
Randomized Approximation of Sobolev Embeddings

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Summary. We study approximation of functions belonging to Sobolev spaces $W_p^r(Q)$ by randomized algorithms based on function values. Here $1 \leq p \leq \infty$, $Q = [0, 1]^d$, and $r, d \in \mathbf{N}$. The error is measured in $L_q(Q)$, with $1 \leq q < \infty$, and we assume $r/d > 1/p - 1/q$, guaranteeing that $W_p^r(Q)$ is embedded into $L_q(Q)$. The optimal order of convergence for the case that $W_p^r(Q)$ is embedded even into $C(Q)$ is well-known. It is $n^{-r/d + \max(1/p - 1/q, 0)}$ (n the number of function evaluations). This rate is already reached by deterministic algorithms, and randomization gives no speedup.

In this paper we are concerned with the case that $W_p^r(Q)$ is not embedded into $C(Q)$ (but, of course, still into $L_q(Q)$). For this situation approximation based on function values was not studied before. We prove that for randomized algorithms the above rate also holds, while for deterministic algorithms no rate whatsoever is possible. Thus, in the case of low smoothness, Monte Carlo approximation algorithms reach a considerable speedup over deterministic ones (up to $n^{-1+\varepsilon}$ for any $\varepsilon > 0$).

We also give some applications to integration of functions and to approximation of solutions of elliptic PDE.

1 Introduction

Denote $\mathbf{N} = \{1, 2, \dots\}$, $\mathbf{N}_0 = \{0, 1, 2, \dots\}$, let $d \in \mathbf{N}$, and let $Q = [0, 1]^d$ be the d -dimensional unit cube. For $1 \leq p \leq \infty$, let $L_p(Q)$ be the space of real-valued p -integrable functions, endowed with the norm

$$\|f\|_{L_p(Q)} = \left(\int_Q |f(x)|^p dx \right)^{1/p}$$

if $p < \infty$, and

$$\|f\|_{L_\infty(Q)} = \operatorname{ess\,sup}_{x \in Q} |f(x)|.$$

For $r \in \mathbf{N}$ the Sobolev space $W_p^r(Q)$ consists of all functions $f \in L_p(Q)$ such that for all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}_0^d$ with $|\alpha| := \sum_{j=1}^d \alpha_j \leq r$, the generalized partial derivative $D^\alpha f$ belongs to $L_p(Q)$. The norm on $W_p^r(Q)$ is defined as

$$\|f\|_{W_p^r(Q)} = \left(\sum_{|\alpha| \leq r} \|D^\alpha f\|_{L_p(Q)}^p \right)^{1/p}$$

if $p < \infty$, and

$$\|f\|_{W_\infty^r(Q)} = \max_{|\alpha| \leq r} \|D^\alpha f\|_{L_\infty(Q)}.$$

Let $C(Q)$ denote the space of continuous functions on Q , equipped with the supremum norm. Let $1 \leq q < \infty$. By the Sobolev embedding theorem (see [Ada75], [Tri78]), for

$$r/d > 1/p - 1/q \tag{1}$$

$W_p^r(Q) \subset L_q(Q)$, and there is a constant $c > 0$ such that for each $f \in W_p^r(Q)$

$$\|f\|_{L_q(Q)} \leq c \|f\|_{W_p^r(Q)}. \tag{2}$$

Consequently, the embedding operator $J_{pq} : W_p^r(Q) \rightarrow L_q(Q)$ defined by $J_{pq}f = f$ is bounded. We shall study optimal approximation of J_{pq} by randomized algorithms which use n function values.

