+\varepsilon} \frac{1}{1+\varepsilon} \right) \int_Q f^{-\varepsilon}(x)\, dx \leq \frac{|Q|}{\varepsilon} (f_Q)^{-\varepsilon}, \quad (5.87)$$

where $\varepsilon > 0$ is so small, that

$$c_1 \equiv \frac{1}{\varepsilon} - \frac{1}{\delta} \frac{1}{1-\theta} \left(\frac{B^{p/(p-1)}}{\theta^{1/(p-1)}} \right)^{1+\varepsilon} \frac{1}{1+\varepsilon} > 0.$$

From (5.87) it follows that

$$\frac{1}{|Q|} \int_Q f^{-\varepsilon}(x)\, dx \leq \frac{1}{c_1 \varepsilon} \left\{ \frac{1}{|Q|} \int_Q f(x)\, dx \right\}^{-\varepsilon}. \quad (5.88)$$

Setting $\varepsilon = \frac{1}{q-1}$, i.e. $q = 1 + \frac{1}{\varepsilon}$, we rewrite (5.88) in the form

$$\frac{1}{|Q|} \int_Q f^{-1/(q-1)}(x)\, dx \leq \frac{1}{c_1 \varepsilon} \left\{ \frac{1}{|Q|} \int_Q f(x)\, dx \right\}^{-1/(q-1)},$$

or, equivalently,

$$\frac{1}{|Q|} \int_Q f(x)\, dx \left\{ \frac{1}{|Q|} \int_Q f^{-1/(q-1)}(x)\, dx \right\}^{q-1} \leq \left(\frac{1}{c_1 \varepsilon} \right)^{q-1} \equiv C,$$

and this completes the proof of Muckenhoupt inequality (5.75). □

5.2.1 One-Dimensional Case

Theorem 5.47 ([43]). *Let f be a non-negative function on $I_0 \subset \mathbb{R}$, satisfying Gurov–Reshetnyak condition* (5.1) *for some ε, $0 < \varepsilon < 2$. Then*

$$\frac{1}{t} \int_0^t |f_*(u) - f_{**}(t)|\, du \leq \varepsilon \cdot f_{**}(t), \quad 0 < t \leq |I_0|. \quad (5.89)$$

Proof. Fix some $t \in (0, |I_0|]$ and let $\alpha = f_{**}(t) \leq f_{I_0}$. Using Lemma 1.18, let us construct a collection of pairwise disjoint intervals I_j, $j = 1, 2, \ldots$, such that

$$\frac{1}{|I_j|} \int_{I_j} f(x)\, dx = \alpha, \quad (5.90)$$

$$f(x) \geq \alpha \quad \text{for almost all} \quad x \in I_0 \setminus E, \tag{5.91}$$

where $E = \cup_{j \geq 1} I_j$. By (5.90),

$$\frac{1}{t} \int_0^t f_*(u)\, du = f_{**}(t) = \frac{1}{|E|} \sum_{j \geq 1} |I_j|\, f_{I_j} =$$

$$= \frac{1}{|E|} \sum_{j \geq 1} \int_{I_j} f(x)\, dx = \frac{1}{|E|} \int_E f(x)\, dx \geq \frac{1}{|E|} \int_0^{|E|} f_*(u)\, du.$$

This implies $|E| \leq t$, provided f_* is monotone. Hence, by (5.91) and (5.1),

$$\int_0^t |f_*(u) - f_{**}(t)|\; du = 2 \int_{\{u:\ f_*(u) < f_{**}(t)\}} (f_{**}(t) - f_*(u))\; du =$$

$$= 2 \int_{\{x \in I_0:\ f(x) < f_{**}(t)\}} (f_{**}(t) - f(x))\; dx = \int_E |f(x) - f_{**}(t)|\; dx =$$

$$= \sum_{j \geq 1} \int_{I_j} \left| f(x) - f_{I_j} \right|\, dx \leq \varepsilon \sum_{j \geq 1} |I_j|\, f_{I_j} = \varepsilon \cdot f_{**}(t) |E| \leq \varepsilon \cdot t \cdot f_{**}(t),$$

and inequality (5.89) follows. □

The next theorem is the analog of Theorem 5.34 for the exact embedding of the Gurov–Reshetnyak class in the Muckenhoupt class.

Theorem 5.48 ([43]). *Let ε, $0 < \varepsilon < 2$, and let $q_0'' = q_0''(\varepsilon) > 1$ be a root of the equation*

$$(q-1) q^{-q/(q-1)} = \frac{\varepsilon}{2}. \tag{5.92}$$

Then

(i) if f is a non-negative function on $I_0 \subset \mathbb{R}$, satisfying Gurov–Reshetnyak condition (5.1) with the given ε, then

$$f_{**}(t) \geq c \cdot f_{**}(|I_0|) \left(\frac{|I_0|}{t} \right)^{-\left(q_0'' - 1\right)}, \quad 0 < t \leq |I_0|, \tag{5.93}$$

where the constant $c > 0$ depends only on ε;

(ii) there exists $f_0 \in L([0,1])$, which satisfies (5.1), and such that

$$(f_0)_{**}(t) \leq c_1 t^{q_0'' - 1}, \quad 0 < t \leq 1, \tag{5.94}$$

where the constant c_1 does not depend on t.

Proof. The function

$$\varphi(q) = (q-1)q^{-q/(q-1)}, \quad q > 1,$$

is continuous on $(1, +\infty)$, $\lim_{q\to 1+0} \varphi(q) = 0$ and $\lim_{q\to +\infty} \varphi(q) = 1$. Moreover, the analysis of the derivative shows that φ is strictly increasing on $(1, +\infty)$. These properties of φ imply that for any ε, $0 < \varepsilon < 2$, equation (5.92) has a unique root $q_0'' = q_0''(\varepsilon) > 1$.

Let us prove (i). Applying Theorem 5.47 and Lemma 2.3 to the function f_*, for $a > 1$ we have

$$f_{**}(t) - f_{**}\left(\frac{t}{a}\right) \le \frac{a}{2}\frac{1}{t}\int_0^t |f_*(u) - f_{**}(t)|\, du \le \frac{a\varepsilon}{2} f_{**}(t), \quad 0 < t \le |I_0|,$$

or, equivalently,

$$f_{**}\left(\frac{t}{a}\right) \ge \left(1 - \frac{a\varepsilon}{2}\right) f_{**}(t), \quad 0 < t \le |I_0|. \tag{5.95}$$

Later we will choose the constant $a > 1$ in such a way, that

$$1 - \frac{a\varepsilon}{2} > 0. \tag{5.96}$$

By (5.95),

$$f_{**}\left(a^{-j}|I_0|\right) \ge \left(1 - \frac{a\varepsilon}{2}\right)^j f_{**}(|I_0|), \quad j = 1, 2, \ldots \tag{5.97}$$

Let $q_0'' > 1$ be the root of equation (5.92). Set $a = (q_0'')^{1/\left(q_0''-1\right)} > 1$. Then

$$1 - \frac{a\varepsilon}{2} = 1 - (q_0'')^{1/\left(q_0''-1\right)}\left(q_0''-1\right)(q_0'')^{-q_0''/\left(q_0''-1\right)} = \frac{1}{q_0''} > 0,$$

so that (5.96) follows. In addition,

$$\left(1 - \frac{a\varepsilon}{2}\right)^{1/\left(q_0''-1\right)} = (q_0'')^{-1/\left(q_0''-1\right)} = \frac{1}{a},$$

and hence (5.97) can be rewritten in the following form

$$f_{**}\left(a^{-j}|I_0|\right) \ge (a^j)^{-\left(q_0''-1\right)} f_{I_0}, \quad j = 1, 2, \ldots \tag{5.98}$$

Now for the given $t \in (0, |I_0|]$ we choose $j \ge 1$ such that $a^{-j}|I_0| < t \le a^{-j+1}|I_0|$. Then from (5.98), by virtue of the monotonicity of f_{**}, we get

$$f_{**}(t) \ge f_{**}\left(a^{-j}|I_0|\right) \ge \left(a^j\right)^{-\left(q_0''-1\right)} f_{I_0} \ge a^{-\left(q_0''-1\right)}\left(\frac{|I_0|}{t}\right)^{-\left(q_0''-1\right)} f_{I_0},$$

which is exactly (5.93) with $c = a^{-\left(q_0''-1\right)}$, i.e., c depends only on ε.

Let us prove (ii). Assume $0 < \varepsilon < 2$ and let $q_0'' > 1$ be defined by (5.92). Set $\alpha = q_0'' - 1 > 0$. Clearly, the function $f(x) = x^\alpha,\ 0 \le x \le 1$ satisfies (5.94). Therefore it remains to show that $f \in GR(\varepsilon)$, i.e., it remains to check the inequality (5.1).

Consider an arbitrary $I \subset [0,1]$. Let us choose $t > 0$ such that $J \equiv [0,t] \supset I$ and $f_J = f_I$. Then, by Property 2.15,

$$(f_I)^{-1}\,\Omega(f;I) \le (f_J)^{-1}\,\Omega(f;J) = \left(f_{[0,1]}\right)^{-1}\Omega(f;[0,1]) =$$

$$= 2\alpha(\alpha+1)^{-(\alpha+1)/\alpha} = 2\left(q_0''-1\right)\left(q_0''\right)^{-q_0''/\left(q_0''-1\right)} = \varepsilon,$$

and this completes the proof of the theorem. □

Part (i) of Theorem 5.48 has the following corollary.

Corollary 5.49 ([43]). *Let f be a non-negative function on $I_0 \subset \mathbb{R}$ such that*

$$\Omega(f;I) \le \varepsilon \cdot f_I, \quad I \subset I_0,$$

for some $\varepsilon < 2$. Then f satisfies the Muckenhoupt inequality

$$\frac{1}{|I|}\int_I f(x)\,dx \left\{\frac{1}{|I|}\int_I f^{-1/(q-1)}(x)\,dx\right\}^{q-1} \le c, \quad I \subset I_0, \tag{5.99}$$

for any $q > q_0''$, where $q_0'' = q_0''(\varepsilon) > 1$ is a root of equation (5.92), *and the constant c depends only on ε and q.*

Proof. Clearly, it is enough to give the proof for the case $I = I_0$. Let $q > q_0''$. Then for $0 < t \le |I_0|$ from (5.93) we obtain

$$f_{**}^{-1/(q-1)}(t) \le a^{\left(q_0''-1\right)/(q-1)}\left(\frac{|I_0|}{t}\right)^{\left(q_0''-1\right)/(q-1)}\left(f_{I_0}\right)^{-1/(q-1)},$$

where $a = a(\varepsilon)$ is defined in the proof of Theorem 5.47. Integrating from 0 to $|I_0|$ we find

$$\int_0^{|I_0|} f_{**}^{-1/(q-1)}(t)\,dt \le c_1\,|I_0|\left(f_{I_0}\right)^{-1/(q-1)},$$

where $c_1 = a^{\left(q_0''-1\right)/(q-1)}(q-1)/\left(q-q_0''\right)$ depends only on q and ε. Therefore

$$\frac{1}{|I_0|}\int_{I_0} f^{-1/(q-1)}(x)\,dx = \frac{1}{|I_0|}\int_0^{|I_0|} f_*^{-1/(q-1)}(t)\,dt \le$$

$$\le \frac{1}{|I_0|}\int_0^{|I_0|} f_{**}^{-1/(q-1)}(t)\,dt \le c_1\left(f_{I_0}\right)^{-1/(q-1)}.$$

The last inequality implies (5.99) with $c = c_1^{q-1}$, which depends only on ε and q. □

Remark 5.50. Part (ii) of Theorem 5.48 implies that for $q = q_0''$ Corollary 5.49 fails.

Remark 5.51. From equation (5.92) it is easy to see, that for $q_0'' \equiv q_0''(\varepsilon)$ and $\varepsilon \to 0$

$$q_0'' - 1 \sim \frac{2\varepsilon}{\mathrm{e}}.$$

Remark 5.52. Set $q = \frac{p}{p-1}$. Then equation (5.92) becomes

$$\frac{p^p}{(p-1)^{p-1}} = \frac{2}{\varepsilon}.$$

This is exactly equation (5.64). It defines the limiting exponent of Gehring class, containing a function which satisfies the Gurov–Reshetnyak condition.

5.2.2 Anisotropic Case

For $d \geq 2$ one can prove the following analog of Theorem 5.47.

Theorem 5.53 ([46]). *Let f be a non-negative function on the segment $R_0 \subset \mathbb{R}^d$, satisfying Gurov–Reshetnyak condition* (5.69) *for some* $\varepsilon,\ 0 < \varepsilon < 2$. *Then*

$$\frac{1}{t}\int_0^t |f_*(u) - f_{**}(t)|\, du \leq \varepsilon \cdot f_{**}(t), \quad 0 < t \leq |R_0|. \tag{5.100}$$

Proof. Essentially it is enough to repeat the proof of Theorem 5.47 with the only difference that now, instead of one-dimensional Lemma 1.18, one has to apply Lemma 1.31, obtaining the following analogs of (5.90) and (5.91) respectively

$$\frac{1}{|R_j|}\int_{R_j} f(x)\, dx = \alpha, \quad j = 1, 2, \ldots, \tag{5.101}$$

$$f(x) \geq \alpha \quad \text{for almost all} \quad x \in R_0 \setminus E. \tag{5.102}$$

Here $E = \cup_{j\geq 1} R_j$ and the interiors of the segments $R_j \subset R_0$ are pairwise disjoint. The rest of the proof just repeats the proof of Theorem 5.47. □

As in the case $d = 1$, Theorem 5.53 implies the following results.

Theorem 5.54 ([46]). *Let* $\varepsilon,\ 0 < \varepsilon < 2$ *be given, and let* $q_0'' = q_0''(\varepsilon) > 1$ *be a root of the equation*

$$(q-1)q^{-q/(q-1)} = \frac{\varepsilon}{2}. \tag{5.103}$$

Then

(i) *if f is a non-negative function on the segment $R_0 \subset \mathbb{R}^d$, satisfying Gurov–Reshetnyak condition* (5.69) *with some given* ε, *then*

$$f_{**}(t) \geq c \cdot f_{**}(|I_0|) \left(\frac{|R_0|}{t} \right)^{-\left(q_0''-1\right)}, \quad 0 < t \leq |R_0|,$$

where the constant $c > 0$ *depends only on* ε*;*

(*ii*) *there exists* $f_0 \in L([0,1]^d)$*, satisfying* (5.69) *such that*

$$(f_0)_{**}(t) \leq c_1 t^{q_0''-1}, \quad 0 < t \leq 1,$$

where c_1 *does not depend on* t*.*

Corollary 5.55 ([46]). *Let* f *be a non-negative function on the segment* $R_0 \subset \mathbb{R}^d$ *such that*

$$\Omega(f;R) \leq \varepsilon \cdot f_R, \quad R \subset R_0,$$

for some $\varepsilon < 2$*. Then* f *verifies the Muckenhoupt inequality*

$$\frac{1}{|R|} \int_R f(x)\,dx \left\{ \frac{1}{|R|} \int_R f^{-1/(q-1)}(x)\,dx \right\}^{q-1} \leq c, \quad R \subset R_0,$$

for any $q > q_0''$*, where* $q_0'' = q_0''(\varepsilon) > 1$ *is a root of equation* (5.103)*, while the constant* c *depends only on* ε *and* q*.*

Remark 5.56. Part (*ii*) of Theorem 5.54 implies that for $q = q_0''$ Corollary 5.55 fails.

A

The Boundedness of the Hardy–Littlewood Maximal Operator from BMO into BLO

In this appendix we prove the boundedness of the Hardy-Littlewood maximal operator which acts from BMO into BLO. First let us give the definition and consider some properties of this operator.

