Chapter 13
On Potential Theory and HS of Harmonic Functions

13.1 Outline of the Chapter

The Newton potential of the earth as well as its anomalous gravity potential are harmonic functions outside the earth body B, therefore the interest of geodesy in spaces of harmonic functions is quite justified. More precisely, from the mathematical point of view we are interested in a situation in which B is an open, simply connected bounded set, with a relatively smooth boundary S and $\overline{B}^c = \Omega$ (the complement of the closure of B) is simply connected too. Let us note explicitly that this prevents B from having holes in it or even single points removed.

We start in Sect. 13.2 building some Hilbert spaces of harmonic polynomials, which, being embedded into polynomial spaces, are indeed finite dimensional. In particular it is proved that these have their own reproducing kernels and, by transforming Cartesian into spherical coordinates, a fundamental relation is found between such reproducing kernels and the sequence $P_n(\cos\psi_{xy})$. Each Legendre polynomial multiplied by $|x|^n \cdot |y|^n$ turns out to be the reproducing kernel of the subspace of harmonic polynomials homogeneous of degree n. By using the properties of reproducing kernel Hilbert spaces, illustrated in Part III, Sect. 12.5, then one finds the famous *summation theroem*.

The approach follows the idea in Krarup (2006), though departing form them in some important steps. For other approaches one can consult (Nikiforov and Uvarov 1988).

In Sect. 13.3 all the machinery of Sect. 13.2 is translated into properties of spherical harmonics. When these are considered as a sequence in $L^2(\sigma)$ (space of functions square integrable on the unit sphere S_1) they are proved not only to be orthonormal but also complete. Thus they are an orthonormal basis in $L^2(S_1)$. Going into the matter of more general spaces of harmonic functions, some classical properties are proved like the maximum principle or the principle of identity of harmonic functions.

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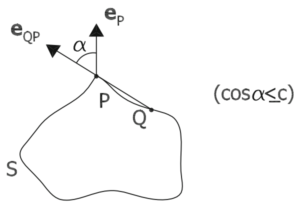


Fig. 13.1 A surface S satisfying the cone condition; $c = \cos\vartheta$

A fundamental result is then established, namely that the sequencies of *internal* as well as that of *external* spherical harmonics, when restricted to any closed bounded and smooth surface S form a complete basis of $L^2(S)$. This implies for instance that any function $f \in L^2(S)$ can be approximated as well as we like, by a finite sum of (external) solid spherical harmonics, i.e. by a *global model*.

The proof that, when we approximate $f \in L^2(S)$ with a sequence of functions harmonic in Ω, i.e. outside S, we also approximate a function u, harmonic in Ω, and that this function, suitably restricted to S, becomes equal to f, is the main purpose of Sect. 13.5. On related matters one can usefully read Fichera (1948).

To do that, the concept of Green's function is introduced and some of its properties are studied. In doing so we create a prototype Hilbert space of harmonic functions, namely that in which potentials in Ω have boundary values in $L^2(S)$.

13.2 Harmonic Functions and Harmonic Polynomials

Recall that in this chapter B is a simply connected open set, as specified at the beginning of Sect. 13.1.

We shall put in the sequel $\overline{B} = B \cup S$, the closure of B and $\Omega = (\overline{B})^c$. We shall assume that S is relatively smooth meaning at least that Gauss' theorem applies to $\overline{B}$, for instance that S satisfies a so-called cone condition, i.e. there is a positive constant $c < 1$ such that for any given point $P \in S$ there is a unit vector $\mathbf{e}_P$ pointing in Ω and a neighborhood $A \subset S$, such that for any other point $Q \in A$ it is $|\mathbf{e}_{QP} \cdot \mathbf{e}_P| \le c$ with $\mathbf{e}_{QP}$ the unit vector in the direction from Q to P. Looking at Fig. 13.1, one sees that if $c = \cos\vartheta$, with ϑ fixed for the whole surface S, the above means that $\alpha \ge \vartheta$ when $Q \in S$ belongs to a suitable neighborhood A of P.

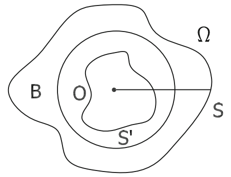
As a matter of fact in the sequel of these notes we shall require a stronger regularity of the boundary S. For instance we shall assume that the exterior normal field $\mathbf{n}_P$ is everywhere defined on S and even that $\mathbf{n}_P$ is Lipschitz continuous, i.e.

$$P, Q \in S; \quad |\mathbf{n}_P - \mathbf{n}_Q| \le c \cdot \overline{PQ};$$

this is basically the same as requiring that S has finite curvature at every point.

Under the above mentioned conditions we can apply an inverse radii transform, sometimes also called Kelvin or Rayleigh transform, which is as follows: put the origin O of $\mathbf{R}^3$ in B and take a spherical coordinate system (r, ϑ, λ), then define

Fig. 13.2 The geometry of Rayleigh transform, with $B' = \mathcal{R}(\Omega)$, $S' = \mathcal{R}(S)$



$$s = \frac{R^2}{r}, \ \vartheta' = \vartheta, \ \lambda' = \lambda \tag{13.1}$$

where R is any radius of a sphere totally inside B. Under (13.1), denoting

$$\xi = (r, \vartheta, \lambda), \ \xi' = (s, \vartheta, \lambda), \ \xi' = \mathcal{R}(\xi), \tag{13.2}$$

we obviously have that, putting

$$S' = \mathcal{R}(S), \ B' = \mathcal{R}(\Omega), \ \Omega' = \mathcal{R}(B), \tag{13.3}$$

then S' is totally inside the sphere $r = s = R$ (see Fig. 13.2), B' is inside S' and contains the origin, while Ω' is outside S'. In particular

$$\mathcal{R}(0) = \infty, \ \mathcal{R}(\infty) = 0. \tag{13.4}$$

Definition 1. A function u is harmonic in classical sense in B, denoted $u \in \mathcal{H}(B)$, if it is continuous with its second derivatives in B, and if $\Delta u = 0$ at any point $P \in B$ (recall that B is open). A function u is harmonic and regular in Ω, $u \in \mathcal{H}(\Omega)$, if it is continuous with its second derivatives in Ω and furthermore

$$\lim_{P \to \infty} u(P) = 0; \tag{13.5}$$

(13.5) means that $\forall \varepsilon > 0, \ \exists R_\varepsilon; \ |u(P)| < \varepsilon$ when $r_P > R_\varepsilon$.

Proposition 1. *Let B, S, Ω and B', S', Ω' be as in (13.3). We can show that if $u \in \mathcal{H}(B)$ then, defining*

$$v(s, \vartheta, \lambda) = \frac{1}{s} u \left(\frac{1}{s}, \vartheta, \lambda \right) = \mathcal{R}(u) \tag{13.6}$$

we have $v \in \mathcal{H}(\Omega')$. In other words if we put $v = \mathcal{R}(u)$, the Rayleigh transform of u, we have $\mathcal{R} : \mathcal{H}(B) \to \mathcal{H}(\Omega')$. Similarly $\mathcal{R} : \mathcal{H}(\Omega) \to \mathcal{H}(B')$. Moreover

$$\mathcal{R}^2(u) = \mathcal{R}(v) = u \tag{13.7}$$

for $\forall u \in \mathcal{H}(B)$ or $u \in \mathcal{H}(\Omega)$.

The above statement basically means that, when useful, we can study properties of spaces of harmonic functions on bounded domains, like B, and then derive the corresponding properties for spaces of regular harmonic for functions in Ω.

Among harmonic functions in B a special role play the polynomials which are also harmonic in B. Having to work with polynomials, it is convenient to adopt a multi-index notation already presented in Example 11 of Sect. 12.2.

Remember now that any polynomial $P_N(\xi)$ is defined everywhere in $\mathbf{R}^3$ and that two polynomials which coincide in a neighborhood of a point ξ_0 coincide everywhere, because then in ξ_0 they will have the same derivatives up to order N (all higher order derivatives are zero); this is the principle of identity of polynomials. The following conclusion can be drawn.

Proposition 2. *Any polynomial harmonic in an open set B is harmonic in the whole of $\mathbf{R}^3$, but of course not regular at ∞, unless it is identically zero.*

Proof. Let $P_N(\xi) = \sum_{n=0}^{N} \sum_{|\alpha|=n} c_\alpha \xi^\alpha$, $\xi = (x, y, z)$, be harmonic in an open set B. Then the polynomial of order $N-2$

$$P_{N-2}(\xi) = \sum_{n=0}^{N} \sum_{|\alpha|=n} c_\alpha (\Delta \xi^\alpha) \equiv 0, \quad \xi \in B$$

and therefore $P_{N-2}(\xi)$ is zero everywhere in $\mathbf{R}^3$. □

Accordingly, we can study the space of harmonic polynomials in R^3, without any specific reference to B. We call it $H\mathcal{P}_N^3$, when only polynomials up to degree N are taken into account. As it is obvious

$$H\mathcal{P}_N^3 \subset \mathcal{P}_N^3,$$

the space of all polynomials, already studied in Examples 1 and 11 in Chap. 12.

Since $\mathcal{P}_N^3$ is a HS of finite dimension, $H\mathcal{P}_N^3$ will also be a finite dimensional HS, under the same scalar product. In particular, to the orthogonal decomposition (see Example 11)

$$\mathcal{P}_N^3 = H_0^3 \oplus H_1^3 \oplus H_2^3 \dots H_N^3, \tag{13.8}$$

where H_k^3 are spaces of polynomials homogeneous in ξ of degree k, there must correspond an analogous decomposition

$$H\mathcal{P}_N^3 = HH_0^3 \oplus HH_1^3 \oplus \dots \oplus HH_N^3 \tag{13.9}$$

where each HH_k^3 contains all the harmonic polynomials, homogeneous of degree k. Note that the orthogonality of HH_ℓ^3 and $HH_k^3, \ell \neq k$, is already guaranteed by (13.8) and the obvious fact that

$$HH_k^3 \subset H_k^3. \tag{13.10}$$

The structure of the reasoning followed here is based on Krarup (2006) and has been used to develop explicit formulas for the traditional spherical harmonics.

Since this will be useful in our construction, we will simultaneously reason on $\mathcal{P}_N^2$, $H\mathcal{P}_N^2$ and HH_k^2.

Our first target will be to count the dimensions of HH_k^2 and HH_k^3.

Definition 2. In order to avoid confusion, let us agree on some notation. We put

$$\begin{aligned}
\boldsymbol{\chi} &= (x, y) \\
\rho &= |\boldsymbol{\chi}| \\
\boldsymbol{\xi} &= (x, y, z) = (\boldsymbol{\chi}, z) \\
r &= |\boldsymbol{\xi}| = \sqrt{\rho^2 + z^2}.
\end{aligned}$$

In particular $\rho = |\boldsymbol{\chi}|$ used in this context should be not be confused with the symbol ρ used sometimes in the text for mass density.

Proposition 3. *We have, with obvious notation,*

$$D_k^2 = \dim HH_k^2 = 2 - \delta_{k0}; \tag{13.11}$$

for each $k \neq 0$, the two homogeneous polynomials are given by the formulas

$$h_k(\boldsymbol{\chi}) = Re(x + iy)^k;\ h_{-k}(\boldsymbol{\chi}) = Im(x + iy)^k; \tag{13.12}$$

furthermore h_k and h_{-k} are orthogonal in H_k^2.

Proof. We note first of all that $HH_0^2 \equiv H_0^2$ with $h_0 \equiv 1$ being the unique linearly independent element, homogenous of degree zero. All the other elements of HH_0^2 are just constant everywhere. Similarly $HH_1^2 \equiv H_1^2$ and all homogenous (and harmonic) polynomials of degree 1 are obtained by combination of $h_1 \equiv x, h_{-1} \equiv y$; all that agrees with the statement of the proposition.

Now let $h(\boldsymbol{\chi}) \in HH_k^2$ with $k \geq 2$; then we can put $x = \rho \cos \lambda$, $y = \rho \sin \lambda$ and we have

$$h(\boldsymbol{\chi}) = \rho^k f_k(\lambda). \tag{13.13}$$

Let us impose to (13.13) to satisfy the Laplace equation in polar coordinates (ρ, λ) in R^2, i.e.

$$\Delta h(\boldsymbol{\chi}) = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\lambda^2}\right) h(\boldsymbol{\chi}) = \rho^{k-2}\left[k^2 f_k(\lambda) + f_k''(\lambda)\right] \equiv 0.$$

This implies that

$$f_k(\lambda) = a_k \cos k\lambda + b_k \sin k\lambda, \tag{13.14}$$

i.e. we have two independent solutions, in polar coordinates, namely

$$k > 0,\ h_k(\rho, \lambda) = \rho^k \cos k\lambda\ ,\ h_{-k}(\rho, \lambda) = \rho^k \sin k\lambda \tag{13.15}$$

Now it is enough to observe that

$$\rho^k \cos k\lambda = Re(x + iy)^k,\ \rho^k \sin k\lambda = Im(x + iy)^k$$

to prove (13.12).

That h_k and h_{-k} are orthogonal to one another derives from the development of $(x + iy)^k$ in a binomial formula; separating the real from the imaginary part we see that they are linear combinations of monomials χ^α which can never be the same. Since such monomials are reciprocally orthogonal (cf. Example 11), we have proved what we wanted. □

Proposition 4. *Let us split H_k^3 into two orthogonal complements*

$$H_k^3 = HH_k^3 \oplus CH_k^3;\ CH_k^3 = (HH_k^3)^\perp, \tag{13.16}$$

then we have

$$CH_k^3 \equiv \{P_k(\xi) = r^2 P_{k-2}(\xi),\ P_{k-2} \in H_{k-2}^3\}; \tag{13.17}$$

furthermore, adopting a notation similar to (13.11),

$$D_k^3 = \dim HH_k^3 = 2k + 1. \tag{13.18}$$

Proof. First we immediately see that CH_k^3 defined by (13.17) is orthogonal to HH_k^3; in fact let $h_k(\xi) \in HH_k^3$, then (cf. (12.80))

$$\forall P_{k-2} \in H_{k-2}^3,\ \ < r^2 P_{k-2}(\xi), h_k(\xi) > \ = P_{k-2}(\partial_\xi)\Delta h_k(\xi)\big|_{\xi=0} \equiv 0. \tag{13.19}$$

We note too, that CH_k^3 is a closed subspace of H_k^3. Now we have to show that CH_k^3 covers the whole complement of HH_k^3; it is enough to show that the orthogonal complement of CH_k^3, defined by (13.17), is in fact HH_k^3.

Since CH_k^3 is closed, $\forall P_k \in H_k^3$ we can make the orthogonal decomposition

$$P_k(\xi) = r^2 P_{k-2}(\xi) + R_k(\xi) \tag{13.20}$$

with $R_k(\xi) \perp CH_k^3$ i.e. R_k belongs to the orthogonal complement of CH_k^3. But this implies

$$< r^2 P_{k-2}, R_k >= P_{k-2}(\partial_\xi)\Delta R_k\big|_{\xi=0} =< P_{k-2}(\xi), \Delta R_k(\xi) >= 0, \tag{13.21}$$

$\forall P_{k-2} \in H^3_{k-2}$. Equation (13.21) then implies $\Delta R_k(\xi) = 0$ because this is a polynomial in H^3_{k-2}. Equation (13.20) is, as a matter of fact, (13.16).

Now, since CH^3_k is one to one with H^3_{k-2}, we have (cf. Exercise 1, Chap. 1)

$$\dim CH^3_k = \dim H^3_{k-2} = \frac{(k-1)k}{2};$$

(13.18) follows from

$$\begin{aligned} D^3_k &= \dim H^3_k - \dim CH^3_k \\ &= \frac{(k+1)(k+2)}{2} - \frac{(k-1)k}{2} = 2k+1. \end{aligned}$$ □

The Exercise 3 is preparatory for the next proposition; the reader is advised at least to read it before continuing.

Proposition 5. *The following decomposition formula holds*

$$P_N(\boldsymbol{\xi}) = h_N(\boldsymbol{\xi}) + r^2 h_{N-2}(\boldsymbol{\xi}) + r^4 h_{N-4}(\boldsymbol{\xi}) + \ldots \tag{13.22}$$

the summation of terms $r^{2k} h_{N-2k}(\boldsymbol{\xi})$ being extended up to $k = \left[\frac{N}{2}\right]$ (the smallest integer $\leq N/2$); as shown in Exercise 3 each $h_{N-2k}(\boldsymbol{\xi})$ is a harmonic polynomial in HH^3_{N-2k}; $h_N(\boldsymbol{\xi})$, which is the orthogonal projection of $P_N(\boldsymbol{\xi})$ onto HH^3_N, is given by the inverse formula

$$h_N(\boldsymbol{\xi}) = P_N(\boldsymbol{\xi}) + q_1 r^2 \Delta P_N(\boldsymbol{\xi}) + q_2 r^4 \Delta^2 P_N(\boldsymbol{\xi}) + \ldots \tag{13.23}$$

where q_k are suitable constants independent of the specific P_N once N is fixed.