For $n \in \mathbf{N}$ we consider the class $\mathcal{A}_n^{\text{ran}}$ of randomized algorithms which are of the form $A = (A_\omega)_{\omega \in \Omega}$, where

$$A_\omega(f) = \varphi_\omega(f(x_{1,\omega}), \dots, f(x_{n,\omega})), \tag{3}$$

$(\Omega, \Sigma, \mathbf{P})$ is a probability space, for each $\omega \in \Omega$, $x_{i,\omega}$ is an element of Q and φ_ω is a mapping from \mathbf{R}^n to $L_q(Q)$, with the property that for each $f \in W_p^r(Q)$, the mapping

$$\omega \in \Omega \rightarrow A_\omega(f) = \varphi_\omega(f(x_{1,\omega}), \dots, f(x_{n,\omega}))$$

is a random variable with values in $L_q(Q)$ (that is, Σ -to-Borel measurable). Since elements of $W_p^r(Q)$ are equivalence classes of functions, relation (3) needs more explanation when $W_p^r(Q)$ is not embedded into $C(Q)$, and hence function values are, in general, not defined for such classes. We impose a further condition on the elements of $\mathcal{A}_n^{\text{ran}}$, let us call it consistency of the algorithm. We assume that whenever f_1 and f_2 are representatives of the same class $f \in W_p^r(Q)$, then

$$\varphi_\omega(f_1(x_{1,\omega}), \dots, f_1(x_{n,\omega})) = \varphi_\omega(f_2(x_{1,\omega}), \dots, f_2(x_{n,\omega})) \quad \mathbf{P} - \text{a.s.}$$

This means that $A_\omega(f_1)$ and $A_\omega(f_2)$ coincide almost surely, and in this sense we can take (3) as the definition of the random variable $A_\omega(f)$. A sufficient condition for consistency is obviously the following: For all i , the mapping

$\omega \rightarrow x_{i,\omega}$ is Lebesgue measurable and for each subset $Q_0 \subset Q$ of Lebesgue measure zero we have

$$\mathbf{P}\{\omega \in \Omega : x_{i,\omega} \in Q_0\} = 0,$$

or equivalently, the distribution of the $x_{i,\omega}$ is absolutely continuous with respect to the Lebesgue measure.

The class $\mathcal{A}_n^{\text{ran}}$ contains the subclass of randomized linear algorithms – they are of the form above with linear φ_ω , thus

$$A_\omega(f) = \sum_{i=1}^n f(x_{i,\omega})\psi_{i,\omega}, \tag{4}$$

for certain $\psi_{i,\omega} \in L_q(Q)$.

Given any $1 \leq s < \infty$, the error of an algorithm $A \in \mathcal{A}_n^{\text{ran}}$ is defined as

$$e^{(s)}(J_{pq}, A, B_{W_p^r(Q)}) = \sup_{f \in B_{W_p^r(Q)}} (\mathbf{E} \|f - A_\omega(f)\|_{L_q(Q)}^s)^{1/s}, \tag{5}$$

where \mathbf{E} is the expectation with respect to \mathbf{P} and $B_{W_p^r(Q)}$ denotes the unit ball of $W_p^r(Q)$. The randomized n -th minimal error is defined as

$$e_n^{\text{ran}}(J_{pq}, B_{W_p^r(Q)}) = \inf_{A \in \mathcal{A}_n} e^{(1)}(J_{pq}, A, B_{W_p^r(Q)}).$$

Hence, no randomized algorithm that uses at most n function values can provide a smaller error than $e_n^{\text{ran}}(J_{pq}, B_{W_p^r(Q)})$. We have chosen the $s = 1$ case for the minimal error, which is convenient for the sequel. Statements for other s can be read from the proofs below.

In terms of information-based complexity theory [TWW88], [Nov88], we consider randomized nonadaptive algorithms using standard information. In terms of [NT06], the e_n^{ran} can be viewed as randomized sampling numbers.