Let f be a summable function on the cube $Q_0 \subset \mathbb{R}^d$. The operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy$$

is called *the Hardy–Littlewood maximal operator* . Here the supremum is taken over all cubes $Q \subset Q_0$ containing the point x. The Hardy–Littlewood maximal operator is very important for the analysis of properties of other operators, in particular, for estimation of the partial sums of Fourier series. The next theorem describes the fundamental property of the operator M(see, for example, [70]).

Theorem A.1 (Hardy, Littlewood). *Let f be a function on the cube $Q_0 \subset \mathbb{R}^d$. Then*

(i) if $f \in L^p(Q_0)$ for $1 \le p \le \infty$, then the function Mf is finite almost everywhere;

(ii) if $f \in L(Q_0)$, then for any $\lambda > 0$

$$|\{x \in Q_0 :\ Mf(x) > \lambda\}| \le \frac{C}{\lambda} \int_{Q_0} |f(x)|\, dx, \tag{A.1}$$

where the constant C depends only on the dimension d (for instance, one can take $C = 5^d$);

(iii) if $f \in L^p(Q_0)$ for $1 < p \le \infty$, then $Mf \in L^p(Q_0)$ and

$$\left\{ \int_{Q_0} (Mf)^p(x)\, dx \right\}^{1/p} \le C_p \left\{ \int_{Q_0} |f(x)|^p\, dx \right\}, \tag{A.2}$$

where C_p depends only on p and d.

The function Mf is called *the Hardy-Littlewood maximal function.* Often inequality (A.1) is called *the inequality of the weak* $(1-1)$*–type* for the Hardy–Littlewood maximal operator, while inequality (A.2) is called *the inequality of the strong* $(p-p)$*–type.* Theorem $A.1$ is called *the Hardy–Littlewood maximal theorem.*

Remark A.2. Lebesgue theorems 1.1 and 1.2 are the consequences of Theorem $A.1$.

The boundedness of the Hardy-Littlewood maximal operator M in BMO was proved in [1]. Following the original proof from [1], we will show that the operator M acts from BMO to BLO and estimate its norm.

Theorem A.3. *Let $f \in BMO(Q_0)$, where $Q_0 \subset \mathbb{R}^d$ is a cube. Then $Mf \in BLO$ and*

$$\|Mf\|_{BLO} \le C\|f\|_*, \tag{A.3}$$

where the constant C depends only on the dimension d.

Proof. Since $M(|f|) = Mf,\ \|\,|f|\,\|_* \le 2\|f\|_*$ we see that in order to prove the theorem it is enough to consider the case $f \ge 0$ on Q_0. Let us denote $F(x) = Mf(x),\ x \in Q_0$. Fix an arbitrary cube $Q \subset Q_0$. Let $3Q$ be another cube, concentric with the cube Q, and such that $l(3Q) = 3l(Q)$. We denote by $\widetilde{Q}$ the smallest cube such that $3Q \cap Q_0 \subset \widetilde{Q} \subset Q_0$. For $x \in Q$ let

$$F_1(x) = \sup\left\{ f_{Q'}:\ Q' \ni x,\ Q' \subset \widetilde{Q} \right\},$$

$$F_2(x) = \sup\left\{ f_{Q'}:\ Q' \ni x,\ Q' \subset Q_0,\ Q' \setminus \widetilde{Q} \ne \emptyset \right\}.$$

Clearly $F(x) = \max\{F_1(x),\ F_2(x)\},\ x \in Q$. Let

$$b_Q = \operatorname*{ess\,inf}_{x \in Q} F(x), \quad E_1 = \{x \in Q:\ F_1(x) \ge F_2(x)\},\ E_2 = Q \setminus E_1.$$

Then

$$\frac{1}{|Q|}\int_Q [F(x) - b_Q]\,dx = \frac{1}{|Q|}\sum_{i=1}^{2}\int_{E_i} [F_i(x) - b_Q]\,dx.$$

So in order to prove the theorem it is enough to show that

$$\int_{E_i} [F_i(x) - b_Q]\,dx \le c|Q|\,\|f\|_* \tag{A.4}$$

for $i = 1, 2$.

Let us consider the case $i = 1$. From the inequality $f_{\widetilde{Q}} \le F(x),\ x \in Q$, we obtain that $f_{\widetilde{Q}} \le b_Q$. Now we apply Calderón–Zygmund lemma 5.6 with the constant b_Q to the function f on $\widetilde{Q}$. As the result we obtain two families of cubes $Q_j \subset Q'_j \subset \widetilde{Q},\ j = 1, 2\ldots$, such that the interiors of Q_j are pairwise disjoint,

$$f_{Q'_j} \le b_Q \le f_{Q_j} \le 2^d b_Q, \quad j = 1, 2, \ldots, \tag{A.5}$$

$$f(x) \le b_Q \quad \text{for almost all} \quad x \in \widetilde{Q} \setminus \left(\bigcup_{j \ge 1} Q_j \right), \tag{A.6}$$

and

$$\left| Q'_j \right| = 2^d \left| Q_j \right|. \tag{A.7}$$

Set $E = \widetilde{Q} \setminus \left(\bigcup_{j \ge 1} Q_j \right)$,

$$b(x) = \sum_{j \ge 1} \left(f(x) - f_{Q'_j} \right) \chi_{Q_j}(x), \quad x \in \widetilde{Q},$$

$$g(x) = \sum_{j \ge 1} f_{Q'_j} \chi_{Q_j}(x) + f(x) \chi_E(x), \quad x \in \widetilde{Q}.$$

Notice that $f(x) = b(x) + g(x)$ for almost all $x \in \widetilde{Q}$. Since the interiors of the cubes Q_j are pairwise disjoint $(A.5)$ and $(A.6)$ imply

$$\|g\|_\infty \le b_Q. \tag{A.8}$$

Further,

$$\|b\|_2 \equiv \left\{ \int_{\widetilde{Q}} b^2(x)\, dx \right\}^{1/2} = \left\{ \sum_{j \ge 1} \int_{Q_j} \left| f(x) - f_{Q'_j} \right|^2 dx \right\}^{1/2} \le$$

$$\le \left\{ \sum_{j \ge 1} \left| Q'_j \right| \varOmega_2^2 \left(f; Q'_j \right) \right\}^{1/2}. \tag{A.9}$$

But, according to Corollary 3.18 of the John–Nirenberg theorem (see also Remark 3.19),

$$\varOmega_2^2 \left(f; Q'_j \right) \le c_1 \|f\|_*^2,$$

where the constant c_1 depends only on d. Since $\left| \widetilde{Q} \right| \le 3^d |Q|$, using $(A.8)$ from $(A.9)$ we get

$$\|b\|_2 \le \left\{ c_1 \|f\|_*^2 \sum_{j \ge 1} \left| Q'_j \right| \right\}^{1/2} = \left\{ 2^d c_1 \|f\|_*^2 \sum_{j \ge 1} \left| Q_j \right| \right\}^{1/2} \le$$

$$\le \left\{ 2^d c_1 \|f\|_*^2 \left| \widetilde{Q} \right| \right\}^{1/2} \le c_2 |Q|^{1/2} \|f\|_*, \tag{A.10}$$

where the constant c_2 depends only on d. The definition of the function F_1 implies

$$F_1(x) = M\left(f\chi_{\widetilde{Q}}\right)(x) = M(b+g)(x) \le Mb(x) + Mg(x), \quad x \in \widetilde{Q},$$

provided the operator M is semi-additive. Now using the Schwartz inequality and Hardy–Littlewood maximal theorem $A.1$, we obtain

$$\int_{E_1} F_1(x)\,dx \le \int_{E_1} Mb(x)\,dx + \|g\|_\infty \cdot |E_1| \le$$

$$\le |E_1|^{1/2}\left\{\int_{E_1}(Mb(x))^2\,dx\right\}^{1/2} + \|g\|_\infty \cdot |E_1| \le c_3\,|E_1|^{1/2}\,\|b\|_2 + \|g\|_\infty \cdot |E_1|\,,$$

where c_3 depends only on d. Substituting $(A.8)$ and $(A.10)$ in the last inequality and taking into account that $E_1 \subset Q$, we get

$$\int_{E_1} F_1(x)\,dx \le c_4|Q|\,\|f\|_* + b_Q\,|E_1|\,,$$

which is exactly $(A.4)$ for $i = 1$.

In order to prove $(A.4)$ for $i = 2$ it is enough to show that

$$F_2(x) - b_Q \le c_5\|f\|_*, \quad x \in E_2. \tag{A.11}$$

Let $x \in E_2$, and let $Q'' \subset Q_0$ be a cube, containing the point x, and such that $Q'' \setminus \widetilde{Q} \ne \emptyset$. Since $x \in Q$ it follows that $|Q''| \ge |Q|$. Let Q''' be the smallest cube that contain both Q'' and Q. Obviously then $|Q'''| \le 2^d\,|Q''|$. Let us show that

$$f_{Q'''} \le b_Q. \tag{A.12}$$

Indeed, for any $y \in Q$

$$f_{Q'''} \le F_2(y) \le \max\{F_1(y),\ F_2(y)\} = F(y).$$

From here, taking the infimum over all $y \in Q$, we obtain $(A.12)$. In its own turn, $(A.12)$ implies

$$f_{Q''} - b_Q \le f_{Q''} - f_{Q'''} \le \frac{1}{|Q''|}\int_{Q''} |f(y) - f_{Q'''}|\;dy \le$$

$$\le \frac{|Q'''|}{|Q''|}\cdot\frac{1}{|Q'''|}\int_{Q'''} |f(y) - f_{Q'''}|\;dy \le 2^d\|f\|_*.$$

Now, taking the supremum over all cubes Q'', we get $(A.11)$. □

Due to inequality (2.37), Theorem $A.3$ has the following immediate corollary.

Corollary A.4 (Bennett, De Vore, Shapley, [1]). *If* $f \in BMO(Q_0)$, *where* $Q_0 \subset \mathbb{R}^d$ *is a cube, then* $Mf \in BMO(Q_0)$ *and*

$$\|Mf\|_* \le C\|f\|_*,$$

where the constant C *depends only on the dimension* d.

B

The Weighted Analogs of the Riesz Lemma and the Gurov–Reshetnyak Theorem

Undoubtedly all results, which we discussed in this book for the Lebesgue measure, can be generalized for any measure, satisfying certain conditions. We did not pay attention to this fact in order to avoid the excessive complication. However, in our opinion it would be useful to consider some of the results, described above, in the weighted case, in particular, for measures that preserve the given proofs. First of all, we mean the multidimensional analog of the Riesz "rising sun lemma" (Lemma 1.30), because it plays the key role in various problems.

B.1 The Weighted Riesz Lemma

Let $d\mu \equiv d\mu(x) = w(x)\,dx$ be a measure and assume that $d\mu$ is absolutely continuous with respect to the Lebesgue measure. We assume in addition that *the weight function* w is non-negative and locally summable (with respect to the Lebesgue measure) on $\mathbb{R}^d$, and denote by $L_\mu(E)$ the class of functions f, summable on E with respect to the measure μ. It is easy to see, that Lemma 1.28 remain valid if we substitute the Lebesgue measure by $d\mu$ and take

$$\mathcal{M}_{\mathcal{B},\mu} f(x) \equiv \sup_{\mathcal{B}\ni J\ni x} \frac{1}{\mu(J)} \int_J |f(y)|\,d\mu(y), \quad x \in R_0.$$

instead of the maximal function $\mathcal{M}_{\mathcal{B}}$. Further, substituting the mean value f_E of the function f on the set E by its *μ-mean value*

$$f_{E,\mu} \equiv \frac{1}{\mu(E)} \int_E f(x)\,d\mu(x),$$

we see that Lemma 1.29 is valid in the weighted case, too. So, also the weighted analog of Lemma 1.30, based on the application of the weighted analog of Lemma 1.29, holds true. Notice, that the absolute continuity of the measure μ provides the equality $f_{R_j,\mu} = \alpha$, which plays the key role in the proof of the weighted analog of Lemma 1.21, and hence in the proof of the weighted analog of Lemma 1.30. Therefore the following result is valid.

Lemma B.1 ([47]). *Let $R_0 \subset \mathbb{R}^d$ be a segment, $d\mu = w(x)\,dx$ be an absolutely continuous measure, and assume $f \in L_\mu(R_0)$ and $\alpha \geq f_{R_0,\mu}$. Then there exists a family of segments $R_j \subset R_0$, $j = 1, 2, \ldots$, with pairwise disjoint interiors such that $f_{R_j,\mu} = \alpha$ and $f(x) \leq \alpha$ for μ-almost every point $x \in R_0 \setminus \left(\bigcup_{j\geq 1} R_j\right)$.*

B.2 The Gurov–Reshetnyak Theorem in the Weighted Case

Now let us consider the weighted analog of Gurov–Reshetnyak theorem 5.4. As before, let $d\mu = w(x)\,dx$ be an absolute continuous measure. The quantity

$$\Omega_\mu(f;Q) = \frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}|\; d\mu(x)$$

is called *the μ-mean oscillation* of the function $f \in L_\mu(Q)$ on the cube $Q \subset \mathbb{R}^d$. For the cube $Q_0 \subset \mathbb{R}^d$ and $0 < \varepsilon < 2$ we denote by $GR_\mu \equiv GR_\mu(\varepsilon) \equiv GR_\mu(\varepsilon, Q_0)$ the class of all non-negative functions $f \in L_\mu(Q_0)$, satisfying *the weighted Gurov–Reshetnyak inequality*

$$\Omega_\mu(f;Q) \leq \varepsilon \cdot f_{Q,\mu}, \quad Q \subset Q_0. \tag{B.1}$$

Substituting the Lebesgue measure dx by $d\mu$ in the proof of Theorem 5.22, we obtain to the following theorem.

Theorem B.2 ([42]). *Let $d\mu = w(x)\,dx$ be an absolute continuous measure on the cube $Q_0 \subset \mathbb{R}^d$, and let $f \in L_\mu(Q_0)$. Then*

(i) if for some ε, $0 < \varepsilon < 2$, the function f satisfies weighted Gurov–Reshetnyak condition $(B.1)$, then for $\varepsilon < \lambda < 2$

$$\mu\left(\left\{x \in Q:\; f(x) > \left(1 - \frac{\varepsilon}{\lambda}\right) \cdot f_{Q,\mu}\right\}\right) \geq \left(1 - \frac{\lambda}{2}\right) \cdot \mu(Q), \quad Q \subset Q_0; \tag{B.2}$$

(ii) if for some σ and θ, $0 < \sigma, \theta < 1$, the function f is such that

$$\mu\left(\{x \in Q:\; f(x) > \sigma \cdot f_{Q,\mu}\}\right) > \theta \cdot \mu(Q), \quad Q \subset Q_0,$$

then

$$\Omega_\mu(f;Q) \leq 2(1 - \sigma\theta) f_{Q,\mu}, \quad Q \subset Q_0.$$

Further, in [61] it was shown, that condition $(B.2)$ implies *the weighted Gehring inequality*

$$\left\{\frac{1}{\mu(Q)}\int_Q f^p(x)\,d\mu(x)\right\}^{\frac{1}{p}} \le c\frac{1}{\mu(Q)}\int_Q f(x)\,d\mu(x), \quad Q\subset Q_0 \qquad (B.3)$$

for some $p>1$. Here the constant c depends only on ε, d and p. Thus Theorem $B.2$ yields the following analog of Corollary 5.23.

