Chapter 5
The Local Modelling of the Gravity Field by Collocation

5.1 Outline of the Chapter

The chapter aims at solving the problem of estimating the residual anomalous potential T_r from all available information, in particular in a certain area. Remember that here residual means that the long wavelength part as well as the short wavelength part of T have been at least reduced by means of the deterministic modelling described in Chaps. 3 and 4.

These models are then applied to data (remove step); from reduced observations we need to find T_r and then the models are added back to this (restore step).

Since the residual part of the potential is small (one has in terms of anomalous height $O\left(\frac{T_r}{\gamma}\right) \cong 2\,\mathrm{m}$), the application of spherical approximation is justified.

This notwithstanding such an approximation remains the harsh limitation of the theory presented in this chapter. This point is explained in Sect. 5.2.

The theory, known in geodesy as *collocation theory*, is introduced here as an optimization problem where a suitable mean square error has to be minimized in a class of estimators invariant under a certain transformation group, acting on the set Ω where the unknown function is defined. Although not so much relevant in geodesy, the case of the circle is on the same time so simple to understand and so complete from the theoretical point of view, that it has been worthwhile to devote Sect. 5.3 to it.

In Sect. 5.4 the same case is treated for the sphere, with the invariance group being that of rotations in R^3. The big theoretical advantage of this approach is that not only the estimation coefficients result as an application of the optimality principle, but also the definition of the covariance function springs out of it in a natural way.

In Sect. 5.4 it is also shown that the formalism set up in the previous paragraphs can be given a stochastic interpretation, to the effect that now T is considered as a random function, obtained by randomly rotating the true T. The formalism is then extended in Sect. 5.5 to the general case, in which we have whatever N observations, corresponding to admissible linear functionals, and we want to predict any other

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admissible linear functional of T. In particular, if we assume, invoking the Runge–Krarup theorem, that T is a function harmonic down to a Bjerhammar sphere, any rotated version of T will continue to be harmonic in the same domain, and the principle above devised, applies.

Since a function harmonic in the exterior of a sphere has a natural representation in terms of spherical harmonics, its coefficients will become random variables, when the field T they represent is random too. The properties of such $\{T_{nm}\}$ as well as their relation to the covariance function of T, are examined in Sect. 5.6. In Sect. 5.7 the item of a local modelling of the covariance function is analyzed and several examples are presented, including those most widely applied in practical computations.

The local computation of a (residual) quasi-geoid from (residual) gravity anomalies is then presented as an example of the so-called *least squares collocation theory*.

Finally, in Sect. 5.9 the optimal combination of a global model, for instance derived from satellite observations, and local data to produce the best local prediction of the geoid, is explicitly solved; a case this that is becoming increasingly important in these years.

5.2 An Introduction to the Problem

Following the developments of Chaps. 3 and 4 we could say that our anomalous gravity potential T has been approximated in the long wavelengths range by a global model T_M and in the very short wavelengths range by the residual terrain correction model T_{RC}, so that a residual anomalous potential

$$T_r = T - T_M - T_{RC} \tag{5.1}$$

has now to be estimated.

This has to be done by using the *residual* observations, which in linearized form are written as

$$\begin{aligned} y_i &= L_i(T_r) + v_i \\ &= L_i(T) + v_i - L_i(T_M) - L_i(T_{RC}) \\ &= Y_i - L_i(T_M) - L_i(T_{RC}), \end{aligned} \tag{5.2}$$

where Y_i are the original observations, y_i the observations reduced by the effects of T_M and T_{RC}, v_i is the observational error. Typical for (5.2), but not the only case considered in the book, is the observation of free air gravity anomalies, for which the relation holds

$$L_i(T) = \left(-\frac{\partial T}{\partial h} + \frac{\gamma'}{\gamma} T\right)\Big|_{P_i}. \tag{5.3}$$

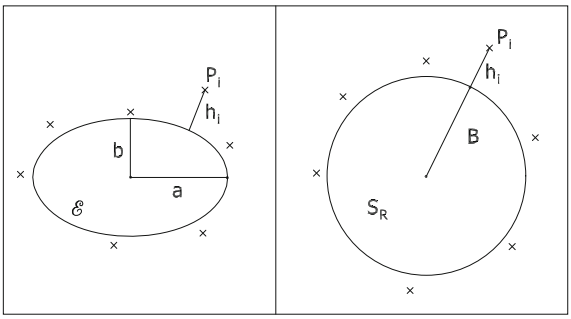

Fig. 5.1 The spherical approximation mapping of the interpolation problem: P_i measurement points, h_i heights over E and S_R

In order to avoid a heavy notation, while developing our methodological apparatus we shall simply put

$$u(P) = T_r(P). \tag{5.4}$$

So, due to all our reductions, $u(P)$ is a harmonic field which, in an ideal case, we expect to be harmonic down to the ellipsoid because the signal caused by the large and smooth density anomalies should be accounted for by T_M and the high frequency signal due to the residual terrain height should be subtracted by means of T_{RC}. Then we could reasonably think of our problem as the one of interpolating the observations (5.2) with a function harmonic down to $\mathcal{E}$.

Since we are approximating the last couple of meters in terms of height anomaly $\zeta = \gamma^{-1}u = \gamma^{-1}T$, we shall accept a spherical approximation set up, for the approximation procedure, in the sense that we map $\mathcal{E}$ to a mean sphere S_R of radius R and we reason with functions harmonic down to S_R (see Fig. 5.1).

Therefore our problem now is to find a function $\widehat{u}$ harmonic down to S_R, such that $y_i - L_i(\widehat{u})$ be small, in the sense of the order of magnitude of ν_i (i.e. of σ_{ν_i}), and as close as possible to u.

It is clear in fact that, as the number of observation points, N, can be very large, but in any event always finite, in principle we can always find many harmonic fields $\widehat{u}$ which in fact interpolate perfectly the data, $L_i(\widehat{u}) = y_i$, as shown very schematically in Fig. 5.2, where the observation points P_i are taken directly on S_R and $L_i(u) = u(P_i)$ is represented in terms of geoid, $\gamma^{-1}u(P_i)$.

Generally speaking, since in nature masses will tend to find a minimum energy configuration (compatibly with the endogenous forces generated by geological processes) and energy is in any way a quadratic positive functional of $u(P)$ which is smaller the smoother is the field, we would prefer an interpolator as smooth as possible, among those that reproduce the data. Even more, if a noise ν is part of our model, we would accept that $L_i(\widehat{u})$ will depart form y_i, with residuals of the order of σ_ν, and on the same time $\widehat{u}$ to be as smooth as possible.

If a smoothness index is taken in terms of a square norm, we are led to the Tikhonov principle which is illustrated and worked out in Part III, Chap. 12. Yet, as one can see in this chapter, the solution does depend quite essentially on the

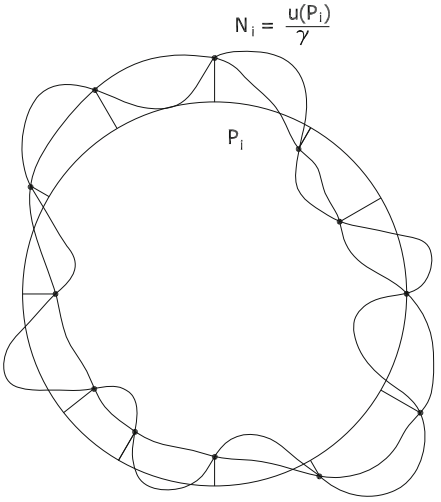

Fig. 5.2 Two different exact interpolations of $\frac{u(P_i)}{\gamma} = N_i$, by two different fields $\widehat{u}(P)$

specific norm chosen to measure the smoothness of $\widehat{u}$, when the norm is represented by a suitable reproducing kernel $K(P, Q)$.

In other words, we have a so-called *norm choice problem* which is absolutely unsolvable on a pure analytical ground. So we shall follow here a different approach which, as we will see, will lead basically to the same solution as that of Sect. 12.4 of Part III but with a precise choice for the reproducing kernel. This solution is based on the choice of an invariant estimator and minimum mean square prediction error, and on its stochastic interpretation.

Notice that in principle we expect $u(P)$ to be harmonic down to $\mathcal{E}$, then approximated by S_R. Yet such condition will never be precisely satisfied; nevertheless by choosing an interpolator $\widehat{u}$ which is authentically harmonic down to S_R we don't prevent ourselves to approximate as closely as we like the true $u(P)$, because of Runge–Krarup theorem (see Sect. 3.5).

In fact, as proved in Part III, Chap. 13, the restrictions of functions $\widehat{u}$ harmonic in $\Omega_R \equiv (r \geq R)$ to the set Ω_e of points exterior to the earth surface S_e, are dense in any reasonable Hilbert space to which we can think that $u(P)$ belongs, for instance in $HL^2(S_e)$, namely the functions harmonic in Ω_e and square integrable on S_e. So, from now on, we shall ignore the problem of the masses between S_e and S_R not perfectly modelled.

5.3 The Principle of Minimum Square Invariant Prediction Error by a Simple Example

In order to select a particular satisfactory solution to our interpolation problem, we have first to define an index expressing analytically our degree of satisfaction, or, if you like, of dissatisfaction, and then to maximize such an index in the former case,

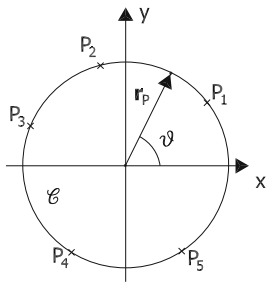

Fig. 5.3 The set up of the interpolation problem on the circle

or, on the contrary, to minimize it in the latter case. This is a problem of optimization theory, where the choice of the target function is always the first fundamental step (see for instance Vapnik 1982, Chap. 2). We choose to minimize a quadratic function of the prediction error, averaged in some suitable sense.

In order to set up our criterion we prefer to start with a simple example where our choice will become very transparent.

Example 1. Assume you have a field $u(P)$ where $P \in C$, a unit circle, so that P can be uniquely identified by a unit vector $\mathbf{r}_P$ or by the angle ϑ of $\mathbf{r}_P$ with respect to the x axis (see Fig. 5.3).

To make things easier we shall assume from the beginning that $u(P)$ has zero mean on C, i.e. that

$$\int_0^{2\pi} u(P)d\vartheta = \int_0^{2\pi} u(\vartheta)d\vartheta = 0. \tag{5.5}$$

Now assume you have observed the values of $u(P)$ at some points P_i

$$y_i = u(P_i), \quad i = 1, 2, \ldots, N \tag{5.6}$$

without any error, and you want to predict $u(P)$ at some other point P. As we see, we have a pure interpolation problem on C.

We note first of all that a *predictor* will be in general a function of the observations $\{y_i\}$ of the points $\{P_i\}$ where the observations are taken and of the prediction point P, in such a way that we are able to compute it when we know $\{y_i\}$ and we fix P;

$$\widehat{u}(P) = F(P, P_1, \ldots, P_N; y_1, \ldots, y_N). \tag{5.7}$$

Since reasoning in a general class of predictors $\{F\}$ is too complicated we shall restrict ourselves to the much simpler class of linear predictors, namely

$$\widehat{u}(P) = F(P, P_1, \ldots, P_N; y_1, \ldots, y_N) = \sum_{i=1}^{N} \lambda_i y_i = \sum_{i=1}^{N} \lambda_i u(P_i). \tag{5.8}$$

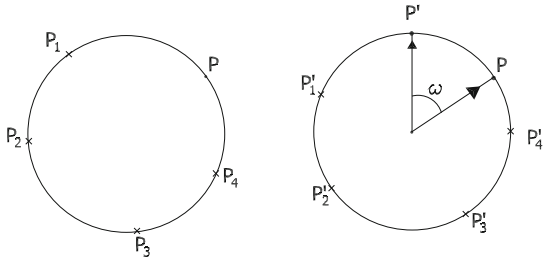

Fig. 5.4 A configuration (P, P_1, P_2, P_3, P_4) and its version $(P', P_1', P_2', P_3', P_4')$ rotated by ω

We observe that (5.8) is a homogeneous linear predictor, i.e. there is not a constant λ_0 in the formula; the reason is that, when we observe $y_1 = y_2 = \ldots y_N = 0$ we prefer the prediction of $u(P)$ to be zero too, i.e. its mean value on the circle, according to the hypothesis (5.5).

We notice also that, in (5.8), λ_i in general will be functions of $P, P_1 \ldots P_N$ but not of $\{y_i\}$, i.e.

$$\lambda_i = \lambda_i(P, P_1, \ldots, P_N) = \lambda_i(\vartheta, \vartheta_1, \ldots, \vartheta_N). \tag{5.9}$$

Whatever $\{\lambda_i\}$ we choose, the corresponding prediction error is

$$\begin{aligned} e(P, P_1, \ldots, P_N) &= u(P) - \widehat{u}(P) \\ &= u(P) - \sum_{i=1}^{N} \lambda_i u(P_i). \end{aligned} \tag{5.10}$$

If we don't have any particular further information on $u(P)$ (for instance that in some regions of C, $u(P)$ is smoother or rougher) it is reasonable to further restrict our class of predictors by requiring that λ_i be invariant under rotation. Namely, take two configuration, $\{P, P_1, \ldots, P_N\}$ and $\{P', P_1', \ldots, P_N'\}$ obtained one from the other by a rotation ω of the circle (see Fig. 5.4);

We claim that if in the first case we have decided that $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ are good coefficients for our prediction job, then the *same* coefficients should work for $\{P', P_1', \ldots, P_N'\}$ because if $(y_1, \ldots, y_N)$ are observed at $(P_1, \ldots, P_N)$ and $\widehat{u}(P)$ is our prediction, then in case we observe again $(y_1, \ldots, y_N)$ at $(P_1', \ldots, P_N')$ we want to make the *same* prediction at P'.

This is translated into analytical terms as follows: let R_ω be a *rotation operator* acting according to the law

$$\begin{aligned} &R_\omega F(P, P_1, \ldots, P_N) = R_\omega F(\vartheta, \vartheta_1, \ldots, \vartheta_N) \\ &= F(P', P_1', \ldots, P_N') = F(\vartheta', \vartheta_1', \ldots, \vartheta_N') \\ &= F(\vartheta + \omega, \vartheta_1 + \omega, \ldots, \vartheta_N + \omega) \end{aligned} \tag{5.11}$$

where F is any function of $(P, P_1, \ldots, P_N)$; then our invariance constraint is

$$\forall \omega, \quad F(\vartheta, \vartheta_1, \ldots, \vartheta_N) \equiv F(\vartheta + \omega, \vartheta_1 + \omega, \ldots, \vartheta_N + \omega) \tag{5.12}$$

A function F satisfying (5.12) must have a particular form, namely

$$F(\vartheta, \vartheta_1, \ldots, \vartheta_N) = G(\vartheta_1 - \vartheta, \vartheta_2 - \vartheta, \ldots, \vartheta_N - \vartheta); \tag{5.13}$$

this derives from (5.12) by choosing $\omega = -\theta$.

So we agree that our prediction coefficients must satisfy (5.11) and (5.13). Accordingly if we apply R_ω to e (cf. (5.10)), we get

$$\begin{aligned} R_\omega e(P, P_1, \ldots, P_N) &= R_\omega u(P) - \sum_{i=1}^{N} \lambda_i R_\omega u(P_i) \\ &= u(\vartheta + \omega) - \sum_{i=1}^{N} \lambda_i u(\vartheta_i + \omega), \end{aligned} \tag{5.14}$$

where λ_i are left unchanged by R_ω because of our invariance hypothesis.

Now observe that due to the very definition of R_ω the identity holds

$$R_\omega\{F^2(P, P_1, \ldots, P_N)\} \equiv \{R_\omega F(P, P_1 \ldots P_N)\}^2. \tag{5.15}$$

Next we define the *mean invariant quadratic prediction error*[1] as

$$\mathcal{E}^2(P, P_1, \ldots, P_N) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\omega R_\omega\{e^2(P, P_1, \ldots, P_N)\}. \tag{5.16}$$

The adjective *invariant* is used for $\mathcal{E}^2$ because it is indeed a rotation invariant function of $(P, P_1, \ldots, P_N)$. In fact, (exploiting also (5.15)),

$$\begin{aligned} &\forall \eta, \quad R_\eta \mathcal{E}^2(P, P_1, \ldots, P_N) = \mathcal{E}^2(R_\eta P, R_\eta P_1, \ldots, R_\eta P_N) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\omega R_\omega\{e^2(R_\eta P, R_\eta P_1, \ldots, R_\eta P_N\} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\omega R_\omega R_\eta\{e^2(P, P_1, \ldots, P_N\} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\omega R_{\omega+\eta}\{e^2(P, P_1, \ldots, P_N\} \\ &= \mathcal{E}^2(P, P_1, \ldots, P_N\}, \end{aligned} \tag{5.17}$$

since integrating in $d\omega$ from 0 to 2π is one and the same thing as integrating from η to $\eta + 2\pi$.

[1] In this chapter we will use $\mathcal{E}^2$ for the mean quadratic prediction error; confusion should not be made with the same symbol $\mathcal{E}$ used elsewhere to denote the ellipsoid.