In the case that $W_p^r(Q)$ is embedded into $C(Q)$, the order of e_n^{ran} is well-known:

$$e_n^{\text{ran}}(J_{pq}, B_{W_p^r(Q)}) \asymp n^{-r/d + \max(1/p - 1/q, 0)}, \tag{6}$$

where we used the following notation: for sequences (a_n) and (b_n) of nonnegative reals we write $a_n \asymp b_n$ if there are constants $c_1, c_2 > 0$ and an $n_0 \in \mathbf{N}$ such that $c_1 a_n \leq b_n \leq c_2 a_n$ for all $n \geq n_0$. Furthermore note that we often use the same symbols c, c_1, \dots for possibly different constants, also in sequences of relations.

The upper bound of (6) can be reached by deterministic methods, for example by piecewise polynomial approximation (see, e.g. [Cia78]). The lower bounds were shown in [Was89] ($p = q = \infty$), [Nov88] ($1 \leq p \leq \infty, q = \infty$),

and [Mat91] ($1 \leq p, q \leq \infty$). Recall from [Ada75] that $W_p^r(Q)$ is embedded into $C(Q)$ iff

$$\left. \begin{array}{l} p = 1 \quad \text{and} \quad r/d \geq 1 \\ \text{or} \\ 1 < p < \infty \quad \text{and} \quad r/d > 1/p \\ \text{or} \\ p = \infty. \end{array} \right\} \tag{7}$$

In this paper we are concerned with the case that (7) does *not* hold. Together with (1), this means we assume

$$\left. \begin{array}{l} 1 \leq p, q < \infty \\ \text{and} \\ 1 - 1/q < r/d < 1 \quad \text{if} \quad p = 1, \\ 1/p - 1/q < r/d \leq 1/p \quad \text{if} \quad 1 < p < \infty. \end{array} \right\} \tag{8}$$

Note that we demanded from the beginning that $W_p^r(Q)$ is embedded into $L_q(Q)$ (otherwise the operator J_{pq} would not even be defined). So here we study the case that $W_p^r(Q)$ is embedded into $L_q(Q)$ but *not* into $C(Q)$. In this case approximation based on function values was not studied before. In the deterministic case there is a certain reason for this: function values are not defined any longer! However, this can easily be overcome. Namely, let us consider approximation on $B_{W_p^r(Q)} \cap C(Q)$, a dense subset of $B_{W_p^r(Q)}$. Now function values *are* defined, and the question arises which rate can be reached on the basis of Sobolev smoothness. It turns out that in the deterministic setting no rate whatsoever is possible on $B_{W_p^r(Q)} \cap C(Q)$. We discuss this issue in section 4.

However, also the analysis of the randomized setting was restricted to the case (7) of embedding into $C(Q)$. In section 2 we present a randomized algorithm which reaches the rate (6) also in the case (8) of non-embedding into $C(Q)$. Hence, in contrast to the situation of embedding (7), randomized algorithms turn out to be superior to deterministic ones. We comment on this in more detail in section 4.

Some applications to randomized integration of functions from $W_p^r(Q)$ and to approximation of solution operators of elliptic partial differential equations are given in Section 3.

2 Randomized Approximation

The following is the main result of this paper. We state it for $1 \leq p \leq \infty$, $1 \leq q < \infty$, since the proof works for all these cases, and thus shows that the method proposed also partly recovers the upper bound from (6) (having in mind though that the new part is the case in which (8) holds).

Theorem 1. *Let $r, d \in \mathbf{N}$, $1 \leq p \leq \infty$, $1 \leq q < \infty$, and assume $r/d > 1/p - 1/q$. Then*

$$e_n^{\text{ran}}(J_{pq}, B_{W_p^r(Q)}) \asymp n^{-r/d + \max(1/p - 1/q, 0)}. \tag{9}$$

We proceed as follows. Under the assumptions of Theorem 1, we develop a general scheme for randomized approximation of functions. Its convergence is then analysed in Proposition 1. Based on this, the proof of Theorem 1 is given at the end of this section.