Equation 13.23 is also known as Pizzetti's formula *in mathematical literature (Dunford and Schwarz 1958; Courant and Hilbert 1962).*

Proof. We just re-write (13.20) in the form

$$P_N(\boldsymbol{\xi}) = h_N(\boldsymbol{\xi}) + r^2 P_{N-2}(\boldsymbol{\xi}), \tag{13.24}$$

where, as we have seen, h_N is harmonic, i.e. $h_N \in HH^3_N$. By iterating (13.24) we get (13.22). Notice that since the degree jumps 2 by 2 from N, we end up with $P_1(\boldsymbol{\xi})$ or $P_0(\boldsymbol{\xi})$, depending whether N is odd or even. But $P_1(\boldsymbol{\xi})$ or $P_0(\boldsymbol{\xi})$ are already harmonic by default.

Now we apply to (13.22) successively $r^2\Delta, r^4\Delta^2 \ldots$ and we get, taking Exercise 3 into account,

$$\begin{aligned} r^2 \Delta P_N &= A_{11} r^2 h_{N-2} + A_{12} r^4 h_{N-4} + A_{13} r^6 h_{N-6} + \dots \\ r^4 \Delta^2 P_N &= A_{22} r^4 h_{N-4} + A_{23} r^6 h_{N-6} + \dots \\ r^6 \Delta^3 P_N &= A_{33} r^6 h_{N-6} + \dots \end{aligned} \tag{13.25}$$

As we see (13.25) can be considered as a triangular system with $(r^{2k} h_{N-2k})$ as unknowns and $(r^{2k} \Delta^k P_N)$ as known terms. Since (cf. Exercise 3)

$$A_{kk} = 2k \cdot (2k-2) \dots 2 \cdot (2N-2k+1) \cdot (2N-2k-1) \dots (2N-4k+3)$$

are always positive (remember that k goes from 1 to $\left[\frac{N}{2}\right]$) the system is invertible. So solving (13.25) and substituting back in (13.22) we get the expression (13.23). We note that q_k can be computed as we suggest in Exercise 4, however here it is only important to strengthen that q_k are independent of P_N, i.e. they are the same $\forall P_N \in H_N^3$. □

At this point we are ready to derive the first important result of this chapter. In fact we note that the elements of HH_N^3, being also elements of H_N^3, enjoy the reproducing property

$$h_N(\boldsymbol{\xi}) = < K_N(\boldsymbol{\xi}, \boldsymbol{\eta}), h_N(\boldsymbol{\eta}) >_{H_k^3} \tag{13.26}$$

with (cf. Example 13)

$$K_N(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{(\boldsymbol{\xi}^t \boldsymbol{\eta})^N}{N!}; \tag{13.27}$$

nevertheless $K_N(\boldsymbol{\xi}, \boldsymbol{\eta})$ is not the reproducing kernel of HH_N^3 because for any fixed $\overline{\eta}$, $K(\boldsymbol{\xi}; \overline{\boldsymbol{\eta}})$ does not belong to HH_N^3, namely it is not harmonic. The next Theorem will provide us with the correct RK of HH_N^3, which is nothing but the orthogonal projection of K_N onto HH_N^3. Hereafter we switch from N to n, to allow for the distinction of the maximum degree of the polynomial, N, from the homogeneous degree n.

Theorem 1. *Each subspace of homogeneous harmonic polynomials HH_n^3 is endowed with a RK, $H_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ given by*

$$H_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = A_n r_{\xi}^n r_{\eta}^n P_n(t) \tag{13.28}$$

where

$$r_\xi = |\boldsymbol{\xi}|, \; r_\eta = |\boldsymbol{\eta}|, \; t = \frac{\xi^t \eta}{r_\xi r_\eta} = \cos \psi_{\xi\eta},$$

$$A_n = \frac{2^n n!}{(2n)!} \tag{13.29}$$

and $P_n(t)$ are exactly the Legendre polynomials of degree n, already seen in Part I, Chap. 3.

Proof. Take $K_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ and apply to it, considered as a function of η, the formula (13.23), so as to define

$$H_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = K_n(\boldsymbol{\xi}, \boldsymbol{\eta}) + \sum_{k=1}^{[\frac{n}{2}]} q_k r_\eta^{2k} \Delta_\eta^k K_n(\boldsymbol{\xi}, \boldsymbol{\eta}). \tag{13.30}$$

By using the following, easy to prove, formula

$$\Delta_\eta^k (\boldsymbol{\xi}^t \boldsymbol{\eta})^n = n(n-1)\dots(n-2k+1) r_\xi^{2k} (\boldsymbol{\xi}^t \boldsymbol{\eta})^{n-2k}$$

in (13.30) we receive

$$H_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{n!} \left\{ (\boldsymbol{\xi}^t \boldsymbol{\eta})^n + \sum_{k=1}^{[\frac{n}{2}]} p_k r_\eta^{2k} r_\xi^{2k} (\boldsymbol{\xi}^t \boldsymbol{\eta})^{n-2k} \right\} \tag{13.31}$$

where we have set

$$p_k = q_k n(n-1)\dots(n-2k+1). \tag{13.32}$$

If we put in evidence in (13.31) r_ξ^n, r_η^n, and we agree that $p_0 = 1$, we can write

$$H_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{r_\xi^n r_\eta^n}{n!} \sum_{k=0}^{[\frac{n}{2}]} p_k t^{n-2k} \equiv r_\xi^n r_\eta^n Q_n(t), \tag{13.33}$$

where $Q_n(t)$ is a polynomial in $t = \cos \psi_{\xi\eta}$ containing only even or odd powers, according to the parity of n. We note that $H_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = H_n(\boldsymbol{\eta}, \boldsymbol{\xi})$, that $H_n(\overline{\boldsymbol{\xi}}, \boldsymbol{\eta})$ is harmonic in η by definition, and therefore $H(\boldsymbol{\xi}, \overline{\boldsymbol{\eta}})$ is harmonic in $\boldsymbol{\xi}$ too, and finally that $H_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ has the reproducing property in HH_n^3 because

$$\begin{aligned} < H_n(\boldsymbol{\xi}, \boldsymbol{\eta}), h_n(\eta) > &= < K_n(\boldsymbol{\xi}, \boldsymbol{\eta}), h_n(\boldsymbol{\eta}) > \\ &\quad + \sum_{k=1}^{[\frac{n}{2}]} p_k r_\xi^{2k} < r_\eta^{2k} (\boldsymbol{\xi}^t \boldsymbol{\eta})^{n-2k},\ h_n(\boldsymbol{\xi}) > \\ &= < K_n(\boldsymbol{\xi}, \boldsymbol{\eta}),\ h_n(\boldsymbol{\eta}) >= h_n(\boldsymbol{\xi}). \end{aligned}$$

In fact all the terms multiplied by a power of r_η^{2k} are orthogonal to all harmonic polynomials.

Observe that $H_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ has to be harmonic in η, whatever is the vector $\boldsymbol{\xi}$, since neither the coefficients q_k nor p_k have to depend on the specific homogenous polynomial $(\boldsymbol{\xi}^t \boldsymbol{\eta})^n$, once n is fixed.

Then we can choose $\overline{\boldsymbol{\xi}} = (0, 0, 1)$, i.e. the unit vector along the z axis; then $t = \cos\vartheta$ and we must have that

$$H_n(\overline{\boldsymbol{\xi}}, \boldsymbol{\eta}) = r_\eta^n Q_n(t) \tag{13.34}$$

is harmonic. On the other hand we already know that $r^n P_n(t)$ is harmonic when $P_n(t)$ is a Legendre polynomial and this means that we must have

$$Q_n(t) = A_n P_n(t). \tag{13.35}$$

In fact, writing the Laplacian first in spherical coordinates (r, ϑ, λ) and then changing ϑ into $t = \cos\vartheta$, one gets

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial\vartheta^2} + \mathrm{ctg}\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\lambda^2}\right] \\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[(1-t^2)\frac{\partial^2}{\partial t^2} - 2t\frac{\partial}{\partial t} + \frac{1}{1-t^2}\frac{\partial^2}{\partial\lambda^2}\right] \end{aligned} \tag{13.36}$$

We can use (13.36) on (13.34) to conclude that

$$r^{n-2}\left[n(n+1)Q_n(t) + (1-t^2)\frac{\partial^2}{\partial t^2}Q_n(t) - 2t\frac{\partial}{\partial t}Q_n(t)\right] \equiv 0. \tag{13.37}$$

If $Q_n(t)$ has to satisfy (13.37) and to be a polynomial with the same parity as n, then Q_n is fixed up to a constant (see the Exercise 5). Since $P_n(t)$ satisfies (13.37), the relation (13.35) must be true, so we have only to find A_n. We note that, (cf. (13.33) and (13.34)) the coefficient of t^n in $Q_n(t)$ is just $\frac{1}{n!}$. The coefficient of t^n in $P_n(t)$ is $\frac{2n!}{2^n(n!)^2}$ (see Exercise 6). So we must have $A_n = 2^n n!/(2n)!$, as it was to be proved. □

Definition 3. For reasons that will become soon clear, we define

$$L_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = r_\xi^n r_\eta^n (2n+1) P_n(\cos\psi_{\xi\eta}) \tag{13.38}$$

so that we have (see (13.35))

$$H_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{A_n}{2n+1} L_n(\boldsymbol{\xi}, \boldsymbol{\eta}). \tag{13.39}$$

Remark 1. Let us remember that $H_n(\boldsymbol{\xi}, \boldsymbol{\eta})$, considered as a family of functions of $\boldsymbol{\eta}$ indexed by $\boldsymbol{\xi}$, is total in HH_n^3 (see Proposition 15, Example 13); the same then must be true for $L_n(\boldsymbol{\xi}, \boldsymbol{\eta})$.

On the other hand we know that HH_n^3 has dimension $2n+1$ (cf. (13.17)), therefore there must be $(2n+1)$ points $\xi_i \neq 0$ such that $\{L_n(\boldsymbol{\xi}_i, \boldsymbol{\eta})\}$ is a basis of HH_n^3. Furthermore, since $L_n(\lambda \boldsymbol{\xi}_i, \boldsymbol{\eta}) = \lambda^n L_n(\boldsymbol{\xi}_i, \boldsymbol{\eta})$, the point $\boldsymbol{\xi}_i$ can be chosen to belong to S_1, the sphere of radius 1.

This means that $\forall h_n \in HH_n^3$ we can put

$$h_n(\cdot) = \sum_{i=1}^{2n+1} \lambda_i L_n(\boldsymbol{\xi}_i, \cdot) \tag{13.40}$$

and that this correspondence is one to one, so that $h_n = 0 \Leftrightarrow \{\boldsymbol{\lambda}\} = 0$. Therefore, using (13.39) and the fact that $H_n(\cdot)$ is a RK,

$$\forall \boldsymbol{\lambda} \neq 0, \ \|h_n\|^2 = \frac{(2n+1)}{A_n} \Sigma_{i,j} \lambda_i \lambda_j L_n(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j) > 0 \tag{13.41}$$

and we see that $\{L_n(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j)\}$ is an invertible matrix.

On the other hand we know that (cf. Part I, (3.188)) $L_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ has also a nice reproducing property when $\boldsymbol{\xi}, \boldsymbol{\eta} \in S_1$ and we adopt an $L^2(S_1)$ scalar product; namely

$$\frac{1}{4\pi} \int_\sigma L_n(\boldsymbol{\xi}, \boldsymbol{\eta}) L_n(\boldsymbol{\xi}', \boldsymbol{\eta}) d\sigma_\eta =< L_n(\boldsymbol{\xi}, \cdot), L_n(\boldsymbol{\xi}', \cdot) >_{L^2(S_1)}= L_n(\boldsymbol{\xi}, \boldsymbol{\xi}').$$

Even more, we know that (cf. Part I, (3.182))

$$\begin{aligned} \frac{1}{4\pi} \int L_n(\boldsymbol{\xi}, \boldsymbol{\eta}) L_m(\boldsymbol{\xi}', \boldsymbol{\eta}) d\sigma_\eta &= < L_n(\boldsymbol{\xi}, \boldsymbol{\eta}), L_m(\boldsymbol{\xi}', \cdot) >_{L^2(S_1)} \\ &= \delta_{nm} L_n(\boldsymbol{\xi}, \boldsymbol{\xi}'). \end{aligned} \tag{13.42}$$

All that allows us to draw a number of conclusions that we state in the form of three Lemmas.

Lemma 1. *Let us introduce the trace operator* $\Gamma_{S_1} : \mathcal{P}_N^3 \to L^2(S_1)$

$$\forall P_N \in \mathcal{P}_N^3; \ \ \Gamma_{S_1} P_N(\boldsymbol{\xi}) = P_N(\boldsymbol{\xi})\big|_{r=1};$$

then the image of HH_n^3 *in* $L^2(S_1)$*, i.e.*

$$\Gamma_{S_1}(HH_n^3) \equiv Span_{\boldsymbol{\xi} \in S_1}\{L_n(\boldsymbol{\xi}, \boldsymbol{\eta})\} \equiv Span_{\boldsymbol{\xi} \in S_1}\{P_n(\cos \psi_{\boldsymbol{\xi}, \boldsymbol{\eta}})\}, \tag{13.43}$$

is isometric to HH_n^3*, up to a constant. Since by combining (13.40) and (13.41) we see that*

$$\forall h_n \in HH_n^3; \ \|h_n\|^2_{HH_n^3} = \frac{2n+1}{A_n} \|\Gamma_{S_1}(h_n)\|^2_{L^2(S_1)}; \tag{13.44}$$

in particular this implies that any two vectors orthogonal in HH_n^3 *with its original scalar product are orthogonal in* $L^2(S_1)$ *too and viceversa.*

Lemma 2. *If we consider the decomposition*

$$H\mathcal{P}_N^3 = HH_1^3 \oplus HH_2^3 \oplus, \ldots, \oplus HH_n^3, \tag{13.45}$$

which is orthogonal in the original topology in HH_n^3, we see that, thanks to (13.41), the same decomposition for the image $\Gamma_{S_1}(H\mathcal{P}_N^3)$ is orthogonal in $L^2(S_1)$ too, and

$$\forall P_N \in H\mathcal{P}_N^3; \ \|P_N\|^2_{H\mathcal{P}_N^3} = \sum_{n=0}^{N} \frac{(2n+1)}{A_n} \|\Gamma_{S_1}(P_N)\|^2_{L^2(S_1)}. \tag{13.46}$$

Equation 13.46 is a norm equivalence for any fixed N, but not when $N \to \infty$. Basically this means that the geometry of $H\mathcal{P}_N^3$ with the original scalar product and with the product of $L^2(S_1)$ is the same.

Lemma 3. *Any element $P_N \in H\mathcal{P}_N^3$ is uniquely determined by its trace on S_1.*

That this occurs for each component $h_n \in HH_n^3$ makes no surprise because $h_n(\lambda \boldsymbol{\xi}) = \lambda^n h_n(\boldsymbol{\xi})$, and then if we give h_n on S_1 we fix it in the whole of $\mathbf{R}^3$. But the Lemma claims that this is the same for all $P_N \in H\mathcal{P}_N^3$. The reason is that the following representation holds

$$\forall P_N \in H\mathcal{P}_N^3; \ P_N(\boldsymbol{\xi}) = \sum_{n=0}^{N} h_n(\boldsymbol{\xi}) = \sum_{n=0}^{N} \sum_{i=1}^{2n+1} \lambda_{ni} L_n(\boldsymbol{\xi}_{ni}, \boldsymbol{\xi}); \tag{13.47}$$

therefore

$$< P_N(\boldsymbol{\xi}), L_m(\boldsymbol{\xi}_{mj}, \boldsymbol{\xi}) >_{L^2(S_1)} = \sum_{i=1}^{2m+1} L_m(\boldsymbol{\xi}_{mi}, \boldsymbol{\xi}_{mj}) \lambda_{mi}, \tag{13.48}$$

$$m = 0, 1, \ldots, N, \ j = 1, 2, \ldots, 2m+1.$$

Equation 13.48 is a set of $N+1$ systems, one for each m, whose solutions exists as a consequence of (13.39). Since the known terms of (13.48) depend only on $P_N(\boldsymbol{\xi})$ on S_1, the Lemma is proved.

In a sense Lemma 3 is nothing but a theorem of existence of the solution of the Dirichlet problem for Laplace equation in polynomial spaces. In fact if we go back to (13.22) we see that $\forall P_N \in \mathcal{P}_N^3$, taking its trace on S_1, i.e. putting $r = \|\xi\| = 1$, we get the same function as the trace of the polynomial $h_N(\boldsymbol{\xi}) + r^2 h_{N-2}(\boldsymbol{\xi}) + \ldots$ and such a trace, as we saw in Lemma 3, is sufficient to know each individual component.

Then, as nicely stated in Krarup (2006): "given any polynomial $P_N(\boldsymbol{\xi})$ in B_1 there is one and only one harmonic polynomial agreeing with it on S_1."