With the help of (5.14) and (5.15) we can indeed perform an explicit computation of $\mathcal{E}^2$, giving

$$\mathcal{E}^2 = \frac{1}{2\pi}\int_0^{2\pi} d\omega u^2(\vartheta+\omega) - 2\sum_{i=1}^{N}\lambda_i \frac{1}{2\pi}\int_0^{2\pi} d\omega u(\vartheta+\omega)u(\vartheta_i+\omega)$$
$$+ \sum_{i,k=1}^{N}\lambda_i\lambda_k \frac{1}{2\pi}\int_0^{2\pi} d\omega u(\vartheta_i+\omega)u(\vartheta_k+\omega) \tag{5.18}$$

It is noteworthy that by introducing the two points function

$$C(\vartheta,\vartheta') = \frac{1}{2\pi}\int_0^{2\pi} d\omega u(\vartheta+\omega)u(\vartheta'+\omega) \tag{5.19}$$

we come to express $\mathcal{E}^2$ in a concise form as

$$\mathcal{E}^2 = C(\vartheta,\vartheta) - 2\sum_{i=1}^{N}\lambda_i C(\vartheta,\vartheta_i) + \sum_{i,k=1}^{N}\lambda_i\lambda_k C(\vartheta_i,\vartheta_k). \tag{5.20}$$

A particularly important remark is that

$$C(\vartheta+\eta,\vartheta'+\eta) = \frac{1}{2\pi}\int_0^{2\pi} d\omega u(\vartheta+\eta+\omega)u(\vartheta'+\eta+\omega) = C(\vartheta,\vartheta')$$

for the same reason used in the proof of (5.17). Therefore $C(\vartheta,\vartheta')$ is also invariant under rotation, namely, with a small abuse of notation,

$$C(\vartheta,\vartheta') = C(\vartheta-\vartheta'). \tag{5.21}$$

The function $C(\vartheta-\vartheta')$ is called a rotation invariant covariance function. In particular it is called a covariance function because it has the typical properties of a covariance; it is symmetric and positive definite.

Such properties are immediately derived from (5.19), but we shall come back to the item at the end of the section.

Minimizing $\mathcal{E}^2$ with respect to $\{\lambda_i\}$ is straightforward and gives the following result: put

$$\begin{cases} \boldsymbol{\lambda} = \{\lambda_i\} \\ i = 1,\dots,N \end{cases} \quad \begin{cases} C = \{C(\vartheta_i-\vartheta_k)\} \\ i,k = 1,\dots,N \end{cases} \quad \begin{cases} \mathbf{C}_\vartheta = \{C(\vartheta-\vartheta_i)\} \\ i = 1,\dots,N \end{cases} \tag{5.22}$$

then

$$\boldsymbol{\lambda} = C^{-1}\mathbf{C}_\vartheta. \tag{5.23}$$

It is interesting to observe that since both the vector $\mathbf{C}_\vartheta$ and the matrix C are rotationally invariant, then so is $\boldsymbol{\lambda}$ too, as it was required form the beginning.

We make a fundamental remark on our solution. Remember that by definition a random field on C (see for instance Rozanov 1982) is a function $\{v(P,\omega)\}$, with $P \in C$ and $\omega \in \Omega$ and with a probability distribution on Ω, satisfying some measurability hypotheses, so that $\forall\{P_1, P_2, \ldots, P_N\}$ we know the probability distribution of the N-vector $\mathbf{v}^t(\omega) = [v(P_1,\omega), \ldots, v(P_N,\omega)]$. Remember also that mean and covariance of $\{v(P,\omega)\}$ are defined as

$$\mu(P) = E\{v(P,\omega)\} = \int_\Omega v(P,\omega)dP(\omega) \tag{5.24}$$

$$\begin{aligned} C(P,P') &= E\{[v(P,\omega)-\mu(P)][v(P',\omega)-\mu(P')\} \\ &= \int_\Omega v(P,\omega)v(P',\omega)dP(\omega) - \mu(P)\mu(P'). \end{aligned} \tag{5.25}$$

Here, as in the rest of the section, it occurs sometimes that the same symbol P is used to mean a point in space and a probability distribution, in which case it is always $P(\omega)$; moreover in this context Ω is an abstract set and not $\overline{B}^c$.

Now let us go back to our field $u(P) = u(\vartheta)$, with $u(\vartheta)$ a periodic function, and define a random field $\{v(\vartheta,\omega)\}$ as

$$v(\vartheta,\omega) = R_\omega u(P) = u(\vartheta+\omega) \tag{5.26}$$

with ω uniformly distributed on C, i.e.

$$\Omega = [0, 2\pi], \qquad dP(\omega) = \frac{d\omega}{2\pi}. \tag{5.27}$$

By applying (5.24) and (5.25) with (5.27), we see that $\mu(P) \equiv 0$ and that $C(P, P')$ is exactly the same covariance that we already defined in (5.19). Moreover if we construct a linear predictor of $v(P,\omega)$ by

$$\widehat{v}(P,\omega) = \sum_{i=1}^{N} \lambda_i v(P_i,\omega) \tag{5.28}$$

and we compute the prediction error

$$e(P,\omega) = v(P,\omega) - \widehat{v}(P,\omega),$$

we end up with the following expression for its variance

$$\begin{aligned} &\sigma^2[e(P,\omega)] \equiv E\{e^2(P,\omega)\} \\ &\equiv C(P,P) - 2\sum_{i=1}^{N} C(P,P_i)\lambda_i + \sum_{i,k=1}^{N} \lambda_i\lambda_k C(P_i,P_k) \\ &\equiv \mathcal{E}^2(P,P_1,\ldots,P_N). \end{aligned} \tag{5.29}$$

Indeed minimizing (5.9) with respect to $\{\lambda_i\}$ is the same problem as minimizing (5.20) and therefore it has the same solution.

This settles the first corner stone of a quite general theorem of equivalence of different approaches, all producing the same type of linear predictors, so that each approach contributes to the theoretical and practical understanding of the collocation theory developed in the next sections.

5.4 On Collocation Theory, or the Wiener-Kolmogorov Principle Applied in Physical Geodesy

We want to generalize the example of the previous section, switching from the circle C to the sphere S_R, from the rotation R_ω on C to a 3D rotation $R_{\boldsymbol{\omega}}$, where $\boldsymbol{\omega}$ now becomes a triple of angles (for instance Euler angles), so as to apply the minimization of a suitably defined invariant quadratic error, or equivalently a minimum prediction error variance principle, to our field $u(P) = T_r(P)$, harmonic outside S_R.

This discussion parallels a similar discussion, already dating back to 1940/1950, among scientists working in signal analysis and stochastic processes theory. In that framework N. Wiener was more stressing the point of view of the invariant estimators, while A. Kolmogorov was more in favour of the pure stochastic interpretation. It is for this reason that we like to label our application in physical geodesy of such a principle after the names of both great scientists.

The method, known in Geodesy as *collocation*, was developed in 1960–1970 by Moritz and Krarup (see Moritz 1980; Krarup 2006, Chap. 4), again one stressing the stochastic, the other the deterministic interpretation. Here we like to follow more the already mentioned point of view of proving the possibility of interpreting in different ways equivalent results, thus giving a clearer perspective to their practical implementation.

The first item we need to settle is to find an analogous of the uniform mean over rotated configurations of N points $\{P_1, \ldots, P_N\}$.

Without going into more difficult mathematical arguments on group theory, for which we refer to literature Moritz (1980) and Sansò and Venuti (2002a), we simply aim at giving a definition, proving that this provides a result with the required properties.

We start by defining the action of the rotation operator R_ω as

$$R_\omega F(P_1, \ldots, P_N) = F(R_\omega P_1, \ldots, R_\omega P_N) \tag{5.30}$$

and we ask ourselves how an invariant F should be made [2]

[2] Often in group theory the inverse rotation matrix R_ω^t is used; since this is irrelevant in the present text and this is not useful, we stick to definition (5.30).

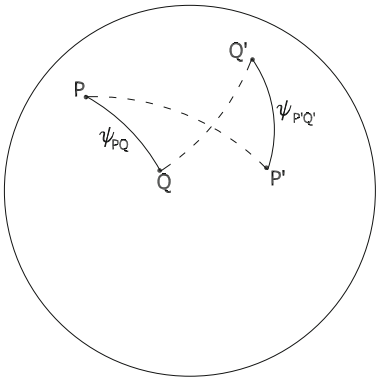

Fig. 5.5 The characterization of a rotation R_ω through the rigid motion of the arc $\overset{\frown}{PQ}$ over a sphere, $(P' = R_\omega P,\ Q' = R_\omega Q)$

Since under R_ω the polyhedron $\{P_1, \ldots, P_N\}$ is rigidly moved to another one $\{P'_1, \ldots, P'_N\}$, leaving the origin of R^3 fixed, we see that the following conditions are satisfied

$$r_{P'_i} = r_{P_i}; \quad \psi_{P'_i P'_j} = \psi_{P_i P_j}, \tag{5.31}$$

where we have denoted as usual with ψ_{PQ} the angle between $\mathbf{r}_P$ and $\mathbf{r}_Q$. It is easy to see that (5.31) is not only necessary but also sufficient for a rigid motion of $(P_1, \ldots, P_N)$ in the three-dimensional space, with the origin fixed in O. Therefore $F(P_1, \ldots, P_N)$ will be invariant under rotation if

$$F(P_1, \ldots, P_N) = F(\ldots r_{P_i} \ldots\ ;\ \ldots \psi_{P_i P_j} \ldots). \tag{5.32}$$

Next we note that in order to characterize a 3D rotation we need only to show how it acts on two points P, Q placed on a sphere.

Namely there is one and only one rotation sending PQ to $P'Q'$ on condition that $\psi_{P'Q'} = \psi_{PQ}$ (see Fig. 5.5) and $r_P = r_Q = r_{P'} = r_{Q'}$.

Since all what we shall really use in the sequel is the average of a two-points function, we concentrate on that, knowing that in any way the definition can be generalized to N points, in case of need. So let $F(P, Q)$ be any regular function of two points defined e.g. on the unit sphere; we put by definition

$$\begin{aligned} &E\{R_\omega[F(P, Q)]\} \\ &= \int dP(\omega) R_\omega F(P, Q) \\ &= A \int d\sigma_{P'} \int_{\psi_{P'Q'} = \psi_{PQ}} F(P', Q') d\alpha_{Q'}, \end{aligned} \tag{5.33}$$

where P' sweeps the whole unit sphere, while, for each fixed P', Q' runs on a circle of spherical radius ψ_{PQ}, occupying all the points of different azimuth α. The variable α ranges from 0 to 2π (see Fig. 5.6).

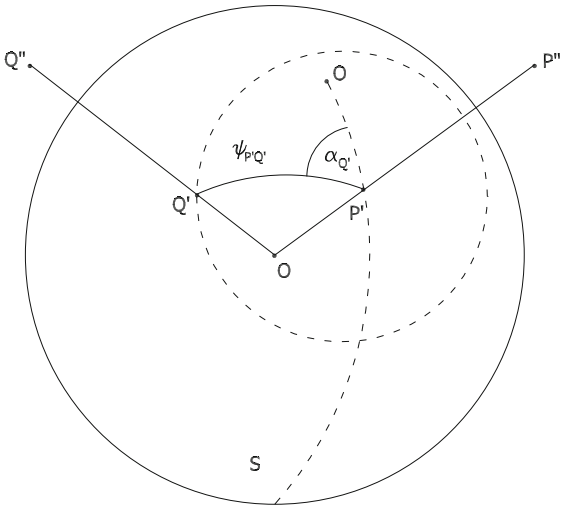

Fig. 5.6 Representation of the integration variable of (5.33): O is the center of S, P'', Q'' are in space, while P', Q' are their projection on S

As it obvious at the end the function (5.33) will depend on P, Q only through ψ_{PQ}, i.e. it will be invariant. Even if the points P, Q were outside the unit sphere, it is clear that (5.33) would depend in the end only on $r_{P''} = r_P, r_{Q''} = r_Q$ and ψ_{PQ} (see Fig. 5.6). So we can say that in general

$$E\{R_\omega[F(P,Q)]\} = C_F(r_P, r_Q, \psi_{PQ}), \tag{5.34}$$

i.e. it is a rotation invariant function. As for the normalization constant A appearing in (5.33), this is determined by considering that $dP(\omega)$ has to be a (uniform) probability distribution, so that one must have

$$E\{1\} = A \int d\sigma_{P'} \int_0^{2\pi} d\alpha_{Q'} = A \cdot 8\pi^2 \equiv 1,$$

implying

$$A = \frac{1}{8\pi^2}. \tag{5.35}$$

Now we can repeat the same reasoning as in Sect. 5.3. Namely if the observations y_i are just $u(P_i)$, $i = 1 \ldots N$, we define a *linear invariant* predictor

$$\widehat{u}(P) = \sum_{i=1}^{N} \lambda_i u(P_i), \tag{5.36}$$

with λ_i such that

$$R_\omega \lambda_i \equiv \lambda_i,$$

and an invariant quadratic prediction error

$$\begin{aligned}\mathcal{E}^2 &= E_\omega\{R_\omega[u(P) - \widehat{u}(P)]^2\} \\ &= C(P,P) - 2\sum_{i=1}^{N}\lambda_i C(P,P_i) + \sum_{i,k=1}^{N}\lambda_i\lambda_k C(P_i,P_k)\end{aligned} \tag{5.37}$$

where we have put

$$C(P,Q) = E\{R_\omega[u(P)u(Q)]\} = \frac{1}{8\pi^2}\int d\sigma_{P'} \int\limits_{\psi_{P'Q'}=\psi_{PQ}} d\alpha_{Q'}u(P')u(Q'), \tag{5.38}$$

also called the covariance function $u(P)$. Just as in (5.23), the minimum of (5.37) is achieved by

$$\lambda_j = \sum_{k=1}^{N} C_{jk}^{(-1)} C(P_k,P) \tag{5.39}$$

and the corresponding value of $\mathcal{E}^2$ is

$$\mathcal{E}^2_{\min} = C(P,P) - \sum_{i,j=1}^{N} C(P,P_i)C_{ij}^{(-1)}C(P_j,P). \tag{5.40}$$

In (5.39) and (5.40) we have used the short notation $C_{ik}^{(-1)}$, to mean the element (i,k) of the matrix C^{-1}, inverse of $C \equiv \{C(P_i,P_k)\}$.

Let us note that again the possibility of using a predictor like (5.39) depends on the availability of the covariance function of u, (5.38); for the moment we just assume it is known and we shall explain later how to estimate it from data.

As in Sect. 5.3 we observe that, if we define a random field v,

$$v(P,\omega) = R_\omega u(P) \tag{5.41}$$

and we postulate a uniform distribution of ω on the 3D rotation group, we receive a totally equivalent problem with the same analytical solution, on condition that

$$\begin{aligned}E_\omega\{v(P,\omega)\} &= \frac{1}{8\pi^2}\int d\sigma_P u(P)\int_0^{2\pi} d\alpha_Q \\ &= \frac{1}{4\pi}\int d\sigma_P u(P) = 0,\end{aligned} \tag{5.42}$$

what we assume to be true, because by hypothesis $u(P) \equiv T_r(P)$ and $T_r(P)$ certainly has a zero mean on any sphere centered at the origin. Note as well that

calling $C(P, Q)$, in (5.38), a covariance function, we are consistent with a standard terminology for random fields.

5.5 The General Collocation Problem

Based on the discussion of Sects. 5.3 and 5.4, from now on we accept the equivalence principle stating that we can proceed with our prediction algorithms either by minimizing the invariant quadratic error in a class of invariant linear estimators or by introducing the model of a random field, as in (5.41), and minimizing the mean square prediction error in a class of linear predictors. Invariant here means invariant with respect to the 3D rotation group, and expectation means averaging over a uniform distribution on the rotation group.

Let us first of all state our problem in the following form: we have observation equations

$$y_i = M_i(u) + v_i, \quad \imath = 1 \dots N \tag{5.43}$$

and we want to predict a functional of u, $L(u)$ by means of a linear homogenous predictor, i.e.

$$L(\widehat{u}) = \sum_{i=1}^{N} \lambda_i y_i; \tag{5.44}$$

to do that we want to apply the Wiener-Kolmogorov (W-K) principle.

To this aim we need to define clearly what is an admissible functional L applied to the random process $v(P, \omega)$.

In fact note that $v(P, \omega) = R_\omega u(P) = u(R_\omega P)$, is a function of two variables and that L will act only on the variable P, so that we expect

$$Y_0 = L_P\{v(P, \omega)\} \tag{5.45}$$

to be a (measurable) function of ω only, i.e. a random variable.

We note that, under suitable regularity conditions,

$$\begin{aligned} E_\omega\{Y_0\} &= E_\omega\{L_P[v(P, \omega)]\} \\ &= \int dP(\omega) L_P\{R_\omega u(P)\} \\ &= L_P\{\int dP(\omega) R_\omega u(P)\} \\ &= L_P\{E_\omega\{v(P, \omega)\}\} = 0, \end{aligned} \tag{5.46}$$

so we expect that all useful random variables of the type (5.45) have zero mean (with respect to ω).

Definition 1 (Admissible functionals). We state that a *functional* $L_P(\,)$ *is admissible*, if the corresponding random variable Y_0 has finite variance.

Namely we require that

$$\begin{aligned} E_\omega\{Y_0^2\} &= \int dP(\omega) L_P[u(R_\omega P)] L_Q[u(R_\omega Q)] \\ &= L_P\{L_Q\{\int dP(\omega) u(R_\omega P) u(R_\omega Q)\}\} \\ &= L_P\{L_Q C(P,Q)\} < +\infty. \end{aligned} \tag{5.47}$$

Covariance propagation. The above computation can be repeated when we need to compute the covariance

$$\begin{aligned} &E\{L_P[v(P,\omega)] M_Q[v(Q,\omega)]\} \\ &\quad = L_P\{M_Q\{E[v(P,\omega)v(Q,\omega)]\}\} \\ &\quad = L_P\{M_Q C(P,Q)\}. \end{aligned} \tag{5.48}$$

Formula (5.48) is in fact the covariance propagation formula for random fields.