Corollary B.3 (the weighted version of the Gurov–Reshetnyak theorem, [42]). *Let $d\mu = w(x)\,dx$ be an absolutely continuous measure, $0<\varepsilon<2$, and let $f\in L_\mu(Q_0)$ be a non-negative function on the cube $Q_0\subset\mathbb{R}^d$ such that weighted Gurov–Reshetnyak condition $(B.1)$ is satisfied. Then there exists $p_0\equiv p_0(\varepsilon,d)>1$ such that for any $p<p_0$ the weighted Gehring inequality $(B.3)$ is satisfied with the constant c, depending only on ε, d and p.*

Notice that similarly to the non-weighted case this way to deduce the weighted version of the Gurov–Reshetnyak theorem does not provide the known exact exponent of summability $p_0(\varepsilon,d)\ge\frac{c(d)}{\varepsilon}$ of f for $\varepsilon\to 0$. We will give now another proof of this theorem with the exact exponent of summability p_0. This proof is based on the application of part (i) of Theorem $B.2$. First we need the following covering lemma.

Lemma B.4 (Mateu, Mattila, Nicolau, Orobitg, [56]). *Let $Q_0\subset\mathbb{R}^d$ be a cube, $d\mu=w(x)\,dx$ be an absolutely continuous measure, and let $E\subset Q_0$ be a μ-measurable set such that $\mu(E)\le\rho\mu(Q_0)$ with $0<\rho<1$. Then there exists at most countable family of cubes Q_j, $j=1,2,\dots$, such that*

(i) $\mu\left(Q_j\bigcap E\right)=\rho\mu\left(Q_j\right)$, $j=1,2,\dots$;

(ii) the family $\{Q_j\}_{j\ge1}$ is almost disjunctive with the constant $B(d)$, i.e. each point of the cube Q_0 is contained in at most $B(d)$ cubes Q_j;

(iii) μ-almost every point of E is contained in $\bigcup_{j\ge1}Q_j$.

We define the non-decreasing equimeasurable rearrangement of the function f on the cube Q_0 with respect to the measure μ by the following equality

$$f_\mu^*(t)=\sup_{e\subset Q_0,\ \mu(e)=t}\ \inf_{x\in e}|f(x)|, \quad 0<t<\mu(Q_0).$$

Set

$$f_\mu^{**}(t)=\frac{1}{t}\int_0^t f_\mu^*(u)\,du, \quad 0<t<\mu(Q_0).$$

Theorem B.5 ([42]). *Let $d\mu=w(x)\,dx$ be an absolutely continuous measure, and $f\in GR_\mu(\varepsilon,Q_0)$ be a non-negative function on the cube $Q_0\subset\mathbb{R}^d$ with $0<\varepsilon<2$. Then for $\varepsilon<\lambda<2$, $\rho<1-\frac{\lambda}{2}$ and $t\le\rho\mu(Q_0)$*

$$f_\mu^{**}(t)\le\left(B(d)\frac{\frac{\lambda}{\rho}+1}{\lambda-\varepsilon}\varepsilon+1\right)f_\mu^*(t). \qquad (B.4)$$

Proof. Fix ε, ρ and t as in the statement of the theorem, and set $E = \left\{x \in Q_0 : \; f(x) > f^*_\mu(t)\right\}$. Then $\mu(E) \le t \le \rho\mu\left(Q_0\right)$. Applying Lemma $B.4$ to the set E with the constant ρ we obtain a family of cubes Q_j, $j = 1, 2, \dots,$ satisfying the conditions of this lemma. Since $\rho < 1 - \frac{\lambda}{2}$ we see that part (i) of the lemma implies

$$\left(f\chi_{Q_j}\right)^*_\mu \left(\left(1 - \frac{\lambda}{2}\right)\mu\left(Q_j\right)\right) \le f^*_\mu(t), \quad j = 1, 2, \dots$$

Hence, according to part (i) of Theorem $B.2$,

$$f_{Q_j,\mu} \le \frac{\lambda}{\lambda - \varepsilon}\left(f\chi_{Q_j}\right)^*_\mu \left(\left(1 - \frac{\lambda}{2}\right)\mu\left(Q_j\right)\right) \le \frac{\lambda}{\lambda - \varepsilon} f^*_\mu(t), \tag{B.5}$$

and so for $j = 1, 2, \dots,$

$$\Omega_\mu\left(f; Q_j\right) \le \frac{\varepsilon\lambda}{\lambda - \varepsilon} f^*_\mu(t). \tag{B.6}$$

Moreover, by Lemma $B.4$ (part (ii)),

$$\sum_{j \ge 1} \mu\left(Q_j \bigcap E\right) \le B(d)\mu(E) \le B(d)t.$$

Using the properties of the equimeasurable rearrangements and inequalities $(B.5)$ and $(B.6)$ we obtain

$$t\left(f^{**}_\mu(t) - f^*_\mu(t)\right) = \int_E \left(f(x) - f^*_\mu(t)\right)\, d\mu(x) \le$$

$$\le \sum_{j\ge 1} \int_{E \bigcap Q_j} \left(f(x) - f^*_\mu(t)\right)\, d\mu(x) \le$$

$$\le \sum_{j\ge 1} \int_{E \bigcap Q_j} \left(f(x) - f_{Q_j,\mu}\right)\, d\mu(x) + \sum_{j \ge 1} \mu\left(E \bigcap Q_j\right)\left(f_{Q_j,\mu} - f^*_\mu(t)\right) \le$$

$$\le \frac{\varepsilon\lambda}{\lambda - \varepsilon} f^*_\mu(t) \sum_{j\ge 1} \mu\left(Q_j\right) + \frac{\varepsilon}{\lambda - \varepsilon} f^*_\mu(t) \sum_{j \ge 1} \mu\left(E \bigcap Q_j\right) \le$$

$$\le B(d)\frac{\frac{\lambda}{\rho} + 1}{\lambda - \varepsilon} \cdot \varepsilon t \cdot f^*_\mu(t),$$

which is equivalent to $(B.4)$. □

From Theorem $B.5$ it follows immediately that the condition $f \in GR_\mu(\varepsilon)$ implies weighted Gehring inequality $(B.3)$ for all $p < 1 + \frac{\lambda - \varepsilon}{B(d)(\lambda/\rho + 1)} \frac{1}{\varepsilon}$. In order to see this it is enough to apply the following lemma to inequality $(B.4)$.

Lemma B.6 (Muckenhoupt, [59]). *Let h be a non-negative non-increasing function on $[0, b]$ such that*

$$\frac{1}{t}\int_0^t h(u)\,du \le c\cdot h(t), \quad 0 \le t \le \frac{b}{20}.$$

Then for any p, $1 \le p < \frac{c}{c-1}$,

$$\left\{\frac{1}{b}\int_0^b h^p(t)\,dt\right\}^{\frac{1}{p}} \le c_1 \frac{1}{b}\int_0^b h(t)\,dt,$$

and the constant c_1 depends only on p and c.

C

Classes of Functions Satisfying the Reverse Hölder Inequality

The defined above Gehring and Muckenhoupt classes G_p and A_q can be considered as the particular cases of a class of functions satisfying the reverse Hölder inequality.

Let $d\mu(x) = w(x)\,dx$ be an absolute continuous measure on the cube $Q_0 \subset \mathbb{R}^d$ such that the weight function w is non-negative and summable on Q_0. Let $\alpha < \beta$ ($\alpha\beta \neq 0$) be two numbers. For $B > 1$ denote by $\widetilde{RH}^{\alpha,\beta}_{d\mu(x)}(B)$ the class of non-negative functions f on Q_0 satisfying the *reverse weighted Hölder inequality*

$$\left\{\frac{1}{\mu(Q)}\int_Q f^\beta(x)\,d\mu(x)\right\}^{\frac{1}{\beta}} \leq B\left\{\frac{1}{\mu(Q)}\int_Q f^\alpha(x)\,d\mu(x)\right\}^{\frac{1}{\alpha}}$$

uniformly over all cubes $Q \subset Q_0$. In the case that the reverse Hölder inequality holds true not only over cubes, but over all segments R contained in some fixed segment $R_0 \subset \mathbb{R}^d$, i.e.,

$$\left\{\frac{1}{\mu(R)}\int_R f^\beta(x)\,d\mu(x)\right\}^{\frac{1}{\beta}} \leq B\left\{\frac{1}{\mu(R)}\int_R f^\alpha(x)\,d\mu(x)\right\}^{\frac{1}{\alpha}},$$

we will denote the corresponding class of functions by $RH^{\alpha,\beta}_{d\mu(x)}(B)$. It is easy to see that $RH^{\alpha,\beta}_{d\mu(x)}(B) \subset \widetilde{RH}^{\alpha,\beta}_{d\mu(x)}(B)$ and this embedding is strict. Clearly $G_p(B) = \widetilde{RH}^{1,p}_{dx}(B)$ ($p > 1$) and $A_q(B) = \widetilde{RH}^{-\frac{1}{q-1},1}_{dx}(B)$ ($q > 1$).

C.1 Estimate of Rearrangements of Functions Satisfying the Reverse Jensen Inequality

For convenience in what follows we will change some of notations previously used. Namely, for non-negative function f we will call the functions

$$f_\mu^{\downarrow}(t) \equiv (f|E)_\mu^{\downarrow}(t) = \sup_{e\subset E,\ \mu(e)=t} \inf_{x\in e} f(x), \quad 0 \le t \le \mu(E)$$

and

$$f_\mu^{\uparrow}(t) \equiv (f|E)_\mu^{\uparrow}(t) = \inf_{e\subset E,\ \mu(e)=t} \sup_{x\in e} f(x), \quad 0 \le t \le \mu(E)$$

the non-increasing and the non-decreasing equimeasurable rearrangements of the function f with respect to the measure $d\mu$ in the μ-measurable set E. If $d\mu(x) = dx$ is the Lebesgue measure then $f_\mu^{\downarrow} = f^*$ and $f_\mu^{\uparrow} = f_*$. The functions $(f|E)_\mu^{\downarrow}(t)$ and $(f|E)_\mu^{\uparrow}(t)$ are μ-equimeasurable with f in E in the sense that for any $\lambda \ge 0$

$$\left|\left\{t \in (0,|E|] : (f|E)_\mu^{\downarrow}(t) > \lambda\right\}\right| = \left|\left\{t \in (0,|E|] :\ (f|E)_\mu^{\uparrow}(t) > \lambda\right\}\right| =$$

$$= \mu\left(\{x \in E :\ f(x) > \lambda\}\right),$$

where $|\cdot|$ denotes the Lebesgue measure. Let Φ be the class of all positive convex downwards functions φ on $(0,+\infty)$ such that $\varphi(0) = \varphi(0+)$ (the values 0 and $+\infty$ in the right-hand are admissible). Since the functions $(f|E)_\mu^{\downarrow}$, $(f|E)_\mu^{\uparrow}$ and f are μ-equimeasurable on the μ-measurable set E we have

$$\int_0^{\mu(E)} \varphi\left((f|E)_\mu^{\downarrow}(t)\right)\, dt = \int_0^{\mu(E)} \varphi\left((f|E)_\mu^{\uparrow}(t)\right)\, dt = \int_E \varphi(f(x))\, d\mu(x)$$

for any $\varphi \in \Phi$.

Recall that for $\varphi \in \Phi$ and for any non-negative function f on E there holds true the so-called *weighted Jensen inequality* (see [26])

$$\varphi\left(\frac{1}{\mu(E)}\int_E f(x)\, d\mu(x)\right) \le \frac{1}{\mu(E)}\int_E \varphi(f(x))\, d\mu(x). \tag{C.1}$$

For any $B > 1$ let us consider the class $RJ^{\varphi}_{d\mu(x)}(B)$ of functions f such that they are non-negative on the segment $R_0 \subset \mathbb{R}^d$ and satisfy the *reverse weighted Jensen inequality*

$$\frac{1}{\mu(E)}\int_E \varphi(f(x))\, d\mu(x) \le B\cdot\varphi\left(\frac{1}{\mu(E)}\int_E f(x)\, d\mu(x)\right) \tag{C.2}$$

uniformly over all segments $R \subset R_0$. It is easy to see that for $\varphi(u) = u^p$ $(p > 1)$ the reverse Jensen inequality becomes the Gehring condition, while for $\varphi(u) = u^{-\frac{1}{q-1}}$ $(q > 1)$ it is the Muckenhoupt condition with appropriate constants.

In this section we give the *exact estimate of the equimeasurable rearrangements of functions satisfying the reverse Jensen inequality.*

Theorem C.1 ([48]). *Let $\varphi \in \Phi$ and let $d\mu$ be an absolute continuous measure on the segment $R_0 \subset \mathbb{R}^d$. Let f be a non-negative function on R_0 satisfying the reverse Jensen inequality (C.2). Then for any interval $I \subset [0, \mu(R_0)]$*

$$\frac{1}{|I|}\int_I \varphi\left(f_\mu^{\downarrow}(t)\right)\,dt \le B\cdot\varphi\left(\frac{1}{|I|}\int_I f_\mu^{\downarrow}(t)\,dt\right), \tag{C.3}$$

$$\frac{1}{|I|}\int_I \varphi\left(f_\mu^{\uparrow}(t)\right)\,dt \le B\cdot\varphi\left(\frac{1}{|I|}\int_I f_\mu^{\uparrow}(t)\,dt\right) \tag{C.4}$$

with the same constant $B > 1$ *as in condition* $(C.2)$.

The estimate of the rearrangements of functions satisfying the reverse Jensen inequality $(C.2)$ over cubes was obtained by C. Sbordone in [68] with some additional assumptions on the function φ. In this case the question about the sharpness of such estimates is quite difficult. Actually we do not know any result concerning the exact estimates of the equimeasurable rearrangements of functions satisfying condition $(C.2)$ over cubes even for some special class of functions φ.

In the one-dimensional case the exact estimate of the equimeasurable rearrangements of functions satisfying the reverse Jensen inequality for $d\mu = dx$ was obtained in [35, 36].

In order to prove Theorem $C.1$ we need some auxiliary results. The key role in the proof will be played by the weighted analog of the Riesz lemma (Lemma $B.1$). Here we give another equivalent formulation of this lemma in the form it will be used later.

Lemma C.2 ([48]). *Let* $R_0 \subset \mathbb{R}^d$ *be a segment,* $d\mu(x) = w(x)\,dx$ *be an absolutely continuous measure,* $f \in L_\mu(R_0)$*, and let* $\alpha \le f_{R_0,\mu}$*. Then there exists a family of pairwise disjoint segments* $R_j \subset R_0,\ j = 1, 2, \dots,$ *such that* $f_{R_j,\mu} = \alpha,\ j = 1, 2, \dots,$ *and* $f(x) \ge \alpha$ *for* μ*-almost every* $x \in R_0 \setminus \left(\bigcup_{j\ge1} R_j\right)$.

The next lemma describes one simple property of convex functions. We well prove it though it becomes completely trivial if one makes an appropriate design.

Lemma C.3 ([35]). *Let* $\varphi \in \Phi$ *and assume that the numbers* $0 \le \gamma_1, \gamma_2 \le 1$, $a \ge b \ge c \ge d > 0$ *are such that*

$$\gamma_1 a + (1-\gamma_1)\,d = \gamma_2 b + (1-\gamma_2)\,c.$$

Then

$$\gamma_1\varphi(b) + (1-\gamma_1)\,\varphi(c) \le \gamma_2\varphi(a) + (1-\gamma_2)\,\varphi(d). \tag{C.5}$$

Proof. Since the function φ is convex

$$\varphi(c) \le \varphi(d) + \frac{\varphi(a)-\varphi(d)}{a-d}(c-d). \tag{C.6}$$

Denote $\xi = \gamma_1 b + (1-\gamma_1)\,c$. Then inequality $(C.6)$ becomes

$$\varphi(c) \le \frac{a-c}{a-d}\left[\varphi(d) + \varphi(a)\frac{(c-d)(a-\xi)}{(a-c)(a-\xi)}\right].$$

It is easy to see that this inequality is equivalent to the following one

$$\varphi(c) + \frac{\varphi(a)-\varphi(c)}{a-c}(\xi - c) \le \varphi(d) + \frac{\varphi(a)-\varphi(d)}{a-d}(\xi - d). \qquad (C.7)$$

Further, by convexity of φ, we have

$$\varphi(b) \le \varphi(c) + \frac{\varphi(a)-\varphi(c)}{a-c}(b-c),$$

which, by $(C.7)$, implies

$$\varphi(c) + \frac{\varphi(b)-\varphi(c)}{b-c}(\xi - c) \le \varphi(c) + \frac{\varphi(a)-\varphi(c)}{a-c}(\xi - c) \le$$

$$\le \varphi(d) + \frac{\varphi(a)-\varphi(d)}{a-d}(\xi - d).$$

Since $\gamma_1 = \frac{\xi-c}{b-c}$ and $\gamma_2 = \frac{\xi-d}{a-d}$ it is easy to see that the last inequality implies $(C.5)$. □

The next lemma is an analog of Property 2.15 of mean oscillations.