Let $(\Omega, \Sigma, \mathbf{P})$ be a probability space, and for each $\omega \in \Omega$ let P_ω be any operator from $C(Q)$ to $L_q(Q)$ of the form

$$P_\omega f = \sum_{j=1}^{\kappa} f(x_{j,\omega}) \psi_{j,\omega} \quad (f \in C(Q))$$

with $x_{j,\omega} \in Q$ and $\psi_{j,\omega} \in L_q(Q)$. We assume that the mappings

$$\omega \rightarrow x_{j,\omega}, \quad \omega \rightarrow \psi_{j,\omega}$$

are random variables, the distributions of the x_j being absolutely continuous with respect to the Lebesgue measure, that

$$(\mathbf{E} |f(x_{j,\omega})|^q)^{1/q} \leq c \|f\|_{W_p^r(Q)} \quad (f \in W_p^r(Q)), \tag{10}$$

$$\text{ess sup}_{\omega \in \Omega} \|\psi_{j,\omega}\|_{L_q(Q)} \leq c \tag{11}$$

for $j = 1, \dots, \kappa$, and that for all $g \in \mathcal{P}_{r-1}(Q)$, the space of polynomials on Q of degree not exceeding $r - 1$,

$$P_\omega g = g \quad \mathbf{P} - \text{a.s.} \tag{12}$$

Families with such properties are easily constructed. For example, fix $0 < \delta < 1$ and let

$$P^{(1)} f = \sum_{j=1}^{\kappa} f(z_j) \psi_j \tag{13}$$

be for $d = 1$ the Lagrange interpolation operator of appropriate degree and for $d > 1$ its tensor product, with $(z_j)_{j=1}^{\kappa}$ the uniform grid on $[0, 1 - \delta]^d$, and $(\psi_j)_{j=1}^{\kappa}$ the respective Lagrange polynomials, considered as functions on \mathbf{R}^d . Put $\Omega_1 = [0, \delta]^d$, Σ_1 the σ -algebra of Lebesgue measurable sets and \mathbf{P}_1 the normalized on $[0, \delta]^d$ Lebesgue measure. For $\omega_1 \in \Omega_1 = [0, \delta]^d$ and $f \in C(Q)$ put

$$x_{j,\omega_1} = z_j + \omega_1, \tag{14}$$

$$\psi_{j,\omega_1}(x) = \psi_j(x - \omega_1) \quad (x \in Q), \tag{15}$$

and

$$(P_{\omega_1}^{(1)} f)(x) = \sum_{j=1}^{\kappa} f(z_j + \omega_1) \psi_j(x - \omega_1). \tag{16}$$

Then we have for $f \in W_p^r(Q)$

$$\begin{aligned} (\mathbf{E} |f(x_{j,\omega_1})|^q)^{1/q} &= \left(\delta^{-d} \int_{[0,\delta]^d} |f(z_j + y)|^q dy \right)^{1/q} \\ &\leq \delta^{-d/q} \|f\|_{L_q(Q)} \leq c \delta^{-d/q} \|f\|_{W_p^r(Q)}, \end{aligned}$$

which shows that (10) is satisfied. It is readily checked that conditions (11) and (12) hold, as well.

Let $l \in \mathbf{N}_0$ and let

$$Q = \bigcup_{i=1}^{2^{dl}} Q_i$$

be the partition of Q into 2^{dl} cubes of sidelength 2^{-l} and of disjoint interior. Let x_i denote the point in Q_i with minimal coordinates. Define the operators E_i and R_i on $L_q(Q)$ by setting for $f \in L_q(Q)$ and $x \in Q$

$$(E_i f)(x) = f(x_i + 2^{-l}x)$$

and

$$(R_i f)(x) = \chi_{Q_i}(x) f(2^l(x - x_i)) = \begin{cases} f(2^l(x - x_i)) & \text{if } x \in Q_i \\ 0 & \text{otherwise.} \end{cases}$$