To simplify formulas, from now on we shall use the short-hand notation (see Krarup 2006, Chap. 15)

$$\begin{cases} L_P C(P,Q) = C(L,Q) \\ L_P\{M_Q C(P,Q)\} = C(L,M). \end{cases} \tag{5.49}$$

Moreover we note that if we take a vector of functionals

$$\mathbf{L} = \begin{vmatrix} L_1(\cdot) \\ L_2(\cdot) \\ \vdots \\ L_N(\cdot) \end{vmatrix} \tag{5.50}$$

and we put

$$\mathbf{Y} = \mathbf{L}\{v(P,\omega)\}, \tag{5.51}$$

then indeed $\mathbf{Y}$ has zero mean,

$$E\{\mathbf{Y}\} = 0,$$

and a covariance matrix $C_{\mathbf{YY}}$ given by

$$\{C_{Y_i Y_k}\} = \{C(L_i, L_k)\} \tag{5.52}$$

which we write in vector form as

$$C_{\mathbf{YY}} = C(\mathbf{L}, \mathbf{L}^t). \tag{5.53}$$

Naturally $C(\mathbf{L}, \mathbf{L}^t)$ is symmetric and positive definite. Similarly the cross-covariance between the vector $\mathbf{Y}$ of (5.51) and $\mathbf{Z} = \mathbf{M}\{v(P, \omega)\}$ is just the matrix

$$C_{\mathbf{YZ}} = E\{\mathbf{Y}\mathbf{Z}^t\} = \{C(L_i, M_k)\} = C(\mathbf{L}, \mathbf{M}^t). \tag{5.54}$$

Now the last thing we need in order to perform our prediction is just to observe that in our models we have two stochastic quantities, the random field $v(P, \omega)$ and the noise vector $\boldsymbol{\nu}$. So we need first of all to represent the stochastic interaction between the two and then we need to warn the reader that when we shall use the expectation symbol $E\{\ \}$, without any particular index, we will mean averaging with respect to all random variables, while we shall use $E_\omega\{\ \}$ or $E_\nu\{\ \}$ when we want to perform an average with respect to a specific random variable.

To complete the hypotheses on the covariance structure of the problem we summarize them as follows:

$$E\{v(P, \omega)\} \equiv 0,\ E\{v(P, \omega)v(Q, \omega)\} = C(P, Q), \tag{5.55}$$

with $C(P, Q)$ a given invariant covariance function and with the propagation rule (5.48) for the covariances of linear functionals of v;

$$E\{\boldsymbol{\nu}\} = 0,\ \ E\{\boldsymbol{\nu}\boldsymbol{\nu}^t\} = C_{\nu\nu}; \tag{5.56}$$

furthermore we shall assume that the noise $\boldsymbol{\nu}$ and the random field v are linearly independent, i.e.

$$E\{v(P, \omega)\nu_i\} = 0,\ \ \forall P,\ \forall i, \tag{5.57}$$

implying also that for any admissible functional L,

$$E\{L_P[v(P, \omega)]\nu_i\} = 0. \tag{5.58}$$

With all these rules of calculus we proceed to establish the W-K principle, namely we start to compute the variance of the prediction error.

Remember that the observation equations and the linear predictor $\widehat{L(v)}$ were defined in (5.43) and (5.44), which we can write in vector form as

$$\mathbf{Y} = \mathbf{M}\{v\} + \boldsymbol{\nu} \tag{5.59}$$

$$L_P[\widehat{v(P}, \omega)] = \boldsymbol{\lambda}^t\mathbf{Y}. \tag{5.60}$$

If $\widehat{L(v)}$ is our predictor, the prediction error is

$$\begin{aligned} e(\omega) &= L(v) - \widehat{L(v)} \\ &= L(v) - \boldsymbol{\lambda}^t\mathbf{Y} \end{aligned} \tag{5.61}$$

and its variance can be computed by

$$\begin{aligned}\mathcal{E}^2 &= E\{e^2(\omega)\} = E\{L(v)^2\} + \qquad (5.62)\\ &\quad - 2E\{\boldsymbol{\lambda}^t \mathbf{Y} L(v)\} + E\{(\boldsymbol{\lambda}^t \mathbf{Y})^2\}\\ &= C(L, L) - 2\boldsymbol{\lambda}^t E\{\mathbf{Y} L(v)\} + \boldsymbol{\lambda}^t C_{\mathbf{YY}} \boldsymbol{\lambda}. \qquad (5.63)\end{aligned}$$

On the other hand

$$\begin{aligned}E\{\mathbf{Y} L(v)\} &= E\{\mathbf{M}(v) L(v)\} + E\{\boldsymbol{\nu} L(v)\} \qquad (5.64)\\ &= C(\mathbf{M}, L);\\ C_{\mathbf{YY}} &= E\{\mathbf{Y}\mathbf{Y}^t\} = E\{[\mathbf{M}(v) + \boldsymbol{\nu}][\mathbf{M}(v) + \boldsymbol{\nu}]^t\}\\ &= E\{\mathbf{M}(v)\mathbf{M}^t(v)\} + E\{\boldsymbol{\nu}\boldsymbol{\nu}^t\} = C(\mathbf{M}, \mathbf{M}^t) + C_{\boldsymbol{\nu}\boldsymbol{\nu}}. \qquad (5.65)\end{aligned}$$

Substituting in (5.62) we can then invoke the W-K principle claiming that the optimal predictor is the one that minimizes $\mathcal{E}^2$, namely the solution of the *normal equation system*

$$C_{\mathbf{YY}} \boldsymbol{\lambda} = C(\mathbf{M}, L) \qquad (5.66)$$

or

$$\boldsymbol{\lambda} = C_{\mathbf{YY}}^{-1} C(\mathbf{M}, L) \qquad (5.67)$$

with $C_{\mathbf{YY}}$ given by (5.65).

Going back to (5.60) we find the W-K predictor

$$\widehat{L(v)} = C(L, \mathbf{M}^t) C_{\mathbf{YY}}^{-1} \mathbf{Y} \qquad (5.68)$$

and substituting into (5.62) we get its squared prediction error as

$$\mathcal{E}^2 = C(L, L) - C(L, \mathbf{M}^t) C_{\mathbf{YY}}^{-1} C(\mathbf{M}, L). \qquad (5.69)$$

Formulas (5.68) and (5.69) are so important that it is worth representing them explicitly in components, namely

$$\widehat{L(v)} = \sum_{k,i=1}^{N} L_P\{M_{P_k} C(P, P_k)\} C_{Y_k Y_i}^{(-1)} Y_i \qquad (5.70)$$

with $C_{Y_k Y_i}^{(-1)}$ the element (k, i) of the inverse of the matrix $C_{\mathbf{YY}}$, i.e.

$$C_{Y_k Y_i} = M_{P_k}\{M_{P_i} C(P_k, P_i)\} + C_{\nu_k \nu_i}; \qquad (5.71)$$

moreover

$$\mathcal{E}^2 = L_P\{L_Q C(P,Q)\} + \tag{5.72}$$

$$-\sum_{k,i=1}^{N} L_P\{M_{P_k} C(P,P_k)\} C^{(-1)}_{Y_k Y_i} L_P\{M_{P_i} C(P_i,P)\}. \tag{5.73}$$

We note that in most cases $C_{v_k v_i}$ is diagonal and, when $M_k(\ )$ are functionals representing the same type of measurement, many times we put $C_{\nu\nu} = \sigma_v^2 I$, although this is not really necessary in our formulas that represent the most general case.

Example 2. We want already here to specify how formulas (5.70), (5.72) work for the most prominent case of this book, namely the prediction of the anomalous potential $T(P)$ (loosely speaking one could say the geoid prediction) from observed pointwise gravity anomalies $\Delta g(P_i)$, $i = 1 \dots N$.

Let us remember that here $T(P)$ and $\Delta g(P)$ mean the residual anomalous potential and the residual gravity anomaly. We mention that in this case $L(\ )$, the functional to be predicted, is just the evaluation of T at the point P,

$$L(T) = ev_P(T) = T(P).$$

As for the gravity anomaly at P, we can usefully reason as follows; first we define a *gravity anomaly operator* A which actually transforms the function $T(P)$ into another function $\Delta g(P)$

$$\Delta g(P) = A(T) \equiv -\frac{\partial T}{\partial h}(P) + \frac{\gamma'}{\gamma} T(P), \tag{5.74}$$

then we evaluate the field $\Delta g(P)$ at a specific measurement point P_k,

$$\begin{aligned} M_k(T) &= ev_{P_k}\{A(T)\} \\ &= \Delta g(P_k). \end{aligned} \tag{5.75}$$

Put in this way we understand that to compute the covariance of M_k, M_i or that of M_k, L one can proceed in two steps. First we define a *covariance function* of $\Delta g(P)$ according to

$$\begin{aligned} C_{\Delta g \Delta g}(P,Q) &= E\{\Delta g(P)\Delta g(Q)\} \\ &= E\{A_P[v(P,\omega)]A_Q[v(Q,\omega)\} \\ &= A_P\{A_Q C(P,Q)\} \end{aligned} \tag{5.76}$$

where

$$v(P,\omega) = R_\omega T(P)\ ; \tag{5.77}$$

then we apply the evaluation at specific measurement points, namely

$$\begin{aligned} C(M_k, M_i) &= ev_{P_k}\{ev_{P_i} C_{\Delta g \Delta g}(P_k, P_i)\} \\ &= C_{\Delta g \Delta g}(P_k, P_i). \end{aligned} \tag{5.78}$$

Accordingly we define the *cross covariance* between the two fields $T(P)$ and $\Delta g(P)$ as

$$C_{T\Delta g}(P, Q) = E\{v(P, \omega) A_Q[v(Q, \omega)]\} = A_Q C(P, Q),$$

with v given by (5.77) and then we evaluate T at a particular point P and Δg at a particular point P_k, thus obtaining

$$\begin{aligned} C(P, M_k) &= ev_P\{ev_{P_k} C_{T\Delta g}(P, P_k)\} \\ &= C_{T\Delta g}(P, P_k). \end{aligned} \tag{5.79}$$

With the above specified rules, the best linear predictor, or collocation predictor of $T(P)$ is (see (5.70))

$$\widehat{T}(P) = \sum_{k,i} C_{T\Delta g}(P, P_k)\{C_{\Delta g \Delta g}(P_k, P_i) + \sigma^2_{\Delta g}\delta_{ik}\}^{(-1)} \Delta g_{\text{obs}}(P_i), \tag{5.80}$$

in (5.80) we have assumed that $C_{v_i v_k} = \sigma^2_{\Delta g}\delta_{ik}$ and we have written $\Delta g_{\text{obs}}(P_i)$ for Y_i.

The corresponding prediction error then becomes (see (5.72)).

$$\begin{aligned} \mathcal{E}^2 = C(P, P) + \\ - \sum_{k,i} C_{T\Delta g}(P, P_k)\{C_{\Delta g \Delta g}(P_k, P_i) + \sigma^2_{\Delta g}\delta_{ik}\}^{(-1)} C_{\Delta g T}(P_i, P) \end{aligned} \tag{5.81}$$

Remark 1. Recalling the definition of covariance of a function $T(P)$ (see (5.38)) namely

$$\begin{aligned} C(P, Q) &= E\{R_\omega T(P) R_\omega T(Q)\} \\ &= \int dP(\omega) T(R_\omega P) T(R_\omega Q) \end{aligned} \tag{5.82}$$

we see that, when T is a regular harmonic function,

$$\Delta_P C(P, Q) = \int dP(\omega) \Delta_P T(R_\omega P) T(R_\omega Q) \equiv 0, \tag{5.83}$$

in fact it is known that the Laplace operator is invariant under rotation, so that if $T(x, y, z)$ is harmonic as function of (x, y, z) and R_ω sends (x, y, z) into (x', y', z') then (see Exercise 1 in Sect. 5.12)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)T(x',y',z')$$
$$\equiv\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right)T(x',y',z')=0.$$

Naturally (5.83) implies $\Delta_Q C(P,Q)=0$ as well, because $C(P,Q)$ is a symmetric function of P and Q.

Now take a general collocation formula with $L_P = ev_P$ and $\{M_k\}$ whatever; similarly to (5.80), if we put

$$\xi_k = \Sigma_i\{C(M_k,M_i)+C_{v_k v_i}\}^{(-1)}Y_i \tag{5.84}$$

we see that the collocation predictor of $T(P)$ can be written as

$$\widehat{T(P)}=\sum_{k=1}^{N}C(P,M_k)\xi_k. \tag{5.85}$$

If we let P free to vary over $\Omega_R\equiv\{r_P\geq R\}$, we can interpret (5.85) more as an approximation of the whole function $T(P)$ than as a pointwise prediction. As such we see that our approximate solution $\widehat{T(P)}$ is automatically harmonic, namely

$$\Delta_P\widehat{T}(P)=\sum_{k=1}^{N}\Delta_P C(P,M_k)\xi_k\equiv 0. \tag{5.86}$$

This is indeed a nice property of our approximation theory.

5.6 Covariance and Spectral Harmonic Calculus

The functions $\widehat{T}(P)$ by which we do approximate the residual potential $T_r(P)$ are all harmonic in Ω_R, as stated in the previous section (Remark 1).

Therefore these functions can be represented by the convergent series

$$T(P)=\sum_{n,m=2}^{+\infty}\sum_{m=-n}^{n}T_{nm}S_{nm}(r_P,\vartheta_P,\lambda_P) \tag{5.87}$$

$$S_{nm}(r_P,\vartheta_P,\lambda_P)=\left(\frac{R}{r_P}\right)^{n+1}Y_{nm}(\vartheta_P,\lambda_P).$$

If we apply to $T(P)$, given by (5.87), the rotation operator we get, with $P'=R_\omega P$,

$$R_\omega T(P)=T(P')=\sum_{n=2}^{+\infty}\sum_{m=-n}^{n}T_{nm}\left(\frac{R}{r_P}\right)^{n+1}Y_{nm}(\vartheta_{P'},\lambda_{P'}); \tag{5.88}$$

on the same time we can state that

$$T(P') = \sum_{n=2}^{+\infty} \sum_{m=-n}^{n} T_{nm}(\omega) \left(\frac{R}{r_P}\right)^{n+1} Y_{nm}(\vartheta_P, \lambda_P), \tag{5.89}$$

because indeed $T(P')$ is also harmonic as a function of P. Naturally the harmonic coefficients of $T(P')$ as function of P, are not the same T_{nm} which appear in (5.87) and in particular they will depend on the relation between P' and P, namely on the specific rotation R_ω applied; this is why we have denoted them $T_{nm}(\omega)$.

We want to study the property of the functionals of T,

$$(P \in S_R), \qquad T_{nm}(\omega) = \frac{1}{4\pi} \int Y_{nm}(\vartheta_P, \lambda_P) T(R_\omega P) d\sigma_P \tag{5.90}$$

and their relation to the original T_{nm}. First of all we notice that, as for all admissible functionals, $E\{T_{nm}(\omega)\} = 0$ and

$$E\{T_{nm}(\omega) T_{jk}(\omega)\} = \frac{1}{(4\pi)^2} \int d\sigma_P \int d\sigma_Q Y_{nm}(\vartheta_P, \lambda_P) Y_{jk}(\vartheta_Q, \lambda_Q) \cdot \\ \cdot E\{T(R_\omega P) T(R_\omega Q)\}. \tag{5.91}$$

On the other hand since the covariance of T is spherically invariant ($P' = R_\omega P$, $Q' = R_\omega Q$),

$$E\{T(R_\omega P) T(R_\omega Q)\} = C(\psi_{P'Q'}) = C(\psi_{PQ}). \tag{5.92}$$

As a function of ψ, $C(\psi)$ is also a function of $\cos\psi$ so that we can write

$$t = \cos\psi; \qquad \overline{C}(t) \equiv C(\psi) = \sum_{n=0}^{+\infty} c_n P_n(t) \\ = \sum_{n=0}^{+\infty} c_n P_n(\cos\psi) \tag{5.93}$$

with (see (3.46))

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} \overline{C}(t) P_n(t) dt \\ = \frac{2n+1}{2} \int_0^{\pi} C(\psi) P_n(\cos\psi) \sin\psi d\psi \tag{5.94}$$

Therefore, recalling the summation rule (3.54), we can substitute in (5.91) and (5.92)

$$C(\psi_{PQ}) = \sum_{p,q=0}^{+\infty} c_p (2p+1)^{-1} Y_{pq}(\vartheta_P, \lambda_P) Y_{pq}(\vartheta_Q, \lambda_Q) \tag{5.95}$$

so that by virtue of the orthogonality of $\{Y_{nm}(\vartheta, \lambda)\}$ we find

$$E\{T_{nm}(\omega) T_{jk}(\omega)\} = \Sigma_{p,q} \frac{c_p}{2p+1} \delta_{pn} \delta_{qm} \delta_{pj} \delta_{qk} = \frac{c_n}{2n+1} \delta_{nj} \delta_{mk}. \tag{5.96}$$

Hence $T_{nm}(\omega)$ are uncorrelated to one another and their variances are the same for all orders in degree n,

$$\sigma^2(T_{nm}) = \sigma_n^2 = \frac{c_n}{2n+1} \tag{5.97}$$

We will call σ_n^2 the degree variances of individual coefficients and c_n the full power degree variances. Although this name has already been used in (3.173) we shall soon see that we are justified in using it here because we will prove that c_n is identical with $\overline{\sigma}_n^2$ given in (3.173).