The above reasoning and Theorem 1 lead us to one of the main results of this chapter, which we propose in the form of a Theorem.

Theorem 2 (Summation theorem). *Given in HH_n^3 any ON set of polynomials $\{\varphi_{nm}(\boldsymbol{\xi})\}$, that we shall call spherical harmonics of degree n and order m, we must have*

$$H_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{m=1}^{2n+1} \varphi_{nm}(\boldsymbol{\xi})\varphi_{nm}(\boldsymbol{\eta}); \tag{13.49}$$

because of (13.38) and (13.39), by simply changing the normalization of $\varphi_{nm}(\boldsymbol{\xi})$, i.e. putting

$$\varphi_{nm}(\boldsymbol{\xi}) = \sqrt{\frac{A_n}{2n+1}}\overline{\varphi}_{nm}(\boldsymbol{\xi}) \tag{13.50}$$

we get

$$L_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{m=1}^{2n+1} \overline{\varphi}_{nm}(\boldsymbol{\xi})\overline{\varphi}_{nm}(\boldsymbol{\eta}); \tag{13.51}$$

$\{\overline{\varphi}_{nm}\}$ are then normalized in $L^2(S_1)$, contrary to $\{\varphi_{nm}\}$ that are normalized in HH_n^3.

Proof. Simply apply Theorem 3 on RKHS. □

13.3 Spherical Harmonics

We can observe that (13.51) holds whatever is the CON system $\{\overline{\varphi}_{nm}(\boldsymbol{\xi})\}$; however there is a particular system of this kind, that we shall study in detail in the next proposition, characterized by the fact that if we express $\boldsymbol{\xi}$ in polar coordinates (r, ϑ, λ) we obtain spherical harmonics in which the three variables separate, in the sense that

$$\overline{\varphi}_{nm}(\boldsymbol{\xi}) = r^n Y_{nm}(\vartheta, \lambda) = r^n f_m(\lambda)\overline{P}_{nm}(\vartheta) \tag{13.52}$$
$$f_m(\lambda) = \cos m\lambda, \;\; f_{-m}(\lambda) = \sin m\lambda, \;\; m = 0, 1, 2 \ldots n$$

Such functions are called, by antonomasia, *inner solid spherical harmonics*. The adjective “inner” refers to the fact that one can apply to (13.52) the Rayleigh transform (see Proposition 1) with respect to the unit sphere, $R = 1$, to obtain the “outer” *solid spherical harmonic*

$$\widetilde{\varphi}_{nm}(\boldsymbol{\xi}) = \frac{1}{r^{n+1}} Y_{nm}(\vartheta, \lambda) \tag{13.53}$$

which are regular harmonic functions in the whole $\mathbf{R}^3$, including the infinity but excluding the origin.

We note that, since $\overline{\varphi}_{nm}(\boldsymbol{\xi})$ are normalized in $L^2(S_1)$, we must have in fact

$$\frac{1}{4\pi}\int Y_{nm}(\vartheta,\lambda)Y_{jk}(\vartheta,\lambda)dSq = \delta_{nj}\delta_{mk}. \tag{13.54}$$

The function $Y_{nm}(\vartheta,\lambda)$ are called *surface spherical harmonics*: they are the trace on S_1 of solid spherical harmonics. The indexes n and m of Y_{nm} are called respectively the *degree* and the *order* of the spherical harmonics.

Now if we use (13.52) in (13.51) we get a very useful, and widely used, Corollary.

Corollary 1. *We have*

$$P_n(\cos\psi_{\xi\eta}) = \frac{1}{2n+1}\sum_{m=-n}^{n} Y_{nm}(\vartheta_\xi,\lambda_\xi)Y_{nm}(\vartheta_\eta,\lambda_\eta), \tag{13.55}$$

where $\boldsymbol{\xi} = (r_\xi,\vartheta_\xi,\lambda_\xi)$, $\boldsymbol{\eta} = (r_\eta,\vartheta_\eta,\lambda_\eta)$ and $\psi_{\xi\eta}$ is the spherical angle between the directions of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

Proposition 6. *For every degree n we find $2n+1$ homogeneous harmonic polynomials of the form (see Definition 2 and formula (13.52) for the notation)*

$$S_{nm}(\rho,\lambda,z) = \rho^{|m|} f_m(\lambda) Q_{n-|m|}(\rho,z) \tag{13.56}$$
$$m = -n,-n+1,\ldots,n-1,n.$$

Note that these $Q_{n-|m|}(\rho,z)$ should not be confused with the Legendre functions of second kind, which by the way are functions of one variable only.

In (13.56) $Q_{n-|m|}(\rho,z)$ is a polynomial homogenous of degree $n-|m|$ in (ρ,z), with a form of the type

$$\begin{aligned} Q_{n-|m|}(\rho,z) &= z^{n-|m|} + q_1 z^{n-|m|-2}\rho^2 + \ldots \\ &= \sum_{k=0}^{I} q_k z^{n-|m|-2k}\rho^{2k} \end{aligned} \tag{13.57}$$

where we have put for the sake of simplicity

$$I = \left[\frac{n-|m|}{2}\right] \tag{13.58}$$

and

$$q_0 = 1. \tag{13.59}$$

The functions $S_{nm}(\rho,\lambda,z)$ are called solid spherical harmonics and when we go to spherical coordinates (r,ϑ,λ) by putting $\rho = r\sin\vartheta, z = r\cos\vartheta$ they get the form

$$S_{nm}(r,\vartheta,\lambda) = r^n f_m(\lambda)\widetilde{P}_{n-|m|}(\vartheta) \tag{13.60}$$

where

$$\widetilde{P}_{n-|m|}(\vartheta) = (\sin\vartheta)^{|m|} Q_{n-|m|}(\sin\vartheta, \cos\vartheta) \tag{13.61}$$
$$= (\sin\vartheta)^{|m|} \sum_{k=0}^{I} q_k (\cos\vartheta)^{n-|m|-2k}(1-\cos^2\vartheta)^k.$$

i.e. $\widetilde{P}_{nm}(\vartheta)$ *is the product of* $(\sin\vartheta)^{|m|}$ *by a polynomial of degree* $n-|m|$ *in* $\cos\vartheta$.

Furthermore the functions $S_{nm}(r,\vartheta,\lambda)$ *are* $L^2(S_1)$ *orthogonal, i.e.*

$$\frac{1}{4\pi}\int_{S_1} S_{n,m}(1,\vartheta,\lambda) S_{n,m'}(1,\vartheta,\lambda) dS_1 = 0 \quad m \neq m'. \tag{13.62}$$

Finally we note that the polynomials $Q_{n-|m|}(\rho, z)$ *in (13.56) and therefore the functions* $\widetilde{P}_{n,m}(\vartheta)$ *in (13.61) are defined up to a proportionality constant which here is fixed by the normalization condition (13.59); we shall see in the sequel other normalization conditions for such functions.*

Proof. We basically must prove that there exist constants $q_0 = 1, q_1, \ldots, q_I$, univocally fixed by the condition that $S_{n,m}(\rho,\lambda,z)$, given by (13.56) and (13.57), satisfies Laplace equation.

First of all we observe that if we put $m = \pm n$ (observe that we claimed $Q_{n-|m|}$ to depend on $|m|$ and not on m as it will be soon justified) in (13.56), we get $Q_0(\rho, z)$ which reduces to $Q_0 \equiv 1$ and

$$S_{n,\pm n} = \rho^n f_{\pm n}(\lambda) \tag{13.63}$$

which are harmonic in (x, y) (cf. (13.66)) and therefore also in (x, y, z).

Furthermore with $m = \pm(n-1)$, (13.56) yields

$$S_{n,\pm(n-1)} = q_1 z \rho^{n-1} f_{\pm(n-1)}(\lambda) \tag{13.64}$$

which is again straightforwardly harmonic in (x, y, z), because it is the first order in z and harmonic in (x, y).

Now take $S_{n,m}$ from (13.56) and impose to it to satisfy Laplace equation which we choose to write in cylindrical coordinates, namely putting

$$\Delta = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \lambda^2}. \tag{13.65}$$

Considering that one has

$$\nabla f_m(\lambda) \cdot \nabla \rho^{|m|} Q_{n-|m|}(\rho, z) \equiv 0$$

because (ρ, λ, z) is an orthogonal coordinate system, one finds

$$\begin{aligned}
&\Delta\left[f_m(\lambda)\rho^{|m|}Q_{n-|m|}(\rho,z)\right] \\
&\quad = \Delta\left[f_m(\lambda)\right]\cdot\rho^{|m|}Q_{n-|m|}(\rho,z) + f_m(\lambda)\Delta\left[\rho^{|m|}Q_{n-|m|}(\rho,z)\right] \\
&\quad = f_m(\lambda)\left\{-m^2\rho^{|m|-2}Q_{n-|m|} + \Delta[\rho^{|m|}Q_{n-|m|}]\right\} = 0
\end{aligned} \tag{13.66}$$

implying that the expression in parenthesis has to be zero. So, we are justified to assume $\rho^{|m|}Q_{n-|m|}$ to depend on $|m|$ as opposed to m. In other words, to both $f_m(\lambda)$ and $f_{-m}(\lambda)$ we can associate the same $\rho^m Q_{n-m}$ with $m \geq 0$. Considering the previous remark (see (13.63) and (13.64)) we can now assume that

$$m = 0, 1, \dots n-1. \tag{13.67}$$

Substituting (13.57) into (13.66) and considering that

$$\begin{aligned}
\Delta z^{n-m-2k}\rho^{m+2k} &= (n-m-2k)(n-m-2k-1)z^{n-m-2k-2}\rho^{m+2k} \\
&\quad + (m+2k)^2 z^{n-m-2k}\rho^{m+2k-2},
\end{aligned}$$

we find

$$\begin{aligned}
&\sum_{k=0}^{I}(n-m-2k)(n-m-2k-1)z^{n-m-2k-2}\rho^{m+2k}q_k \\
&\qquad + \sum_{k=0}^{I}4k(m+k)z^{n-m-2k}\rho^{m+2k-2}q_k = 0
\end{aligned} \tag{13.68}$$

We explicitly note that the last term of the first summation is always zero because we have (remember (13.58))

$$(n-m-2I)(n-m-2I-1) \equiv 0,$$

while the first term of the second summation is also zero because of the factor k. As a result the two sums in (13.68) contain exactly the same monomials and equating the corresponding coefficients we get the recursive relation

$$q_k = -\frac{(n-m-2k+2)(n-m-2k+1)}{4k(m+k)}q_{k-1}. \tag{13.69}$$

The equation (13.69) fixes all q_k when the normalization (13.59) is assumed.

The second claim of proposition is elementary in nature because

$$S_{nm}(1,\vartheta,\lambda) = f_m(\lambda)\widetilde{P}_{n-|m|}(\vartheta)$$

and, for $m' \neq m$,

$$< S_{nm}(1,\vartheta,\lambda),\ S_{nm'}(1,\vartheta,\lambda) >_{L^2(S_1)}$$
$$= \frac{1}{4\pi}\int_0^{\pi} d\vartheta \sin\vartheta \widetilde{P}_{n-|m|}(\vartheta)\widetilde{P}_{n-|m'|}(\vartheta)\int_0^{2\pi} f_m(\lambda) f_{m'}(\lambda) d\lambda = 0,$$

since the Fourier functions $\{f_m(\lambda)\}$ are well-known to be orthogonal in $L^2([0,2\pi])$. □

Remark 2. As we have already recalled, there are other functions of ϑ which are used in geodetic literature to form spherical harmonics, namely, with $t = \cos\vartheta$ and $m = 0, 1, \ldots n$,

$$P_{nm}(\vartheta) = (1-t^2)^{m/2} D_t^m P_n(t) \tag{13.70}$$

$$\overline{P}_{nm}(\vartheta) = k_{nm} P_{nm}(\vartheta) \tag{13.71}$$

$$k_{nm} = \sqrt{(2-\delta_{m0})(2n+1)\frac{(n-m)!}{(n+m)!}} \tag{13.72}$$

The $P_{nm}(\vartheta)$ are known as *Legendre associated functions* of the first kind. They were found by studying Laplace equation in spherical coordinates, imposing that

$$\Delta[r^n f_m(\lambda) P_{nm}(\vartheta)] = 0. \tag{13.73}$$

By using formula (13.36) we find for P_{nm} as functions of $t = \cos\vartheta$, the equation

$$(1-t^2)P''_{nm}(t) - 2tP'_{nm}(t) + \left[n(n+1) - \frac{m^2}{1-t^2}\right] P_{nm}(t) = 0 \tag{13.74}$$

which is also known as *Legendre equation*.

It has to be underlined that if we put $m = 0$ in (13.70) we get

$$P_{n0}(t) = P_n(t), \tag{13.75}$$

i.e. the associated Legendre functions of order zero are simply the Legendre polynomials. This agrees with the fact that if we put $m = 0$ into (13.74) we go back to (13.37), i.e. the equation that is satisfied by $P_n(t)$. One can prove that P_{nm}, as given by (13.70), do satisfy (13.74) and then (13.73) too.

On the other hand we see from (13.70) that $P_{nm}(\vartheta)$ is $(\sin\vartheta)^m$ multiplied by a polynomial of degree $(n-m)$ in $\cos\vartheta = t$, i.e. it has exactly the same form of $\widetilde{P}_{nm}(\vartheta)$ (cf. (13.61)). Since by Proposition (13.6) it has been proved that $\widetilde{P}_{nm}(\vartheta)$ are unique, up to a multiplicative constant, we conclude that $\widetilde{P}_{nm}$ are the same as the Legendre functions with a different normalization, i.e.

$$\widetilde{P}_{nm}(\vartheta) = A_{nm} P_{nm}(\vartheta). \tag{13.76}$$

The constant A_{nm} are easy to find by comparing the coefficients of maximum degree in t in $D^m P_n$ and in $Q_{n-m}(\sin\vartheta, \cos\vartheta)$, but they are really not needed here. What is important is that (13.76) holds. Finally, a different normalization is in fact used in all practical computations as well as in theoretical formulas, namely that of the functions $\overline{P}_{nm}(\vartheta)$, also called *normalized Legendre functions*.

The normalization condition of $\overline{P}_{nm}(\vartheta)$ is derived from the request that the surface spherical harmonics

$$Y_{nm}(\vartheta, \lambda) = f_m(\lambda)\overline{P}_{nm}(\vartheta) \tag{13.77}$$

have norm one in $L^2(S_1)$, namely

$$\frac{1}{4\pi}\int Y_{nm}^2(\vartheta, \lambda)d\sigma = 1. \tag{13.78}$$

If we use the relations (remember the definition (13.52) of $f_m(\lambda)$)

$$\int_0^{2\pi} f_m^2(\lambda)d\lambda = (1 + \delta_{m0})\pi$$

into (13.78), i.e.

$$\begin{aligned}\frac{1}{4\pi}\int_0^{2\pi} d\lambda f_m^2(\lambda)\cdot\int_0^{\pi} d\vartheta \sin\vartheta\,\overline{P}_{nm}(\vartheta)^2 \\ = \frac{1+\delta_{m0}}{4}\int_{-1}^{1}\overline{P}_{nm}(t)^2 dt = 1,\end{aligned}$$

we see that the constants k_{nm} have to be computed from

$$\frac{1+\delta_{m0}}{4}\cdot k_{nm}^2\int_{-1}^{1} P_{nm}^2(t)dt = 1, \tag{13.79}$$

i.e.

$$m \neq 0 \quad k_{nm} = 2\left\{\int_{-1}^{1} P_{nm}^2(t)dt\right\}^{-1/2} \tag{13.80}$$

$$m = 0 \quad k_{n0} = \sqrt{2}\left\{\int_{-1}^{1} P_{n0}^2(t)dt\right\}^{-1/2}. \tag{13.81}$$

In particular we have

$$k_{n0} = \frac{1}{\sqrt{2n+1}},\ \overline{P}_{n0}(t) = \sqrt{2n+1}P_n(t). \tag{13.82}$$

We conclude that the spherical harmonics, $\{Y_{nm}(\vartheta, \lambda\}$, here precisely defined are the same functions anticipated by the definition (13.52) and therefore they satisfy the summation theorem (13.55).

Now that we have set up an explicit construction of a CON system in HH_n^3, namely $\{r^n Y_{nm}(\vartheta, \lambda)\}$, when we endow this space with the $L^2(S_1)$ product, we have to find suitable numerical methods for an efficient computation of the spherical harmonic functions $\{Y_{nm}(\vartheta, \lambda)\} = \overline{P}_{nm}(\vartheta) f_m(\lambda)$, i.e. of the associated Legendre functions $\overline{P}_{nm}(\vartheta)$. This is done by establishing recursive relations, among which two are relatively simple and widely used in practice.