Lemma C.4 ([35]). *Let g be a non-negative monotone summable function on the interval $[a,b]$ and let the interval $[\alpha,\beta] \subset [a,b]$ be such that*

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt = \frac{1}{b-a}\int_a^b g(t)\,dt.$$

Then for any $\varphi \in \Phi$

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(g(t))\,dt \le \frac{1}{b-a}\int_a^b \varphi(g(t))\,dt. \qquad (C.8)$$

Proof. For definiteness we assume that g does not decrease on $[a,b]$. Let

$$a' = \inf\left\{t \in [a,b] : \ g(t) \ge g_{[\alpha,\beta]}\right\}, \quad b' = \sup\left\{t \in [a,b] : \ g(t) \le g_{[\alpha,\beta]}\right\}.$$

Clearly $a' \le b'$. If $a' \le \alpha$ or $b' \ge \beta$, then we set $c = \frac{\alpha+\beta}{2}$; otherwise we set $c = \frac{a'+b'}{2}$. In both cases $g(t) \le g_{[\alpha,\beta]}$ for $t \le c$ and $g(t) \ge g_{[\alpha,\beta]}$ for $t \ge c$. So,

$$\int_\alpha^c \left(g_{[\alpha,\beta]} - g(t)\right)\,dt = \int_c^\beta \left(g(t) - g_{[\alpha,\beta]}\right)\,dt. \qquad (C.9)$$

Let us partition the non-degenerate interval $[\alpha,c]$ into the non-degenerated intervals $\Delta_i^{(l)}$ $(i = 1,\ldots,n)$. For every $\Delta_i^{(l)}$ let us construct the interval $\Delta_i^{(r)} \subset [c,\beta]$ such that

$$\int_{\Delta_i^{(l)}} \left(g_{[\alpha,\beta]} - g(t)\right) \, dt = \int_{\Delta_i^{(r)}} \left(g(t) - g_{[\alpha,\beta]}\right) \, dt,$$

so that the interiors of the intervals $\Delta_i^{(r)}$, as well as the interiors of $\Delta_i^{(l)}$, are pairwise disjoint and

$$\bigcup_{i=1}^{n} \Delta_i^{(l)} = [\alpha, c], \quad \bigcup_{i=1}^{n} \Delta_i^{(r)} = [c, \beta].$$

Such a construction is possible due to $(C.9)$. We obtain that for $i = 1, \ldots, n$

$$\frac{\left|\Delta_i^{(l)}\right|}{\left|\Delta_i^{(l)}\right| + \left|\Delta_i^{(r)}\right|} \cdot \frac{1}{\left|\Delta_i^{(l)}\right|} \int_{\Delta_i^{(l)}} g(t) \, dt + \frac{\left|\Delta_i^{(r)}\right|}{\left|\Delta_i^{(l)}\right| + \left|\Delta_i^{(r)}\right|} \cdot \frac{1}{\left|\Delta_i^{(r)}\right|} \int_{\Delta_i^{(r)}} g(t) \, dt = g_{[\alpha,\beta]}.$$

Moreover, if we denote $\Delta^{(l)} = [a, \alpha]$, $\Delta^{(r)} = [\beta, b]$, then from the condition $g_{[\alpha,\beta]} = g_{[a,b]}$ we get

$$\frac{\left|\Delta^{(l)}\right|}{\left|\Delta^{(l)}\right| + \left|\Delta^{(r)}\right|} \cdot \frac{1}{\left|\Delta^{(l)}\right|} \int_{\Delta^{(l)}} g(t) \, dt + \frac{\left|\Delta^{(r)}\right|}{\left|\Delta^{(l)}\right| + \left|\Delta^{(r)}\right|} \cdot \frac{1}{\left|\Delta^{(r)}\right|} \int_{\Delta^{(r)}} g(t) \, dt = g_{[\alpha,\beta]}.$$

Notice that, by monotonicity of g, for any $i = 1, \ldots, n$

$$\frac{1}{\left|\Delta^{(l)}\right|} \int_{\Delta^{(l)}} g(t) \, dt \le \frac{1}{\left|\Delta_i^{(l)}\right|} \int_{\Delta_i^{(l)}} g(t) \, dt \le$$

$$\le \frac{1}{\left|\Delta_i^{(r)}\right|} \int_{\Delta_i^{(r)}} g(t) \, dt \le \frac{1}{\left|\Delta^{(r)}\right|} \int_{\Delta^{(r)}} g(t) \, dt.$$

The application of Lemma $C.3$ yields

$$\frac{\left|\Delta_i^{(l)}\right|}{\left|\Delta_i^{(l)}\right| + \left|\Delta_i^{(r)}\right|} \varphi\left(\frac{1}{\left|\Delta_i^{(l)}\right|} \int_{\Delta_i^{(l)}} g(t) \, dt\right) +$$

$$+ \frac{\left|\Delta_i^{(r)}\right|}{\left|\Delta_i^{(l)}\right| + \left|\Delta_i^{(r)}\right|} \varphi\left(\frac{1}{\left|\Delta_i^{(r)}\right|} \int_{\Delta_i^{(r)}} g(t) \, dt\right) \le$$

$$\le \frac{\left|\Delta^{(l)}\right|}{\left|\Delta^{(l)}\right| + \left|\Delta^{(r)}\right|} \varphi\left(\frac{1}{\left|\Delta^{(l)}\right|} \int_{\Delta^{(l)}} g(t) \, dt\right) +$$

$$+ \frac{\left|\Delta^{(r)}\right|}{\left|\Delta^{(l)}\right| + \left|\Delta^{(r)}\right|} \varphi\left(\frac{1}{\left|\Delta^{(r)}\right|} \int_{\Delta^{(r)}} g(t) \, dt\right)$$

for $i = 1, \ldots, n$. Summing up these inequalities over all i and applying Jensen inequality $(C.1)$ we get

$$\sum_{i=1}^{n} \left[\left|\Delta_i^{(l)}\right| \varphi \left(\frac{1}{\left|\Delta_i^{(l)}\right|} \int_{\Delta_i^{(l)}} g(t)\, dt \right) + \left|\Delta_i^{(r)}\right| \varphi \left(\frac{1}{\left|\Delta_i^{(r)}\right|} \int_{\Delta_i^{(r)}} g(t)\, dt \right) \right] \leq$$

$$\leq \frac{\beta - \alpha}{\left|\Delta^{(l)}\right| + \left|\Delta^{(r)}\right|} \left[\int_a^\alpha \varphi(g(t))\, dt + \int_\beta^b \varphi(g(t))\, dt \right]. \qquad (C.10)$$

If the function φ is monotone, then the sum σ in the left-hand side of $(C.10)$ is bounded by the lower and upper Darboux sums of the integral $\int_\alpha^\beta \varphi(g(t))\, dt$ that correspond to the partition of the interval $[\alpha, \beta]$ by intervals $\Delta_i^{(l)}$ and $\Delta_i^{(r)}$ $(i = 1, \ldots, n)$. Moreover, if the side-lengths of the intervals $\Delta_i^{(l)}$ tend to zero, it follows that the side-lengths of the intervals $\Delta_i^{(r)}$ tend to zero, too. Therefore $(C.10)$ implies

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \varphi(g(t))\, dt \leq \frac{1}{(\alpha - a) + (b - \beta)} \left[\int_a^\alpha \varphi(g(t))\, dt + \int_\beta^b \varphi(g(t))\, dt \right]. \qquad (C.11)$$

Otherwise, if the function φ is not monotone, then in order to prove $(C.11)$ it is enough to present the convex downwards function φ as a sum of two monotone convex downwards functions and then prove $(C.11)$ for each component of such a representation.

It is easy to see that $(C.11)$ implies $(C.8)$. $\square$

The next two lemmas are of great importance for the proof of Theorem $C.1$.

Lemma C.5 ([35]). *Let f be a non-negative function on $E \bigcup \widehat{E}$ such that*

$$\frac{1}{\mu(E)} \int_E f(x)\, d\mu(x) = \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} f(x)\, d\mu(x) \equiv A \qquad (C.12)$$

and

$$f(x) \leq A, \quad x \notin E \bigcap \widehat{E}, \qquad (C.13)$$

$$f(x) \leq f(y), \quad x \in \widehat{E} \setminus E,\ y \in E. \qquad (C.14)$$

Then for any $\varphi \in \Phi$

$$\frac{1}{\mu(E)} \int_E \varphi\left(f(x)\right)\, d\mu(x) \leq \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} \varphi\left(f(x)\right)\, d\mu(x). \qquad (C.15)$$

Proof. Let us consider the non-trivial case when f is not μ-equivalent to a constant A. First let us show that

$$\mu\left(\widehat{E}\right) \le \mu(E). \tag{$C.16$}$$

Indeed, by $(C.12)$ and $(C.13)$,

$$\int_{E\setminus\widehat{E}} (A - f(x))\, d\mu(x) = \int_{\widehat{E}\setminus E} (A - f(x))\, d\mu(x). \tag{$C.17$}$$

Using condition $(C.14)$ we can choose some c such that

$$f(x) \le c \le f(y), \quad x \in \widehat{E} \setminus E,\ y \in E.$$

Then

$$\int_{E\setminus\widehat{E}} (A - f(x))\, d\mu(x) \le (A - c)\mu\left(E \setminus \widehat{E}\right),$$

$$\int_{\widehat{E}\setminus E} (A - f(x))\, d\mu(x) \ge (A - c)\mu\left(\widehat{E} \setminus E\right).$$

Since $c < A$ these two inequalities together with $(C.17)$ imply

$$\mu\left(\widehat{E} \setminus E\right) \le \mu\left(E \setminus \widehat{E}\right),$$

which is equivalent to $(C.16)$.

Now let us construct the sets E' and E'' such that

$$E' \bigcup E'' = E \setminus \widehat{E}, \quad E' \bigcap E'' = \emptyset, \quad \mu\left(E'\right) = \mu\left(\widehat{E} \setminus E\right)$$

and $f(x) \le f(y)$ for any $x \in E''$, $y \in E'$. Chose some integer k and partition the sets E', E'' and $\widehat{E} \setminus E$ into pairwise disjoint subsets in the following way. Denote

$$g^{(1)}(t) = \left(f|E'\right)^{\downarrow}_{\mu}(t), \quad 0 \le t \le \mu\left(E'\right),$$

$$g^{(2)}(t) = \left(f|E''\right)^{\downarrow}_{\mu}(t), \quad 0 \le t \le \mu\left(E''\right),$$

$$g^{(3)}(t) = \left(f|\left(\widehat{E} \setminus E\right)\right)^{\downarrow}_{\mu}(t), \quad 0 \le t \le \mu\left(\widehat{E} \setminus E\right),$$

and set $\alpha^{(j)}_{0,k} = g^{(j)}(0)$, $(j = 1, 2, 3)$, $\tau^{(j)}_{0,k} = 0$, $(j = 1, 2)$. Assume that we have already constructed the numbers

$$\tau^{(1)}_{0,k} < \tau^{(1)}_{1,k} < \cdots < \tau^{(1)}_{s,k}, \quad \tau^{(2)}_{0,k} < \tau^{(2)}_{1,k} < \cdots < \tau^{(2)}_{s,k},$$

$$\alpha^{(1)}_{0,k} \ge \alpha^{(1)}_{1,k} \ge \ldots \ge \alpha^{(1)}_{s,k} \ge \alpha^{(2)}_{0,k} \ge \alpha^{(2)}_{1,k} \ge \ldots \ge \alpha^{(2)}_{s,k} \ge \alpha^{(3)}_{0,k} \ge \alpha^{(3)}_{1,k} \ge \ldots \ge \alpha^{(3)}_{s,k}$$

and the pairwise disjoint sets

$$E'_{l,k} \subset E', \quad E''_{l,k} \subset E'', \quad \widehat{E}_{l,k} \subset \widehat{E} \setminus E, \quad l = 1, \ldots, s$$

such that

$$\max\left(\alpha^{(1)}_{l,k}, \alpha^{(1)}_{l-1,k} - \frac{1}{k}\right) \le f(x) \le \alpha^{(1)}_{l-1,k}, \quad x \in E'_{l,k},$$

$$\max\left(\alpha^{(2)}_{l,k}, \alpha^{(2)}_{l-1,k} - \frac{1}{k}\right) \le f(x) \le \alpha^{(2)}_{l-1,k}, \quad x \in E''_{l,k},$$

$$\alpha^{(3)}_{l,k} \le f(x) \le \alpha^{(3)}_{l-1,k}, \quad x \in \widehat{E}_{l,k},$$

$$\mu\left(E'_{l,k}\right) = \mu\left(\widehat{E}_{l,k}\right) = \tau^{(1)}_{l,k} - \tau^{(1)}_{l-1,k}, \tag{C.18}$$

$$\mu\left(E''_{l,k}\right) = \tau^{(2)}_{l,k} - \tau^{(2)}_{l-1,k},$$

$$\int_{E'_{l,k}} f(x)\, d\mu(x) - \int_{\widehat{E}_{l,k}} f(x)\, d\mu(x) = \int_{E''_{l,k}} (A - f(x))\, d\mu(x). \tag{C.19}$$

If

$$\mu\left(E'\right) = \sum_{l=1}^{s} \mu\left(E'_{l,k}\right),$$

then the partition of the sets E', E'' and $\widehat{E} \setminus E$ can be stopped here. Otherwise we can construct $\alpha^{(j)}_{s+1,k}$, $\tau^{(j)}_{s+1,k}$, $E'_{s+1,k}$, $E''_{s+1,k}$ and $\widehat{E}_{s+1,k}$ in the following way. Consider the strictly increasing continuous functions

$$\eta(\tau) = \int_{\tau^{(1)}_{s,k}}^{\tau} \left(g^{(1)}(t) - g^{(3)}(t)\right) dt, \quad \tau^{(1)}_{s,k} \le \tau \le \mu\left(E'\right),$$

and

$$\zeta(\xi) = \int_{\tau^{(2)}_{s,k}}^{\xi} \left(A - g^{(2)}(t)\right) dt, \quad \tau^{(2)}_{s,k} \le \xi \le \mu\left(E''\right).$$