In fact the following remarkable result holds (see also Moritz 1980).

Lemma 1. *The distribution of* $\mathbf{T}_n \equiv \{T_{nm}\}$ *in* R^{2n+1} *(remember that we have* $2n+1$ *orders for each degree* n*) is singular, its support is the sphere with squared radius*

$$|\mathbf{T}_n(\omega)|^2 = \sum_{m=-n}^{n} T_{nm}^2(\omega) = c_n \tag{5.98}$$

and in fact $\mathbf{T}_n(\omega)$ *is uniformly distributed on this sphere.*

There are two consequences of this lemma: the first is that if we know even approximate values for T_{nm}, we can directly estimate $C(\psi_{PQ})$, given by (5.93), with $c_n = \Sigma_m T^2{}_{nm}$.

Namely the harmonic coefficients of one particular function given on S_R, provide us the degree variances of the process generated by randomly rotating this function.

We notice here as well that the formula $c_n = \Sigma_m T^2{}_{nm}$ justifies the name given to c_n of full power degree variances, in fact we can verify now that $c_n = \overline{\sigma}_n^2$ according to the previous definition on (3.173).

The other consequence is that the Lemma gives an answer to a guess popping up from times to times in geodesy, that the distribution of T, and then for instance of $\{T_{nm}\}$ too, could be normal (cf. Jekeli 1991). Indeed this is not possible in a strict sense, as observed long ago by Lauritzen (see Lauritzen 1973), because then $\{T_{nm}\}$ for fixed n would all be independent with zero mean and variance σ_n^2, what would imply that

$$\Sigma_m T^2{}_{nm}(\omega) \sim \sigma_n^2 \chi_{2n+1}^2, \tag{5.99}$$

i.e. it cannot be a constant with respect to ω. Yet (5.99) shows that this variable has a variance tending to zero. In fact (5.99) implies

$$\sigma^2[\Sigma_m T_{nm}^2(\omega)] = \sigma_n^4 2 \cdot (2n+1)$$

which must tend to zero since

$$\Sigma_n \sigma_n^2 (2n+1) = \Sigma_n \Sigma_m T_{nm}^2 < +\infty \tag{5.100}$$

by hypothesis.

Indeed (5.100) implies $\sigma_n^2(2n+1) \to 0$ and, a fortiori, $\sigma_n^4(2n+1) \to 0$. So from the practical point of view the field T, at least above a certain degree, could still be approximately normal.

The use of (5.97) simplifies the calculation of various covariances and cross-covariances for fields which have an easy spectral representation, as we show in the next example.

Example 3. As we have seen in (5.80), to apply the present theory to the determination of a gravimetric quasi-geoid we need $C_{T\Delta g}(P, Q)$ and $C_{\Delta g \Delta g}(P, Q)$. If we apply the spherical approximation formula (cf. (5.101))

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r} T$$

that, in terms of harmonic coefficients translates into

$$\Delta g_{nm} = \frac{n-1}{R} T_{nm}, \tag{5.101}$$

we get straightforwardly

$$E\{T_{nm} \Delta g_{jk}\} = \delta_{nj} \delta_{mk} \frac{n-1}{R} \sigma_n^2(T) \tag{5.102}$$

and

$$E\{\Delta g_{nm} \Delta g_{jk}\} = \delta_{nj} \delta_{mk} \frac{(n-1)^2}{R^2} \sigma_n^2(T). \tag{5.103}$$

implying

$$c_n(\Delta g) = \frac{(n-1)^2}{R^2} c_n(T). \tag{5.104}$$

With these rules we can put

$$C_{T\Delta g}(P,Q) = \Sigma_{n,m}\frac{(n-1)}{R}\sigma_n^2(T)S_{nm}(r_P,\vartheta_P,\lambda_P)S_{nm}(r_Q,\vartheta_Q,\lambda_Q)$$
$$= \Sigma_n\frac{(n-1)}{R}\sigma_n^2(T)\left(\frac{R^2}{r_P r_Q}\right)^{n+1}(2n+1)P_n(\cos\psi_{PQ}) \tag{5.105}$$

and

$$C_{\Delta g\Delta g}(P,Q) = \Sigma_{n,m}\frac{(n-1)^2}{R^2}\sigma_n^2(T)S_{nm}(r_P,\vartheta_Q,\lambda_P)S_{nm}(r_Q,\vartheta_Q,\lambda_Q)$$
$$= \Sigma_n\frac{(n-1)^2}{R^2}\sigma_n^2(T)\left(\frac{R^2}{r_P r_Q}\right)^{n+1}(2n+1)P_n(\cos\psi_{PQ}). \tag{5.106}$$

Let us note that in particular (5.106) coincides, in spherical approximation, with (5.76).

It is useful to observe that not all the fields that can be derived from T possess a spherical invariant covariance, although the spectral calculus, when applicable, facilitates the calculations as the next example shows.

Example 4. We want to compute the covariance of $T_\lambda = \frac{\partial T}{\partial\lambda}$. Note that this quantity is just the eastern deflection of the vertical η multiplied by $r\sin\vartheta$. To this aim let us observe that, according to our definition of $Y_{nm}(\vartheta,\lambda)$ (cf. (3.50) and (3.51)) we have

$$\frac{\partial}{\partial\lambda}Y_{nm}(\vartheta,\lambda) = -mY_{n,-m}(\vartheta,\lambda). \tag{5.107}$$

But then

$$T_\lambda(P) = \Sigma_{n,m}(-m)T_{nm}S_{n,-m}$$
$$= \Sigma_{n,m}mT_{n,-m}S_{n,m}$$

or

$$(T_\lambda)_{nm} = mT_{n,-m}. \tag{5.108}$$

The last relation implies

$$E\{(T_\lambda)_{nm}^2\} = m^2\sigma_n^2(T) \tag{5.109}$$

so that T_λ has not *degree variances*, i.e. the variances of $(T_\lambda)_{nm}$ are not the same for all orders m.

It is useful here to observe that the covariance of T_λ can also be derived directly from $C(P, Q)$ with the following formula

$$C_{T_\lambda T_\lambda}(P, Q) = E\{T_\lambda(P)T_\lambda(Q)\} = \frac{\partial^2}{\partial\lambda_P \partial\lambda_Q} C(P, Q). \tag{5.110}$$

If we put

$$C(P, Q) = C(r_P, r_Q, \psi_{PQ}) = \overline{C}(r_P, r_Q, \cos\psi_{PQ}) \tag{5.111}$$

and we note that

$$\cos\psi_{PQ} = \sin\vartheta_P \sin\vartheta_Q \cos(\lambda_P - \lambda_Q) + \cos\vartheta_P \cos\vartheta_Q$$

so that

$$\frac{\partial}{\partial\lambda_Q} \cos\psi_{PQ} = \sin\vartheta_P \sin\vartheta_Q \sin(\lambda_P - \lambda_Q)$$

and

$$\frac{\partial^2}{\partial\lambda_P \partial\lambda_Q} \cos\psi_{PQ} = \sin\vartheta_P \sin\vartheta_Q \cos(\lambda_P - \lambda_Q),$$

we can compute (5.110).

Put

$$\overline{C}' = \frac{\partial}{\partial t}\overline{C}(r_P, r_Q, t)$$
$$\overline{C}'' = \frac{\partial^2}{\partial t^2}\overline{C}(r_P, r_Q, t);$$

then you find

$$C_{T_\lambda T_\lambda} = \overline{C}'(r_P, r_Q, \cos\psi_{PQ}) \sin\vartheta_P \sin\vartheta_Q \cos(\lambda_P - \lambda_Q) + \\ - \overline{C}''(r_P, r_Q, \cos\psi_{PQ})(\sin\vartheta_P \sin\vartheta_Q \sin(\lambda_P - \lambda_Q))^2,$$

which is not a function of ψ_{PQ} only, i.e. it is not a rotation invariant function.

Remark 2. In order to perform the covariance calculus of horizontal derivatives, a simple approach is, after fixing the two point P and Q, to compute the full covariance of the derivatives along the great circle connecting P and Q and orthogonal to it. The result can then be rotated to produce covariances of derivatives in any direction (Tscherning and Rapp 1974).

To get acquainted with the covariance spectral calculus we propose to the reader Exercise 2 at the end of the chapter.

5.7 The Estimate of Global Covariance Functions

The whole building of collocation theory rests on the assumption that there is a covariance function of the unknown $T(P), C_{TT}(P, Q)$, and that this function be known in some way. Since there is no theoretical a priori model for it we can only rely on data themselves to obtain an estimate of $C_{TT}(P, Q)$. Naturally the best theoretical framework to do that, would be a unified *estimation theory* where both $T(P)$ and $C_{TT}(P, Q)$ are optimally estimated together from data.

At this point indeed the problem becomes highly non-linear and, although some theoretical work has been done in this direction, no numerical experiments have been performed for the moment (Sansò and Venuti 2002a). So in practice we have to live with a two-steps procedure in which we first *estimate* $C_{TT}(P, Q)$, with an admissible model, and then we use it to apply the rest of collocation theory. This parallels very much what we are doing in the ordinary least squares theory (Koch, 1987) where we have to estimate both the vector of the parameters and the covariance matrix of the observable variables. In least squares theory however this practice is justified because we can prove that a variation of such covariance matrix induces a second order variation into the estimator of the parameters. Fortunately here we have again a similar situation as it has been proved in Sansò et al. (2000). So there is a reasonable argument to accept the two-step procedure. Yet the question is open on how to estimate practically $C_{TT}(P, Q)$ from data (see also Part II, Chap. 7).

We have two formulas relating the covariance function to observable quantities: one is its definition (5.92) that writes more explicitly as

$$P, Q \in S_Q,\ C_{TT}(\psi) = \frac{1}{4\pi} \int d\sigma_P T(P) \int_{\psi_{PQ}=\psi} d\alpha_Q T(Q); \qquad (5.112)$$

the other one is

$$C_{TT}(\psi_{PQ}) = \sum_{n=2}^{+\infty} c_n P_n(\cos \psi_{PQ}) \qquad (5.113)$$

with

$$c_n = \sum_{m=-n}^{n} T_{nm}^2. \qquad (5.114)$$

Both formulas require the knowledge of T on S_R (directly in (5.112) and through T_{nm} in (5.114)); both express $C_{TT}(P, Q)$ when $P, Q \in S_R$ and then can be harmonically continued in $P, Q \in \Omega_R$ by

$$C_{TT}(P, Q) = \sum_{n=1}^{+\infty} \left(\frac{R^2}{r_P r_Q} \right)^{n+1} c_n P_n(\cos \psi_{PQ}). \qquad (5.115)$$

Yet, since the quantity related to T that we know best at present, at the level of the ellipsoid, here approximated by S_R, is Δg, averaged in blocks, as explained in Chap. 3, the model (5.106) has been rather used, namely

$$C_{\Delta g \Delta g}(P, Q) = \sum_{n=2}^{+\infty} \left(\frac{R^2}{r_P r_Q} \right)^{n+1} c_n(\Delta g) P_n(\cos \psi_{PQ}). \tag{5.116}$$

where

$$c_n(\Delta g) = \frac{(n-1)^2}{R^2} c_n(T) = \frac{(n-1)^2}{R^2} \sum_{m=-n}^{n} T_{nm}^2. \tag{5.117}$$

Naturally with our finite data set we can only estimate $c_n(\Delta g)$ up to some maximum degree $N_{\max}$. It is by interpolating the empirical spectrum of Δg, i.e. (5.117), and then extrapolating it above $N_{\max}$ that we can have some model extending to all degrees up to infinity. The idea is similar to what we presented in Sect. 3.8, but with much more refined models which, beyond giving a better interpolation of empirical data, have also the advantage that the series (5.115) and (5.117) can be added providing us with closed analytical forms, more manageable from the numerical point of view. The argument and the relative models will be taken up in more details in the next section. What is interesting at this point is to underline two facts. The first is that all models include in both $c_n(T)$ and $c_n(\Delta g)$ an exponential factor which can therefore interpreted as $\left(\frac{R_B}{R}\right)^{2(n+1)}$, meaning that our kernel $C_{TT}(P, Q)$ will be harmonic down to a smaller sphere than S_R, in fact down to the Bjerhammar radius R_B, which in the most famous of such models (cf. Tscherning and Rapp 1974), has a value $R_B \cong 6{,}370\,\text{km}$. Note that R_B is different from the mean earth radius $R \cong 6{,}371\,\text{km}$, by 1 km only. The second is that, despite its usefulness, the degree variances of this global covariance function above $N_{\max}$ cannot well represent the local physical reality of our gravity field. In fact at the scales of 100 km down to 1 km the actual gravity field displays features so diverse from one part of the globe to the other that putting them all together into a unique covariance function prevents us from the construction of a very fine approximation of T, and then of the geoid, as required nowadays.

This argument calls for another step in our approximation road, where the local features of T or Δg are accounted for. We could say another step zooming into a smaller data area A and applying some kind of multi-resolution analysis concept. This will be achieved by means of the so-called local covariance functions.

We conclude the section with still another Example that will become useful in the sequel. This answers in the affirmative to the question: is it possible to have isotropic covariances on the bounding sphere that have a finite support, i.e. a $C(\psi)$ and a fixed arc $\Delta < \pi$ such that $C(\psi) = 0$ for $\forall \psi \geq \Delta$? In the example, we will construct one of such covariances, $M_\Delta(\psi)$, so that, recalling that the product of two covariance functions is again a covariance function, we can then construct for every $C(\psi)$ a finite support counterpart just by taking $C_\Delta(\psi) = M_\Delta(\psi) \cdot C(\psi)$.

Example 5. Let us recall that if we take at the north pole a function equal to 1 just when the colatitude ϑ is such that $\vartheta \leq \Delta$, and is equal to zero outside,

$$\chi_\Delta(\vartheta) = \begin{cases} 1 & \vartheta \leq \Delta \\ 0 & \vartheta > \Delta, \end{cases}$$

one can write

$$\chi_\Delta(\vartheta) = \sum_{n=0}^{+\infty} \beta_n P_n(\cos\vartheta)$$

where the so-called *Meissel's coefficients* β_n are given explicitly by (see also Sect. 3, A.4)

$$(t = \cos\vartheta) \qquad \beta_n = \frac{2n+1}{2} \int_{\cos\Delta}^{1} P_n(t)dt$$
$$= \frac{1}{2}[P_{n-1}(\cos\Delta) - P_{n+1}(\cos\Delta)].$$

Note that the relation between β_n and the coefficients of the moving average operator, defined in Sect. 3, A.4 is

$$\beta_n = \frac{1}{4}(2n+1)(1-\cos\Delta)M_n(\Delta).$$

Recalling that $Y_{n0} = \sqrt{2n+1}\,P_n(\cos\vartheta)$ we can write also

$$\chi_\Delta(\vartheta) = \sum_{n=0}^{+\infty} \frac{\beta_n}{\sqrt{2n+1}} Y_{n,0}(\vartheta)$$

If we consider this function as a potential on the sphere and we compute its covariance in spectral from (cf. (5.113) and (5.114)) we find

$$M_\Delta(\psi) = \sum_{n=0}^{+\infty} \frac{\beta_n^2}{2n+1} P_n(\cos\psi)$$

On the other hand if we compute the same covariance by (5.112) we see that we must fix P in the cap $D(O,\Delta)$ of radius Δ around the north pole O, we must fix a radius ψ and then take the product of $\chi_\Delta(\vartheta_P)$ by the average of $\chi_\Delta(\vartheta_Q)$ on the circle of radius ψ around P; finally we integrate in P over $D(P,\Delta)$. Note that when P is outside $D(O,\Delta)$, the integrand in (5.112) is automatically zero.

Now if P is in $D(O,\Delta)$ and on the same time $\psi > 2\Delta$, the circle of radius ψ and centre P, will not intercept anymore $D(O,\Delta)$ and, as result, we will have

$$M_\Delta(\psi) = 0, \ \forall \psi > 2\Delta.$$

The situation is illustrated in Fig. 5.7.

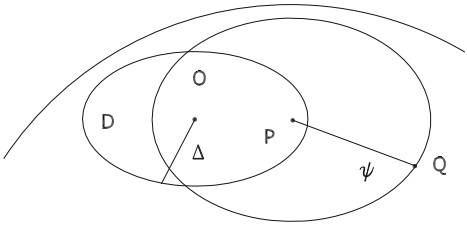

Fig. 5.7 Domains of integration used in the computation of the covariance of $T = \chi_\Delta(\vartheta_P)$

Let us observe explicitly that although we can construct covariances of finite support on the spherical boundary, as soon as we go to an external sphere, $r > R$, $C(P, Q)$ cannot be anymore zero on any part of the sphere of positive measure, otherwise as a harmonic function it should be zero everywhere (see Sacerdote and Sansò 1991).