Proposition 7. *The following recursive relation on the degree n for $\overline{P}_{nm}(t), \overline{P}'_{nm}(t)$ (as functions of $t = \cos\vartheta$) holds*

$$-\begin{vmatrix} \overline{P}_{n+1,m}(t) \\ \overline{P}'_{n+1,m}(t) \end{vmatrix} = A_{nm} \begin{vmatrix} t & 0 \\ 1 & t \end{vmatrix} \begin{vmatrix} \overline{P}_{nm}(t) \\ \overline{P}'_{nm}(t) \end{vmatrix} - B_{nm} \begin{vmatrix} \overline{P}_{n-1,m}(t) \\ \overline{P}'_{n-1,m} \end{vmatrix}, \tag{13.83}$$

where

$$A_{nm} = \left[\frac{(2n+1)(2n+3)}{(n+1-m)(n+1+m)} \right]^{1/2}$$
$$B_{nm} = \left[\frac{(2n+3)(n+m)(n-m)}{(2n-1)(n+1-m)(n+1+m)} \right]^{1/2};$$

for every $m \neq 0$ such relations can start from

$$\begin{cases} \overline{P}_{mm}(t) = k_{mm}(1-t^2)^{m/2} = k_{mm}(\sin\vartheta)^m \\ \overline{P}'_{mm}(t) = -k_{mm} m (1-t^2)^{m/2-1} t \end{cases} \tag{13.84}$$

and

$$\overline{P}_{m-1,m} \equiv 0, \ \overline{P}'_{m-1,m} \equiv 0; \tag{13.85}$$

for $m = 0$ we can start from

$$\begin{aligned} &\overline{P}_{00} = 1, \ \overline{P}_{10} = \sqrt{3}t \\ &\overline{P}'_{00} = 0, \ \overline{P}'_{10} = \sqrt{3}. \end{aligned} \tag{13.86}$$

Proof. Of the two relations (13.83) we need proving only the first one, as the second is just the derivative with respect to t of the first.

We take the recursive relation (cf. Part I, (3.24)) for $P_n(t)$

$$(n+1)P_{n+1}(t) = (2n+1)tP_n - nP_{n-1}$$

and apply D^m to obtain

$$(n+1)D^m P_{n+1} \\ = (2n+1)tD^m P_n + (2n+1)mD^{m-1}P_n - nD^m P_{n-1}. \qquad (13.87)$$

Then we remember that (see Part I, (3.37))

$$P_n = \frac{1}{2n+1}\left[P'_{n+1} - P'_{n-1}\right]$$

so that

$$D^{m-1}P_n = \frac{1}{2n+1}[D^m P_{n+1} - D^m P_{n-1}];$$

substituting back in (13.87) and re-ordering, we get

$$(n+1-m)D^m P_{n+1} = (2n+1)tD^m P_n - (n+m)D^m P_{n-1}. \qquad (13.88)$$

Now we multiply (13.88) by $(1-t^2)^{m/2}$ arriving at

$$P_{n+1,m} = \frac{2n+1}{n+1-m}tP_{n,m} - \frac{n+m}{n+1-m}P_{n-1,m}. \qquad (13.89)$$

We note here that during the step (13.87) whenever it happens that $n<m$ we can put

$$n<m,\ D^m P_n \equiv 0 \Rightarrow P_{n,m} \equiv 0, \qquad (13.90)$$

because P_n is a polynomial of degree n in t; this already justifies (13.85).

Finally in (13.89) we can multiply and divide each P_{nm} by k_{nm} (cf. (13.71) and (13.72)), i.e. we can put

$$P_{\ell,m} = k^{-1}_{\ell,m}\overline{P}_{\ell,m},\ \ell = n+1, n, n-1$$

and simplify, to obtain the first of (13.83). The relation (13.84) is just the definition of $\overline{P}_{nm}$ and its derivative; (13.85) is already justified.

For $m = 0$, we never have $n<m$, but we can initialize (13.84) with (13.86), which are again just definitions. □

Proposition 8. *The following recursive relations, on the order m, hold*

$$P_{n,m+1} = \frac{2t}{\sqrt{1-t^2}}mC_{nm}\overline{P}_{nm} - C_{nm}D_{nm}\overline{P}_{n,m-1} \qquad (13.91)$$

$$C_{nm} = \left[\frac{1}{(n-m)(n-m+1)}\right]^{1/2},\quad m<n$$

$$D_{nm} = [(n+m)(n-m+1)]^{1/2} \cdot \sqrt{1+\delta_{m1}},$$
$$\overline{P}'_{nm} = \frac{t}{1-t^2} m \overline{P}_{nm} - \frac{D_{nm}}{\sqrt{1-t^2}} \overline{P}_{n,m-1}. \tag{13.92}$$

We note that, although we could indeed put (13.92) in a form where $\overline{P}'_{n,m+1}$ *is given as a combination of* $\overline{P}'_{nm}$ *and* $\overline{P}'_{n,m-1}$*, such equation which can be computed in sequence after (13.91), has a simpler form which we prefer.*

We note also that (13.91) and (13.92) can be triggered by a previous computation of $\overline{P}_{n0}$*, for all the degrees needed, and then*

$$\begin{cases} \overline{P}_{n1} = \sqrt{\dfrac{2}{n(n+1)}} (1-t^2)^{1/2-1} P_{n0} \\ \overline{P}'_{n1} = \dfrac{t}{1-t^2} \overline{P}_{n1} - \dfrac{\sqrt{2n(n+1)}}{\sqrt{1-t^2}} \overline{P}_{n0}. \end{cases} \tag{13.93}$$

Proof. We start from the notable relation, proved in Exercise 13.9,

$$(1-t^2) P_n^{(m+2)} = 2(m+1) t P_n^{(m+1)} - (n-m)(n+m+1) P_n^{(m)}, \tag{13.94}$$

where

$$P_n^{(m)}(t) = D^m P_n(t).$$

We substitute $(m-1)$ to (m) and multiply it by $(1-t^2)^{m/2}$, to get

$$P_{n,m+1} = \frac{2t}{\sqrt{1-t^2}} m P_{nm} - (n+m)(n-m+1) P_{n,m-1}. \tag{13.95}$$

Substituting

$$\overline{P}_{nj} = k_{nj} P_{nj}, \ j = m+1, m, m-1 \tag{13.96}$$

in (13.95) and simplifying, we get (13.91). Now we go back to the definition

$$P_{nm} = (1-t^2)^{m/2} D^m P_n$$

and differentiate, obtaining

$$P'_{nm} = -m \frac{t}{1-t^2} P_{nm} + \frac{1}{\sqrt{1-t^2}} P_{n,m+1}. \tag{13.97}$$

Using (13.95) in (13.97) and normalizing with (13.96) we find (13.92).

Finally the first of (13.93) is just the definition of $\overline{P}_{n1}$, while the second is (13.92) with $m = 1$. □

Remark 3. Whatever recursive relations are used to compute

$$\overline{P}_{nm}(t), \overline{P}'_{nm}(t) = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\overline{P}_{nm}(\vartheta),$$

we can then easily compute the second derivative $\overline{P''}_{nm}(t)$ by exploiting the equation (13.74) suitably normalized, i.e.

$$\overline{P}''_{nm} = \frac{2t}{1-t^2}\overline{P}'_{nm} - \frac{1}{1-t^2}\left[n(n+1) - \frac{m^2}{1-t^2}\right]\overline{P}_{nm}. \tag{13.98}$$

Remark 4. It is possible to see that for $m \neq 0$, $\overline{P}_{nm}(t) \to 0$, when $n \to \infty$. Therefore when $Y_{nm}(\vartheta, \lambda)$ are used with sums up to very high degree and order, for instance several thousands, the relative error in the calculus of such harmonics can increase significantly, and recursive relations on the degree n, starting from $\overline{P}_{mm}(t)$, are not any more providing reliable results, specially when $\overline{P}_{mm}$ itself is already very small.

So if one has to compute a function like

$$f(\vartheta, \lambda) = \sum_{n=0}^{N}\sum_{m=-n}^{n} f_{nm}Y_{nm}(\vartheta, \lambda) \tag{13.99}$$

for N equal to several thousands, one has to use (13.91) and (13.92), which for low orders give a good approximation and when they start giving bad results one can truncate the summation, because the functions $\overline{P}_{nm}(t)$ are in any case so small that they contribute little to sums like (13.99).

Among others, this is also the reason why we start (13.91) at $m = 0$ instead of $m = n$.

Alternatively one can still use the recursion on the degree n, however one starts from a $\widetilde{P}_{nm} = H_{nm}P_{nm}$ suitably re-normalized, and in the end one divides again the result by H_{nm}. This simple trick allows the accurate computation of P_{nm} for all degrees and orders up to some thousands.

A longer number of interesting relations like those presented above are known in literature, including different forms of the summation theorem (Martinec 1998).

13.4 Hilbert Spaces of Harmonic Functions and First Theorems of Potential Theory

It is now time to abandon the use of simple harmonic polynomials, which implies working only in finite dimensional spaces, and rather go to HS of harmonic functions, namely to transform sums into series and to study limit properties when the dimension of the space goes to infinity.

The material of this section is covered by several books in geodesy, like Moritz (1980), Krarup (2006), and Heiskanen and Moritz (1967). A recent simple mathematical book on the subject is Axler et al. (2001).

The first result to be presented is so important that we state it in the form of a Theorem.

Theorem 3. *The sequence of normalized spherical harmonics*

$$\{Y_{nm}(\vartheta, \lambda);\ |m| \le n,\ n = 0, 1, 2 \ldots\} \tag{13.100}$$

is a CON system in $L^2(S_1)$ that is $\forall f(\vartheta, \lambda) \in L^2(S_1)$ we have

$$\begin{cases} f(\vartheta, \lambda) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} f_{nm} Y_{nm}(\vartheta, \lambda) \\ f_{nm} = \frac{1}{4\pi} \int_{S_1} f(\vartheta, \lambda) Y_{nm}(\vartheta, \lambda) d\sigma \end{cases}, \tag{13.101}$$

the series being convergent in $L^2(S_1)$; furthermore

$$\| f \|^2_{L^2(S_1)} = \frac{1}{4\pi} \int_{S_1} f^2(\vartheta, \lambda) d\sigma = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} f^2_{nm}. \tag{13.102}$$

Proof. That $\{Y_{nm}\}$ is an orthonormal system in $L^2(S_1)$ we already know; we have to prove that it is complete.

We could just invoke Proposition 12 here, but to prepare further results (theorem 7) we prefer to prove directly that $\{Y_{nm}\}$ is total in $L^2(S_1)$ and then use Proposition 9. So we need to prove that

$$\forall f \in C^1(S_1); \quad < f, Y_{nm} >_{L^2(S_1)} = 0, \forall n, m \Rightarrow f = 0. \tag{13.103}$$

First note that (13.103) implies

$$\forall n,\ \forall P \in S_1 \quad < f(Q), P_n(\psi_{PQ}) >_{L^2(S_1)} \equiv 0, \tag{13.104}$$

because

$$< f(Q), P_n(\psi_{PQ}) >_{L^2(S_1)} = \sum_{m=-n}^{n} (2n+1)^{-1} Y_{nm}(P) < f, Y_{nm} >_{L^2(S_1)} . \tag{13.105}$$

Now consider the single-layer potential

$$V(P) = \frac{1}{4\pi} \int \frac{f(Q)}{\ell_{PQ}} d\sigma_Q = \frac{1}{4\pi} \int_{S_1} \frac{f(Q)}{\sqrt{1 + r_P^2 - 2 r_P \cos\psi_{PQ}}} d\sigma_Q; \tag{13.106}$$

if we take $r_P < 1$ or $r_P > 1$ we have, respectively

$$r_P < 1, \quad \frac{1}{\ell_{PQ}} = \sum_{n=0}^{+\infty} r_P^n P_n(\cos\psi_{PQ}), \tag{13.107}$$

$$r_P > 1, \quad \frac{1}{\ell_{PQ}} = \sum_{n=0}^{+\infty} \frac{1}{r_P^{n+1}} P_n(\cos\psi_{PQ}). \tag{13.108}$$

The two series (13.107) and (13.108) converge uniformly in ψ_{PQ} because

$$|P_n(\cos\psi)| \le 1, \ \forall\psi$$

so that we can substitute them in (13.106) and exchange summation and integral to find

$$r_P < 1, \quad V(P) = \sum_{n=0}^{+\infty} r_P^n \left\{ \frac{1}{4\pi} \int f(Q) P_n(\cos\psi_{PQ}) d\sigma_Q \right\} = 0,$$

$$r_P > 1, \quad V(P) = \sum_{n=0}^{+\infty} \frac{1}{r_P^n} \left\{ \frac{1}{4\pi} \int f(Q) P_n(\cos\psi_{PQ}) d\sigma_Q \right\} = 0.$$

In other words, since the potential of an L^2 single layer admits almost everywhere on S, radial limits, (cf. Miranda 1970), we find

$$V(P) \equiv 0 \tag{13.109}$$

everywhere in $\mathbf{R}^3$.

On the other hand remember that the following jump relations for the normal derivatives, taken across S_1, hold (cf. Part I, (1.54))

$$f(P) = -\frac{1}{4\pi} \left\{ \left(\frac{\partial V}{\partial \nu} \right)_+ - \left(\frac{\partial V}{\partial \nu} \right)_- \right\} \tag{13.110}$$

so that we find, because of (13.109), that it is also

$$f(P) \equiv 0, \tag{13.111}$$

as it was to be proved.

The relation (13.102) is just Parseval's identity for this specific case. □

Example 1. The following is the Dirichlet problem for a ball B_R of radius R. Let a function $f(\vartheta, \lambda)$ be given on S_R, the boundary of B_R, and for the sake of definiteness we assume $f \in L^2(S_R)$; we want to find a $h(r, \vartheta, \lambda)$ which is harmonic in B_R and agrees, in a suitable sense, to be here defined, with f on S_R

$$u(R, \vartheta, \lambda) = f(\vartheta, \lambda). \tag{13.112}$$

Let us state the convention that, at the level of notation, when we represent a series of spherical harmonics without specifying the summation limits, we implicitly mean that we add over all degrees and orders, namely

$$\sum_{n,m} f_{nm} Y_{nm}(\vartheta, \lambda) \equiv \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} f_{nm} Y_{nm}(\vartheta, \lambda).$$

Because of Theorem 3 we know that we can put

$$P \in S_R, \ f(P) = \sum_{n,m} < f(Q), Y_{nm}(Q) > Y_{nm}(P) = \sum_{n,m} f_{nm} Y_{nm}(P);$$

since for each degree and order we know that

$$f_{nm} S_{nm}(r, \vartheta, \lambda) = f_{nm} \left(\frac{r}{R}\right)^n Y_{nm}(\vartheta, \lambda),$$

is indeed harmonic and agrees with

$$f_{nm} Y_{nm}(\vartheta, \lambda) \text{ on } S_R,$$

we guess that the sought solution is given by

$$u(r, \vartheta, \lambda) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} f_{nm} \left(\frac{r}{R}\right)^n Y_{nm}(\vartheta, \lambda). \tag{13.113}$$

The problem is whether the series is convergent and we are allowed to apply the Laplace operator term-wise, so that from $\Delta \left[\left(\frac{r}{R}\right)^n Y_{nm}(\vartheta, \lambda)\right] = 0$ we can deduce $\Delta u(r, \vartheta, \lambda) = 0$. Remember that if we put $\vartheta = \vartheta', \lambda = \lambda'$ in (13.55), i.e. $\xi = \eta$ and $\psi_{\xi\eta} = 0$, we have

$$\frac{1}{2n+1} \sum_{m=-n}^{n} Y_{nm}^2(\vartheta, \lambda) = P_n(1) = 1. \tag{13.114}$$

This implies that (see also Martinec 1998)

$$|Y_{nm}(\vartheta, \lambda)| \leq \sqrt{2n+1}. \tag{13.115}$$

Then, from (13.113), by using Schwarz inequality, we have

$$|u(r,\vartheta,\lambda)| \le \sum_{n=0}^{+\infty}\left(\frac{r}{R}\right)^n \left\{\sum_m f_{nm}^2 \cdot \sum_m Y_{nm}^2\right\}^{1/2}$$
$$\le \sum_{n=0}^{+\infty}\sqrt{2n+1}\left(\frac{r}{R}\right)^n \cdot \sqrt{\sum_m \cdot f_{nm}^2}$$
$$\le A\cdot\sum_{n=0}^{+\infty}\sqrt{2n+1}\left(\frac{r}{R}\right)^n ; \qquad (13.116)$$

the last step in (13.116) is justified because from (13.102) we know that $\sum_{m=-n}^{n} f_{nm}^2 \to 0$, so that there must be a constant A such that

$$\forall n \ge 0, \quad \sqrt{\sum_{m=-n}^{n} f_{nm}^2} \le A.$$

The relation (13.116) shows that our series is absolutely and uniformly convergent in every ball strictly contained in B_R and concentric with that. Since the multiplication of $\left(\frac{r}{R}\right)^n$ by any (fixed) polynomial in n does not modify the convergence of (13.116), we deduce that we can apply termwise such operators as $\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$ and $r^{-2}\Delta_\sigma$ and verify that in fact (13.113) is harmonic.