Clearly $\eta\left(\tau^{(1)}_{s,k}\right) = \zeta\left(\tau^{(2)}_{s,k}\right) = 0$ and $\eta\left(\mu\left(E'\right)\right) = \zeta\left(\mu\left(E''\right)\right)$. Due to the strict monotonicity of the function $\zeta(\xi)$ for any $\tau \in \left[\tau^{(1)}_{s,k}, \mu\left(E'\right)\right]$ there exists a unique value $\xi = \xi(\tau)$ such that

$$\zeta(\xi(\tau)) = \eta(\tau), \tag{C.20},$$

and the function $\xi(\tau)$ is continuous. Denote

$$\tau^{(1)}_{s+1,k} = \sup\left\{\tau \in \left(\tau^{(1)}_{s,k}, \mu\left(E'\right)\right] :\right.$$

$$\left.\max\left[g^{(1)}\left(\tau^{(1)}_{s,k}\right) - g^{(1)}(\tau),\ g^{(2)}\left(\tau^{(2)}_{s,k} + 0\right) - g^{(2)}(\xi(\tau))\right] < \frac{1}{k}\right\},$$

$$\tau^{(2)}_{s+1,k} = \xi\left(\tau^{(1)}_{s+1,k}\right).$$

Notice that the domain of definition of the supremum in not empty because the non-increasing rearrangement is continuous from the right. Set $\alpha^{(1)}_{s+1,k} = g^{(1)}\left(\tau^{(1)}_{s+1,k}\right)$, $\alpha^{(2)}_{s+1,k} = g^{(2)}\left(\tau^{(2)}_{s+1,k}\right)$, $\alpha^{(3)}_{s+1,k} = g^{(3)}\left(\tau^{(1)}_{s+1,k}\right)$. Then we can construct the sets

$$E'_{s+1,k} \subset E' \setminus \left(\bigcup_{l=1}^{s} E'_{l,k}\right),\ E''_{s+1,k} \subset E'' \setminus \left(\bigcup_{l=1}^{s} E''_{l,k}\right),$$

$$\widehat{E}_{s+1,k} \subset \left(\widehat{E} \setminus E\right) \setminus \left(\bigcup_{l=1}^{s} \widehat{E}_{l,k}\right)$$

such that

$$\max\left(\alpha^{(1)}_{s+1,k}, \alpha^{(1)}_{s,k} - \frac{1}{k}\right) \le f(x) \le \alpha^{(1)}_{s,k}, \quad x \in E'_{s+1,k}, \tag{C.21}$$

$$\max\left(\alpha^{(2)}_{s+1,k}, \alpha^{(2)}_{s,k} - \frac{1}{k}\right) \le f(x) \le \alpha^{(2)}_{s,k}, \quad x \in E''_{s+1,k}, \tag{C.22}$$

$$\alpha^{(3)}_{s+1,k} \le f(x) \le \alpha^{(3)}_{s,k}, \quad x \in \widehat{E}_{s+1,k}.$$

In view of this construction equality $(C.20)$ reads

$$\int_{E'_{s+1,k}} f(x)\, d\mu(x) - \int_{\widehat{E}_{s+1,k}} f(x)\, d\mu(x) = \int_{E''_{s+1,k}} (A - f(x))\, d\mu(x).$$

If

$$\mu(E') = \sum_{l=1}^{s+1} \mu\left(E'_{l,k}\right),$$

then from $(C.17)$ we get

$$\mu(E'') = \sum_{l=1}^{s+1} \mu\left(E''_{l,k}\right), \quad \mu\left(\widehat{E} \setminus E\right) = \sum_{l=1}^{s+1} \mu\left(\widehat{E}_{l,k}\right),$$

and so the partition of the sets E', E'' and $\widehat{E} \setminus E$ is finished. Otherwise we have

$$\max_{j=1,2}\left(\alpha^{(j)}_{s,k} - \alpha^{(j)}_{s+1,k}\right) \ge \frac{1}{k},$$

which implies that the number s of the steps cannot grow infinitely, i.e., there exists s_k such that

$$E' = \bigcup_{l=1}^{s_k} E'_{l,k}, \quad E'' = \bigcup_{l=1}^{s_k} E''_{l,k}, \quad \widehat{E} \setminus E = \bigcup_{l=1}^{s_k} \widehat{E}_{l,k}.$$

For $l = 1, \ldots, s_k$ denote

$$b_{l,k} = \frac{1}{\mu\left(E'_{l,k}\right)} \int_{E'_{l,k}} f(x)\, d\mu(x), \quad c_{l,k} = \frac{1}{\mu\left(E''_{l,k}\right)} \int_{E''_{l,k}} f(x)\, d\mu(x),$$

$$d_{l,k} = \frac{1}{\mu\left(\widehat{E}_{l,k}\right)} \int_{\widehat{E}_{l,k}} f(x)\, d\mu(x).$$

Then equality $(C.19)$ can be rewritten in the following form

$$b_{l,k}\mu\left(E'_{l,k}\right) + c_{l,k}\mu\left(E''_{l,k}\right) = d_{l,k}\mu\left(\widehat{E}_{l,k}\right) + A\mu\left(E''_{l,k}\right).$$

Taking into account $(C.18)$, we have

$$\frac{\mu\left(E'_{l,k}\right)}{\mu\left(E'_{l,k}\right) + \mu\left(E''_{l,k}\right)} b_{l,k} + \frac{\mu\left(E''_{l,k}\right)}{\mu\left(E'_{l,k}\right) + \mu\left(E''_{l,k}\right)} c_{l,k} =$$

$$= \frac{\mu\left(\widehat{E}_{l,k}\right)}{\mu\left(\widehat{E}_{l,k}\right) + \mu\left(E''_{l,k}\right)} d_{l,k} + \frac{\mu\left(E''_{l,k}\right)}{\mu\left(\widehat{E}_{l,k}\right) + \mu\left(E''_{l,k}\right)} A.$$

Since $A \geq b_{l,k} \geq c_{l,k} \geq d_{l,k}$ the application of Lemma $C.3$ with $a = A$, $b = b_{l,k}$, $c = c_{l,k}$, $d = d_{l,k}$ and $\gamma_1 = \gamma_2 = \mu\left(\widehat{E}_{l,k}\right) / \left(\mu\left(\widehat{E}_{l,k}\right) + \mu\left(E''_{l,k}\right)\right)$ yields

$$\mu\left(E'_{l,k}\right)\varphi\left(b_{l,k}\right) + \mu\left(E''_{l,k}\right)\varphi\left(c_{l,k}\right) \leq \mu\left(\widehat{E}_{l,k}\right)\varphi\left(d_{l,k}\right) + \mu\left(E''_{l,k}\right)\varphi(A).$$

Applying now Jensen inequality $(C.1)$ to the second term in the right-hand side we obtain

$$\mu\left(E'_{l,k}\right)\varphi\left(b_{l,k}\right) + \mu\left(E''_{l,k}\right)\varphi\left(c_{l,k}\right) \leq$$

$$\leq \int_{\widehat{E}_{l,k}} \varphi(f(x))\, d\mu(x) + \mu\left(E''_{l,k}\right)\varphi(A), \quad l = 1, \ldots, s_k.$$

Hence

$$\int_{E \bigcap \widehat{E}} \varphi(f(x))\, d\mu(x) + \sum_{l=1}^{s_k} \left(\mu\left(E'_{l,k}\right)\varphi\left(b_{l,k}\right) + \mu\left(E''_{l,k}\right)\varphi\left(c_{l,k}\right)\right) \leq$$

$$\leq \int_{E \bigcap \widehat{E}} \varphi(f(x))\, d\mu(x) + \int_{\widehat{E} \setminus E} \varphi(f(x))\, d\mu(x) + \mu\left(E''\right)\varphi(A) =$$

$$= \int_{\widehat{E}} \varphi(f(x))\, d\mu(x) + \mu\left(E''\right)\varphi(A) =$$

$$= \frac{\mu(E)}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} \varphi(f(x))\, d\mu(x) + \left(1 - \frac{\mu(E)}{\mu\left(\widehat{E}\right)}\right) \int_{\widehat{E}} \varphi(f(x))\, d\mu(x) + \mu\left(E''\right)\varphi(A). \tag{C.23}$$

But since

$$\mu(E) - \mu\left(\widehat{E}\right) = \mu\left(E'\right) + \mu\left(E''\right) - \mu\left(\widehat{E} \setminus E\right) = \mu\left(E''\right)$$

it is easy to see that Jensen inequality $(C.1)$ together with $(C.12)$ implies

$$\left(1 - \frac{\mu(E)}{\mu\left(\widehat{E}\right)}\right) \int_{\widehat{E}} \varphi(f(x))\, d\mu(x) + \mu\left(E''\right)\varphi(A) \leq$$

$$\leq \left(\mu\left(\widehat{E}\right) - \mu\left(E\right)\right) \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} \varphi(f(x))\, d\mu(x) +$$

$$+\mu\left(E''\right) \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} \varphi(f(x))\, d\mu(x) =$$

$$= \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} \varphi(f(x))\, d\mu(x) \left(\mu\left(\widehat{E}\right) - \mu(E) + \mu\left(E''\right)\right) = 0.$$

Therefore $(C.23)$ is equivalent to the inequality

$$\frac{1}{\mu(E)} \left[\int_{E \bigcap \widehat{E}} \varphi(f(x))\, d\mu(x) + \sum_{l=1}^{s_k} \left(\mu\left(E'_{l,k}\right)\varphi\left(b_{l,k}\right) + \mu\left(E''_{l,k}\right)\varphi\left(c_{l,k}\right)\right)\right] \leq$$

$$\leq \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} \varphi(f(x))\, d\mu(x). \tag{C.24}$$

Setting

$$f_k(x) = f(x)\chi_{E \bigcap \widehat{E}}(x) + \sum_{l=1}^{s_k} \left(b_{l,k}\chi_{E'_{l,k}}(x) + c_{l,k}\chi_{E''_{l,k}}(x)\right), \quad x \in E.$$

we can rewrite inequality $(C.24)$ as follows

$$\frac{1}{\mu(E)} \int_{E} \varphi\left(f_k(x)\right)\, d\mu(x) \leq \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} \varphi(f(x))\, d\mu(x). \tag{C.25}$$

Since, by $(C.21)$ and $(C.22)$,

$$|f(x) - f_k(x)| \leq \frac{1}{k}$$

for μ-almost all $x \in E$ it follows that the sequence of functions $\varphi(f_k)$ converges to $\varphi(f)$ μ-almost everywhere, provided the convex function φ is monotone. Therefore the application of the Fatou lemma to $(C.25)$ yields $(C.15)$. □

Lemma C.6 ([35]). *Let f be a non-negative function on $E \bigcup \widehat{E}$ such that*

$$\frac{1}{\mu(E)} \int_E f(x)\, d\mu(x) = \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} f(x)\, d\mu(x) \equiv A,$$

$$f(x) \geq A, \quad x \notin E \bigcap \widehat{E},$$

$$f(x) \geq f(y), \quad x \in \widehat{E} \setminus E,\ y \in E.$$

Then for any $\varphi \in \Phi$

$$\frac{1}{\mu(E)} \int_E \varphi(f(x))\, d\mu(x) \leq \frac{1}{\mu\left(\widehat{E}\right)} \int_{\widehat{E}} \varphi(f(x))\, d\mu(x).$$

The proof of this lemma is analogous to the proof of Lemma $C.5$ and we omit it here.

Proof of Theorem C.1. Since the equality

$$f_\mu^{\downarrow}(t) = f_\mu^{\uparrow}(\mu(R_0) - t)$$

holds true in all points of continuity of the rearrangements, i.e., almost everywhere on $[0, \mu(R_0)]$, inequalities $(C.3)$ and $(C.4)$ are equivalent.

Fix an interval $I \subset [0, \mu(R_0)]$. If

$$\frac{1}{|I|} \int_I f_\mu^{\downarrow}(t)\, dt \geq \frac{1}{\mu(R_0)} \int_{R_0} f(x)\, d\mu(x),$$

then there exists $T \in [0, \mu(R_0)]$ such that

$$\frac{1}{|I|} \int_I f_\mu^{\downarrow}(t)\, dt = \frac{1}{T} \int_0^T f_\mu^{\downarrow}(t)\, dt.$$

In this case denote $J = [0, T]$. Otherwise, if

$$\frac{1}{|I|} \int_I f_\mu^{\downarrow}(t)\, dt < \frac{1}{\mu(R_0)} \int_{R_0} f(x)\, d\mu(x),$$

then we can choose $T \in [0, \mu(R_0)]$ such that

$$\frac{1}{|I|} \int_I f_\mu^{\downarrow}(t)\, dt = \frac{1}{T} \int_{\mu(R_0)-T}^{\mu(R_0)} f_\mu^{\downarrow}(t)\, dt$$

and denote $J = [\mu(R_0) - T, \mu(R_0)]$. In both cases we have

$$\frac{1}{|I|}\int_I f_\mu^\downarrow(t)\,dt = \frac{1}{|J|}\int_J f_\mu^\downarrow(t)\,dt.$$

Moreover, since $f_\mu^\downarrow$ is monotone $J \supset I$. Setting $[a,b] = J$, $[\alpha,\beta] = I$ and $g = f_\mu^\downarrow$ in Lemma $C.4$ we get

$$\frac{1}{|I|}\int_I \varphi\left(f_\mu^\downarrow(t)\right)\,dt \leq \frac{1}{|J|}\int_J \varphi\left(f_\mu^\downarrow(t)\right)\,dt.$$

Therefore it is enough to proof inequality ($C.3$) only for the interval J. In other words, ($C.3$) and ($C.4$) follow from the pair of inequalities

$$\frac{1}{T}\int_0^T \varphi\left(f_\mu^\downarrow(t)\right)\,dt \leq B\cdot\varphi\left(\frac{1}{T}\int_0^T f_\mu^\downarrow(t)\ ,dt\right),\quad 0\leq T\leq \mu\left(R_0\right),\quad (C.26)$$

$$\frac{1}{T}\int_0^T \varphi\left(f_\mu^\uparrow(t)\right)\,dt \leq B\cdot\varphi\left(\frac{1}{T}\int_0^T f_\mu^\uparrow(t)\,dt\right),\quad 0\leq T\leq \mu\left(R_0\right).\quad (C.27)$$

In order to prove ($C.26$) and ($C.27$) let us fix some $T\in[0,\mu\left(R_0\right)]$ and denote

$$A^\downarrow = \frac{1}{T}\int_0^T f_\mu^\downarrow(t)\,dt \geq \frac{1}{\mu\left(R_0\right)}\int_{R_0} f(x)\,d\mu(x),\qquad (C.28)$$

$$A^\uparrow = \frac{1}{T}\int_0^T f_\mu^\uparrow(t)\,dt \leq \frac{1}{\mu\left(R_0\right)}\int_{R_0} f(x)\,d\mu(x).$$

By Lemmas $B.1$ and $C.2$, we can construct two collections of pairwise disjoint segments $R_j^\downarrow\subset R_0$ and $R_k^\uparrow\subset R_0$ such that

$$\frac{1}{\mu\left(R_j^\downarrow\right)}\int_{R_j^\downarrow} f(x)\,d\mu(x) = A^\downarrow,\quad j=1,2,\ldots,\qquad (C.29)$$

$$f(x)\leq A^\downarrow \text{ for } \ \mu-\text{almost all } x\in R_0\setminus\left(\bigcup_{j\geq 1} R_j^\downarrow\right),$$

$$\frac{1}{\mu\left(R_k^\uparrow\right)}\int_{R_k^\uparrow} f(x)\,d\mu(x) = A^\uparrow,\quad k=1,2,\ldots,\qquad (C.30)$$

$$f(x)\geq A^\uparrow \text{ for } \ \mu-\text{almost all } x\in R_0\setminus\left(\bigcup_{k\geq 1} R_k^\downarrow\right).$$

Let $\widehat{E}^\downarrow = \bigcup_{j\geq 1} R_j^\downarrow$, $\widehat{E}^\uparrow = \bigcap_{k\geq 1} R_k^\uparrow$. Then

$$\mu\left(\left\{x\in R_0:\ f(x) > A^\downarrow\right\}\setminus\widehat{E}^\downarrow\right) = 0,\qquad (C.31)$$

$$\mu\left(\left\{x \in R_0 : \ f(x) < A^{\uparrow}\right\} \setminus \widehat{E}^{\uparrow}\right) = 0.$$