5.8 The Estimate of Local Covariance Functions

As defined in (5.38), with the further specification of definition (5.76) we can say that the covariance function of the gravity anomaly field $\Delta g(P)$, at the level of the mean earth sphere, S_{R_e}, is given by

$$\begin{aligned} &P, Q \in S_{R_e}, \\ &C_{\Delta g \Delta g}(P, Q) = E\{\Delta g(P)\Delta g(Q)\} \\ &\quad = \frac{1}{8\pi^2} \int d\sigma_{P'} \int_{\psi_{P'Q'}=\psi_{PQ}} d\alpha_{Q'} \Delta g(P')\Delta g(Q') = C_{\Delta g \Delta g}(\psi_{PQ}); \end{aligned} \tag{5.118}$$

analogous formulas hold for $C_{TT}(P, Q)$ and $C_{T\Delta g}(P, Q)$ which are the main ingredients needed to derive the estimates (5.80) and (5.81).

The relation between the three functions is given by (5.76) and (5.79) in the ordinary geometric space and by (5.105) and (5.106) in the spectral domain. Although we derived them for the residual potential, represented by the random field $v(P, \omega) = T_r(R_\omega P)$, they basically hold for any random field similarly defined by means of its values on the sphere S_{R_e}, with the help of a uniform distribution on the rotation group, and harmonically continued in $\Omega_{R_e} \equiv \{r \geq R_e\}$. So in order to be close to the applications considered in this book we shall reason in this section on the covariance of Δg_r, with the understanding that the same arguments apply to any random field having an isotropic covariance function.

Moreover, such a remark will be used in next sections.

From (5.118) and a set of observed values

$$Y_i = \Delta g(P_i) + \nu_i, \quad i = 1, 2 \ldots N \tag{5.119}$$

with ν_i independent noises of equal variance σ_ν^2, we can reasonably build an estimator of the covariance in a very similar way of what is done with random processes, with respect to a time variable.

In fact, consider the following expression

$$\widehat{C}_{\Delta g\Delta g}(\overline{\psi}) = \frac{1}{N(\overline{\psi}, \Delta)} \Sigma_{\{i,k\}} Y_i Y_k, \tag{5.120}$$

where the summation is extended only to the pair of points $\{i, k\}$ such that

$$\overline{\psi} - \Delta < \psi_{P_i P_k} \leq \overline{\psi} + \Delta \tag{5.121}$$

and $N(\overline{\psi}, \Delta)$ is the number of such pairs.

Observe that, recalling also (5.55), (5.56) and (5.57),

$$\begin{aligned} E_{\omega,v}\{Y_i Y_k\} &= E_\omega\{\Delta g(P_i)\Delta g(P_k)\} + \sigma_v^2 \delta_{ik} \\ &= C_{\Delta g\Delta g}(\psi_{P_i P_k}) + \sigma_v^2 \delta_{ik}. \end{aligned} \tag{5.122}$$

As far as $\overline{\psi} - \Delta \geq 0$, i.e. $\psi_{P_i P_k} > 0$, we always have $\delta_{ik} = 0$ in (5.122), so that from (5.120) we find again, denoting $\{i, k\}$ the set of pairs satisfying (5.121),

$$E_{\omega,v}\{\widehat{C}_{\Delta g\Delta g}(\overline{\psi})\} = \frac{1}{N(\overline{\psi}, \Delta)} \Sigma_{\{i,k\}} C_{\Delta g\Delta g}(\psi_{P_i P_k}). \tag{5.123}$$

Now, if we assume that the observation points $\{P_i\}$ are well distributed, so that $\psi_{P_i P_k}$ sweeps in a fairly homogeneous way the interval $[\overline{\psi} - \Delta, \overline{\psi} + \Delta]$ and if we further agree that Δ is such that $N(\overline{\psi}, \Delta)$ is large enough e.g. at least larger than 10, and on the same time small enough, to allow $C_{\Delta g\Delta g}(\psi)$ to be almost linear in the interval $[\overline{\psi} - \Delta, \overline{\psi} + \Delta]$, we deduce from (5.123)

$$E_{\omega,v}\{\widehat{C}_{\Delta g\Delta g}(\overline{\psi})\} \approx C_{\Delta g\Delta g}(\overline{\psi}), \tag{5.124}$$

namely $\widehat{C}_{\Delta g\Delta g}(\overline{\psi})$ is a quasi-unbiased estimator of $C_{\Delta g\Delta g}(\overline{\psi})$.

Furthermore we note that (5.120) can be considered as well as a discretization of formula (5.112) or its analogous for Δg.

Accordingly, once the value of Δ has been fixed, what is in fact one of the very issues for the data analyzer, we can derive estimates $\widehat{C}_{\Delta g\Delta g}(\overline{\psi})$ for

$$\overline{\psi} = \Delta, 3\Delta, 5\Delta \ldots (\ell m + 1)\Delta. \tag{5.125}$$

Furthermore, by taking $i = k$ in (5.122), we derive

$$E\left\{\frac{1}{N}\sum_{i=1}^{N} Y_i^2\right\} = C_{\Delta g\Delta g}(0) + \sigma_v^2, \tag{5.126}$$

i.e.

$$S_y^2 = \frac{1}{N}\sum_{i=1}^{N} Y_i^2 \tag{5.127}$$

is an unbiased estimator of $C_{\Delta g \Delta g}(0) + \sigma_\nu^2$. All together the values

$$S_y^2;\ \widehat{C}_{\Delta g \Delta g}(\Delta);\ \widehat{C}_{\Delta g \Delta g}(3\Delta) \ldots \widehat{C}_{\Delta g \Delta g}((2m+1)\Delta) \tag{5.128}$$

constitute what is called the *empirical covariance function*; when Δg is the residual gravity anomaly Δg_r and the points $\{P_i\}$ are taken from a *local* area A only, we have a local empirical covariance function.

Note that, in order that such empirical covariance function could be further used in the prediction process, some conditions have to be fulfilled at least approximately. We already said about the choice of Δ, but we also have to assume that when data come from a local area A, $(2m+1)\Delta$ (see (5.118)) be significantly smaller than the size of A, identified with its diameter when A is a cap or with its side if A is a squared geographic block; at the same time $C_{\Delta g \Delta g}((2m+1)\Delta)$ and the other tail values of $C_{\Delta g \Delta g}$ beyond $(2m+1)\Delta$, should be small enough to make the correlation with observations beyond this distance negligible; moreover the size of A should be big enough to let the field Δg to have a zero average on it, i.e.

$$\frac{1}{N}\sum_{i=1}^{N} \Delta g(P_i) \approx 0, \tag{5.129}$$

as otherwise we could not write a covariance estimator in the form (5.120).

In reality, having an empirical average significantly different from zero on A would mean just that there is an important correlation of Δg_r in A with Δg_r outside A, so that we cannot hope to derive a good local estimate of T in A because we are lacking essential information.

One further concern is that the height of the points P_i should not have too strong a variation in A; in fact we see (cf. (5.116)) that if all points have the same height h, then the degree variances of Δg are just modified by a factor $\left(\frac{R_e}{R_e+h}\right)^{2n+4}$, that can be accounted for in modelling the covariance, while if $r_i = R_e + h_i$ is quite variable, then the covariance of the signal coming from h_i will enter into the empirical values $\widehat{C}_{\Delta g \Delta g}(\overline{\psi})$.

Finally we remind that our estimate (5.120) is relevant only if the residual $\Delta g_r(P)$ has a behaviour statistically homogeneous and isotropic in A; in other words there should not be in $\Delta g_r(P)$ features that make one part of A to look statistically very different from another one. This is typically achieved if the remove step for the model and for the residual terrain correction components has been correctly performed and the area A is suitably selected by the analyzer.

We get hold of an empirical covariance function that we need to transform into a model covariance function, namely into a function possessing the correct properties of symmetry and positive definiteness, without which the collocation prediction formulae loose any significance. This is the case if we impose to the model covariance to satisfy the relation (5.116), namely

$$C_{\Delta g\Delta g}(P,Q)=\sum_{n=2}^{+\infty}c_n(\Delta g)\left(\frac{R^2}{r_P r_Q}\right)^{n+2}P_n(\cos\psi_{PQ}),\tag{5.130}$$

with positive full power degree variances $c_n(\Delta g)$.

Now the point is how to model $c_n(\Delta g)$, taking also into account that we are talking about Δg_r, so that we expect $c_n(\Delta g)$ to have a different meaning when $n \le M$ (M being the maximum degree of our global model $T_M(P)$) than when $n > M$.

In fact if we write for the coefficients $T_{nm}^{(M)}$ of the global model the relation

$$T_{nm}^{(M)}=T_{nm}+\tau_{nm}\tag{5.131}$$

with τ_{nm} the estimation error for the coefficient T_{nm}, we see that in the low frequency band (cf. (5.101)),

$$(n\le M),\quad \Delta g_{r,nm}=\frac{n-1}{R}\tau_{nm}\tag{5.132}$$

so that

$$(n\le M),\quad c_n(\Delta g)=\frac{(n-1)^2}{R^2}\sum_{m=-n}^{n}\tau_{nm}^2,\tag{5.133}$$

according to (5.104).

Now (5.133) expresses the full power degree variances of the estimation errors $\{\tau_{nm}\}$, when the average is taken over the full rotation group. If we further average (5.133) with respect to the random variables τ_{nm}, which represent the propagation of the observation (and model) errors from original data to the estimates $T_{nm}^{(M)}$, we can define what are called *error degree variances*, namely

$$(n\le M),\quad \varepsilon_n(\Delta g_r)=E_\tau\{c_n(\Delta g_r)\}=\sum_{m=-n}^{n}\sigma^2(\tau_{nm}).\tag{5.134}$$

The variances $\sigma^2(\tau_{nm})$ are available from least squares estimates up to degrees of a few hundreds, or are derived by noise propagation through quadrature formulas (see Rapp 1997a; Pavlis et al. 2008), so we can claim that ε_n are known at least up to the specific degree M, which is useful in the present context (see Remark 3 below).

As for higher degrees, $n > M$, the full power degree variances are usually modelled by means of some parametric form. Typical are formulas of the type

$$c_n(\Delta g)=C_0h^{n+2}\frac{A(n)}{B(n)}\tag{5.135}$$

where

$$0<h<1\tag{5.136}$$

and $A(n)$, $B(n)$ are polynomials in n such that $B(n)$ has no zeroes for integer values larger than 1. The big advantage of the form (5.135) is that in many cases it becomes possible to add the series (5.130) obtaining an explicit analytic expression which is then quite comfortable to be used in further computations (see Sect. 5.9).

Remark 3. Let us put

$$h = \frac{R_B^2}{R^2},\ (R_B < R) \tag{5.137}$$

in (5.135) and substitute it back into (5.130); we find then

$$C_{\Delta g \Delta g}(P, Q) = \sum_{n=2}^{+\infty} \frac{A(n)}{B(n)} \left(\frac{R_B^2}{r_P r_Q} \right)^{n+2} P_n(\cos \psi_{PQ}). \tag{5.138}$$

Since $|P_n(\cos \psi)| \leq 1$, it is clear that (5.138) is converging in $r_P, r_Q > R_B$, whatever be the polynomials A and B; therefore any collocation solution that uses this covariance will be harmonic down to a sphere with radius R_B. As already mentioned at the end of Sect. 5.6, the constant R_B is called a *Bjerhammar radius* after the work of A. Bjerhammar (see for instance Bjerhammar 1987); whence the index B.

Summarizing the previous general discussion, we arrive at a model of local covariance function that can be expressed as

$$C^{\text{Mod}}_{\Delta g \Delta g}(P, Q) = a \sum_{n=2}^{M} \varepsilon_n \frac{(n-1)^2}{R^2} \left(\frac{R^2}{r_P r_Q} \right)^{n+2} P_n(\cos \psi_{PQ}) + C_r(P, Q) \tag{5.139}$$

$$C_r(P, Q) = \sum_{n=M+1}^{+\infty} c_n(\Delta g) \left(\frac{R^2}{r_P r_Q} \right)^{n+2} P_n(\cos \psi_{PQ}) \tag{5.140}$$

$$c_n(\Delta g) = C_0 h^{n+2} \frac{A(n)}{B(n)}. \tag{5.141}$$

Parameters of the representation (5.139), (5.140) and (5.141) are: the calibration constant a, the degree M used in the specific remove-restore procedure, the constant C_0, the Bjerhammar radius R_B, i.e. the value of h, the coefficients of the polynomials $A(n)$, $B(n)$ which however can be normalized to have the zero degree coefficients equal to 1, namely $a_0 = b_0 = 1$.

By using all these parameters one can interpolate the empirical covariance function, using only the values outside the origin $\widehat{C}_{\Delta g \Delta g}(\Delta), \ldots, \widehat{C}_{\Delta g \Delta g}((2m+1)\Delta)$.

In this covariance modelling process it is important to use M as a parameter because the experience shows that many times the use of R_B only does not allow to reach the right shape of the covariance in the first (and most important) part of $C_{\Delta g \Delta g}(\psi)$, typically decreasing from the value $C_{\Delta g \Delta g}(0)$.

The value S_y^2 (cf. (5.127)) is then used to estimate σ_v^2,

$$\widehat{\sigma}_v^2 = S_y^2 - \widehat{C}_{\Delta g \Delta g}(0). \tag{5.142}$$

As it is obvious one must have

$$\widehat{\sigma}_v^2 \geq 0 \tag{5.143}$$

for this estimate to be acceptable; therefore (5.143) acts as a constraint for the model

$$C_{\Delta g \Delta g}^{\text{Mod}}(0) \leq S_y^2. \tag{5.144}$$

All in all, this estimation procedure casts so to say into a theoretically acceptable form the statistical behaviour of Δg_r in the specific area A, captured by the empirical estimates (5.120). Therefore, despite its global appearance, $C_{\Delta g \Delta g}^{\text{Mod}}$ represents in fact the physical correlation of Δg_r in the area A and in general it should not be used for another area. This reflects, to some extent, the multi-resolution character of the solution we are elaborating, step after step.

Example 6. It is important to understand that the transition from Δg to Δg_r removes power from $C_{\Delta g \Delta g}$, namely it damps its value at the origin and at the same time it reduces the correlation length, i.e. the smallest value ψ_c for which the relation

$$C_{\Delta g \Delta g}(\psi) = \frac{1}{2} C_{\Delta g \Delta g}(0) \tag{5.145}$$

is satisfied. More properly one could say that the transition from Δg to Δg_r reduces the index $\dfrac{C_{\Delta g \Delta g}(0)}{\psi_c}$, that could be taken as an indicator of the smoothness of the covariance. In this respect, it is interesting to observe the sequence of the covariance functions for the full signal of free air Δg over the area $6^\circ \leq \lambda \leq 20$, $36^\circ \leq \varphi \leq 47^\circ$ corresponding to a domain A covering the Italian region (Fig. 5.8), and the covariance function of the reduced Δg_r over the same region (Fig. 5.9). Finally in Fig. 5.9 we show as well the covariance from the Tschering–Rapp family (see formula (7.16) in Part II, Chap. 7) that interpolates $\widehat{C}_{\Delta g_r \Delta g_r}$.

Notice that in the chosen land area the gravity signal is quite variable, due to the complex geological structure of the region. So the covariance of the global gravity field, reflecting a mean behaviour for the whole earth, suggests a behaviour smoother than that implied by the local covariance in Fig. 5.8. On the other hand the covariance of Δg_r is both less powerful and smoother than that of the free air anomalies.

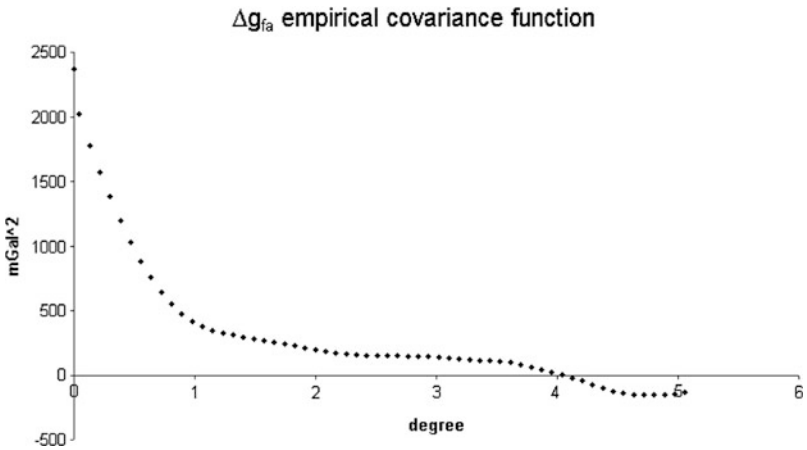

Fig. 5.8 The free air gravity anomaly empirical covariance over the Italian area

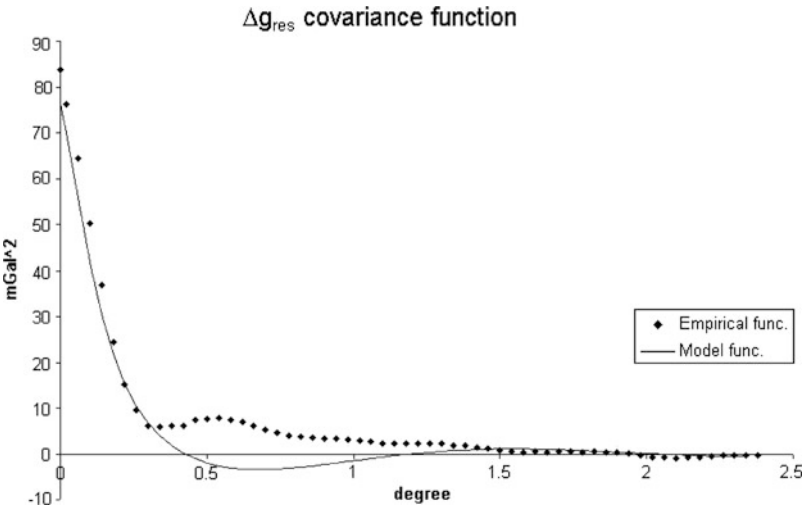

Fig. 5.9 The empirical covariance of the reduced gravity anomaly over the Italian area and the best fitting Tscherning–Rapp model

5.9 Covariance Parametric Models

As we have seen in the two previous sections, an estimation procedure for the covariance function of T or Δg passes through the adaptation of a parametric model to suitable empirical covariance values.