As for the Dirichlet boundary condition (13.112) contrary to intuition, it is not enough to put $r = R$ in (13.113) and then verify that we are left with the harmonic development of $f(P)$, because this series is not really convergent in a pointwise sense but only in $L^2(S_1)$, i.e. in a mean square sense over S_1.

The correct definition is as follows: we take the trace of $u(r,\vartheta,\lambda)$ at any sphere with $r = R - \delta$ and we take the difference

$$f(\vartheta,\lambda) - u(R-\delta,\vartheta,\lambda) = \sum_{n,m} f_{nm}\left[1 - \left(\frac{R-\delta}{R}\right)^n\right] Y_{nm}(\vartheta,\lambda);$$

we evaluate the $L^2(S_1)$ norm of such difference, namely

$$\|f(\vartheta,\lambda) - u(R-\delta,\vartheta,\lambda)\|^2_{L^2(S_1)} = \sum_{n,m} f_{nm}^2\left[1 - \left(1 - \frac{\delta}{R}\right)^n\right]^2 . \qquad (13.117)$$

Since each term of the positive series (13.114) is bounded above by f_{nm}^2, because indeed

$$\left[1 - \left(1 - \frac{\delta}{R}\right)^n\right]^2 \le 1, \; n = 0, 1, \ldots$$

we can pass to the limit for $\delta \to 0$ under the series and find

$$\lim_{\delta \to 0} \| f(\vartheta, \lambda) - u(R - \delta, \vartheta, \lambda) \|^2_{L^2(S_1)} = 0. \tag{13.118}$$

Remark 5. This interpretation of the Dirichlet boundary condition has been introduced by Cimmino (1952) and it has been taken up in geodesy in a number of works on BVP (Cimmino 1955; Sansò and Venuti 1998). As a matter of fact, the theory holds in general, with suitable changes, for smooth surfaces, S, and for functions $f(P)$ square integrable over S, as we shall soon see.

Example 2 (Poisson integral). In this example we continue Example 1, giving to the solution of Dirichlet problem for the sphere (13.113) the form of a Poisson integral.

In fact, substituting

$$f_{nm} = \frac{1}{4\pi} \int f(\vartheta', \lambda') Y_{nm}(\vartheta', \lambda') d\sigma'$$

into (13.113) and recalling the convergence result of such series, we can claim that

$$\begin{aligned} \forall r < 1, \quad u(r, \vartheta, \lambda) = & \\ = \frac{1}{4\pi} \int \left\{ \sum_{n,m} Y_{nm}(\vartheta, \lambda) \left(\frac{r}{R}\right)^n Y_{nm}(\vartheta' \lambda') \right\} f(\vartheta', \lambda') d\sigma' & \\ = \frac{1}{4\pi} \int \left\{ \sum_{n=0}^{+\infty} \left(\frac{r}{R}\right)^n (2n+1) P_n(\cos\psi) \right\} f(\vartheta', \lambda') d\sigma', & \end{aligned} \tag{13.119}$$

ψ being the spherical angle between the direction of $P(\vartheta, \lambda)$ and $P'(\vartheta', \lambda')$ on the unit sphere.

Now recall that

$$G(s, t) = \frac{1}{\{1 + s^2 - 2st\}^{1/2}} = \sum_{n=0}^{+\infty} s^n P_n(t)$$

and observe that

$$2s \frac{\partial}{\partial s} G(s, t) + G(s, t) = \sum_{n=0}^{+\infty} (2n+1) s^n P_n(t). \tag{13.120}$$

Using (13.120) in (13.119), with $s = \frac{r}{R}$ and $t = \cos\psi$, one receives, after re-arranging,

$$\begin{cases} u(r, \vartheta, \lambda) = \dfrac{1}{4\pi} \displaystyle\int \Pi_{Ri}(r, \psi) f(\vartheta', \lambda') d\sigma \\ \Pi_{Ri}(r, \psi) = \frac{R(R^2 - r^2)}{\left[R^2 + r^2 - 2rR\cos\psi\right]^{3/2}}. \end{cases} \tag{13.121}$$

The function $\Pi_{Ri}(r, \psi)$ is the so-called *internal Poisson kernel* for the ball of radius R. We note that Π_{Ri} is a function of the point $P \equiv (r, \vartheta, \lambda)$ and $P' \equiv (R, \vartheta', \lambda')$ in the sense that

$$\Pi_{Ri}(P, P') = \frac{R(R^2 - |\mathbf{r}_P|^2)}{|\mathbf{r}_{P'} - \mathbf{r}_P|^3}. \tag{13.122}$$

With the notation of (13.122) and observing that $R^2 d\sigma \equiv dS$, the area element of the sphere of radius R, we can re-write (13.121) as

$$u(P) = \frac{1}{4\pi R^2} \int_S \Pi_{Ri}(P, P') f(P') dS_{P'}. \tag{13.123}$$

With the aid of (13.123) it becomes quite evident that $u(P)$, inside any sphere of radius $R' < R$, is in fact continuous with all its derivatives, i.e. $u \in C^\infty(B_R)$, because the Poisson kernel enjoys the same property.

We conclude with some properties of the Poisson kernel. Since, with $f(P') \equiv 1$ we are to find $u(P) \equiv 1$ in (13.123), we see that

$$\frac{1}{4\pi R^2} \int \Pi_{Ri}(P, P') dS_{P'} \equiv 1;$$

moreover $\prod_{Ri}(P, P') > 0$ when $r < R$ so that

$$\int \left| \Pi_{Ri}(P, P') \right| dS_{P'} < \text{const},$$

in fact it is equal to 1; furthermore, taking once $\psi \neq 0$ and then $\psi = 0$ in (13.121) we find

$$\begin{aligned} \psi \neq 0 \quad & \lim_{r_P \to R} \Pi_{Ri}(P, P') = 0 \\ \psi = 0 \quad & \lim_{r_P \to R} \Pi_{Ri}(P, P') = +\infty. \end{aligned}$$

The above four properties are enough to guarantee that, $\forall \varphi(P') \in C(S_R)$ and $P_0 \in S_R$

$$\lim_{\substack{P \to P_0 \\ (r_P \to R)}} \frac{1}{4\pi R^2} \int \Pi_{Ri}(P, P') \varphi(P') dS_{P'} = \varphi(P_0). \tag{13.124}$$

Proposition 9 (Mean value property). *Let $u(P)$ be harmonic in a domain B and let $B_R(P_0)$ be a ball of centre P_0 and radius R such that $B_R(P_0) \subset B$; put*

$$M_{B_{P_OR}}\{u\} = \frac{1}{\frac{4}{3}\pi R^3}\int_{B_{P_0R}} u(Q)dB \quad (13.125)$$

$$M_{S_{P_OR}}\{u\} = \frac{1}{4\pi R^2}\int_{S_{P_0R}} u(Q)dS; \quad (13.126)$$

i.e. the mean value of $u(P)$ over B_{P_0R} or over S_{P_0R} respectively; then we have

$$M_{B_{P_0R}}\{u\} = M_{S_{P_0R}}\{u\} = u(P_0). \quad (13.127)$$

Proof. $u(P)$ has to be continuous on S_{P_0R} because this surface is contained in B; then $u(P)|_{S_{P_0R}}$ is also in $L^2(S_{P_0R})$ and we can apply (13.122) and (13.123) with $P = P_0, |\mathbf{r}_P| = |\mathbf{r}_{P_0}| = 0,\ |\mathbf{r}_{P'} - \mathbf{r}_{P_0}| = R$ so that $\prod_{Ri}(P_0, P') \equiv 1$; the result is the second equality of (13.127). Moreover if we multiply both members of (13.126) by R^2dR and integrate, we find

$$\int_0^{\overline{R}} R^2dR \cdot u(P_0) = \frac{1}{4\pi}\int_0^{\overline{R}}\int_{S_{P_0R}} u(Q)dSdR = \frac{1}{4\pi}\int_{B_{P_0\overline{R}}} u(Q)dB; \quad (13.128)$$

noting that $\int_0^{\overline{R}} R^2dR = \frac{1}{3}\overline{R}^3$ and dividing both members of (13.128) by such quantity, we get the first equality of (13.127). □

Proposition 10. *This is the inverse of Proposition 9; namely, let $u(P)$ be defined and continuous up to second derivatives in B; assume further that (13.127) holds for $u(P)$, $\forall R$ such that $B_{P_0R} \subset B$, then $u(P)$ is harmonic in B*

$$\Delta u(P) = 0 \ \textit{ in } B. \quad (13.129)$$

Proof. Fix P_0 and let $\boldsymbol{\xi}$ be a vector of constant length R, such that $P_0 + \boldsymbol{\xi} \subset B$; we can write

$$u(P_0 + \boldsymbol{\xi}) = u(P_0) + \boldsymbol{\xi} \cdot \nabla u(P_0) + \frac{1}{2}\boldsymbol{\xi}^t D^2u(P_0)\boldsymbol{\xi} + o_2(\boldsymbol{\xi}), \quad (13.130)$$

where we have put

$$D^2u(P_0) = \left\{ \frac{\partial^2}{\partial\xi_i\partial\xi_k} u(P_0 + \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}=0} \right\}.$$

Note that, by direct computation, one has

$$M_{S_{P_0R}}\{\boldsymbol{\xi}\} = 0,\ M_{S_{P_0R}}\{\boldsymbol{\xi}\ \boldsymbol{\xi}^t\} = \frac{R^2}{3}I. \quad (13.131)$$

Take $M_{S_{P_0R}}$ of both members of (13.130) and use (13.127) to find

$$u(P_0) = u(P_0) + \frac{1}{2} TrD^2u(P_0) M_{S_{P_0R}}\{\boldsymbol{\xi}\boldsymbol{\xi}^t\} + o_2(R),$$

i.e., using (13.131),

$$\frac{R^2}{6} TrD^2u(P_0) + o_2(R) = 0.$$

Dividing by R^2 and letting $R \to 0$ we see then that

$$TrD^2u(P_0) = \Delta u(P_0) = 0. \qquad \square$$

Theorem 4 (Maximum principle). *Let $\{u(P)\}$ be harmonic in B and continuous up to the boundary, then, unless $u(P)$ is constant everywhere in B,*

$$\forall P \in B, \quad \min_{Q\in S} u(Q) < u(P) < \max_{Q\in S} u(Q). \tag{13.132}$$

Proof. Since it is a continuous function on the bounded closed set $\overline{B} = B \cup S$, $u(P)$ attains a minimum and a maximum value in such set. Let $\overline{P}$ be a point of absolute maximum; we prove that either $\overline{P} \in S$ or $u(P)$ is constant. In fact if $\overline{P} \in B$, then it is also a relative maximum so that, taking a suitable ball $B_{\overline{P},R}$, one has to find

$$u(\overline{P}) = M_{B_{\overline{P}R}}\{u(P)\} \le u(\overline{P}); \tag{13.133}$$

equality in (13.133) can be achieved only if $u(P) \equiv u(\overline{P})$, $\forall P \in B_{\overline{P},R}$, i.e. if $u(P)$ is constant in $B_{\overline{P},R}$. By the principle of identity of harmonic functions that will soon be proved, we should then have $u(P) = \text{const}$ everywhere in B. Otherwise (13.133) becomes impossible, i.e $\overline{P} \in S$. The same reasoning holds for the minimum. $\square$

Corollary 2. *Also the derivatives of $\{u(P)\}$ are controlled by their extreme values on the boundary, on condition that we stay away from S with P. More precisely, let K be any bounded closed set such that $K \subset B$ and let δ be the distance between K and S*

$$\delta = \min_{\substack{P\in S\\ Q\in K}} |\mathbf{r}_P - \mathbf{r}_Q|;$$

then there is a constant A such that

$$\max_{P\in K} \left| \frac{\partial u}{\partial x_i}(P) \right| \le A \cdot \delta^{-1} \max_{P\in S} |u(P)|.$$

Proof. First note that (13.127) implies

$$\max_{P\in B}|u(P)| \le \max_{P\in S}|u(P)|.$$

Then, since $\frac{\partial u}{\partial x_i}$ is a harmonic function too, write the mean value property for any $P_0 \in K$. As, it is (denoting with $\mathbf{e}_i$ the unit vector in the direction of the i-th axis)

$$\frac{\partial u}{\partial x_i}(P_0) = \frac{1}{\frac{4}{3}\pi R^3}\int_{B_{P_0R}} \frac{\partial u}{\partial x_i} dB = \frac{1}{\frac{4}{3}\pi R^3}\left\{\int_{S_{P_0R}} \mathbf{e}_i \cdot \mathbf{n} u dS\right\},$$

then

$$P_0 \in K, \quad \left|\frac{\partial u}{\partial x_i}(P_0)\right| \le \frac{3}{R}\,\frac{1}{4\pi R^2}\int_{S_{P_0R}} |u(Q)|dS \le \frac{3}{R}\max_{P\in S}|u(P)|. \quad (13.134)$$

Now we note that once K is fixed R can always be extended to became $R = \delta$, so that (13.134) implies

$$\max_{P_0\in K}\left|\frac{\partial u}{\partial x_i}(P_0)\right| \le 3\delta^{-1}\max_{P\in S}|u(P)|,$$

as it was to be proved. □

Remark 6. We can observe that the argument on extremal values in B holds even if $u(P)$ is not continuous up to boundary. For instance if $u(P)$ is only bounded on S, we can establish (13.132) in the weaker form

$$\sup_{P\in B}|u(P)| \le \sup_{P\in S}|u(P)|. \quad (13.135)$$

We notice too that (13.135) guarantees that the Dirichlet problem for functions harmonic in B and bounded in $\overline{B}$ has a unique solution. Furthermore if $\{u_n(P)\}$ is harmonic in P and uniformly convergent to $f(P)$ on S then $\{u_n(P)\}$ converges uniformly to some function $u(P)$ in $\overline{B}$ such that $u(P)|_S \equiv f(P)$ and even more $u(P)$ is harmonic too. In fact, going to the limit in

$$\lim_{n\to\infty} u_n(P_0) = u(P_0) = \lim_{n\to\infty} M_{B_{P_0R}}\{u_n\} = M_{B_{P_0R}}\{u\},$$

we see that $u(P)$ has to satisfy the mean value property and then it is harmonic by dint of Proposition 10.

As a matter of fact, all that means that if we take a space of harmonic continuous functions in $\overline{B}$ with norm $\|u\|_{C(\overline{B})}$ and the corresponding space of continuous functions on S which are traces on S of the former, $f(P) \equiv u(P)|_S$, then the two spaces are in a one-to-one correspondence one to the other, and in addition this correspondence is isometric, i.e.

$$\|f\|_{C(S)} = \|u\|_{C(\overline{B})}.$$

That such correspondence is onto, i.e. that

$$\{f;\ f = u|_S\,, \Delta u = 0 \text{ in } B,\ u \in C(\overline{B})\} \equiv C(S),$$

when S is smooth, as we have assumed, is basically a theorem that is classical in potential theory (cf. Kellog 1953; Miranda 1970).

We state it here, without proof, in a form that will be used in the sequel.

Theorem 5. *When S is a closed smooth surface (for instance with a normal field $\mathbf{n}_P$ such that $|\mathbf{n}_P - \mathbf{n}_Q| \le c \cdot \overline{PQ}$, i.e. it is Lipschitz continuous), the traces of harmonic functions $u(P) \in C(\overline{B})$ cover the whole $C(S)$, so that the problem of Dirichlet*

$$\begin{cases} \Delta u = 0 \textit{ in } B \\ u|_S = f \textit{ given on } S \end{cases}$$

has one and only one solution, for every $f \in C(S)$.

In addition if $f(P)$ has λ-Hölder continuous derivatives along the boundary, i.e., with ∇_t denoting the tangential component of the gradient,

$$|\nabla_t f(P) - \nabla_t f(Q)| \le c \cdot (\overline{PQ})^\lambda,\ \forall PQ \in S$$

then also $u(P)$ has λ-Hölder continuous derivatives in $\overline{B}$, i.e.

$$|\nabla u(P)| \le c', \quad |\nabla u(P) - \nabla u(Q)| \le c''(\overline{PQ})^\lambda,\ \forall P, Q \in \overline{B},$$

for suitable constants c' and c''. In particular the normal derivative of $u(P)$ on S is continuous and therefore bounded.

Definition 4. Following (Krarup 2006), we define the radius of convergence of a series of spherical harmonics, $\sum_{n,m} u_{nm} r^n Y_{nm}(\vartheta, \lambda)$, as

$$R_c = \sup\left\{r;\ \Sigma u_{nm}^2 r^{2n} < +\infty\right\}. \tag{13.136}$$

We note that indeed if we take any R, such that,

$$R < R_c, \tag{13.137}$$

then the series results to be convergent inside the ball with boundary S_R, i.e. for $r < R$.