For $\omega \in \Omega$ set

$$P_{l,\omega} f = \sum_{i=1}^{2^{dl}} R_i P_\omega E_i f = \sum_{i=1}^{2^{dl}} \sum_{j=1}^{\kappa} f(x_i + 2^{-l}x_{j,\omega}) R_i \psi_{j,\omega} \tag{17}$$

(observe that we use the same random variables $x_{j,\omega}$ for all i). It easily follows from the assumptions on $(P_\omega)_{\omega \in \Omega}$ that $(P_{l,\omega})_{\omega \in \Omega}$ is an algorithm from $\mathcal{A}_m^{\text{ran}}$, where $m = \kappa 2^{dl}$. In fact it is a linear algorithm.

Proposition 1. *Let $r, d \in \mathbf{N}$, $1 \leq p \leq \infty$, $1 \leq q < \infty$, and assume $r/d > 1/p - 1/q$. Let $(P_\omega)_{\omega \in \Omega}$ be as above satisfying (10), (11), (12), and let $(P_{l,\omega})_{\omega \in \Omega}$ for $l \in \mathbf{N}_0$ be given by (17). Then there is a constant $c > 0$ such that for all $l \in \mathbf{N}_0$ and $f \in W_p^r(Q)$*

$$(\mathbf{E} \|f - P_{l,\omega} f\|_{L_q(Q)}^q)^{1/q} \leq c 2^{-rl + \max(1/p - 1/q, 0)dl} \|f\|_{W_p^r(Q)}. \tag{18}$$

Proof. It follows from (10) and (11) that for $f \in W_p^r(Q)$

$$\begin{aligned} (\mathbf{E} \|P_\omega f\|_{L_q(Q)}^q)^{1/q} &\leq \left(\mathbf{E} \left(\sum_{j=1}^{\kappa} |f(x_{j,\omega})| \|\psi_{j,\omega}\|_{L_q(Q)} \right)^q \right)^{1/q} \\ &\leq c \sum_{j=1}^{\kappa} (\mathbf{E} |f(x_{j,\omega})|^q)^{1/q} \leq c \|f\|_{W_p^r(Q)}. \end{aligned} \tag{19}$$

We denote

$$|f|_{r,p,Q} = \left(\sum_{|\alpha|=r} \|D^\alpha f\|_{L_p(Q)}^p \right)^{1/p}$$

if $p < \infty$ and

$$|f|_{r,\infty,Q} = \max_{|\alpha|=r} \|D^\alpha f\|_{L_\infty(Q)}.$$

Next we apply Theorem 3.1.1 from [Cia78]: there is a constant $c > 0$ such that for all $f \in W_p^r(Q)$

$$\inf_{g \in \mathcal{P}_{r-1}(Q)} \|f - g\|_{W_p^r(Q)} \leq c|f|_{r,p,Q}. \tag{20}$$

It follows from (2), (12), (19), and (20) that

$$\begin{aligned} (\mathbf{E} \|f - P_\omega f\|_{L_q(Q)}^q)^{1/q} &= \inf_{g \in \mathcal{P}_{r-1}(Q)} (\mathbf{E} \|(f - g) - P_\omega(f - g)\|_{L_q(Q)}^q)^{1/q} \\ &\leq c \inf_{g \in \mathcal{P}_{r-1}(Q)} \|f - g\|_{W_p^r(Q)} \leq c|f|_{r,p,Q}. \end{aligned} \tag{21}$$