For this purpose let us note that if we accept the model (5.135) and we put

$$s = \frac{R_B^2}{r_P r_Q},\ t = \cos\psi \tag{5.146}$$

our target is to sum a series of the form

$$C_{\Delta g \Delta g}(s,t) = \sum_{n=2}^{+\infty} \rho_{\Delta g}(n) s^{n+2} P_n(t) \tag{5.147}$$

with $\rho(n)$ a rational function of n.

Since it is convenient in the present context, we shall however start from the covariance of T, that in this case, with the notation (5.146), can be written

$$C_{TT}(s,t) = \sum_{n=2}^{+\infty} \rho_T(n) s^{n+1} P_n(t). \tag{5.148}$$

In performing our calculus we shall need a few relations that we list for the comfort of the reader. We start by recalling (see (3.16) and (3.17)) the definition of generating function

$$G(s,t) = \sum_{n=0}^{+\infty} s^n P_n(t) = \frac{1}{\sqrt{1+s^2-2st}} \tag{5.149}$$

and the obvious relation

$$\sum_{n=2}^{+\infty} s^n P_n(t) = G(s,t) - 1 - st. \tag{5.150}$$

Then we have

$$\begin{cases} \frac{\partial}{\partial s} G^{-1}(s,t) = (s-t) G(s,t) \\ \frac{\partial}{\partial s} G(s,t) = -(s-t) G^3(s,t). \end{cases} \tag{5.151}$$

Furthermore, as one can verify by direct differentiation, one has

$$\int_0^s G(\sigma,t)\,d\sigma = \log \frac{s - t + G^{-1}(s,t)}{1-t}; \tag{5.152}$$

note that when $s \to 0$ both members tend to zero.

Moreover we observe that, for any $F(s,t)$,

$$-\frac{\partial}{\partial r_P} F(s,t) = \frac{s}{r_P} \frac{\partial}{\partial s} F(s,t) \tag{5.153}$$

and similarly for $-\frac{\partial}{\partial r_Q} F(s,t)$.

With such tools a number of intermediate results are derived in the exercises at the end of the chapter, that the reader is invited to make.

We continue the section by concentrating on one of the covariance models that are most widely used in modelling gravity covariances. Before doing so we underline again that such a model can be used for both, global and local covariance modelling. In fact any global model of which we know the sum in analytical form, namely

$$C(s,t) = \sum_{n=0}^{+\infty} c_n s^{n+2} P_n(t) \tag{5.154}$$

can be turned into a truncated form of the type

$$\begin{aligned} C_M(s,t) &= \sum_{n=M+1}^{+\infty} c_n s^{n+2} P_n(t) \\ &= C(s,t) - \sum_{n=0}^{M} c_n s^{n+2} P_n(t), \end{aligned} \tag{5.155}$$

which is easily computed because $C(s,t)$ has a closed form and the second term in (5.155) is just a finite sum up to a few hundred terms.

The Tscherning–Rapp model. This model (see Tscherning and Rapp 1974) has, in its classical formulation, the general form (5.130) and (5.135), parameterizing the gravity full power degree variances as

$$c_n(\Delta g) = A \left(\frac{R_B^2}{R^2} \right)^{n+2} \cdot \frac{n-1}{(n-2)(n+B)}, \quad n \geq 3, \tag{5.156}$$

or, what amounts to the same, the form (5.138) with

$$\frac{A(n)}{B(n)} = \frac{A(n-1)}{(n-2)(n+B)}, \quad n \geq 3. \tag{5.157}$$

For reasons that are explained in Appendix A.2, the parameter B is restricted to integer values.

The computation of $C_{\Delta g \Delta g}(s,t)$ corresponding to the choices (5.157) is fully worked out in Appendix A.2. The result can be cast into the form

$$C_{\Delta g \Delta g}(s,t) = A \left\{ \frac{B+1}{B+2} K_B(s,t) + \frac{1}{B+2} K_{-2}(s,t) \right\} \tag{5.158}$$

and the algorithms to compute $K_B(s,t)$ and $K_{-2}(s,t)$ have to be found in Appendix A.2.

With similar arguments one can compute as well the covariance function of T and the cross-covariance of T and Δg which are essential to perform the prediction of T from Δg and compute the corresponding prediction error.

We have

$$C_{TT}(s,t) = AR^2 \left\{ \frac{1}{(B+2)} \frac{1}{s} K_{-2}(s,t) + \right. \tag{5.159}$$
$$+ \frac{1}{(B+1)(B+2)} \frac{1}{s} K_B(s,t) - \frac{1}{B+1} [s - s^2 t - sG^{-1}(s,t) +$$
$$\left. - s^2 + \log \frac{1 - st + G^{-1}(s,t)}{2} - s^3 P_2(t) \right] \Bigg\}$$

and

$$C_{T\Delta g}(s,t) = A \frac{R^2}{r_P (B+2)} \left\{ \frac{1}{s} K_{-2}(s,t) - \frac{1}{s} K_B(s,t) \right\} . \tag{5.160}$$

Note that in (5.160) Δg is evaluated at P while T is evaluated at Q and we have here $s = \frac{R_B}{r_P r_Q}$, $t = \cos \psi_{PQ}$.

5.10 The Least Squares Collocation (l.s.c.) Solution

By *solution* we mean here computing the predictor (5.68) with its prediction error variance (5.69), when the problem at hand is fully general. When we have to predict T from Δg, we have to utilize formulas (5.80) and (5.81). When we apply the latter formulas to a local data set, $\{P_i\} \in A$, of residual gravity anomalies, $\Delta g_r^{\text{obs}}(P_i)$, then we can predict *local* values of the residual anomalous potential $\widehat{T}_r(P)$.

A l.s.c. solution is exactly one such solution when a local covariance function is used in formula (5.80) and (5.81).

We notice here that there seems to be a certain degree of contradiction in applying the W-K principle of Sect. 5.4 to the present local context. In fact, by definition the covariance function of Sect. 5.4 is obtained by averaging on the full sphere, or better on the full rotation group; on the contrary the local covariance function used in a l.s.c. solution is derived only for the area A where we have data and it would be different for the true earth in another area.

Since the formula for the isotropic covariance function, (5.38), was in fact obtained from the minimum quadratic invariant error principle (5.37), it seems interesting to ask whether there is an analogous minimum quadratic error principle, valid for the data in the area A only, leading us to the use of a *local* covariance function. A rigorous answer to this question would be in the negative sense. However it is feasible to build a local theory implying a definition of a local

covariance function that is only approximately isotropic and is close to what is suggested by the estimation formula (5.118).

Yet this goes beyond the scope of this presentation and here we limit ourselves to some more elementary considerations.

Basically our solution would be justified at least in a mean square sense, if the field T_r we want to estimate had, outside the area A and over all the rest of the sphere, the same statistical behaviour. If we impose such a hypothesis by definition, we will have a prediction which is optimal for this *virtual* field and on the same time it agrees with ours, at least in terms of observations, in the area A.

So the question is not whether the local covariance is good for the whole sphere (which is not) but rather what is the region in space where our *local* approximation procedure gives valid answers.

Fortunately collocation theory helps by giving us the tool to compute the prediction error (see (5.72) and (5.81)) and we can decide to go with the prediction point as far as possible till the prediction error reaches a predefined threshold. In this sense it is useful to observe that sometimes it is convenient to fix a threshold for the relative prediction error, namely, if $T(P)$ is the predicted functional,

$$\mathcal{E}_r(P) = \left\{ \frac{\mathcal{E}(P)^2}{C_{TT}(P,P)} \right\}^{(1/2)} \tag{5.161}$$

$$= \left\{ 1 - \frac{\sum_{i,k=1}^{N} C_{T,T}(P,M_i) C^{(-1)}_{Y_i Y_k} C_{TT}(P,M_k)}{C_{TT}(P,P)} \right\}^{(1/2)} .$$

This expresses the ratio of the prediction error to the signal we want to predict and can be fixed to levels like 1%, and 5% or others. For instance, one can decide to estimate a residual geoid of 1 m, r.m.s., with an error of 1 cm.

A warning has to be done at this point: when formulas like (5.81) or (5.161) are used in an extrapolation mode, i.e. for points P outside the area A, they give us always optimistic values because outside A the actual residual gravity field might not be well-represented, as for its statistical behaviour, by the same local covariance that has been estimated form values in A only. As a matter of fact this is of no great concern because numerical experience shows that already inside A, close to its boundary, $\mathcal{E}^2(P)$ and $\mathcal{E}_r^2(P)$ increase to unacceptable values and the prediction has to stop.

Remark 4. The above phenomenon can be understood qualitatively on the basis of the following reasoning. Remember that the local covariance function is estimated from empirical values and we have agreed that those have to become small at angular distance $\psi > \Delta$ for some Δ much smaller than the size of A. Accordingly, exploiting the possibility illustrated in the Example 5, we can model the theoretical

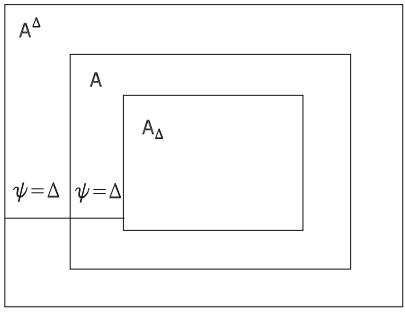

Fig. 5.10 A the set covered by data; A^{Δ} the set where data give some information; A_{Δ} the set where a good prediction can be performed

local covariance to have a finite support, i.e. to go strictly to zero on the sphere, when $\psi > \Delta$.

So, assume one has to perform a prediction at P, on the sphere, from observed values $T(P_i)$.

We see that outside the set $A^{\Delta} \equiv \{P \; ; \; \psi_{PQ} \leq \Delta \text{ for some } Q \in A\}$ (see Fig. 5.10) the l.s.c. predictor of $\widehat{T}(P)$ is $\widehat{T}(P) = 0$. In fact if the observation points P_i are all in A and P is outside A^{Δ}, $\psi_{PP_i} > 0, \forall i$ and then $\widehat{T}(P)$, written in the form

$$\widehat{T}(P) = \sum_{i=1}^{N} \xi_i C(\psi_{PP_i}) \tag{5.162}$$

is indeed zero. On the contrary, if we are well inside A, depending on the density of data and on the signal to noise ratio, we can have a good prediction of T. Let's reason now on a belt for instance of width Δ in A, i.e. in $A \backslash A_{\Delta}$, with $A_{\Delta} \equiv \{P \in A \; ; \; \psi_{PQ} < \Delta \Rightarrow Q \in A\}$. We expect that important information for the prediction of $T(P)$ is lost when $P \in A \backslash A_{\Delta}$ and correspondingly the prediction error becomes higher (see Fig. 5.10).

The above reasoning, though not rigorous, gives an idea of what happens in reality. A few exercises at the end of the section will be useful to the reader to enter into the subject.

Now that we have roughly agreed how to settle the prediction domain in a horizontal direction, we have to address the problem of the vertical dimension of this domain. The following trivial example can help in grasping the problem.

Example 7. Assume that $T(P)$ has covariance function

$$C(P, Q) = \sum_{n=m}^{+\infty} c_n \left(\frac{R^2}{r_P r_Q} \right)^{n+1} P_n(\cos \psi_{PQ}) \; ;$$

assume that at Q, with $r_Q = R$, we have observed $T(Q)$ without noise and we want to predict $T(P)$ along the radius passing through Q. By applying (5.70) with

evaluation functionals and one observation only we get (note that $\psi_{PQ} = 0$ and $P_n(1) = 1$)

$$\begin{aligned}\widehat{T}(P) &= C(P,Q)C^{-1}(Q,Q)T(Q)\\ &= \frac{\sum_{n=m}^{+\infty} c_n \left(\frac{R}{r_P}\right)^{n+1}}{\sum_{n=m}^{+\infty} c_n} T(Q).\end{aligned} \tag{5.163}$$

and the corresponding relative error from (5.161) is

$$\mathcal{E}_r^2(P) = 1 - \frac{\left[\sum_{n=m}^{+\infty} c_n \left(\frac{R}{r_P}\right)^{n+1}\right]^2}{\sum_{n=m}^{+\infty} c_n \cdot \sum_{n=m}^{+\infty} c_n \left(\frac{R}{r_P}\right)^{2n+2}} \tag{5.164}$$

If we take the limit for $r_P \to \infty$ of (5.164), we receive

$$\lim_{r_P \to \infty} \mathcal{E}_r^2(P) = 1 - \frac{c_m}{\sum_{n=m}^{+\infty} c_n}. \tag{5.165}$$

Then we expect $\mathcal{E}_r(P)$ to be close to 1 when r_P increases, i.e. P moves to the zenith of Q. For instance take for c_n the simple model

$$c_n = h^n$$

with h close to 1, then we see from (5.165) that

$$\mathcal{E}_r^2(P) \to h$$

i.e. the relative error becomes almost 100%. So if we fix a threshold for $\mathcal{E}_r$ then we will find an upper limit for the height where our solution is acceptable.

The phenomenon, highlighted in the Example 7, has general character and is basically related to the fact that if Q_i are observation points with $r_{Q_i} = R$ and P is taken on a higher sphere, $r_P > R$, then $C(P, Q_i)$ is modified by multiplying c_n by the factor $\left(\frac{R}{r_P}\right)^{n+1}$; this corresponds to giving more weight to the low frequencies and to damp the high frequencies so that the shape of the covariances is flattened. In turn this implies that we need more measurements distant from the prediction area, to perform a good prediction job. Accordingly we understand that data on a larger

area are needed to make a prediction with fixed relative error. Or, equivalently, when we rise in height the area of valid predictions has to be reduced.

Remark 5. Another way to approach the "localization" of the approximation to T is to push even further our simplification of reference model to arrive to the so-called *planar approximation*, where the reference gravity vector is always pointing to a parallel direction. Also in this case the collocation concept can be applied with the advantage of having available the Fourier transform machine (see Chap. 10 of this Part II). An interesting connection can then be established between planar and spherical covariance functions (see Forsberg 1987).

5.11 On the Optimal Combination of Global Coefficients and Local Observations

The procedure of removing from the anomalous potential, and all the corresponding observables, a global model T_M and then applying to the residual part T_r the collocation prediction, based on data in a local area A only, as explained in Sects. 5.8 and 5.10, is not strictly rigorous. As a matter of fact one should apply the W.K. principle to a full combination of the available information, namely the local data *and* the global model coefficients. Beyond the rigor, one of the advantages of proceeding along this line is that we can overcome the request that $T_r(P)$ be of zero average on A; such a request in fact is sometimes restrictive, specially if we have to predict the potential with high accuracy in a small area.

So we assume we have performed only a smoothing for the high frequency residual terrain correction and we call again $T(P)$ the remaining unknown potential. Then we consider as given information a set of *local* observations

$$\begin{aligned} Y_i &= M_i(T) + v_i, \\ 1 &= 1, 2 \dots J \end{aligned} \tag{5.166}$$

with T a random field with a global covariance

$$C_{TT}(P, Q) = C(P, Q) = \sum_{n=2}^{+\infty} c_n(T) \left(\frac{R^2}{r_P r_Q} \right)^{n+1} P_n(\cos \psi_{PQ}) \tag{5.167}$$

which for the moment we consider as known. As usual v_i are observation noises with zero mean and a known covariance matrix C_v, moreover v_i are independent of T. In vector from we write (5.166) as

$$\mathbf{Y} = \mathbf{M}(T) + \boldsymbol{v} \tag{5.168}$$

with first moments specified as usual by

$$E\{\mathbf{Y}\} = 0,\ C_{YY} = C(\mathbf{M}, \mathbf{M}^t) + C_{vv}. \tag{5.169}$$

In addition we shall assume to know the harmonic coefficients of T to some degree N, namely

$$\begin{aligned} T_{nm}^M &= T_{nm} + \tau_{nm} \\ -n \le m &\le n\ ;\ n = 2, \ldots, N. \end{aligned} \tag{5.170}$$

In (5.170) T_{nm} are the *true* harmonic coefficients of T, that we write as linear functionals

$$T_{nm} = \frac{1}{4\pi} \int T(R, \vartheta, \lambda) Y_{nm}(\vartheta, \lambda) d\sigma = H_{nm}(T) \tag{5.171}$$

and τ_{nm} are the *errors* of the known coefficients on the nature of which we shall comment later on. We find it convenient to vectorize (5.170) as $N - 1$ vector equations, namely

$$\mathbf{T}_n^M = \mathbf{T}_n + \boldsymbol{\tau}_n = \mathbf{H}_n(T) + \boldsymbol{\tau}_n. \tag{5.172}$$

The error vectors $\boldsymbol{\tau}_n$ are assumed to be of zero average and to have covariance matrices

$$G_n = E\{\boldsymbol{\tau}_n \boldsymbol{\tau}_n^t\}\ ; \tag{5.173}$$

moreover, though not essential, we shall assume that

$$E\{\boldsymbol{\tau}_n \boldsymbol{\tau}_\ell^t\} = \delta_{n\ell} G_n, \tag{5.174}$$

i.e. $\boldsymbol{\tau}_n$ and $\boldsymbol{\tau}_\ell$ referring to different degrees are uncorrelated.