In fact

$$\left|\sum_{n,m} u_{nm} r^n Y_{nm}(\vartheta, \lambda)\right|^2 \le \left(\sum_{n,m} u_{nm}^2 R^{2n}\right)\left(\sum_n \left(\frac{r}{R}\right)^{2n} (2n+1)\right)$$

and the first series is convergent in force of definition (13.136) and condition (13.137), while the second one is convergent $\forall r < R$.

Proposition 11. *Let $u(P)$ be harmonic in B, take any P_0 and let δ_{P_0} be its distance from the boundary S; then $u(P)$ can be developed into a series of spherical harmonics with convergence radius*

$$R_c \geq \delta_{P_0}. \tag{13.138}$$

Proof. Indeed let $R < \delta_{P_0}$; then we can write in $B_{P_0 R}$

$$\begin{aligned} u(P) &= \frac{1}{4\pi R^2} \int_{S_{P_0 R}} \Pi_{Ri}(P, P')u(P')dS_{P'} \\ &= \sum_{n,m} u_{nm}(P_0, R) \left(\frac{r_P}{R}\right)^n Y_{nm}(\vartheta_P, \lambda_P) \end{aligned}$$

with

$$\sum_{n,m} u_{nm}(P_0, R)^2 = \frac{1}{4\pi R^2} \int_{S_{P_0 R}} u^2(P')dS$$

which is finite because on $S(P_0, R), u(P)$ is continuous.

Since this is true $\forall R < \delta_{P_0}$ we have at least $R_c = \delta_{P_0}$; i.e. (13.138) is proved.

□

Note that it can very well happen that $R_c > \delta_{P_0}$ and this means that $u(P)$ can be extended as a harmonic function to a region larger then B.

Theorem 6 (Principle of identity of harmonic functions). *Remember that by hypothesis B is a bounded simply (arcwise) connected set with "smooth" boundary. Let $u(P), v(P)$ be two functions harmonic in B and P_0 a point in B; if, for some $R_0 > 0$,*

$$u(P) \equiv v(P), \ P \in B_{P_0 R_0}, \tag{13.139}$$

i.e. the two functions coincide in a neighborhood of P_0, then

$$u(P) \equiv v(P), \quad \forall P \in B. \tag{13.140}$$

Proof. Note that taking $u - v$ instead of u and 0 instead of v, we have to prove that

$$\{\exists R_0; u(P) = 0, \ \forall P \in B_{P_0 R_0}\} \Rightarrow \{u(P) \equiv 0, \ \forall P \in B\}. \tag{13.141}$$

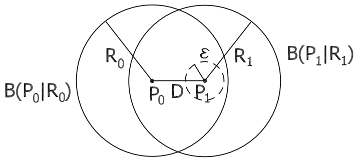


Fig. 13.3 Extension of $u(P)$ from $B_{P_0R_0}$ to $B_{P_1R_1}$

First we show that if $u(P) \equiv 0$ in $B_{P_0R_0}$ and if we take any P_1 with $\overline{P_0P_1} = D < R_0$ and $u(P)$ is harmonic in $B_{P_1R_1}$ with $R_1 > R_0 - D$ (see Fig. 13.3), then

$$u(P) \equiv 0, \quad P \in B_{P_1R_1}. \tag{13.142}$$

In fact under the above hypothesis we can put in $B_{P_1R_1}$, thanks to Proposition 11,

$$u(P) = \sum_{n,m} u_{n,m}(P_1, R_1) \left(\frac{r}{R_1}\right)^n Y_{nm}(\vartheta, \lambda); \tag{13.143}$$

On the other hand, since $D < R_0$, we have for a sufficiently small $\varepsilon > 0$, (cf. Fig. 13.3)

$$B_{P_1\varepsilon} \subset B_{P_0R_0}.$$

But then we can write

$$\forall n, m,\ u_{nm}(P_1, R_1)\left(\frac{\varepsilon}{R_1}\right)^n = \frac{1}{4\pi\varepsilon^2}\int_{S_\varepsilon} Y_{nm}(P')u(P')dS_{P'} = 0,$$

because $u(P')$ is identically zero in $B_{P_0R_0}$.

This implies $u_{nm}(P_1, R_1) = 0,\ \forall(n, m)$, and then (13.141).

Note that if we take any two spheres B_0, B_1 partially overlapping, one can always find a third sphere B' which is in the same position as discussed above, with respect to each of them (see Fig. 13.4). This implies that

$$u(P) \equiv 0 \text{ in } B_0 \Rightarrow u(P) \equiv 0 \text{ in } B' \Rightarrow u(P) \equiv 0 \text{ in } B_1.$$

Now take any $\overline{P} \in B$; according to our hypothesis we have an arc of finite length $L_{P_0\overline{P}}$, joining P_0 to $\overline{P}$ such that

$$L_{P_0\overline{P}} \in B;$$

let then

$$\begin{aligned}\delta &= \text{dist}\,(L_{P_0\overline{P}}, S)\\ &= \inf_{\substack{P\in L_{P_0\overline{P}}\\ Q\in S}} |\mathbf{r}_P - \mathbf{r}_Q| > 0.\end{aligned}$$

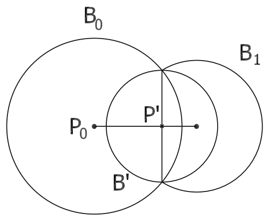


Fig. 13.4 Propagation of $u(P) \equiv 0$ from B_0 to B' to B_1. We may conclude then that if $u(P) \equiv 0$ in B_0, then the same happens in B' and also in B_1

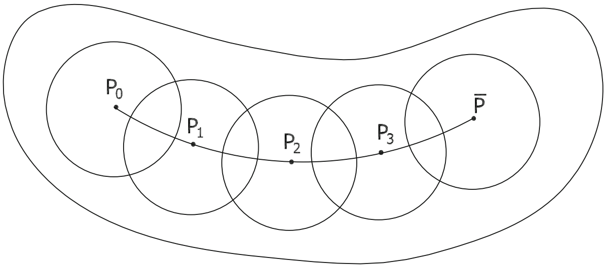


Fig. 13.5 Note that $B_{P_k R} \subset B$ by the choice of R

It is now obvious that we can join P_0 to $\overline{P}$ with a finite number of spheres $B_{P_k \delta}$ with

$$P_k \in L_{P_0 \overline{P}}, \; B_{P_k \delta} \subset B$$

and each $B_{P_k \delta}$ partially overlapping with $B_{P_{k-1} \delta}$ and $B_{P_{k+1} \delta}$ (cf. Fig. 13.5). By using the above argument then we see that $u(P)$ is necessarily zero in each $B_{P_k \delta}$ and then also in $\overline{P}$. □

We are now ready to prove one of the fundamental results of this chapter.

Theorem 7. *Let us consider the traces of solid spherical harmonics $S_{nm}(P) = S_{nm}(r, \vartheta, \lambda) = r^n Y_{nm}(\vartheta, \lambda)$ on the boundary S, $S_{nm}(Q)|_S$. This system of functions is complete in $L^2(S)$, i.e. if $f \in L^2(S)$ and*

$$\int_S f(Q) r_Q^n Y_{nm}(\vartheta_Q, \lambda_Q) dS_Q = 0, \; \forall n, m \qquad (13.144)$$

then $f = 0$ a.e. on S.

Proof. We go for a direct proof, showing that $(S_{nm}|_S)$ is total in $L^2(S)$ and then recall Proposition 9.

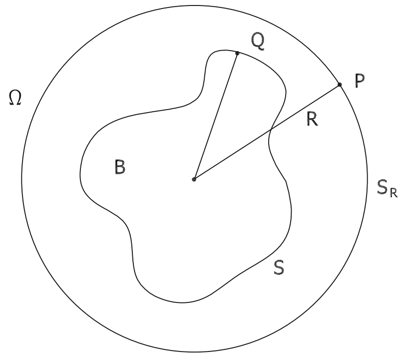


Fig. 13.6 Note that $Q \in S, r_Q < r_P = R$

Consider the single layer potential

$$V(P) = \int_S \frac{f(Q)}{\ell_{PQ}} dS_Q \tag{13.145}$$

and note that $V(P)$ is continuous everywhere in $R^3 \backslash S$ and harmonic in both B and Ω (see Fig. 13.6). Now take R so that $S_R \subset \Omega$ and take $Q \in S, P \in S_R$; we have then

$$\begin{aligned} \frac{1}{\ell_{PQ}} &= \sum_n \frac{r_Q^n}{R^{n+1}} P_n(\cos \psi_{PQ}) \\ &= \sum_{n,m} \frac{r_Q^n}{R^{n+1}} \frac{Y_{nm}(\vartheta_P, \lambda_P) Y_{nm}(\vartheta_Q, \lambda_Q)}{2n+1} \end{aligned}$$

and the series converges uniformly on S_R (Fig. 13.6). But then

$$P \in S_R, \; V(P) = \sum_{n,m} \frac{Y_{nm}(\vartheta_P, \lambda_P)}{(2n+1)R^{n+1}} \int_S f(Q) r_Q^n Y_{nm}(\vartheta_Q, \lambda_Q) dS \equiv 0.$$

The same reasoning indeed holds for any sphere outside S_R; in other words $V(P)$ is identically zero outside S_R, but then, by dint of Theorem 6, $V(P) \equiv 0$ in Ω. Since $V(P)$ has limits almost everywhere on S along the normal (see Miranda 1970, Chap. II Sect. 14), it has to be too

$$V(P) \equiv 0, \; P \in S. \tag{13.146}$$

On the other hand $V(P)$ is continuous and harmonic in B too and therefore (13.146), by the uniqueness of the solution of the Dirichlet problem (see Remark 6), implies that $V(P)$ is identically zero in B. Now it is enough to use the jump relations (see (13.110))

$$f(Q) = -\frac{1}{4\pi} \left\{ \left(\frac{\partial V}{\partial \nu} \right)_+ - \left(\frac{\partial V}{\partial \nu} \right)_- \right\}$$

to see that one has

$$f(Q) \equiv 0,\ Q \in S$$

as it was to be proved. □

Remark 7. It has to be clear that the systems $\{r^n Y_{nm}|_S\}\left\{\frac{1}{r^{n+1}} Y_{nm}|_S\right\}$ are complete in $L^2(S)$ but not orthogonal in this space, unless S is itself a sphere. This means that if one wants to approximate any $f(Q) \in L^2(S)$ by means of a finite combination of $S_{nm}(P)$ (internal or external), then one cannot use a simple projection argument by using orthogonality relations.

The orthogonal projection of $f(P)$ on

$$\text{Span}\{S_{nm}, n \le N\} \equiv \left\{\sum_{n=0}^{N} \sum_{m=-n}^{n} \lambda_{nm} r_P^n Y_{nm}(\vartheta_P, \lambda_P)\right\}$$

has to be found by solving a Galerkin system (cf. Sect. 15.5), namely

$$\sum_{n=0}^{N} \sum_{m=-n}^{n} \lambda_{nm} \left\{\frac{1}{4\pi} \int_S r^{n+j} Y_{nm}(\vartheta_P, \lambda_P) Y_{jk}(\vartheta_P, \lambda_P) dS_P\right\}$$
$$= \frac{1}{4\pi} \int_S f(P) r_P^j Y_{jk}(\vartheta_P, \lambda_P) dS_P, \tag{13.147}$$

providing the function of $\text{Span}\{S_{nm}, n \le N\}$ which is closest to $f(Q)$ in the $L^2(S)$ norm. The system (13.147) can indeed become very large, having as many as $(N+1)^2$ unknowns and its "normal" matrix is fully populated when S has not any particular symmetry. So its numerical solution can be sought more easily by iterative methods rather than by exact methods, like Cholesky.

It is worth noting the strict analogy of (13.147) with the standard least squares normal system, so widely used in Geodesy.

13.5 Green's Function and Krarup's Theorem

The reason why we are so interested in establishing the completeness of $\{S_{nm}\}$ in $L^2(S)$ is that we hope that while simple potentials like

$$u_N(P) = \sum_{n=0}^{N} \sum_{m=-n}^{n} u_{nm} S_{nm}(P)$$

do approach a given $L^2(S)$ function $f(P)$ on the boundary, on the same time inside B they do converge to some harmonic function $u(P)$ which then could be considered as solution of the Dirichlet problem with $f(P)$ as given boundary value, at least in some suitable sense.

In order to get a result of this kind we need to introduce the classical concept of Green's function and its use in potential theory.

Proposition 12 (Green's function). *Given B with a smooth boundary S, as above specified, there is a function $G(P,Q)$ (called Green's function of B) of two points $P, Q \in B$, such that, for fixed $P \in B$,*

$$\Delta_Q G(P,Q) = -4\pi\delta(P,Q) \tag{13.148}$$

$$G(P,Q)|_{Q\in S} = 0. \tag{13.149}$$

The Green function $G(P,Q)$ of B enjoys the following properties:

(a) Put

$$v(P) = \frac{1}{4\pi}\int_B G(P,Q)g(Q)dB_Q, \tag{13.150}$$

with g a measurable bounded function in B, then, at least in distribution sense,

$$\begin{cases} \Delta v(P) = -g(P) \text{ in } B \\ v(P)|_S = 0; \end{cases} \tag{13.151}$$

moreover $v(P)$ results to be continuous with its first derivatives in $\overline{B}$,

(b) Put

$$u(P) = -\frac{1}{4\pi}\int_S G_{n_Q}(P,Q)f(Q)dS_Q, \tag{13.152}$$

where $G_{n_Q}(P,Q)$ is the normal derivative of $G(P,Q)$ at $Q \in S$, $f \in C(S)$, then

$$\begin{cases} \Delta u(P) = 0 \qquad \text{in } B \\ u(P)|_S = f(P), \end{cases} \tag{13.153}$$

(c)

$$G(P,Q) \geq 0, \;\; P, Q \in B, \tag{13.154}$$

$$-G_{n_Q}(P,Q) \geq 0, \;\; Q \in S \tag{13.155}$$

(d) $G(P,Q)$ is symmetric, i.e.

$$G(Q,P) = G(P,Q)$$

Proof. Since

$$\Delta_Q \frac{1}{\ell_{PQ}} = -4\pi\delta(P,Q),$$

it is clear that if we define $h(P,Q)$ for every P fixed in B, such that

$$h(P,Q) : \begin{cases} \Delta_Q h(P,Q) = 0 \\ h(P,Q)|_{Q\in S} = \frac{1}{\ell_{PQ}} \end{cases} \tag{13.156}$$

and we put

$$G(P,Q) = \frac{1}{\ell_{PQ}} - h(P,Q), \tag{13.157}$$

we satisfy (13.148) and (13.149).

That $h(P,Q)$ exists $\forall P \in B$ (remember that B is open), is a consequence of Theorem 5. From the same theorem we derive that $h(P,Q)$, as well as $G(P,Q)$, has λ-Hölder continuous derivatives in $\overline{B}$; in particular it will be

$$Q \in S, \quad -G_{n_Q}(P,Q) = \left|G_{n_Q}(P,Q)\right| \le C, \tag{13.158}$$

as far as P is fixed in B, which implies that also $\frac{1}{\ell_{PQ}}$ is continuous and with Lipschitz continuous derivatives on the boundary S.

Property (a) is proved by using the definition of Laplacian in distribution sense, namely by recalling that (remember that $\mathcal{D}(B)$ is the linear space of functions that are C^∞ in B and that are identically equal to zero outside a closed, bounded set $K \subset B$)

$$\forall\varphi \in \mathcal{D}(B), \int \varphi(P)\Delta v(P)dB_P = \int \Delta\varphi(P)v(P)dB_P$$

and by interchanging the integration order when we use a $v(P)$ as in (13.150). Then (13.148), together with the symmetry of $G(P,Q)$, (point d)), means exactly that

$$\int \Delta\varphi(P)\left(-\frac{1}{4\pi}G(P,Q)\right)dB_Q = \varphi(Q),$$

and we find, for every smooth $\varphi(P)$

$$\int \varphi(P)\Delta v(P)dB_P = -\int \varphi(Q)g(Q)dB_Q,$$

i.e. (13.151). That $v(P)|_S = 0$, comes from symmetry of $G(P, Q)$ and from (13.149).

We don't prove here that $v(P)$ is continuous with its first derivatives in $\overline{B}$.

To prove (b), start with the function u that is solution of (13.153) and assume it is continuous with its first derivatives in $\overline{B}$, i.e. $u \in C^1(\overline{B})$, so that we can write (cf. Part I, (1.61))

$$P \in B, \quad u(P) = \frac{1}{4\pi}\int_S \left\{u_n(Q)\frac{1}{\ell_{PQ}} - u(Q)\partial_{n_Q}\frac{1}{\ell_{PQ}}\right\} dS_Q. \quad (13.159)$$

On the other hand, when $Q \in S$, $\frac{1}{\ell_{PQ}} \equiv h(P, Q)$, so that using the identity

$$\int_S u_{n_Q}(Q)h(P, Q)dS_Q = \int_S u(Q)h_{n_Q}(P, Q)dS_Q,$$

we receive from (13.159)

$$\begin{aligned} u(P) &= \frac{1}{4\pi}\int_S u(Q)\left\{h_{n_Q}(P, Q) - \partial_{n_Q}\frac{1}{\ell_{PQ}}\right\} dS_Q \\ &= -\frac{1}{4\pi}\int_S G_{n_Q}(P, Q)u(Q)dS_Q; \end{aligned} \quad (13.160)$$

since $u(P)$ is continuous in $\overline{B}$ and $u(Q)|_S \equiv f(Q)$, and (13.152) and (13.153) are proved.