Now, if we prove inequalities

$$\frac{1}{T}\int_0^T \varphi\left(f_\mu^{\downarrow}(t)\right)\,dt \le \frac{1}{\mu\left(\widehat{E}^{\downarrow}\right)}\int_{\widehat{E}^{\downarrow}} \varphi(f(x))\,d\mu(x), \tag{C.32}$$

$$\frac{1}{T}\int_0^T \varphi\left(f_\mu^{\uparrow}(t)\right)\,dt \le \frac{1}{\mu\left(\widehat{E}^{\uparrow}\right)}\int_{\widehat{E}^{\uparrow}} \varphi(f(x))\,d\mu(x), \tag{C.33}$$

then we immediately get (C.26) and (C.27). Indeed, by (C.32), (C.1) and (C.29),

$$\frac{1}{T}\int_0^T \varphi\left(f_\mu^{\downarrow}(t)\right)\,dt \le \left(\sum_{j\ge 1}\mu\left(R_j^{\downarrow}\right)\right)^{-1}\sum_{j\ge 1}\int_{R_j^{\downarrow}} \varphi(f(x))\,d\mu(x) \le$$

$$\le \sup_{j\ge 1}\frac{1}{\mu\left(R_j^{\downarrow}\right)}\int_{R_j^{\downarrow}} \varphi(f(x))\,d\mu(x) \le$$

$$\le B\cdot\sup_{j\ge 1}\varphi\left(\frac{1}{\mu\left(R_j^{\downarrow}\right)}\int_{R_j^{\downarrow}} f(x)\,d\mu(x)\right) = B\cdot\varphi\left(A^{\downarrow}\right),$$

i.e., (C.26). Similarly, inequality (C.27) follows from (C.33), (C.1) and (C.30). So, it remains to prove (C.32) and (C.33). For this let us construct the sets $E^{\downarrow}$ and $E^{\uparrow}$ such that

$$\mu\left(E^{\downarrow}\right) = \mu\left(E^{\uparrow}\right) = T$$

and

$$f(x) \ge f_\mu^{\downarrow}(T), \quad x \in E^{\downarrow},$$

$$f(x) \le f_\mu^{\uparrow}(T), \quad x \in E^{\uparrow}.$$

Let us show that the sets $E = E^{\downarrow}$ and $\widehat{E} = \widehat{E}^{\downarrow}$ satisfy the conditions of Lemma C.5. Indeed, by (C.28) and (C.29),

$$\frac{1}{\mu\left(E^{\downarrow}\right)}\int_{E^{\downarrow}} f(x)\,d\mu(x) = \frac{1}{T}\int_0^T f_\mu^{\downarrow}(t)\,dt = A^{\downarrow} =$$

$$= \left(\sum_{j\ge 1}\mu\left(R_j^{\downarrow}\right)\right)^{-1}\sum_{j\ge 1}\int_{R_j^{\downarrow}} f(x)\,d\mu(x) = \frac{1}{\mu\left(\widehat{E}^{\downarrow}\right)}\int_{\widehat{E}^{\downarrow}} f(x)\,d\mu(x),$$

which is (C.12) for $A = A^{\downarrow}$. Further, (C.31) implies that the embedding

$$\left\{x \in R_0 : \ f(x) > A^{\downarrow}\right\} \subset \widehat{E}^{\downarrow}$$

holds true up to a set of μ-mesure zero. In its own turn the embedding

$$\left\{x \in R_0 : \ f(x) > A^{\downarrow}\right\} \subset E^{\downarrow}$$

follows from the definition of the equimeasurable rearrangement $f_{\mu}^{\downarrow}$. Clearly (C.13) follows from these two embeddings. Finally, inequality (C.14) is simply a property of the rearrangement $f_{\mu}^{\downarrow}$. Similarly one can show that the sets $E = E^{\uparrow}$ and $\widehat{E} = \widehat{E}^{\uparrow}$ satisfy conditions of Lemma C.6. Applying Lemmas C.5 and C.6, we obtain the following inequalities

$$\frac{1}{\mu\left(E^{\downarrow}\right)} \int_{E^{\downarrow}} \varphi(f(x))\, d\mu(x) \le \frac{1}{\mu\left(\widehat{E}^{\downarrow}\right)} \int_{\widehat{E}^{\downarrow}} \varphi(f(x))\, d\mu(x),$$

$$\frac{1}{\mu\left(E^{\uparrow}\right)} \int_{E^{\uparrow}} \varphi(f(x))\, d\mu(x) \le \frac{1}{\mu\left(\widehat{E}^{\uparrow}\right)} \int_{\widehat{E}^{\uparrow}} \varphi(f(x))\, d\mu(x),$$

which are equivalent to (C.32) and (C.33) respectively. □

C.2 About the Exact Extension of the Reverse Weighted Hölder Inequality

The following theorem describes the fundamental properties of Gehring and Muckenhoupt classes (see [59, 18, 8]).

Theorem C.7 (Coifman, Fefferman, [8]).

a) For any $q > 1,\ B > 1$ *there exist* $q_1 \equiv q_1(q, B, d) > q$ *and* $p_1 \equiv p_1(q, B, d) > 1$ *such that*

$$G_q(B) \subset G_{q'}\left(B_1\right), \tag{C.34}$$

$$G_q(B) \subset A_{p'}\left(B_2\right) \tag{C.35}$$

for all $q < q' < q_1,\ p' > p_1,$ *where* $B_1 \equiv B_1\left(q, B, q', d\right),\ B_2 \equiv B_2\left(q, B, p', d\right).$

b) For any $p > 1,\ B > 1$ *there exist* $p_2 \equiv p_2(p, B, d) < p\ \ (p_2 > 1)$ *and* $q_2 \equiv q_2(p, B, d) > 1$ *such that*

$$A_p(B) \subset A_{p''}\left(B_3\right), \tag{C.36}$$

$$A_p(B) \subset G_{q''}\left(B_4\right) \tag{C.37}$$

for all $p'' > p_2,\ 1 < q'' < q_2,$ *where* $B_3 \equiv B_3\left(p, B, p'', d\right),\ B_4 \equiv B_4\left(p, B, q'', d\right).$

Embeddings (C.34) and (C.36) describe the so-called "self-improvement of exponents" property of the Gehring and Muckenhoupt classes, while (C.35) and (C.37) illustrate their interconnection. There are a lot of publications that study the properties described by Theorem C.7. For instance, Bojarski and Wik in [4, 5, 79] found the exact asymptotic behavior of $q_1(q, B, d)$ for

$B \to 1+0$. D'Apuzzo and Sbordone in [67, 10] calculated the maximal value of $q_1(q, B, 1)$ for the subclass of $G_q(B)$, consisting of non-increasing functions of one variable. In [35] it was shown that in the case $d = 1$ it is enough to consider only monotone functions to find the extremal values of p_i and q_i $(i = 1, 2)$ in Theorem $C.7$. This is why the maximal value of $q_1(q, B, 1)$, which was found in [10] for monotone functions, remains the same in the general case. In [35] it was also found the minimal value of $p_2(p, B, 1)$. The results of [10] and [35] were generalized by Popoli in [63]. Namely, in [63] it was shown that in the one-dimensional case the condition $f \in RH_{dx}^{\alpha,\beta}(B)$ implies the summability of the function f^s, where

a) $\beta \le s < \beta_0$, if $\alpha \cdot \beta > 0$,

b) $\alpha_0 < s \le \alpha$, if $\alpha \cdot \beta < 0$,

and the extremal values β_0 in the *a*)-case and α_0 in the *b*)-case are the roots of the equation

$$\left(\frac{x}{x-\beta}\right)^{\frac{1}{\beta}} = B \cdot \left(\frac{x}{x-\alpha}\right)^{\frac{1}{\alpha}}.$$

Essentially this result is a natural generalization of exact embeddings $(C.34)$ and $(C.36)$ in the case $d = 1$. Further, in [62] Popoli proved the weighted analog of embedding $(C.34)$ with the maximal exponent $q_1(q, B, 1)$. As the corollary of this result he obtained $(C.36)$ with the minimal exponent $p_2(p, B, 1)$. The best values of $p_1(q, B, 1)$ and $q_2(p, B, 1)$ in $(C.35)$ and $(C.37)$ respectively were found by Malaksiano in [54, 55]. Moreover, Vasiunin in [75] found not only the extremal values of $p_2(p, B, 1)$ and $q_2(p, B, 1)$ for embeddings $(C.36)$ and $(C.37)$, but also the best values of the constants B_3 and B_4. The limiting exponent of summability for a $RH_{dx}^{1,\beta}(B)$-function for any $d \ge 1$ was obtained by Kinnunen in [31].

So, in the one-dimensional case the extremal values p_i and q_i $(i = 1, 2)$ in Theorem $C.7$ are known for all embeddings $(C.34)$ - $(C.37)$. Using these results in the case $d = 1$ it is not difficult to find the best "improvements" for each exponent of the embedding of the class $RH_{dx}^{\alpha,\beta}(B)$ into another class $RH_{dx}^{\alpha',\beta'}(B')$. However it worth to mention that the authors of the publications cited above proposed rather different methods to find the extremal exponents. Indeed, in [10, 35, 63, 62] the proofs are based on the application of various versions of the Hardy inequalities; in [54, 55] the author compare a given function with the one, which is assumed extremal; in [75] the proof is based on the application of the Bellman function. Here we realize the uniform approach for the analysis of the embeddings of the class $RH_{d\mu(x)}^{\alpha,\beta}(B)$, which does not depend on the signs of its parameters α and β and works in the multidimensional case, too. Namely, as in [35], using the exact estimate of the equimeasurable rearrangements of functions (Theorem $C.1$) we reduce the problem to the case of a monotone function of one variable. Then, as in [10] and the successive works [35, 63, 62], applying the appropriate weighted analogs of the Hardy inequalities we prove the following theorem *about the exact embedding of* $RH_{d\mu(x)}^{\alpha,\beta}(B)$*-classes.*

Theorem C.8 ([49]). *Let $d \geq 1$ and assume that the non-negative function f is such that*

$$\left\{\frac{1}{\mu(R)}\int_R f^\beta(x)\,d\mu(x)\right\}^{\frac{1}{\beta}} \leq B\left\{\frac{1}{\mu(R)}\int_R f^\alpha(x)\,d\mu(x)\right\}^{\frac{1}{\alpha}} \qquad (C.38)$$

uniformly over all segments $R \subset R_0$, where $R_0 \subset \mathbb{R}^d$ is a fixed segment, $d\mu(x) = w(x)\,dx$ is an absolutely continuous measure on R_0, the constants $\alpha < \beta$ are different from zero and $B > 1$. Then for every $\gamma \in (-\infty, \min(0,\alpha)) \cup (\max(0,\beta), +\infty)$ such that

$$\left(1-\frac{\alpha}{\gamma}\right)^{\frac{1}{\alpha}} > B\cdot\left(1-\frac{\beta}{\gamma}\right)^{\frac{1}{\beta}}, \qquad (C.39)$$

there exist positive constants $B' \equiv B'(\alpha,\beta,B,\gamma)$ and $B'' \equiv B''(\alpha,\beta,B,\gamma)$ such that on every segment $R \subset R_0$

$$\frac{1}{B'}\left\{\frac{1}{\mu(R)}\int_R f^\alpha(x)\,d\mu(x)\right\}^{\frac{1}{\alpha}} \leq \left\{\frac{1}{\mu(R)}\int_R f^\gamma(x)\,d\mu(x)\right\}^{\frac{1}{\gamma}} \leq$$

$$\leq B''\left\{\frac{1}{\mu(R)}\int_R f^\beta(x)\,d\mu(x)\right\}^{\frac{1}{\beta}}. \qquad (C.40)$$

Moreover, if $d\mu(x) = dx$ and $\gamma \in (-\infty, \min(0,\alpha)) \cup (\max(0,\beta), +\infty)$ does not satisfy (C.39), then in general one of two inequalities (C.40) fails. More precisely, the left inequality fails if $\gamma < \alpha$, while the right one fails if $\gamma > \beta$.

We will prove this theorem later (see p.181). First let us consider some auxiliary statemets.

Remark C.9. Fix $\alpha,\ \beta$ such that $\alpha \cdot \beta \neq 0$ and $\alpha < \beta$ and let

$$\Psi_{\alpha,\beta}(\gamma) = \left(1-\frac{\alpha}{\gamma}\right)^{\frac{1}{\alpha}} \cdot \left(1-\frac{\beta}{\gamma}\right)^{-\frac{1}{\beta}}.$$

This function is defined for all $\gamma \in (-\infty, \min(0,\alpha)) \bigcup (\max(0,\beta), +\infty)$, it is continuous and strictly increasing from 1 to $+\infty$ on $(-\infty, \min(0,\alpha))$ and strictly decreasing from $+\infty$ to 1 on $(\max(0,\beta), +\infty)$. Hence the equation $\Psi_{\alpha,\beta}(\gamma) = B$ has exactly two roots $\gamma^- < \min(0,\alpha)$ and $\gamma^+ > \max(0,\beta)$ for any $B > 1$. Observe that inequality (C.40) for $\gamma \in [\alpha,\beta] \setminus \{0\}$ is the standard Hölder inequality with $B' = B'' = 1$. Thus Theorem C.8 states that $f \in RH^{\alpha,\beta}_{d\mu(x)}(B)$ implies (C.40) for all $\gamma \in (\gamma^-, \gamma^+) \setminus \{0\}$ and in general the values γ^- and γ^+ cannot be improved.

Remark C.10. We do not consider the limit cases $-\infty,\ 0,\ +\infty$ for the parameters $\alpha,\ \beta,\ \gamma$.

Remark C.11. From Theorem C.8 in particular if follows that the limiting positive and negative exponents of summability of $f \in RH^{\alpha,\beta}_{d\mu(x)}(B)$ do not depend on the dimension d. Indeed, condition $(C.38)$ for the function f to be of class $RH^{\alpha,\beta}_{d\mu(x)}(B)$ is assumed to be satisfied over all segments. We do not know the exact limiting exponents of summability for $\widetilde{RH}^{\alpha,\beta}_{d\mu(x)}(B)$-functions for $d \geq 2$ even in some particular non-trivial case.

C.2.1 Hardy's Inequalities

Let the numbers $q,\ p$ be different from zero and let v be a weighted function, i.e., we assume that v is a non-negative function, summable on the interval $[a,b] \subset \mathbb{R}$. Let g be a non-negative function on $[a,b]$. Denote

$$\langle g \rangle_{q,p,v(t)\,dt,[a,b]} = \{v([a,b])\}^{-\frac{1}{p}} \left\{ \int_a^b ((v[a,t]))^{\frac{q}{p}-1}\, g^q(t) v(t)\, dt \right\}^{\frac{1}{q}},$$

where $v([a,t]) = \int_a^t v(\tau)\,d\tau\ (a \leq t \leq b)$. It is easy to see that for $r \neq 0$

$$\langle g^r \rangle_{q,p,v(t)\,dt,[a,b]} = \langle g \rangle^r_{rq,rp,v(t)\,dt,[a,b]}\,. \qquad (C.41)$$

Observe that the following equality

$$\frac{p}{p-q}\left[\langle g \rangle^q_{q,p,v(t)\,dt,[a,b]} - \langle g \rangle^q_{q,q,v(t)\,dt,[a,b]}\right] = \left\langle\, \langle g \rangle_{q,q,v(\tau)\,d\tau,[a,\cdot]}\, \right\rangle^q_{q,p,v(t)\,dt,[a,b]} \qquad (C.42)$$

holds true for all p and q such that $p \neq q,\ pq \neq 0$. It can be easily checked by integration by parts. If we assume $\langle \cdot \rangle^0_{0,0,v(t)\,dt,[a,b]} = 1$, then the equality

$$\langle g \rangle_{q,q,g^p(t)v(t)\,dt,[a,b]} = \langle g \rangle^{-\frac{p}{q}}_{p,p,v(t)\,dt,[a,b]} \cdot \langle g \rangle^{1+\frac{p}{q}}_{q+p,q+p,v(t)\,dt,[a,b]} \qquad (C.43)$$

holds true for all p and q different from zero.