Furthermore we assume that all $\boldsymbol{\tau}_n$ are not correlated with the random field $T(P)$, $E\{T(P)\boldsymbol{\tau}_n\} = 0$.

In addition, although it is possible that the same observations $\mathbf{Y}$ have been used too in the estimate of $\mathbf{T}_n^M$, since in this case they are mixed with a much larger data set coming from everywhere on the earth, outside A, we shall assume that the correlation of $\boldsymbol{\tau}_n$ and $\mathbf{Y}$ is zero, namely

$$E\{\mathbf{Y}\boldsymbol{\tau}_n^t\} = 0. \tag{5.175}$$

In principle predicting by collocation any functional $L(T)$ of T is nothing new, however the specific form of the functionals $\mathbf{H}_n$ and their covariance and cross-covariances with $\mathbf{Y}$ are such as to provide the solution in a very suggestive form.

So deciding to limit ourselves to $L_P(T) = T(P)$ and so to search the predictors in the form

$$\widehat{T}(P) = \boldsymbol{\lambda}^t \mathbf{Y} + \sum_{n=2}^{N} \boldsymbol{\alpha}_n^t \mathbf{T}_n^M \tag{5.176}$$

we can construct directly the normal system for the unknowns $\boldsymbol{\lambda}$ and $\{\boldsymbol{\alpha}_n\}$. To do so it is convenient first to compute some cross-covariances. For the sake of convenience, to follow the vectorized notation (5.172) we can put

$$T(P) = \sum_{n=2}^{+\infty} \sum_{m=-n}^{n} T_{nm} S_{nm}(r\vartheta, \lambda) = \sum_{n=2}^{+\infty} \mathbf{T}_n^t \mathbf{S}_n(P), \tag{5.177}$$

implicitly defining $\mathbf{S}_n$.

Then we have, recalling that $\sigma_n^2 = \frac{c_n(T)}{2n+1}$,

$$E\{\left(\mathbf{T}_n^M\right)\left(\mathbf{T}_\ell^M\right)^t\} = C(\mathbf{H}_n, \mathbf{H}_\ell^t) + G_n \delta_{n\ell} \tag{5.178}$$

$$= E\{\mathbf{T}_n \mathbf{T}_\ell^t\} + G_n \delta_{n\ell} = (\sigma_n^2 I + G_n)\delta_{n\ell},$$

$$E\{\mathbf{Y}\left(\mathbf{T}_n^M\right)^t\} = C(\mathbf{M}, \mathbf{H}_n^t) = E\{\mathbf{M}(T)\mathbf{T}_n^t\} \tag{5.179}$$

$$= E\left\{\sum_{\ell=2}^{+\infty} \mathbf{M}(\mathbf{S}_\ell^t)\mathbf{T}_\ell \mathbf{T}_n^t\right\} = \sigma_n^2 \mathbf{M}(\mathbf{S}_n^t),$$

$$E\{\mathbf{Y}T(P)\} = C(\mathbf{M}, P), \tag{5.180}$$

$$E\{\mathbf{T}_n^M T(P)\} = C(\mathbf{H}_n, P) = \sigma_n^2 \mathbf{S}_n(P). \tag{5.181}$$

Since the normal equation system has general form

$$\begin{cases} C_{YY}\boldsymbol{\lambda} + \sum_{\ell=2}^{N} C_{YT_\ell^M}\boldsymbol{\alpha}_\ell = C_{YT} \\ C_{T_n^M Y}\boldsymbol{\lambda} + \sum_{\ell=2}^{N} C_{T_n^M T_\ell^M}\boldsymbol{\alpha}_\ell = C_{T_n^M T} \end{cases} \tag{5.182}$$

$$(n = 2, \ldots, N),$$

by using the specifications (5.178) through (5.181) we find

$$C_{YY}\boldsymbol{\lambda} + \sum_{\ell=2}^{N} \sigma_\ell^2 \mathbf{M}(\mathbf{S}_\ell^t)\boldsymbol{\alpha}_\ell = C(\mathbf{M}, P) \tag{5.183}$$

$$\sigma_n^2[\mathbf{M}(\mathbf{S}_n^t)]^t\boldsymbol{\lambda} + (\sigma_n^2 I_n + G_n)\boldsymbol{\alpha}_n = \sigma_n^2 \mathbf{S}_n(P). \tag{5.184}$$

The partitioned form of this system suggests to solve (5.184) with respect to $\boldsymbol{\alpha}_n$ and then substitute back into (5.183). In this way, posing

$$\Gamma_n = \sigma_n^2(\sigma_n^2 I_n + G_n)^{-1}, \tag{5.185}$$

$$\boldsymbol{\alpha}_n = \Gamma_n \mathbf{S}_n(P) - \Gamma_n[\mathbf{M}(\mathbf{S}_n^t)]^t \boldsymbol{\lambda}, \tag{5.186}$$

we find

$$(C_{YY} - \sum_{\ell=2}^{n} \sigma_\ell^2 \{[\mathbf{M}(\mathbf{S}_\ell^t)]\Gamma_\ell[\mathbf{M}(\mathbf{S}_\ell^t)]^t\})\boldsymbol{\lambda} \tag{5.187}$$

$$= C(\mathbf{M}, P) - \sum_{\ell=2}^{N} \sigma_\ell^2 \mathbf{M}(\mathbf{S}_\ell^t)\Gamma_\ell \mathbf{S}_\ell(P).$$

As we see, we have now a unique equation in $\boldsymbol{\lambda}$, i.e. (5.187). In order to better understand its meaning we set in clear the components of the relevant matrices and vectors. We have

$$\{[\mathbf{M}(\mathbf{S}_\ell^t)]\Gamma_\ell[\mathbf{M}(\mathbf{S}_\ell^t)]^t\}_{ij} \tag{5.188}$$

$$= \sum_{k,h=-\ell}^{\ell} M_i\{S_{\ell k}(P_i)\}\Gamma_{\ell,kh} M_j[S_{\ell h}(P_j)]$$

$$= M_i \left\{ M_j \left\{ \sum_{k,h=-\ell}^{\ell} \Gamma_{\ell,kh} S_{\ell k}(P_i) S_{\ell h}(P_j) \right\} \right\}.$$

So, if we call

$$C_\Gamma(P, Q) = \sum_{\ell=2}^{N} \sigma_\ell^2 \mathbf{S}_\ell^t(P)\Gamma_\ell \mathbf{S}_\ell(Q) \tag{5.189}$$

we can state that

$$\sum_{\ell=2}^{N} \sigma_\ell^2 [\mathbf{M}(\mathbf{S}_\ell^t)\Gamma_\ell[\mathbf{M}(\mathbf{S}_\ell^t)]^t = C_\Gamma(\mathbf{M}, \mathbf{M}^t). \tag{5.190}$$

Similarly

$$\sum_{\ell=2}^{N} \sigma_\ell^2 \mathbf{M}(\mathbf{S}_\ell^t)\Gamma_\ell \mathbf{S}_\ell(P) = C_\Gamma(\mathbf{M}, P), \tag{5.191}$$

so that (5.187) becomes

$$[C_{yy} - C_\Gamma(\mathbf{M}, \mathbf{M}^t)]\boldsymbol{\lambda} = C(\mathbf{M}, P) - C_\Gamma(\mathbf{M}, P). \tag{5.192}$$

To further elaborate on (5.192) we find

$$C_{YY} - C_\Gamma(\mathbf{M}, \mathbf{M}^t) = C(\mathbf{M}, \mathbf{M}^t) - C_\Gamma(\mathbf{M}, \mathbf{M}^t) + C_v, \tag{5.193}$$

The (5.193) suggests the introduction of the *reduced* covariance

$$\begin{aligned} C(P, Q) - C_\Gamma(P, Q) &= \sum_{\ell=2}^{+\infty} \sigma_\ell^2 \mathbf{S}_\ell^t(P)\mathbf{S}_\ell(Q) - \sum_{\ell=2}^{+\infty} \sigma_\ell^2 \mathbf{S}_\ell^t(Q)\Gamma_\ell \mathbf{S}_\ell(Q) \\ &= \sum_{\ell=2}^{+\infty} \mathbf{S}_\ell^t(P)\sigma_\ell^2(I - \Gamma_\ell)\mathbf{S}_\ell(Q) = \overline{C}(P, Q), \end{aligned} \tag{5.194}$$

where (5.194) we have implicitly introduced the convention that

$$\Gamma_\ell \equiv 0, \ell > N \tag{5.195}$$

so as to extend directly the summation to infinity.

Another remark on (5.194) is that $\overline{C}(P, Q)$ is a true covariance function because the matrices $\sigma_\ell^2(I - \Gamma_\ell)$ are positive definite.

In fact, recalling (5.185),

$$\begin{aligned} \sigma_\ell^2(I - \Gamma_\ell) &= \sigma_\ell^2[(\sigma_\ell^2 I_\ell + G_\ell)^{-1}(\sigma_\ell^2 I + G_\ell) - \sigma_\ell^2(\sigma_\ell^2 I + G_\ell)^{-1}] \\ &= \sigma_\ell^2(\sigma_\ell^2 I + G_\ell)^{-1} G_\ell = \Gamma_\ell G_\ell. \end{aligned} \tag{5.196}$$

Since $I - \Gamma_\ell$ is symmetric and Γ_ℓ, G_ℓ too, one has that $\Gamma_\ell G_\ell = G_\ell \Gamma_\ell$ implying that (5.195) can be written as

$$\sigma_\ell^2(I - \Gamma_\ell) = G_\ell^{(1/2)} \Gamma_\ell G_\ell^{(1/2)}; \tag{5.197}$$

thus showing the positive definiteness of $I - \Gamma_\ell$.

With the help of (5.196) and (5.194) gets the form

$$\overline{C}(P, Q) = \sum_{\ell_2}^{N} \mathbf{S}_\ell^t(P) G_\ell \Gamma_\ell \mathbf{S}_\ell(Q) + \sum_{\ell=N+1}^{+\infty} \sigma_\ell^2 \mathbf{S}_\ell^t(P)\mathbf{S}_\ell(Q) \tag{5.198}$$

Remark 6. Let us assume that the errors of the model coefficients, $\boldsymbol{\tau}_\ell$, have further covariances that are proportional to the identity, i.e. these errors have the same variance per degree and are independent, then one can put

$$G_\ell = \sigma_{\tau\ell}^2 I = \frac{\varepsilon_\ell}{2\ell+1} I \ , \ \sigma_\ell^2 = \frac{c_\ell}{2\ell+1}, \ \Gamma_\ell = \rho_\ell I, \ \rho_\ell = \frac{\sigma_\ell^2}{\sigma_\ell^2 + \sigma_{\tau\ell}^2}$$

and one finds

$$\overline{C}(P,Q) = \sum_{\ell=2}^{N} \rho_\ell \varepsilon_\ell \left(\frac{R^2}{r_P r_Q}\right)^{\ell+1} P_\ell(\cos\psi_{PQ}) + \sum_{\ell=N+1}^{+\infty} c_\ell(T) \left(\frac{R^2}{r_P r_Q}\right)^{\ell+1} P_\ell(\cos\psi_{PQ}). \tag{5.199}$$

This is an almost perfect counterpart of (5.139) and (5.140) with the difference that here we are using the reduced covariance of T, there the local covariance of Δg.

The most remarkable difference between (5.199) and (5.139) is in the factors $\rho_\ell = \frac{\sigma_\ell^2}{\sigma_\ell^2+\sigma_{\tau\ell}^2}$ multiplying the error degree variances.

On account of the identity

$$\rho_\ell \varepsilon_\ell = \frac{\sigma_\ell^2 \sigma_{\tau\ell}^2 (2\ell+1)}{\sigma_\ell^2 + \sigma_{\tau\ell}^2} = \frac{\sigma_{\tau\ell}^2}{\sigma_\ell^2 + \sigma_{\tau\ell}^2} c_\ell = \chi_\ell c_\ell$$

we see that (5.199) can be written as well as

$$\overline{C}(P,Q) = \sum_{\ell=2}^{+\infty} \chi_\ell c_\ell(\tau) \left(\frac{R^2}{r_P r_Q}\right)^{\ell+1} P_\ell(\cos\psi_{PQ}) \tag{5.200}$$

if we agree that $\chi_\ell \equiv 1$ when $\ell > N$. The form (5.200) shows clearly that the role of the error $\tau_{\ell m}$ is to turn down the degree variances of T when the ratio signal to noise is high while it leaves c_ℓ unaltered for the high degrees of the model where $\sigma_{\tau\ell}^2$ becomes larger. Note however that if we stop the model at N such that $\sigma_{\tau\ell}^2 = \sigma_\ell^2$, when $\ell = N$, then we have $\chi_N = \frac{1}{2}$.

Another remark is that the degrees above N in (5.199) can be modelled on the basis of local data as described in Sect. 5.8 of this chapter.

In terms of $\overline{C}$ our reduced normal system (5.192) becomes

$$(\overline{C}(M,M) + C_v)\boldsymbol{\lambda} = \overline{C}(\mathbf{M},P), \tag{5.201}$$

implying the solution of a classical collocation normal system with covariance $\overline{C}(P,Q)$. Once $\boldsymbol{\lambda}$ is found form (5.201), we can go back to (5.186) and we can write

$$\boldsymbol{\alpha}_n^t = \mathbf{S}_n^t(P)\Gamma_n - \boldsymbol{\lambda}^t \mathbf{M}(\mathbf{S}_n^t)\Gamma_n. \tag{5.202}$$

Therefore (5.176) gives

$$\widehat{T}(P) = \boldsymbol{\lambda}^t \mathbf{Y} + \sum_{n=2}^{N} \mathbf{S}_n^t(P) \Gamma_n \mathbf{T}_n^M + \\ -\boldsymbol{\lambda}^t \mathbf{M} \left(\sum_{n=2}^{N} \mathbf{S}_n^t \Gamma_n \mathbf{T}_n^M \right). \tag{5.203}$$

This suggests to introduce a modified model

$$T_\Gamma(P) = \sum_{n=2}^{+\infty} \mathbf{S}_n^t(P) \Gamma_n \mathbf{T}_n^M \tag{5.204}$$

so that (5.203) writes

$$\begin{aligned} \widehat{T}(P) &= \boldsymbol{\lambda}^t \mathbf{Y} - \boldsymbol{\lambda}^t \mathbf{M}(T_\Gamma) + T_\Gamma(P) \\ &= \boldsymbol{\lambda}^t \{\mathbf{M}(T - T_\Gamma) + \boldsymbol{\nu}\} + T_\Gamma(P). \end{aligned} \tag{5.205}$$

So our optimal solution is in fact the result of a remove-restore procedure, where the optimal model to be used however is not simply

$T^M(P) = \sum_{n=2}^{N} \left(\mathbf{T}_n^M\right)^t \mathbf{S}_n(P)$, but rather $T_\Gamma(P)$.

It is noteworthy that in accordance with this interpretation, the normal equation for $\boldsymbol{\lambda}$, (5.201), can be viewed as an ordinary collocation equation if we observe that $\overline{C}(P, Q)$ is in reality the covariance function of $T - T_\Gamma = v(P)$. In fact

$$v(P) = T - T_\Gamma = \sum_{n=2}^{N} \mathbf{S}_n^t(P)(I - \Gamma_n)\mathbf{T}_n + \\ -\sum_{n=2}^{N} \mathbf{S}_n^t(P) \Gamma_n \boldsymbol{\tau}_n + \sum_{n=N+1}^{+\infty} \mathbf{S}_n^t(P) \mathbf{T}_n \tag{5.206}$$

so that, by covariance propagation

$$C_{vv}(P, Q) = \sum_{n=2}^{N} \mathbf{S}_n^t(P) \sigma_n^2 (I - \Gamma_n)^2 \mathbf{S}_n(Q) \\ + \sum_{n=2}^{N} \mathbf{S}_n^t(P) \Gamma_n G_n \Gamma_n \mathbf{S}_n(Q) + \sum_{n=N+1}^{+\infty} \sigma_n^2 \mathbf{S}_n^t(P) \mathbf{S}_n(Q). \tag{5.207}$$

With the help of (5.196), it is not difficult to prove that

$$\sigma_n^2(I-\Gamma_n)^2+\Gamma_nG_n\Gamma_n=\Gamma_nG_n(I-\Gamma_n)+\Gamma_nG_n\Gamma_n$$
$$=\Gamma_nG_n=G_n\Gamma_n, \tag{5.208}$$

so that (5.207) is identical with (5.198). Let us observe that the covariance $\overline{C}(P,Q)$ in general is not isotropic unless the conditions $G_\ell=\sigma_{\tau\ell}^2 I$, studied in Remark 6, are satisfied.

Therefore $\overline{C}(P,Q)$, in the low degrees components, should not be empirically estimated in the usual way if the mentioned conditions are not fulfilled. In fact, if we do so, we loose information on the stochastic structure of $\boldsymbol{\tau}_\ell$.