Clearly,

$$\|R_i f\|_{L_q(Q)} = 2^{-dl/q} \|f\|_{L_q(Q)} \quad (f \in L_q(Q)). \tag{22}$$

From (21) and (22) we obtain for all $f \in W_p^r(Q)$,

$$\begin{aligned} (\mathbf{E} \|f - P_{l,\omega} f\|_{L_q(Q)}^q)^{1/q} &= \left(\mathbf{E} \left\| \sum_{i=1}^{2^{dl}} (R_i E_i f - P_\omega P_\omega E_i f) \right\|_{L_q(Q)}^q \right)^{1/q} \\ &= \left(\mathbf{E} \sum_{i=1}^{2^{dl}} \|R_i (E_i f - P_\omega E_i f)\|_{L_q(Q)}^q \right)^{1/q} \\ &= \left(2^{-dl} \sum_{i=1}^{2^{dl}} \mathbf{E} \|E_i f - P_\omega E_i f\|_{L_q(Q)}^q \right)^{1/q} \\ &\leq c \left(2^{-dl} \sum_{i=1}^{2^{dl}} |E_i f|_{r,p,Q}^q \right)^{1/q} \\ &\leq c 2^{\max(1/p-1/q,0)dl} \left(2^{-dl} \sum_{i=1}^{2^{dl}} |E_i f|_{r,p,Q}^p \right)^{1/p} \end{aligned}$$

and furthermore,

$$\begin{aligned} \left(2^{-dl} \sum_{i=1}^{2^{dl}} |E_i f|_{r,p,Q}^p \right)^{1/p} &= \left(2^{-dl} \sum_{i=1}^{2^{dl}} \sum_{|\alpha|=r} \int_Q |D^\alpha f(x_i + 2^{-l}x)|^p dx \right)^{1/p} \\ &= 2^{-rl} \left(\sum_{i=1}^{2^{dl}} \sum_{|\alpha|=r} \int_{Q_i} |D^\alpha f(y)|^p dy \right)^{1/p} \\ &= 2^{-rl} |f|_{r,p,Q} \leq 2^{-rl} \|f\|_{W_p^r(Q)}. \end{aligned}$$

(with the usual modifications for $p = \infty$). Combining the last two inequalities gives

$$(\mathbf{E} \|f - P_{l,\omega} f\|_{L_q(Q)}^q)^{1/q} \leq c 2^{-rl + \max(1/p - 1/q, 0)dl} \|f\|_{W_p^r(Q)},$$

which concludes the proof.

Proof of Theorem 1. Let $n \in \mathbf{N}$ and put

$$l = \left\lceil \frac{\log_2 n}{d} \right\rceil. \tag{23}$$

Then $(P_{l,\omega})_{\omega \in \Omega}$ belongs to $\mathcal{A}_m^{\text{ran}}$ with $m = \kappa 2^{dl} \leq cn$, so Proposition 1 together with (23) gives the upper bound in (9). The lower bound follows from standard techniques of information-based complexity (reduction to the average case on subsets formed by smooth bump functions) and is identical to that given in [Mat91] (see also [Nov88], [Hei93]). We omit it here.

3 Some Applications

First we consider integration. Let $Q = [0, 1]^d$ and let $I : W_p^r(Q) \rightarrow \mathbf{R}$ be the integration operator

$$If = \int_Q f(x) dx.$$

Corollary 1. *Let $r, d \in \mathbf{N}$, $1 \leq p < \infty$, and put $\bar{p} = \min(p, 2)$. Then*

$$e_n^{\text{ran}}(I, B_{W_p^r(Q)}) \asymp n^{-r/d - 1 + 1/\bar{p}}.$$

This result was shown by Novak for $p \geq 2$ and for $p < 2$ and $r/d \geq 1/p - 1/2$, that is, for the case that $W_p^r(Q)$ is embedded into $L_2(Q)$, see [Nov88], 2.2.9, and also the references therein for previous work. Our analysis supplies the remaining cases and a new technique: Novak used a result of [EZ60] on stochastic quadratures. For spaces embedded into $C(Q)$, another proof

was given in [Hei93] using deterministic approximation as variance reduction (separation of the main part). What we present below might be viewed as a stochastic analogon of the latter.