The next lemma is the *weighted analog of the Hardy inequality* ([26]). For $p \geq q \geq 1,\ p > 1$ it was proved in [62]. In the other case $(p < 0,\ q < 0)$ the proof of this lemma is analogues to the proof of the Hardy inequality in [26].

Lemma C.12. *Let either* $p \geq q \geq 1,\ p > 1$ *or* $p < 0,\ q < 0$. *Then*

$$\left\langle\, \langle g \rangle_{1,1,v(\tau)\,d\tau,[a,\cdot]}\, \right\rangle^q_{q,p,v(t)\,dt,[a,b]} \leq \left(\frac{p}{p-1}\right)^q \langle g \rangle^q_{q,p,v(t)\,dt,[a,b]}\,. \qquad (C.44)$$

Proof. We can rewrite $(C.44)$ as follows

$$\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} \left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1} \int_a^t g(\tau)v(\tau)\,d\tau\right)^q v(t)\,dt \leq$$

$$\leq \left(\frac{p}{p-1}\right)^q \int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} g^q(t)v(t)\,dt.$$

We prove this inequality by integration by parts. We have

$$\frac{q}{p}\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} \left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^q v(\tau)\,d\tau =$$

$$= \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}} \left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^q \Bigg|_{t=a}^{t=b} +$$

$$+q\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} \left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^q v(t)\,dt-$$

$$-q\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} g(t)\left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^{q-1} v(t)\,dt =$$

$$= \left(\int_a^b v(t)\,dt\right)^{\frac{q}{p}} \left(\left(\int_a^b v(t)\,dt\right)^{-1}\int_a^b g(t)v(t)\,dt\right)^q +$$

$$+q\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} \left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^q v(t)\,dt-$$

$$-q\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} g(t)\left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^{q-1} v(t)\,dt \geq$$

$$\geq q\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} \left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^q v(t)\,dt-$$

$$-q\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} g(t)\left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^{q-1} v(t)\,dt.$$

Applying now the Hölder inequality with the exponent q and the weighted function $\left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} v(t)$ we obtain

$$q\left(1-\frac{1}{p}\right)\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} \left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^q v(t)\,dt \leq$$

$$\leq q\int_a^b \left(\int_a^t v(\tau)\,d\tau\right)^{\frac{q}{p}-1} g(t)\left(\left(\int_a^t v(\tau)\,d\tau\right)^{-1}\int_a^t g(\tau)v(\tau)\,d\tau\right)^{q-1} v(t)\,dt \leq$$

$$\leq q \left\{ \int_a^b \left(\int_a^t v(\tau)\, d\tau \right)^{\frac{q}{p}-1} g^q(t) v(t)\, dt \right\}^{\frac{1}{q}} \times$$

$$\times \left\{ \int_a^b \left(\int_a^t v(\tau)\, d\tau \right)^{\frac{q}{p}-1} \left(\left(\int_a^t v(\tau)\, d\tau \right)^{-1} \int_a^t g(\tau) v(\tau)\, d\tau \right)^q v(t)\, dt \right\}^{1-\frac{1}{q}}.$$

Notice that the last inequality holds true for $q \geq 1$ as well as for $q < 0$. Since $p \in (-\infty, 0) \cup (1, +\infty)$ it follows that $1 - \frac{1}{p} > 0$ and

$$q \left\{ \int_a^b \left(\int_a^t v(\tau)\, d\tau \right)^{\frac{q}{p}-1} \left(\left(\int_a^t v(\tau)\, d\tau \right)^{-1} \int_a^t g(\tau) v(\tau)\, d\tau \right)^q v(t)\, dt \right\}^{\frac{1}{q}} \leq$$

$$\leq \frac{pq}{p-1} \left\{ \int_a^b \left(\int_a^t v(\tau)\, d\tau \right)^{\frac{q}{p}-1} g^q(t) v(t)\, dt \right\}^{\frac{1}{q}}.$$

Now if we divide this inequality by q, raise the both parts to the q-th power and multiply the result by $\left\{ \int_a^b v(t)\, dt \right\}^{-\frac{q}{p}}$, we obtain exactly $(C.44)$. □

The next lemma is the *the weighed analog* of another well-known *Hardy inequality* [25]. The proof of this lemma can be find in [62].

Lemma C.13. *Let g be a non-increasing function on $[a, b]$ and let $r \geq 1$. Then*

$$\langle g \rangle_{r,r,v(t)\, dt,[a,b]} \leq \frac{1}{r} \langle g \rangle_{1,r,v(t)\, dt,[a,b]}. \qquad (C.45)$$

Proof. First of all we have

$$g^r(t) \int_a^t v(\tau)\, d\tau \leq \int_a^t g^r(\tau) v(\tau)\, d\tau, \quad a \leq t \leq b,$$

provided the function g^r is non-incresing. Further, since the function $\psi(z) = \frac{1}{r} z^{\frac{1}{r}-1}$ is non-increasing on $(0, +\infty)$ this inequality implies

$$\frac{1}{r} \left(\int_a^t g^r(\tau) v(\tau)\, d\tau \right)^{\frac{1}{r}-1} g^r(t) v(t) \leq \frac{1}{r} \left(g^r(t) \int_a^t v(\tau)\, d\tau \right)^{\frac{1}{r}-1} g^r(t) v(t).$$

The integration over $[a, b]$ yields

$$\int_a^b \frac{d}{dt} \left[\left(\int_a^t g^r(\tau) v(\tau)\, d\tau \right)^{\frac{1}{r}} \right] dt \leq \frac{1}{r} \int_a^b \left(\int_a^t v(\tau)\, d\tau \right)^{\frac{1}{r}-1} g(t) v(t)\, dt.$$

Hence

$$\left\{ \int_a^b g^r(\tau) v(\tau)\, d\tau \right\}^{\frac{1}{r}} \leq \frac{1}{r} \int_a^b \left(\int_a^t v(\tau)\, d\tau \right)^{\frac{1}{r}-1} g(t) v(t)\, dt,$$

which is equivalent to $(C.45)$. □

C.2.2 The Proof of the Main Theorem

First let us consider some particular cases of Theorem $C.8$ for monotone functions of one variable. For the function g defined on the interval $[a,b]$ the condition $g \in RH^{\alpha,\beta}_{v(t)\,dt}(B)$ can be written in the following form:

$$\langle g\rangle_{\beta,\beta,v(t)\,dt,[a',b']} \le B \cdot \langle g\rangle_{\alpha,\alpha,v(t)\,dt,[a',b']}\,, \quad [a',b'] \subset [a,b]. \tag{C.46}$$

We recall that the condition (see $(C.39)$)

$$\left(1-\frac{\alpha}{\gamma}\right)^{\frac{1}{\alpha}} > B\left(1-\frac{\beta}{\gamma}\right)^{\frac{1}{\beta}} \tag{C.47}$$

describes the relation between the parameters of Theorem $C.8$.

Lemma C.14. *Let α, β be different from zero and such that $\alpha < \beta$. Assume that the function h does not increase on $[a,b]$ and*

$$\langle h\rangle_{\beta,\beta,v(\tau)\,d\tau,[a,t]} \le B \cdot \langle h\rangle_{\alpha,\alpha,v(\tau)\,d\tau,[a,t]}\,, \quad a \le t \le b. \tag{C.48}$$

Then for any $\gamma > \max(0,\beta)$, satisfying condition $(C.47)$, there exists B'' such that

$$\langle h\rangle_{\gamma,\gamma,v(t)\,dt,[a,b]} \le B'' \cdot \langle h\rangle_{\beta,\beta,v(t)\,dt,[a,b]}\,. \tag{C.49}$$

Proof. First we consider the case $\beta > 0$. Setting $q=\beta,\ p=\gamma,\ g=h$ in $(C.42)$ and using $(C.48)$ and $(C.41)$, we have

$$\frac{\gamma}{\gamma-\beta}\cdot\langle h\rangle^{\beta}_{\beta,\gamma,v(t)\,dt,[a,b]} - \frac{\gamma}{\gamma-\beta}\cdot\langle h\rangle^{\beta}_{\beta,\beta,v(t)\,dt,[a,b]} =$$

$$= \left\langle \langle h\rangle_{\beta,\beta,v(\tau)\,d\tau,[a,\cdot]}\right\rangle^{\beta}_{\beta,\gamma,v(t)\,dt,[a,b]} \le$$

$$\le B^{\beta}\cdot\left\langle \langle h\rangle_{\alpha,\alpha,v(\tau)\,d\tau,[a,\cdot]}\right\rangle^{\beta}_{\beta,\gamma,v(t)\,dt,[a,b]} =$$

$$= B^{\beta}\cdot\left\langle \langle h\rangle^{\beta}_{\alpha,\alpha,v(\tau)\,d\tau,[a,\cdot]}\right\rangle_{1,\frac{\gamma}{\beta},v(t)\,dt,[a,b]} =$$

$$= B^{\beta}\cdot\left\langle \langle h^{\alpha}\rangle^{\frac{\beta}{\alpha}}_{1,1,v(\tau)\,d\tau,[a,\cdot]}\right\rangle_{1,\frac{\gamma}{\beta},v(t)\,dt,[a,b]} =$$

$$= B^{\beta}\cdot\left\langle \langle h^{\alpha}\rangle_{1,1,v(\tau)\,d\tau,[a,\cdot]}\right\rangle^{\frac{\beta}{\alpha}}_{\frac{\beta}{\alpha},\frac{\gamma}{\alpha},v(t)\,dt,[a,b]}.$$

Now we apply Lemma $C.12$ with $q=\frac{\beta}{\alpha},\ p=\frac{\gamma}{\alpha},\ g=h^{\alpha}$, taking into account that if $\alpha>0$, then $p>q>1$, and if $\alpha<0$, then $p<0,\ q<0$, so that the conditions of Lemma $C.12$ are satisfied. Then, by $(C.41)$,

$$\frac{\gamma}{\gamma-\beta}\,\langle h\rangle^{\beta}_{\beta,\gamma,v(t)\,dt,[a,b]} - \frac{\gamma}{\gamma-\beta}\,\langle h\rangle^{\beta}_{\beta,\beta,v(t)\,dt,[a,b]} \le$$

$$\le B^{\beta}\left(\frac{\gamma}{\gamma-\alpha}\right)^{\frac{\beta}{\alpha}}\langle h^{\alpha}\rangle_{\frac{\beta}{\alpha},\frac{\gamma}{\alpha},v(t)\,dt,[a,b]} = B^{\beta}\left(\frac{\gamma}{\gamma-\alpha}\right)^{\frac{\beta}{\alpha}}\langle h\rangle^{\beta}_{\beta,\gamma,v(t)\,dt,[a,b]}\,.$$

Hence

$$\left[\frac{\gamma}{\gamma-\beta} - B^{\beta}\left(\frac{\gamma}{\gamma-\alpha}\right)^{\frac{\beta}{\alpha}}\right]\langle h\rangle^{\beta}_{\beta,\gamma,v(t)\,dt,[a,b]} \le \frac{\gamma}{\gamma-\beta}\,\langle h\rangle^{\beta}_{\beta,\beta,v(t)\,dt,[a,b]}\,.$$

If γ satisfies (C.47), then the expression in the square brackets in the left-hand side is positive, i.e.,

$$\langle h\rangle^{\beta}_{\beta,\gamma,v(t)\,dt,[a,b]} \le C\cdot\langle h\rangle^{\beta}_{\beta,\beta,v(t)\,dt,[a,b]}\,,$$

or, by (C.41),

$$\left\langle h^{\beta}\right\rangle_{1,\frac{\gamma}{\beta},v(t)\,dt,[a,b]} \le C\cdot\langle h\rangle^{\beta}_{\beta,\beta,v(t)\,dt,[a,b]}\,.$$

Applying (C.45) with $r=\frac{\gamma}{\beta}>1$ and the non-increasing function $g=h^{\beta}$ we get

$$\left\langle h^{\beta}\right\rangle_{\frac{\gamma}{\beta},\frac{\gamma}{\beta},v(t)\,dt,[a,b]} \le \frac{\beta}{\gamma}\left\langle h^{\beta}\right\rangle_{1,\frac{\gamma}{\beta},v(t)\,dt,[a,b]} \le C'\cdot\langle h\rangle^{\beta}_{\beta,\beta,v(t)\,dt,[a,b]}\,,$$

which, in view of (C.41), is equaivalent to

$$\langle h\rangle^{\beta}_{\gamma,\gamma,v(t)\,dt,[a,b]} \le C'\cdot\langle h\rangle^{\beta}_{\beta,\beta,v(t)\,dt,[a,b]}\,.$$

Since $\beta>0$ inequality (C.49) follows.

It remains to consider the case $\beta<0$. Set $v_0=h^{\beta}v$. Then $v=h^{-\beta}v_0$ and, in view of (C.43), condition (C.48) becomes

$$\langle h\rangle^{1-\frac{\beta}{\alpha}}_{-\beta,-\beta,v_0(\tau)\,d\tau,[a,t]} \le B\cdot\langle h\rangle^{1-\frac{\beta}{\alpha}}_{\alpha-\beta,\alpha-\beta,v_0(\tau)\,d\tau,[a,t]}\,,\quad a\le t\le b.$$

Moreover, since $1-\frac{\beta}{\alpha}>0$ we have

$$\langle h\rangle_{-\beta,-\beta,v_0(\tau)\,d\tau,[a,t]} \le B^{\frac{\alpha}{\alpha-\beta}}\,\langle h\rangle_{\alpha-\beta,\alpha-\beta,v_0(\tau)\,d\tau,[a,t]}\,,\quad a\le t\le b.$$

Let $\alpha_0=\alpha-\beta,\ \beta_0=-\beta,\ B_0=B^{\frac{\alpha}{\alpha-\beta}}$. Then $\alpha_0<0<\beta_0$ and so we have reduced this case to the one we have already considered. Indeed, as we have seen, if $\gamma_0>\beta_0$ are such that

$$\left(1-\frac{\alpha_0}{\gamma_0}\right)^{\frac{1}{\alpha_0}} > B_0\left(1-\frac{\beta_0}{\gamma_0}\right)^{\frac{1}{\beta_0}}\,, \qquad (C.50)$$

then

$$\langle h\rangle_{\gamma_0,\gamma_0,v_0(t)\,dt,[a,b]} \le B_0'' \cdot \langle h\rangle_{\beta_0,\beta_0,v_0(t)\,dt,[a,b]} \,.$$

Setting $\gamma = \gamma_0+\beta$ and taking into account $v_0 = h^\beta v$, we rewrite this inequality in the following way

$$\langle h\rangle_{\gamma-\beta,\gamma-\beta,h^\beta(t)v(t)\,dt,[a,b]} \le B_0'' \cdot \langle h\rangle_{-\beta,-\beta,h^\beta(t)v(t)\,dt,[a,b]} \,,$$

which, by $(C.43)$, is equivalent to

$$\langle h\rangle^{\frac{\gamma}{\gamma-\beta}}_{\gamma,\gamma,v(t)\,dt,[a,b]} \le B_0'' \cdot \langle h\rangle^{\frac{\gamma}{\gamma-\beta}}_{\beta,\beta,v(t)\,dt,[a,b]} \,.$$

Since $\frac{\gamma}{\gamma-\beta} > 0$ it follows that the above inequality is equivalent to $(C.49)$. Now to conclude the proof in this case it is enough to recall that $(C.50)$ is equivalent to $(C.47)$. $\square$