Although the ideas presented in this section have been formulated since some years their implementation in numerical tests is relatively recent (Pail et al. 2010). These however have given good results in both cases, the estimation of global enhanced models or the prediction of very local geoid models. In this respect we have confirmed the guess that the hypothesis of zero local mean value for Δg_r is not required in the present situation.

A final point is worth mentioning, on the interpretation of $\boldsymbol{\tau}_\ell$, i.e. errors in the model coefficients. These errors have been usually interpreted as the propagation to $\mathbf{T}_\ell^M$ of the noise present in the observations used in their estimation. This certainly accounts for the difference of $\mathbf{T}_\ell^M$ with respect to the true T_ℓ. This point of view has been taken up in Sect. 5.8.

However when we model a local covariance function and we compare the statistical behaviour of the low degrees coefficients between their global definition and their local appearance in the area A, we might find a considerable difference between the two, specially on account of the dimension of A. In this respect, consider that an area of $10^\circ\times 10^\circ$ is just $\frac{1}{648}$ times the area of the whole sphere. Although there are in literature examples of attempts to model even globally non homogenous covariances (Rummel and Schwarz 1977) we feel that the subject is far from being settled. So we just state here that, the way in which this kind of variability, that is reflected into a *localization error* for $\mathbf{T}_\ell$, could be included and accounted for into our data analysis, will be object of future research.

5.12 Exercises

Exercise 1. Let $(\mathbf{r})=(x_1,x_2,x_3)$ be a Cartesian coordinate system and $(\mathbf{r}')=(x_1',x_2',x_3')$ another Cartesian system rotated with respect to the first. Assume that $T(\mathbf{r})=T(x_1,x_2,x_3)$ is a harmonic function in an open set Ω, that the rotation transforms into the open Ω'. Put

$$v(x_1',x_2',x_3')=T[x_1(\mathbf{r}'),x_2(\mathbf{r}'),x_3(\mathbf{r}')]\,;$$

prove that $v(x_1',x_2',x_3')$ is harmonic in Ω'.

(**Hint**: note that

$$v[\mathbf{r}'(\mathbf{r})] \equiv T(\mathbf{r})$$

and observe that

$$x_i' = \Sigma_k R_{ik} x_k,$$

where $R \equiv [R_{ik}]$ is the rotation matrix between $(\mathbf{r})$ and $(\mathbf{r}')$. Recall that $R^t R = I$. Compute $\Sigma_i \frac{\partial^2}{\partial x_i^2} T$ by using the chain rule and prove that

$$\Sigma_i \frac{\partial^2 T}{\partial x_i^2} = \Sigma_k \frac{\partial^2 v}{\partial x_k'^2})$$

Exercise 2. Compute in spectral form and in spherical approximation the following covariances and cross-covariances

$$C_{\delta g \delta g}(P,Q),\ C_{\delta g \Delta g}(P,Q),\ C_{T \delta g}(P,Q),$$
$$C_{T_{rr} T_{rr}}(P,Q),\ C_{T_{rr} \Delta g}(P,Q).$$

Furthermore, put $T_\vartheta = \frac{\partial}{\partial \vartheta} T(P)$ and compute $C_{T_\vartheta T_\vartheta}(P,Q)$, following the last calculation of Example 4.

Exercise 3. Recalling the definition (5.148), assume that

$$\rho_T(n) = \frac{C_T}{n+1}\ ; \tag{5.209}$$

show that the corresponding degree variances of T and Δg are

$$c_n(T) = \frac{C_T}{(n+1)} \left(\frac{R_B^2}{R^2}\right)^{n+1}$$
$$c_n(\Delta g) = \frac{C_T}{R_B^2} \frac{(n-1)^2}{(n+1)} \cdot \left(\frac{R_B^2}{R^2}\right)^{n+2}.$$

(**Hint:** compare (5.130), (5.135) and (5.138) with (5.147) and recall the relation (5.104)).

Exercise 4. Consider the covariance function of T when (5.209) holds; prove that

$$C_{TT}(s,t) = C_T \left[\log \frac{s - t + G^{-1}(s,t)}{1-t} - s - \frac{1}{2} t s^2\right]. \tag{5.210}$$

(**Hint:** note that $\frac{s^{n+1}}{n+1} = \int_0^s \sigma^n d\sigma$; use this in (5.148), exchange integration and summation and use (5.149) and (5.152)).

Exercise 5. Prove that, with the covariance (5.210),

$$C_{T\delta g}(P,Q) = -\frac{\partial}{\partial r_P} C_{TT}(s,t) \tag{5.211}$$
$$= C_T \frac{s}{r_P}[G(s,t) - 1 - ts]$$
$$C_{\delta g \delta g}(P,Q) = -\frac{\partial}{\partial r_Q} C_{T\delta g}(s,t) \tag{5.212}$$
$$= \frac{C_T}{R_B^2} s^2 [(1-ts)G^3(s,t) - 1 - 2ts] \; ;$$

then find the corresponding crosscovariances and covariances $C_{T\Delta g}, C_{\Delta g \Delta g}$, by propagation through the linear relation

$$\Delta g(P) = \delta g(P) - \frac{2}{r_P} T(P). \tag{5.213}$$

Exercise 6. Put into (5.210) $r_P = r_Q = R = 6,371$ and $R_B = 6,361$; moreover, compute the covariance at the origin, i.e. $\psi = 0 \Rightarrow t = 1$, and impose that

$$C_{TT}(s,1) = \sigma^2(T) = \gamma^2 \sigma^2(N)$$
$$= 978^2\,\mathrm{Gal}^2 \cdot 1^2\,\mathrm{m}^2 \cong 0.956 \cdot 10^6\,\mathrm{Gal}^2\,\mathrm{m}^2$$

show that in this case

$$C_T = 0.224 \cdot 10^6\,\mathrm{Gal}^2\,\mathrm{m}^2.$$

By using this value in (5.212) show that

$$C_{\delta g \delta g}(s,1) \cong 559 \cdot 10^{-6}\,\mathrm{Gal}^2$$

i.e.

$$\sigma(\delta g) \cong 23.6\,\mathrm{mGal}.$$

In other words a mean square geoid of 1m with the spectrum implied by (5.209) corresponds to a mean square gravity disturbance of 23.5 mGal.

The reader is warned that these numbers do not refer to the true gravity field but they are just realistic.

(**Hint:** note that if one puts $t = 1$ in (5.210) one gets the indefinite form $\frac{0}{0}$. Therefore the limit for $t \to 1$ has to be computed by the de l'Hopital rule.)

Exercise 7. Assume that two values of geoid N_{-1}, N_1 are observed without noise at -1 km and 1 km from the origin respectively (see Fig. 5.11).

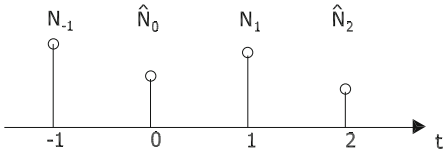

Fig. 5.11 Observed and predicted values according to the exercise

Assume that the covariance of N along the axis t (cf. Fig. 5.11) is given by

$$C(t_1, t_2) = q^{|\tau|} = e^{-\alpha|\tau|}$$
$$\tau = t_1 - t_2, \quad \alpha = \log \frac{1}{q},\ q < 1.$$

Prove that the optimal prediction $\widehat{N}(t)$ at $t = 0$ and $t = 2$ is given by

$$\widehat{N}(0) = \frac{9}{1 - q^4}(N_{-1} + N_1)$$
$$\widehat{N}(2) = qN_1$$

and the corresponding quadratic prediction errors are

$$\mathcal{E}^2(0) = 1 - \frac{2q^2}{1 + q^2}$$
$$\mathcal{E}^2(2) = 1 - q^2.$$

Note that $\mathcal{E}^2(2) > \mathcal{E}^2(0)$ because the extrapolation error is larger than the interpolation error. For instance, with $q^2 = \frac{1}{2}$ one has $\mathcal{E}^2(0) = \frac{1}{3}$, $\mathcal{E}^2(2) = \frac{1}{2}$.

Exercise 8. Assume that the geoid $N(t)$ along a section (line) has covariance

$$C(t_1, t_2) = e^{-\alpha\tau^2}$$
$$\tau = t_1 - t_2.$$

Assume that one has observed at $t = 0$ both the geoid $N_0 = N(0)$ and its derivative $\varepsilon_0 = \frac{dN}{dt}(0)$, i.e. basically the deflection of the vertical changed of sign. The observation noises have respectively standard deviations σ_N and σ_ε.

Compute the prediction $N(t)$ and the corresponding prediction error for every t and verify that

$$\widehat{N}(t) = e^{-\alpha\tau^2}\left[\frac{1}{1 + \sigma_N^2} N_0 + \frac{2\alpha}{2\alpha + \sigma_\varepsilon^2}\tau\varepsilon_0\right]$$
$$\mathcal{E}^2(t) = 1 - e^{-2\alpha\tau^2}\left[\frac{1}{1 + \sigma_N^2} + \frac{4\alpha^2}{2\alpha + \sigma_\varepsilon^2}\right].$$

(**Hint**: first compute for any t_1, t_2 the functions $C(t_1, t_2)$, $\frac{\partial}{\partial t_2} C(t_1, t_2)$, $\frac{\partial^2}{\partial t_1 \partial t_2} C(t_1, t_2)$ and then put $t_1 = t_2 = 0$.

Note that in this way $C(N(0), \varepsilon(0)) = \frac{\partial}{\partial t_2} C(t_1, t_2)\Big|_{t_1=t_2=0} = 0$.).

Exercise 9. We use the same symbols and the same covariances of Exercise 5. Assume one has measured without noise δg at a point Q, put $t = \cos \psi_{PQ}, r_P = R$, and predict $\widehat{N}(P)$ for every P. In particular prove that, choosing $P = Q$ (i.e. $t = 1$) one has

$$\widehat{N}(Q) = \frac{\widehat{T}(Q)}{\gamma} = \frac{R_B}{\gamma} \frac{(1-s)}{[1-(1-s)^2(1+2s)]} \delta g(Q)$$

$$\mathcal{E}^2(Q) = C_T \left\{ \log \frac{1}{1-s} - s - \frac{1}{2}s^2 - \frac{s^6}{[1-(1-s)^2(1+2s)]} \right\}$$

Appendix

A.1

We want to prove the relation (5.98), sending the interested reader to the literature Moritz (1980) and Sansò (1986) for the distribution of the vector $\mathbf{T}(\omega)$.

We have

$$\sum_{m=-n}^{n} T_{nm}^2(\omega) = \frac{1}{(4\pi)^2} \int d\sigma_P T(R_\omega P) \int d\sigma_Q T(R_\omega Q) \cdot \sum_{m=-n}^{n} Y_{nm}(\vartheta_P, \lambda_P) Y_{nm}(\vartheta_Q, \lambda_Q) \tag{5.214}$$

$$= \frac{2n+1}{(4\pi)^2} \int d\sigma_P \int d\sigma_Q T(R_\omega P) T(R_\omega Q) P_n(\cos \psi_{PQ})$$

$$= \frac{2n+1}{(4\pi)^2} \int d\sigma_{P'} \int d\sigma_{Q'} T(P') T(Q') P_n(\cos \psi_{P'Q'}) ;$$

the last equality is justified because $\psi_{PQ} = \psi_{P'Q'}$ and the double integral over the sphere can be performed with any angular coordinates giving always the same result.

Now we organize the double integral in (5.214) as follows; first fix P' and let Q' circulate around P' at a distance $\psi_{P'Q'} = \psi$; then integrate in $d\sigma_{P'}$; then we finally let ψ to vary from 0 to π. We get, putting $d\sigma_{Q'} = \sin \psi d\psi d\alpha$ into (5.214), recalling also the definition (5.38) and using (5.94),

$$\sum_{m=-n}^{n} T_{nm}^2(\omega) = \frac{(2n+1)}{2}\int_0^\pi d\psi \sin\psi P_n(\cos\psi)\cdot \tag{5.215}$$
$$\cdot\frac{1}{8\pi^2}\int d\sigma_{P'}\int_{\psi_{P'Q'}=\psi} T(P')T(Q')d\alpha_{Q'}$$
$$= \frac{2n+1}{2}\int_0^\pi d\psi \sin\psi P_n(\cos\psi)C(\psi) = c_n,$$

as it was to be proved.

A.2

We want to prove formula (5.156), providing the explicit form of $K_B(s,t)$ and $K_{-2}(s,t)$. We first expand (5.157) into the sum of fractions, with the identity

$$\frac{n-1}{(n-2)(n+B)} \equiv \frac{B+1}{B+2}\frac{1}{n+B} + \frac{1}{B+2}\frac{1}{n-2}$$

so that we can write

$$C_{\Delta g\Delta g}(s,t) = A\left\{\frac{B+1}{B+2}\sum_{n=3}^{+\infty}\frac{s^{n+2}}{n+B}P_n(t) + \frac{1}{B+2}\sum_{n=3}^{+\infty}\frac{s^{n+2}}{n-2}P_n(t)\right\}$$
$$= A\left\{\frac{B+1}{B+2}K_B(s,t) + \frac{1}{B+2}K_{-2}(s,t)\right\} \tag{5.216}$$

We compute at first the last term:

$$K_{-2}(s,t) = s^4\sum_{n=3}^{+\infty}\frac{s^{n-2}}{n-2}P_n(t) \tag{5.217}$$
$$= s^4\int_0^s\sum_{n=3}^{+\infty}\sigma^{n-3}P_n(t)d\sigma$$
$$= s^4\int_0^s\frac{1}{\sigma^3}\left\{\sum_{n=0}^{+\infty}\sigma^n P_n(t) - 1 - \sigma t - \sigma^2 P_2(t)\right\}d\sigma$$
$$= s^4\int_0^s\frac{1}{\sigma^3}\left\{G(\sigma,t) - 1 - \sigma t - \sigma^2 P_2(t)\right\}d\sigma$$
$$= \frac{s^2}{2}[1 + 2ts - (3ts+1)G^{-1}(s,t)] - s^4 P_2(t)\log\frac{1 - st + G^{-1}(s,t)}{2}$$
$$+s^4\frac{7t^2-1}{4}.$$

The last integral is calculated with the help of mathematical tables adjusting the integration constant in such a way that both members of (5.217), multiplied by s^{-4}, tend to 0 when s tends to 0. As for the first term one writes, assuming $B > 0$,

$$\begin{aligned} K_B(s,t) &= s^{2-B}\sum_{n=3}^{+\infty}\frac{s^{n+B}}{n+B}P_n(t) \\ &= s^{2-B}\int_0^s\sum_{n=3}^{+\infty}\sigma^{n+B-1}P_n(t)d\sigma \\ &= s^{2-B}\int_0^s\sigma^{B-1}\left\{\sum_{n=0}^{+\infty}\sigma^n P_n(t) - 1 - \sigma t - \sigma^2 P_2(t)\right\}d\sigma \\ &= s^{2-B}\int_0^s\sigma^{B-1}G(\sigma,t)d\sigma - \frac{s^2}{B} - \frac{s^3}{B+1}t - \frac{s^4}{B+2}P_2(t). \end{aligned} \tag{5.218}$$

Now the integrals

$$I_B = \int_0^s \sigma^{B-1}G(\sigma,t)d\sigma \tag{5.219}$$

can be computed, for integer values of B, by exploiting a recursive relation, namely

$$I_{k+1} = \frac{s^{k-1}}{k}G^{-1}(s,t) + \frac{(2k-1)}{k}tI_k - \frac{k-1}{k}I_{k-1} \tag{5.220}$$

which is derived from the identity

$$\frac{\partial}{\partial s}\left[s^{k-1}G^{-1}(s,t)\right] = \left[ks^k - (2k-1)ts^{k-1} + (k-1)s^{k-2}\right]G(s,t), \tag{5.221}$$

integrating both members from 0 to s and re-arranging. In order to trigger (5.220) we need two initial values of I_k, for instance I_1, I_2. But I_1 has already been given in (5.152) and I_2 is easy to compute since, recalling (5.151),

$$\begin{aligned} I_2 &= \int_0^s \sigma G(\sigma,t)d\sigma = \int_0^s(\sigma - t)G(\sigma,t)d\sigma + t\int_0^s G(\sigma,t)d\sigma \\ &= G^{-1}(s,t) - 1 + tI_1. \end{aligned} \tag{5.222}$$

The relations (5.216), (5.217), (5.220), (5.152) and (5.222) all together give the explicit form of the covariance function of Δg for every integer B. For a global use of this covariance the model (3.181) coming from the best fit of EGM08 degree variances between degrees 180 and 1,800, can be used, with the only warning that

in (3.181) one has $\overline{\sigma}_\ell^2 = c_\ell\left(\frac{T}{\gamma}\right)$, whereas we treat here $c_n(\Delta g)$ related to the former by the relation

$$c_n(\Delta g) = \frac{(\ell-1)^2}{\overline{R}^2} c_n(T) = \frac{(\ell-1)^2}{\overline{R}^2}\left(\frac{GM}{\overline{R}^2}\right)^2 \overline{\sigma}_\ell^2. \tag{5.223}$$

We notice by the way that also the improved model (3.178) transformed according to (5.223) can be added by applying exactly the same methods presented in the Appendix and the decomposition

$$\frac{\ell-1}{(\ell-2)(\ell+4)(\ell+17)} = \frac{1}{114}\frac{1}{\ell-2} + \frac{5}{78}\frac{1}{\ell+4} - \frac{18}{247}\frac{1}{\ell+17}. \tag{5.224}$$