The restrictive condition $u(P) \in C^1(\overline{B})$ is eliminated by taking a sequence $f_n(P) \in C^1(S)$ such that $f_n(P) \to f(P)$ uniformly on S; then by the maximum principle $u_n(P) \to u(P)$ inside B, and (13.159) holds for $u(P)$ and $f(P)$ because $G_{n_Q}(P, Q)$ is a bounded function when $P \in B$.

Note that, since $u(P)$ is continuous up to the boundary, (13.160) tells us that (cf. Fig. 13.8)

$$P_0 \in S, \quad u(P_0) = \lim_{\delta\to 0} u(P_0 - \delta\mathbf{n}) = \lim_{\delta\to 0} -\frac{1}{4\pi}\int_S G_n(P_0 - \delta\mathbf{n}, Q)u(Q)dS_Q$$

meaning exactly that $\{-\frac{1}{4\pi}G_\nu(P_0 - \delta\mathbf{n}, Q)dS_Q\}$ tends to a measure of mass one concentrated in P_0 as a measure on S, when $\delta \to 0$.

Point c) is a consequence of the maximum principle; in fact fix P in B and a small sphere $B_\varepsilon(P)$ around P all contained in B (cf. Fig. 13.7),

then

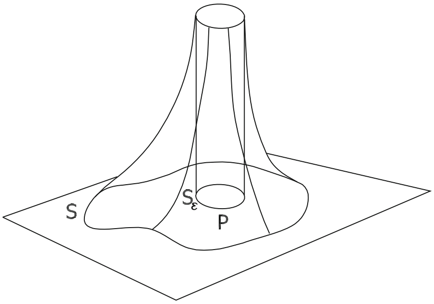


Fig. 13.7 An image of $G(P, Q)$

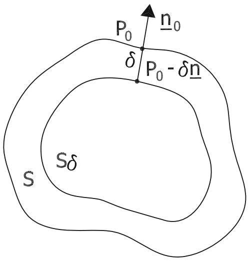


Fig. 13.8 The surface S and its internal translate at distance δ. This exists for sufficiently small δ

$$\varepsilon < \min_{Q\in S} \ell_{PQ}.$$

On the other hand $h(P, Q)$ for P fixed in B and Q variable, has extremes on S, so that

$$0 =< \min_{Q\in S} \frac{1}{\ell_{PQ}} \leq h(P, Q) \leq \max_{Q\in S} \frac{1}{\ell_{PQ}} < \frac{1}{\varepsilon}.$$

But then on S_ε

$$Q \in S_\varepsilon,\ G(P, Q) = \frac{1}{\varepsilon} - h(P, Q) > 0.$$

Since $G(P, Q)$ is harmonic between S_ε and S, where $G(P, Q)$ is zero, and ε is arbitrary, we find that (13.154) has to hold.

Moreover, if you put

$$V(\delta) = G(P, Q_0 - \delta\mathbf{n}), \quad Q_0 \in S,$$

you see that

$$-V(\delta) \leq 0,\ V(0) = 0$$

so that you find

$$\frac{\partial}{\partial \delta} V(\delta)\Big|_{\delta=0} = -G_{n_Q}(P, Q_0) \geq 0,$$

i.e. (13.155).

Finally, to prove point d) one needs to prove only that $h(P, Q)$ is symmetric. Write the identity (cf. (13.152), (13.157) and the second of (13.156))

$$\begin{aligned} h(P,Q) &= \frac{1}{4\pi} \int_S h(P,Q') \left\{ h_{n_{Q'}}(Q,Q') - \partial_{n_{Q'}} \frac{1}{\ell_{Q,Q'}} \right\} dS_{Q'} \\ &= \frac{1}{4\pi} \int_S h(P,Q') h_{n_{Q'}}(Q,Q') dS_{Q'} - \frac{1}{4\pi} \int_S \frac{1}{\ell_{PQ'}} \partial_{n_{Q'}} \frac{1}{\ell_{QQ'}} dS_{Q'}. \end{aligned} \tag{13.161}$$

The first integral is symmetric because one can move $\frac{\partial}{\partial n_{Q'}}$ from $h(Q, Q')$ and apply it to $h(P, Q')$, as a consequence of the second Green identity applied to two harmonic functions (cf. Part I, (1.57)). As for the second term note that, for $P \neq Q$,

$$\begin{aligned} &\frac{1}{4\pi} \int_S \left\{ \frac{1}{\ell_{PQ'}} \partial_{n_{Q'}} \frac{1}{\ell_{QQ'}} - \frac{1}{\ell_{QQ'}} \partial_{n_{Q'}} \frac{1}{\ell_{PQ'}} \right\} dS_{Q'} \\ &= \frac{1}{4\pi} \int_B \left\{ \frac{1}{\ell_{PQ'}} \Delta \frac{1}{\ell_{QQ'}} - \frac{1}{\ell_{QQ'}} \Delta \frac{1}{\ell_{PQ'}} \right\} dB_{Q'} \\ &= -\int_B \left\{ \frac{1}{\ell_{PQ'}} \delta(Q,Q') - \frac{1}{\ell_{QQ'}} \delta(P,Q') \right\} dB_{Q'} = -\left(\frac{1}{\ell_{PQ}} - \frac{1}{\ell_{PQ}} \right) = 0; \end{aligned}$$

therefore also the second integral in (13.161) is symmetric, as it was to be proved. □

We are now ready to prove a theorem which extends (13.152) and (13.153) to any $f \in L^2(S)$.

Theorem 8. *Given any $f \in L^2(S)$ we find a unique solution of the Dirichlet problem, i.e. we can extend the Green operator (13.152) to $L^2(S)$ by continuity, in the sense that we find a sequence of functions harmonic in B, $\{u^{(N)}\}$ such that $u^{(N)}\big|_S = f^{(N)}$, with $f^{(N)} \in C(S)$ and $u^N(P) \to u(P)$, uniformly in any closed subset of B, and simultaneously $f^N \to f$ in $L^2(S)$; $u(P)$ is related to f by (13.152), for any P in B open, and therefore it is harmonic in B.*

Proof. Assume we have proven a majorization of the type

$$\int_B u^2(P) dB_P \leq C \int_S u^2(Q) dS_Q, \tag{13.162}$$

at least $\forall u \in C(\overline{B})$.

Then we can take any $f^{(N)} \in C(S)$ and such that $\| f^{(N)} - f \|_{L^2(S)} = \int_S (f^{(N)} - f)^2 dS$ tends to zero and define the corresponding $u^{(N)}$ through (13.152); obviously $u^{(N)}$ satisfties (13.153). On the same time, due to (13.162), $u^{(N)}$ has an $L^2(B)$ limit in B, i.e. there is $u(P) \in L^2(B)$ such that

$$\|u - u^{(N)}\|_{L^2(B)} \to 0. \tag{13.163}$$

Now, take any closed set $K \subset B$; then there is a $\delta > 0$ such that

$$\underset{\substack{P\in K\\ Q\in S}}{\mathrm{Dist}}(P, Q) \geq \delta > 0. \tag{13.164}$$

Since $u^{(N)}(P)$ are harmonic they satisfy the mean value property, i.e. $\forall P \in K$, taken the sphere $B_\delta(P)$ one has $B_\delta(P) < B$ and

$$u^{(N)}(P) = \frac{1}{4/3\pi\delta^3} \int_{B_\delta} u^{(N)}(Q) dB_Q;$$

so, $\forall P \in K$,

$$\begin{aligned}
&\left|u^{(N+k)}(P) - u^{(N)}(P)\right| \\
&\quad \leq \frac{1}{4/3\pi\delta^3} \int_{B_\delta(P)} \left|u^{(N+k)}(Q) - u^{(N)}(Q)\right| dB_Q \\
&\quad \leq \sqrt{\frac{1}{4/3\pi\delta^3}} \|u^{(N+k)} - u^{(N)}\|_{L^2(B_\delta(P))} \\
&\quad \leq \sqrt{\frac{1}{4/3\pi\delta^3}} \|u^{(N+k)} - u^{(N)}\|_{L^2(B)}.
\end{aligned} \tag{13.165}$$

Since (13.165) holds uniformly in k and δ is fixed we have that $\{u^{(N)}(P)\}$ converges uniformly in K and $u(P)$ is then continuous in every $K \subset B$, i.e. in the whole B.

Furthermore

$$u(P) = \lim_{N\to\infty} -\int_S G_n(P, Q) f^{(N)}(Q) dS_Q = -\int_S G_n(P, Q) f(Q) dS_Q \tag{13.166}$$

the limit being justified by the fact that the distance of P from S is positive and then $G_n(P, Q)$ is continuous and bounded for $Q \in S$. The relation (13.166) proves that $u(P)$ is harmonic in B. The same conclusion can be derived from the fact that $u(P)$ has to satisfy the mean value property too and then, on account of Proposition 10, $u(P)$ has to be harmonic.

The inequality (13.162) is proved as follows: first assume $u(P)$ to be continuous with its gradient in $\overline{B}$ and apply the Green identity to $u^2(Q)$ and $G(P, Q)$, for any fixed P in B. Recalling that $G(P, Q) = 0$, when $Q \in S$, and that $\Delta u^2 = 2|\nabla u|^2$

$$\begin{aligned}
&\int_B 2|\nabla u|^2 G(P, Q)dB_Q + 4\pi u^2(P) \\
&= \int_B \left\{\left[\Delta u^2(Q)\right] G(P, Q) - u^2(Q)\Delta G(P, Q)\right\} dB_Q \\
&= \int_S \left\{\left[\partial_n u^2(Q)\right] G(P, Q) - u^2(Q)G_{n_Q}(P, Q)\right\} dS_Q \qquad (13.167) \\
&= -\int u^2(Q)G_{n_Q}(P, Q)dS_Q.
\end{aligned}$$

Since $G(P, Q) > 0$, when $P, Q \in B$, (13.167) implies

$$u^2(P) \le -\frac{1}{4\pi}\int_S G_{n_Q}(P, Q)u^2(Q)dS_Q \ . \qquad (13.168)$$

Integrating over B one gets

$$\int_B u^2(P)dB_P \le \int_S \left[-\frac{1}{4\pi}\int_B G_{n_Q}(P, Q)dB_P\right]u^2(Q)dS_Q. \qquad (13.169)$$

On the other hand

$$\begin{aligned}
V(Q) &= -\frac{1}{4\pi}\int_B G(P, Q)dB_P \\
&= -\frac{1}{4\pi}\int_B G(Q, P)dB_P
\end{aligned}$$

is a function of the type (13.150), which is then continuous up to the boundary with its first derivatives. But then

$$\left|\ \partial_{n_Q} V(Q)\right|\Big|_S \le C,$$

which inserted into (13.169) gives (13.162). □

Corollary 3. *The harmonic function*

$$u(P) = -\frac{1}{4\pi}\int_S G_{n_Q}(P, Q)f(Q)dS_Q \qquad (13.170)$$

with $f(Q)$ in $L^2(S)$ admits in fact $f(Q)$ as trace on the boundary S in the sense that, taken a sufficiently small δ as in Fig. 13.8, one has

$$\lim_{\delta\to 0}\int_S [u(P_0-\delta\mathbf{n})-f(P_0)]^2 dS_{P_0}=0. \tag{13.171}$$

Proof. For the proof of (13.171) see Cimmino (1952, 1955) and Sansò and Venuti (1998). Here we make only a small reasoning which satisfies our intuition that $f(P)$ has to be given by the values attained by $u(P)$ on the boundary S. In fact consider that the harmonic polynomials $\{r^n P_{nm}(\vartheta,\lambda)|_S\}$ do form a total system in $L^2(S)$; therefore they can always be orthonormalized in $L^2(S)$ (see Remark 7) providing so a basis of polynomials, that we shall call $h_N(P)$, such that $\{h_N(P)|_S\}$ is a CON system in $L^2(S)$. We shall have

$$h_N(P)=\sum_{n=0}^{N}\sum_{m=-n}^{n} a_{nm}r^n Y_{nm}(\vartheta,\lambda),$$

so that $h_N(P)$ are harmonic and certainly continuous in $\overline{B}$. So if we put

$$f^{(N)}(P)=\sum_{k=0}^{N} f_k h_k(P)$$

we get a sequence such that, for suitable fixed coefficients $\{f_k\}$,

$$\lim_{N\to\infty}\|f(P)-f^{(N)}(P)\|^2_{L^2(S)}=0,$$

i.e.

$$f(P)=\sum_{k=0}^{+\infty} f_k h_k(P),\quad P\in S. \tag{13.172}$$

On the other hand the functions $f^{(N)}(P)$ are well-defined and harmonic throughout all B so that we can take

$$u^{(N)}(P)=f^{(N)}(P),\ P\in B;$$

furthermore the sums $u^{(N)}(P)$ do converge uniformly to $u(P)$ in every closed set $K\subset B$

$$u(P)=\sum_{k=0}^{+\infty} f_k h_k(P). \tag{13.173}$$

Now if we simply take $P \in S$ in (13.173), we find that

$$u(P) = f(P), \ P \in S,$$

such equality meaning that the series (13.173) is $L^2(S)$ convergent to $f(P)$.

This rather heuristic proof can be made more rigorous, but cannot substitute (13.171), which has to be proved by a further specific analysis. □

Proposition 13. *The set of functions $\{u(P)\}$ which are harmonic in B and such that $\| u(P)|_S \|_{L^2(S)}$ is finite, is a Hilbert space with scalar product*

$$< u, v >_{L^2(S)} = \int_S u(P)v(P)dS_P; \tag{13.174}$$

both, scalar products and norms, have to be understood as limits of similar expressions from inside; for instance (13.174) means

$$< u, v >_{L^2(S)} = \lim_{\delta \to 0} \int u(P_0 - \delta \mathbf{n})v(P_0 - \delta \mathbf{n})dS_{P_0}; \tag{13.175}$$

we call this Hilbert space $HL^2(S)$.

Proof. That limits like (13.175) do exists is in fact consequence of the Corollary of Theorem 8.

That $HL^2(S)$ is a Hilbert space descends from the fact that the correspondence

$$u \in HL^2(S) \Leftrightarrow f = u\big|_S \in L^2(S)$$

is one-to-one thanks to Theorem 8, and isometric in the sense that

$$\|u\|_{HL^2(S)} = \|f\|_{L^2(S)}.$$

So any convergent sequence in one space corresponds to a convergent sequence in the other; moreover $L^2(S)$ *is* a Hilbert space, i.e. it is complete and so the same is true for $HL^2(S)$. □

We are able now to prove a very important theorem which is known in geodetic literature with the name of Runge-Krarup theorem, sometimes also associated to the name of Keldysh-Laurentiev (cf. Krarup 2006; Moritz 1980).

As a matter of fact this piece of theory, specially in the formulation of Krarup, is very general, however we will provide here a version which is adapted to that part of potential theory that is explored in these notes and, in particular, to the case of potentials which are in $HL^2(S)$.

Theorem 9. *Let B be an open domain as specified at the beginning of Sect. 2.2 and B_0 another open domain, satisfying similar hypotheses and such that*

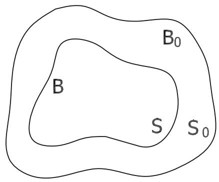


Fig. 13.9 The two nested domains B and B_0. Note that, due to condition (13.175), S and S_0 can never touch each other

$$B_0 \supset \overline{B} \tag{13.176}$$

(see Fig. 13.9).

Denote with S_0, S the boundaries of B_0, B and with $HL^2(S_0), HL^2(S)$ two Hilbert spaces of functions harmonic in B_0 and B respectively. Define the restriction operator $\mathcal{R}_B$ so that to any $u_0(P) \in HL^2(S_0)$ we associate the same function but restricted to the domain B; it is clear that such a function will be harmonic in B and even more it will be in $HL^2(S)$ because S is completely included in B_0 and $u_0(P)$ is then continuous on S; formally

$$\mathcal{R}_B : HL^2(S_0) \to HL^2(S);\ \mathcal{R}_B(u_0) = u_0(P)\big|_B ; \tag{13.177}$$

then the set

$$\mathcal{R}_B[HL^2(S_o)] \equiv \{u \in HL^2(S); u = \mathcal{R}_B u_0, u_0 \in HL^2(S_0)\}$$

is dense in $HL^2(S)$. This means that

$$\forall u \in HL^2(S),\ \exists\{u_N\} \in HL^2(S_0)$$
$$\Rightarrow \|u - u_N\|_{HL^2(S)} = \left\{\int_S [u(P) - u_N(P)]^2 dS_P\right\}^{1/2} \to 0.$$

Proof. The proof is straightforward. We just note that $\{S_{nm}(r, \vartheta, \lambda)\} \in HL^2(B_0)$ and on the other hand this sequence is total in $HL^2(S)$.