Lemma C.15. *Let α, β be different from zero and such that $\alpha < \beta$, and assume that the function h is non-decreasing on $[a,b]$ and such that*

$$\langle h\rangle_{\beta,\beta,v(\tau)\,d\tau,[a,t]} \le B \cdot \langle h\rangle_{\alpha,\alpha,v(\tau)\,d\tau,[a,t]} \,, \quad a \le t \le b. \tag{C.51}$$

Then for any $\gamma < \min(0,\alpha)$, satisfying condition $(C.47)$, there exists $B' > 0$ such that

$$\langle h\rangle_{\gamma,\gamma,v(t)\,dt,[a,b]} \ge \frac{1}{B'}\,\langle h\rangle_{\alpha,\alpha,v(t)\,dt,[a,b]} \,. \tag{C.52}$$

Proof. As in the proof of Lemma $C.14$, first we consider the case $\alpha < 0$. Setting $q = \alpha,\ p = \gamma,\ g = h$ in $(C.42)$ and using $(C.51)$ and $(C.41)$, we have

$$\begin{aligned}
\frac{\gamma}{\gamma-\alpha}\,\langle h\rangle^\alpha_{\alpha,\gamma,v(t)\,dt,[a,b]} - \frac{\gamma}{\gamma-\alpha}\,\langle h\rangle^\alpha_{\alpha,\alpha,v(t)\,dt,[a,b]} &= \\
= \Big\langle \langle h\rangle_{\alpha,\alpha,v(\tau)\,d\tau,[a,\cdot]} \Big\rangle^\alpha_{\alpha,\gamma,v(t)\,dt,[a,b]} &\le \\
\le B^{-\alpha} \Big\langle \langle h\rangle_{\beta,\beta,v(\tau)\,d\tau,[a,\cdot]} \Big\rangle^\alpha_{\alpha,\gamma,v(t)\,dt,[a,b]} &= \\
= B^{-\alpha} \Big\langle \langle h\rangle^\alpha_{\beta,\beta,v(\tau)\,d\tau,[a,\cdot]} \Big\rangle_{1,\frac{\gamma}{\alpha},v(t)\,dt,[a,b]} &= \\
= B^{-\alpha} \Big\langle \langle h^\beta\rangle^{\frac{\alpha}{\beta}}_{1,1,v(\tau)\,d\tau,[a,\cdot]} \Big\rangle_{1,\frac{\gamma}{\alpha},v(t)\,dt,[a,b]} &= \\
= B^{-\alpha} \Big\langle \langle h^\beta\rangle_{1,1,v(\tau)\,d\tau,[a,\cdot]} \Big\rangle^{\frac{\alpha}{\beta}}_{\frac{\alpha}{\beta},\frac{\gamma}{\beta},v(t)\,dt,[a,b]} &.
\end{aligned}$$

Now we apply Lemma $C.12$ with $q = \frac{\alpha}{\beta},\ p = \frac{\gamma}{\beta},\ g = h^\beta$, taking into account that if $\beta < 0$, then $p > q > 1$, and if $\beta > 0$, then $p < 0,\ q < 0$, so that the conditions of Lemma $C.12$ are fulfilled. Then, by $(C.41)$,

$$\frac{\gamma}{\gamma-\alpha}\,\langle h\rangle^{\alpha}_{\alpha,\gamma,v(t)\,dt,[a,b]}-\frac{\gamma}{\gamma-\alpha}\,\langle h\rangle^{\alpha}_{\alpha,\alpha,v(t)\,dt,[a,b]}\le$$

$$\le B^{-\alpha}\left(\frac{\gamma}{\gamma-\beta}\right)^{\frac{\alpha}{\beta}}\left\langle h^{\beta}\right\rangle_{\frac{\alpha}{\beta},\frac{\gamma}{\beta},v(t)\,dt,[a,b]}=B^{-\alpha}\left(\frac{\gamma}{\gamma-\beta}\right)^{\frac{\alpha}{\beta}}\langle h\rangle^{\alpha}_{\alpha,\gamma,v(t)\,dt,[a,b]}\,.$$

Therefore

$$\left[\frac{\gamma}{\gamma-\alpha}-B^{-\alpha}\left(\frac{\gamma}{\gamma-\beta}\right)^{\frac{\alpha}{\beta}}\right]\langle h\rangle^{\alpha}_{\alpha,\gamma,v(t)\,dt,[a,b]}\le\frac{\gamma}{\gamma-\alpha}\,\langle h\rangle^{\alpha}_{\alpha,\alpha,v(t)\,dt,[a,b]}\,.$$

If γ satisfies $(C.47)$, then the expression in the square brackets in the left-hand side is positive. Hence

$$\langle h\rangle^{\alpha}_{\alpha,\gamma,v(t)\,dt,[a,b]}\le C\cdot\langle h\rangle^{\alpha}_{\alpha,\alpha,v(t)\,dt,[a,b]}\,,$$

which, by virtue of $(C.41)$, is equivalent to

$$\langle h^{\alpha}\rangle_{1,\frac{\gamma}{\alpha},v(t)\,dt,[a,b]}\le C\cdot\langle h\rangle^{\alpha}_{\alpha,\alpha,v(t)\,dt,[a,b]}\,.$$

Applying now $(C.45)$ with $r=\frac{\gamma}{\alpha}>1$ and the non-increasing function $g=h^{\alpha}$ we obtain

$$\langle h^{\alpha}\rangle_{\frac{\gamma}{\alpha},\frac{\gamma}{\alpha},v(t)\,dt,[a,b]}\le\frac{\alpha}{\gamma}\,\langle h^{\alpha}\rangle_{1,\frac{\gamma}{\alpha},v(t)\,dt,[a,b]}\le C'\cdot\langle h\rangle^{\alpha}_{\alpha,\alpha,v(t)\,dt,[a,b]}\,,$$

i.e., by $(C.41)$,

$$\langle h\rangle^{\alpha}_{\gamma,\gamma,v(t)\,dt,[a,b]}\le C'\cdot\langle h\rangle^{\alpha}_{\alpha,\alpha,v(t)\,dt,[a,b]}\,.$$

Since $\alpha<0$ inequality $(C.52)$ follows.

It remains to consider the case $\alpha>0$. Let $v_0=h^{\alpha}v$. Then $v=h^{-\alpha}v_0$ and using $(C.43)$ we can rewrite condition $(C.51)$ in the following way

$$\langle h\rangle^{1-\frac{\alpha}{\beta}}_{\beta-\alpha,\beta-\alpha,v_0(\tau)\,d\tau,[a,t]}\le B\cdot\langle h\rangle^{1-\frac{\alpha}{\beta}}_{-\alpha,-\alpha,v_0(\tau)\,d\tau,[a,t]}\,,\quad a\le t\le b.$$

Since $1-\frac{\alpha}{\beta}>0$ we have

$$\langle h\rangle_{\beta-\alpha,\beta-\alpha,v_0(\tau)\,d\tau,[a,t]}\le B^{\frac{\beta}{\beta-\alpha}}\,\langle h\rangle_{-\alpha,-\alpha,v_0(\tau)\,d\tau,[a,t]}\,,\quad a\le t\le b.$$

Denote $\alpha_0=-\alpha,\ \beta_0=\beta-\alpha,\ B_0=B^{\frac{\beta}{\beta-\alpha}}$. Then $\alpha_0<0<\beta_0$ and we come back to the case which we have already considered. In other words, if $\gamma_0<\alpha_0$ satisfies

$$\left(1-\frac{\alpha_0}{\gamma_0}\right)^{\frac{1}{\alpha_0}}>B_0\left(1-\frac{\beta_0}{\gamma_0}\right)^{\frac{1}{\beta_0}}\,,\tag{C.53}$$

then

$$\langle h\rangle_{\gamma_0,\gamma_0,v_0(t)\,dt,[a,b]}\ge\frac{1}{B_0'}\,\langle h\rangle_{\alpha_0,\alpha_0,v_0(t)\,dt,[a,b]}\,.$$

Setting $\gamma = \gamma_0 + \alpha$ and taking into account the equality $v_0 = h^\alpha v$ we can rewrite the last inequality as follows

$$\langle h \rangle_{\gamma-\alpha,\gamma-\alpha,h^\alpha(t)v(t)\,dt,[a,b]} \geq \frac{1}{B_0'} \langle h \rangle_{-\alpha,-\alpha,h^\alpha(t)v(t)\,dt,[a,b]} \,,$$

which, in view of (C.43), is equivalent to

$$\langle h \rangle^{\frac{\gamma}{\gamma-\alpha}}_{\gamma,\gamma,v(t)\,dt,[a,b]} \geq \frac{1}{B_0'} \langle h \rangle^{\frac{\gamma}{\gamma-\alpha}}_{\alpha,\alpha,v(t)\,dt,[a,b]} \,.$$

Since $\frac{\gamma}{\gamma-\alpha} > 0$ the last inequality is equivalent to (C.52). Recalling now that (C.53) is equivalent to (C.47) we conclude the analysis of the case $\alpha > 0$. $\square$

In the proof of Theorem C.8 in the general case we will use that fact that for any μ-measurable set E

$$\int_E f^p(x)\,d\mu(x) = \int_0^{\mu(E)} \left[(f|E)^{\downarrow}_{\mu}(t)\right]^p dt = \int_0^{\mu(E)} \left[(f|E)^{\uparrow}_{\mu}(t)\right]^p dt \qquad (C.54)$$

for any real p.

Proof of Theorem C.8. Theorem C.1 implies that for any segment $R \subset R_0$

$$\left\langle\, (f|R)^{\downarrow}_{\mu} \,\right\rangle_{\beta,\beta,d\tau,[0,t]} \leq B \cdot \left\langle\, (f|R)^{\downarrow}_{\mu} \,\right\rangle_{\alpha,\alpha,d\tau,[0,t]} \,, \quad 0 \leq t \leq \mu(R), \qquad (C.55)$$

$$\left\langle\, (f|R)^{\uparrow}_{\mu} \,\right\rangle_{\beta,\beta,d\tau,[0,t]} \leq B \cdot \left\langle\, (f|R)^{\uparrow}_{\mu} \,\right\rangle_{\alpha,\alpha,d\tau,[0,t]} \,, \quad 0 \leq t \leq \mu(R). \qquad (C.56)$$

Fix some segment $R \subset R_0$. Then, by (C.55), the function $h \equiv (f|R)^{\downarrow}_{\mu}$ satisfies the condition of Lemma C.14 with $[a,b] = [0,\mu(R)]$ and $v \equiv 1$. Hence if $\gamma > \max(0,\beta)$ satisfies (C.39), we have (C.49), which, by (C.54), is equivalent to the right inequality in (C.40). Similarly (C.56) and Lemma C.15 yield the left inequality in (C.40) for $\gamma < \min(0,\alpha)$, which satisfies (C.39).

It remains to show that if $d\mu(x) = dx$, then the left inequality in (C.40) fails for $\gamma \leq \gamma^-$, while the right inequality in (C.40) fails for $\gamma \geq \gamma^+$. Here γ^- and γ^+ are the parameters defined in Remark C.9. Let $d = 1$ and let, for example, $\gamma \geq \gamma^+$. Define $f_0(x) = x^{-\frac{1}{\gamma^+}}$, $x \in [0,1]$. It is easy to see that the right inequality in (C.40) fails for f_0. So, it remains to show that

$$f_0 \in RH^{\alpha,\beta}_{dx}(B), \qquad (C.57)$$

where

$$B = \left(1 - \frac{\alpha}{\gamma^+}\right)^{\frac{1}{\alpha}} \cdot \left(1 - \frac{\beta}{\gamma^+}\right)^{-\frac{1}{\beta}} . \qquad (C.58)$$

If $\beta > 0$, then take $g = f_0^\alpha$ and rewrite (C.57) as follows

$$\frac{1}{|I|} \int_I g^{\frac{\beta}{\alpha}}(x)\,dx \leq B^\beta \left(\frac{1}{|I|} \int_I g(x)\,dx\right)^{\frac{\beta}{\alpha}} , \quad I \subset [0,1].$$

Fix some interval $I \subset [0,1]$ and choose $b > 0$ (possibly greater than 1) such that

$$\frac{1}{b}\int_0^b g(x)\,dx \equiv \left(1-\frac{\alpha}{\gamma^+}\right)^{-1} b^{-\frac{\alpha}{\gamma^+}} = \frac{1}{|I|}\int_I g(x)\,dx.$$

Since $\frac{\beta}{\alpha} \in (-\infty,0)\cup(1,+\infty)$ the function $\varphi(t) = t^{\frac{\beta}{\alpha}}$ is convex downwards. Therefore the inequality

$$\frac{1}{|I|}\int_I g^{\frac{\beta}{\alpha}}(x)\,dx \le \frac{1}{b}\int_0^b g^{\frac{\beta}{\alpha}}(x)\,dx$$

implies (see Lemma $C.4$)

$$\frac{1}{|I|}\int_I g^{\frac{\beta}{\alpha}}(x)\,dx \le \left(1-\frac{\beta}{\gamma^+}\right)^{-1} b^{-\frac{\beta}{\gamma^+}} =$$

$$= B^\beta\left(\left(1-\frac{\alpha}{\gamma^+}\right)^{-1} b^{-\frac{\alpha}{\gamma^+}}\right)^{\frac{\beta}{\alpha}} = B^\beta\left(\frac{1}{|I|}\int_I g(x)\,dx\right)^{\frac{\beta}{\alpha}},$$

where the parameter B is defined by $(C.58)$. Thus $(C.57)$ holds true for $\beta > 0$.

If $\beta < 0$, then we take $g = f_0^\beta$ and rewrite $(C.57)$ in the following form

$$\frac{1}{|I|}\int_I g^{\frac{\alpha}{\beta}}(x)\,dx \le B^{-\alpha}\left(\frac{1}{|I|}\int_I g(x)\,dx\right)^{\frac{\alpha}{\beta}}, \quad I \subset [0,1].$$

Fixing some interval $I \subset [0,1]$ we can choose b such that

$$\frac{1}{b}\int_0^b g(x)\,dx \equiv \left(1-\frac{\beta}{\gamma^+}\right)^{-1} b^{-\frac{\beta}{\gamma^+}} = \frac{1}{|I|}\int_I g(x)\,dx.$$

Since $\alpha < \beta < 0$ the function $\varphi(t) = t^{\frac{\alpha}{\beta}}$ $(t > 0)$ is convex downwards. Hence, applying again the inequality

$$\frac{1}{|I|}\int_I g^{\frac{\alpha}{\beta}}(x)\,dx \le \frac{1}{b}\int_0^b g^{\frac{\alpha}{\beta}}(x)\,dx,$$

according to Lemma $C.4$ we obtain

$$\frac{1}{|I|}\int_I g^{\frac{\alpha}{\beta}}(x)\,dx \le \left(1-\frac{\alpha}{\gamma^+}\right)^{-1} b^{-\frac{\alpha}{\gamma^+}} =$$

$$= B^{-\alpha}\left(\left(1-\frac{\beta}{\gamma^+}\right)^{-1} b^{-\frac{\beta}{\gamma^+}}\right)^{\frac{\alpha}{\beta}} = B^{-\alpha}\left(\frac{1}{|I|}\int_I g(x)\,dx\right)^{\frac{\alpha}{\beta}},$$

where the parameter B is defined by $(C.58)$. So, $(C.57)$ holds true for $\beta < 0$, too.

Analogously one can show that if $d = 1$, then also the left inequality in $(C.40)$ fails for $\gamma \le \gamma^-$. Finally, the case $d \ge 2$ can be easily reduced to the one-dimensional case if we consider the functions that are constant with respect to all variables but one. □

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