So we have simultaneously

$$\text{Span}\{S_{nm}(r, \vartheta, \lambda)\} \subset HL^2(S_0);\ \mathcal{R}_B \text{Span}\{S_{nm}(r, \vartheta, \lambda)\} \subset HL^2(S).$$

Then, by taking the closure of the second relation in $HL^2(S)$, one has

$$HL^2(S) = \overline{\mathcal{R}_B \text{Span}\{S_{nm}\}}; \tag{13.178}$$

at the same time, by the first relation

$$\mathcal{R}_B \text{Span}\{S_{nm}\} \subseteq \mathcal{R}_B HL^2(S_0) \subseteq HL^2(S)$$

which, closed in $HL^2(S)$, yields

$$HL^2(S) = \overline{\mathcal{R}_B \text{Span}\{S_{nm}\}} \subseteq \overline{\mathcal{R}_B HL^2(S_0)} \subseteq HL^2(S). \tag{13.179}$$

(13.178) and (13.179) together prove the theorem. □

Remark 8. Since B_0 in the previous theorem is arbitrary, one can use as B_0 a ball and indeed instead of $HL^2(B_0)$ one can use any Hilbert space of functions harmonic in B_0, such that all the $S_{nm}(r, \vartheta, \lambda)$ do belong to it.

For instance, take B_0 to be a ball of radius R, such that $B_0 \supset \overline{B}$, and take the Hilbert space HK with reproducing kernel (cf. Theorem 3)

$$\begin{aligned} K(P, Q) &= \sum_{n,m=0}^{+\infty} k_n \left(\frac{r_P}{R}\right)^n \left(\frac{r_Q}{R}\right)^n Y_{nm}(\vartheta_Q, \lambda_P) Y_{nm}(\vartheta_Q, \lambda_Q) \\ &= \sum_{n,m=0}^{+\infty} k_n S_{nm}(P) S_{nm}(Q) \qquad (k_n > 0,\ \forall n) \end{aligned} \tag{13.180}$$

That HK is a Hilbert space is easy to verify, that it contains all the solid spherical harmonics is a consequence of (13.180) and in particular of the condition $k_n > 0$; in fact recalling Theorem 3, formula (13.180) tells us that $\{\sqrt{k_n} S_{nm}(P)\}$ is a CON system in HK.

That the functions in HK are harmonic in B_0 is also clear from the shape of $K(P, Q)$ and the fact that by definition

$$f(P) =< K(P, Q), f(Q) >_{HK} . \tag{13.181}$$

Finally, in order that (13.180) be not a pure formal expression, one needs to impose some convergence conditions to the coefficients $\{k_n\}$. Observing that (13.180) can be written as

$$K(P, Q) = \sum_{n=0}^{+\infty} (2n + 1) k_n \left(\frac{r_P r_Q}{R^2}\right)^n P_n(\cos \psi_{PQ}) \tag{13.182}$$

and recalling that $|P_n(t)| \le 1$, one immediately sees that under the condition

$$\sum_{n=0}^{+\infty} k_n (2n + 1) < +\infty, \tag{13.183}$$

the series (13.180) is uniformly convergent up to the boundary, i.e. up to the sphere of radius R.

The Theorem 9 is so relevant to the understanding of physical geodesy, that we restate it, in the form of a Corollary, in its outer version, which holds automatically true by virtue of the inverse radii transformation (see Proposition 1).

Corollary 4. *Let B, S be as in Theorem 9 and let $\Omega = (\overline{B})^c$, be the space exterior to S; let now B_0, with boundary S_0, be such that*

$$\overline{B}_0 \subset B \tag{13.184}$$

and $\Omega_0 = (\overline{B}_0)^c$, so that

$$\Omega_0 \supset \overline{\Omega}; \tag{13.185}$$

let $H_e L^2(S)$ be the Hilbert space of functions harmonic in Ω, regular at infinity endowed with the norm

$$\|u\|^2_{H_e L^2(S)} = \int_S u^2(P) dS_P, \tag{13.186}$$

and let $H_e L^2(S_0)$ be the similar space for Ω_0.

Note that we have added an index e to signify that here we are dealing with functions harmonic in the external domains (Ω, Ω_0) as opposed to the case discussed in Theorem 9. Let us define $\mathcal{R}_\Omega$ as the operator of restriction to Ω, applied to functions in $H_e L^2(S_0)$; then we have

$$\overline{\mathcal{R}_\Omega[H_e L^2(S_0)]} = H_e L^2(S); \tag{13.187}$$

i.e. $\forall u \in H_e L^2(S)$ there is a sequence $\{u_N(P)\} \in H_e L^2(S_0)$, harmonic in Ω_0 such that

$$\lim_{N\to\infty} \int_S [u(P) - u_N(P)]^2 dS = 0 \tag{13.188}$$

and that consequently $u_N(P) \to u(P)$ pointwise in Ω and even uniformly in every closed bounded set contained in Ω.

Moreover if, in analogy with Remark 8, we consider the case that B_0 is a ball of radius R and S_0 a so-called Bjerhammar sphere*, and the Hilbert space of functions harmonic in Ω_0, HK_e, endowed with the reproducing kernel*

$$\begin{aligned} K_e(P, Q) &= \sum_{n,m} k_n \left(\frac{R}{r_P}\right)^{n+1} \left(\frac{R}{r_Q}\right)^{n+1} Y_{nm}(\vartheta_P, \lambda_P) Y_{nm}(\vartheta_Q, \lambda_Q) \\ &= \sum_{n=0}^{+\infty} (2n+1) k_n \left(\frac{R^2}{r_P r_Q}\right)^{n+1} P_n(\cos\psi_{PQ}) \qquad (k_n > 0,\ \forall n), \end{aligned} \tag{13.189}$$

we still have

$$\overline{R_{\Omega}[HK_e]} = H_e L^2(S) \tag{13.190}$$

and (13.188) holds with $\{u_N(P)\} \in HK_e$, i.e. harmonic in Ω_0, down to the Bjerhammar sphere S_0.

13.6 Exercises

Exercise 1. Prove Proposition 1 by showing, with the use of spherical coordinates and assuming $R = 1$, that

$$\begin{aligned}\Delta_s v(s, \vartheta, \lambda) &= \frac{\partial^2 v}{\partial s^2} + \frac{2}{s}\frac{\partial v}{\partial s} + \frac{1}{s^2}\Delta_\sigma v \\ &= \frac{1}{s^5}\left[u''\left(\frac{1}{s}, \vartheta, \lambda\right) + 2su'\left(\frac{1}{s}, \vartheta, \lambda\right) + s^2 \Delta_\sigma u\left(\frac{1}{s}, \vartheta, \lambda\right)\right] \\ &= \frac{1}{s^5}\Delta_r u(r, \vartheta, \lambda) = 0,\end{aligned}$$

where $u'(r, \vartheta, \lambda) = \frac{\partial}{\partial r} u(r, \vartheta, \lambda)$.

Exercise 2. Compute $h_m(x, y), h_{-m}(x, y)$ directly for $m = 2, m = 3$ and prove that they give the same result as those computed from (13.12), namely

$$\begin{aligned} &h_2 = x^2 - y^2,\ h_{-2} = xy \\ &h_3 = x^3 - 3xy^2,\ h_{-3} = y^3 - 3x^2 y. \end{aligned}$$

Exercise 3. Let h_{N-2k} be a harmonic polynomial in HH^3_{N-2k}; prove the formula

$$\begin{cases} \Delta^m r^{2k} h_{N-2k} = A_{mk} r^{2k-2m} h_{N-2k},\ m \leq k, \\ A_{mk} = 2k(2k-2)\ldots(2k-2m+2) \\ \cdot(2N-2k+1)(2N-2k-1)\ldots(2N-2k-2m+3) \end{cases} . \tag{13.191}$$

For this purpose first prove that

$$\Delta r^{2\ell} h_{N-2k} = 2\ell(2N + 2\ell - 4k + 1) r^{2\ell-2} h_{N-2k} \tag{13.192}$$

and then apply the Laplace operator m times to $r^{2k} h_{N-2k}$, using iteratively such a relation.

(**Hint**: to prove the second of the above relations, use $\Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f\Delta g$; note that $\Delta r^{2\ell} = 2\ell(2\ell+1)r^{2\ell-2}$, $\nabla r^{2\ell} = 2\ell r^{2\ell-2}\xi$ (with $\boldsymbol{\xi} = \frac{\mathbf{r}}{r}$) and for any function f homogeneous of degree α we have

$$\boldsymbol{\xi} \cdot \nabla f(\boldsymbol{\xi}) = \alpha f(\boldsymbol{\xi})\ ).$$

Exercise 4. Since we already know that formula (13.23) holds true, one can compute q_k just by imposing that it has to be $\Delta h_N = 0$. Prove that

$$q_1 = -\frac{1}{2(2N-1)},\ q_2 = \frac{1}{2\cdot 4(2N-1)(2N-3)}\dots \tag{13.193}$$

(**Hint**: prove, by using the same argument as in Exercise 3, that

$$\Delta(r^{2k}\Delta^k P_N) = 2k(2N-2k+1)r^{2k-2}\Delta^k P_N + r^{2k}\Delta^{k+1}P_N,$$

then impose $\Delta h_N = 0$, considering $r^{2k}\Delta^{k+1}P_N$ as independent variables).

Exercise 5. Prove that if

$$\Delta[r^n(t^n + a_1 t^{n-2} + a_2 t^{n-4} + \dots)] = 0$$

then a_1, a_2 are univocally determined.

(**Hint**: by using (13.36) prove that

$$\Delta\left[r^n t^{n-2\ell}\right] = r^{n-2}\left\{2\ell(2n-2\ell+1)t^{n-2\ell} + (n-2\ell)(n-2\ell-1)t^{n-2\ell-2}\right\}\Big).$$

Exercise 6. Prove that the coefficient c_n of t^n in $P_n(t)$ is

$$c_n = \frac{(2n!)}{2^n(n!)^2} \tag{13.194}$$

(**Hint**: recall that

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t)$$

and derive the recursive relation

$$(n+1)c_{n+1} = (2n+1)c_n.$$

Observe that from the expression of c_n for $n = 0,\ n = 1$, we correctly obtain $c_0 = c_1 = 1$ and then prove that c_n satisfies the above recursive relation).

Exercise 7. Compute the spherical harmonics $r^n Y_{nm}(\vartheta, \lambda)$ for all orders and degrees 2 and 3 and transform them back to polynomials in (x, y, z), verifying their harmonicity.

Exercise 8. Verify the summation rule (13.55) for degree 2 and 4. (Warning: note that for (13.55) to hold it is necessary to use fully-normalized spherical harmonics.)

Exercise 9. Prove that (13.74) is satisfied by the functions (13.70).
(**Hint**: first call L the Legendre operator

$$L\cdot = D_t(1-t^2)D_t\cdot$$

and remember that (cf. (13.37))

$$LP_n(t) = -n(n+1)P_n(t),$$

or

$$(1-t^2)D^2P_n = 2tDP_n - n(n+1)P_n \tag{13.195}$$

Recalling also that $P_{nm} = (1-t^2)^{m/2}\ P_n^{(m)}$, $P_n^{(m)} = D_n^m$, prove that

$$\begin{aligned} LP_{nm}(t) = L\left[(1-t^2)^{m/2}P_n^{(m)}\right] &= m[mt^2-(1-t^2)](1-t^2)^{\frac{m}{2}-1}P_n^{(m)} + \\ -2t(m+1)(1-t^2)^{m/2}P_n^{(m+1)} &+ (1-t^2)^{\frac{m}{2}+1}P_n^{(m+2)}; \end{aligned} \tag{13.196}$$

then from (13.195), by applying D^m to both members and recalling that

$$D^m(fg) = \sum_{k=0}^{m} \binom{m}{k} D^k(f)\cdot D^{m-k}(g),$$

derive

$$\begin{aligned} &D^m[(1-t^2)D^2P_n] \\ &= (1-t^2)P_n^{(m+2)} - 2mtP_n^{(m+1)} - m(m-1)P_n^{(m)} \\ &= 2tP_n^{(m+1)} + 2mP_n^{(m)} - n(n+1)P_n^{(m)}; \end{aligned}$$

rearranging the last equality you get

$$(1-t^2)P_n^{(m+2)} = 2t(m+1)P_n^{(m+1)} + [m(m+1)-n(n+1)]P_n^{(m)};$$

then substitute back in LP_{nm}).

Exercise 10. Derive the normalization constant

$$k_{n0} = \sqrt{2n+1}$$

from the reproducing relation (cf. Part I, (3.188)).

$$\frac{1}{4\pi}\int_{S_1} P_m(\cos\psi_{PQ})P_n(\cos\psi_{P'Q})d\sigma = (2n+1)^{-1}P_n(\cos\psi_{PP'})$$

(**Hint**: put $P = P'$ at the North Pole).

Exercise 11. Compute $\overline{P}_{nm}(t)$, up to degree and order 4, using (13.83) and (13.91).

Exercise 12. Repeat the reasoning of Example 2 for the exterior Dirichlet problem proving that, $\forall f(P) \in L^2(S_R)$,

$$\begin{cases} u(P) = \sum_{n,m} f_{nm}\left(\frac{R}{r_P}\right)^{n+1} Y_{nm}(\vartheta_P, \lambda_P) \\ f_{nm} = \frac{1}{4\pi}\int f(\vartheta', \lambda')Y_{nm}(\vartheta', \lambda')d\sigma' \end{cases}$$

and that, accordingly

$$r_P > R, \quad u(P) = \frac{1}{4\pi}\int \Pi_{Re}(P, P')f(P')d\sigma_{P'}$$

with $\prod_{Re}(P, P')$, the external Poisson kernel,

$$r > R, \quad \Pi_{Re}(P, P') = \frac{R(r^2 - R^2)}{[r^2 + R^2 - 2rR\cos\psi]^{3/2}}.$$

Exercise 13. Using a complementary argument to that of Theorem 7 and a small sphere inside B, prove that the sequence of outer spherical harmonics $\left\{\frac{1}{r^{n+1}}Y_{nm}(\vartheta, \lambda\right\}$ restricted to S forms again a complete system in $L^2(S)$.

Exercise 14. Prove that the Green function of the sphere with radius R is given by

$$G(P, Q) = \frac{1}{\sqrt{r_P^2 + r_Q^2 - 2r_P r_Q\cos\psi_{PQ}}} - \frac{1}{\sqrt{R^2 + \frac{r_P^2 r_Q^2}{R^2} - 2r_P r_Q\cos\psi_{PQ}}};$$

moreover verify that

$$-\frac{\partial}{\partial r_Q}G(P, Q)\bigg|_{r_Q=R} = -\frac{1}{R^2}\Pi_{Ri}(P, Q)$$

as it has to be in view of (13.123) and (13.152)

(**Hint**: that $G(P,Q)|_{r_Q=R}=0$ is obvious. You need only to prove that $\left(R^2+\frac{r_P^2 r_Q^2}{R^2}-2r_P r_Q\cos\psi_{PQ}\right)^{-1/2}=h(P,Q)$ is harmonic in $Q\in B$; this is clear if one observes that, with $\mathbf{r}_P^*=\frac{R^2}{r_P^2}\mathbf{r}_P$, implying $|r_P^*|>R$, one can write

$$h(P,Q)=\frac{r_P}{R}\left(r_P^{*2}+r_P^2-2r_Q^* r_Q\cos\psi\right)^{-1/2}=\frac{r_P}{R}|\mathbf{r}_P^*-\mathbf{r}_Q|^{-1})$$

Exercise 15. Prove that, when B is a ball of radius R, then the inequality (13.162) can be put in the rather expressive form (with $|B|=\frac{4}{3}\pi R^3,\ |S|=4\pi R^2$)

$$\frac{1}{|B|}\int_B u^2 dB\le\frac{1}{|S|}\int_S u^2 dS.$$

Similarly when we use a regular potential u which is harmonic in Ω, the space outside a sphere of radius R, one can write the inequality

$$\int_\Omega u^2\frac{1}{r^2}d\Omega\le R\int_\sigma u^2 d\sigma$$

(**Hint**: use just the two representations

$$u(P)=\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}u_{nm}\left(\frac{r}{R}\right)^n Y_{nm}(\vartheta,\lambda),\qquad r\le R$$

$$u(P)=\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}u_{nm}\left(\frac{R}{r}\right)^{n+1} Y_{nm}(\vartheta,\lambda),\qquad r\ge R$$

and compute directly $\int_S u^2 dS,\int_B u^2 dB$ in the first case and $\int_\Omega u^2\frac{1}{r^2}d\Omega$ in the second case. Remember that $\{Y_{nm}(\vartheta,\lambda)\}$ is orthonormal with

$$\frac{1}{4\pi}\int Y_{nm}(\vartheta,\lambda)^2 d\sigma=1).$$