Proof of Corollary 1. Fix $0 < \delta < 1$ and let $(\Omega_1, \Sigma_1, \mathbf{P}_1)$ be as defined above, following relation (13). Let $n \in \mathbf{N}$, let l be given by (23), and let $P_{l, \omega_1}^{(1)}$ for $\omega_1 \in \Omega_1$ be as defined in (13-17), that is, for $f \in W_p^r(Q)$ and $x \in Q$,

$$\left(P_{l, \omega_1}^{(1)} f\right)(x) = \sum_{i=1}^{2^{dl}} \sum_{j=1}^{\kappa} f(x_i + 2^{-l}(z_j + \omega_1)) \chi_{Q_i}(x) \psi_j(2^l(x - x_i) - \omega_1). \tag{24}$$

Finally, let y_{k, ω_2} ($k = 1, \dots, n$) be independent, uniformly distributed on Q random variables over some probability space $(\Omega_2, \Sigma_2, \mathbf{P}_2)$. Define the algorithm $(A_{\omega_2}^{(2)})_{\omega_2 \in \Omega_2}$ to be the usual Monte Carlo method

$$A_{\omega_2}^{(2)}(g) = \frac{1}{n} \sum_{k=1}^n g(y_{k, \omega_2}). \tag{25}$$

It is known that for $g \in L_p(Q)$

$$\left(\mathbf{E}_{\omega_2} |I g - A_{\omega_2}^{(2)}(g)|^{\bar{p}}\right)^{1/\bar{p}} \leq 2^{2/\bar{p}-1} n^{-1+1/\bar{p}} \|g\|_{L_p(Q)} \tag{26}$$

(see [Hei93] for the case $1 \leq p < 2$). Now we put

$$(\Omega, \Sigma, \mathbf{P}) = (\Omega_1, \Sigma_1, \mathbf{P}_1) \times (\Omega_2, \Sigma_2, \mathbf{P}_2)$$

and define an algorithm $A = (A_\omega)_{\omega \in \Omega}$ by setting for $\omega = (\omega_1, \omega_2)$ and $f \in W_p^r(Q)$

$$A_\omega(f) = I P_{l, \omega_1}^{(1)} f + A_{\omega_2}^{(2)}(f - P_{l, \omega_1}^{(1)} f).$$

On the basis of (24) and (25) measurability and consistency readily follow, and we have $A \in \mathcal{A}_m^{\text{ran}}$ for $m = \kappa 2^{dl} + n \leq cn$. Moreover,

$$I f - A_\omega(f) = I(f - P_{l, \omega_1}^{(1)} f) - A_{\omega_2}^{(2)}(f - P_{l, \omega_1}^{(1)} f).$$

Using Fubini's theorem, (26), and Proposition 1 for $q = p$, we derive

$$\begin{aligned} & \left(\mathbf{E} |I f - A_\omega(f)|^{\bar{p}}\right)^{1/\bar{p}} \\ &= \left(\mathbf{E}_{\omega_1} \mathbf{E}_{\omega_2} \left| I \left(f - P_{l, \omega_1}^{(1)} f \right) - A_{\omega_2}^{(2)} \left(f - P_{l, \omega_1}^{(1)} f \right) \right|^{\bar{p}}\right)^{1/\bar{p}} \\ &\leq cn^{-1+1/\bar{p}} \left(\mathbf{E}_{\omega_1} \left\| f - P_{l, \omega_1}^{(1)} f \right\|_{L_p(Q)}^{\bar{p}}\right)^{1/\bar{p}} \\ &\leq cn^{-1+1/\bar{p}} \left(\mathbf{E}_{\omega_1} \left\| f - P_{l, \omega_1}^{(1)} f \right\|_{L_p(Q)}^p\right)^{1/p} \\ &\leq cn^{-1+1/\bar{p}-r/d} \|f\|_{W_p^r(Q)}, \end{aligned}$$

concluding the proof of the upper bound. The lower bound is already contained in [Nov88], 2.2.9, Proposition 1.

Now we turn to elliptic problems. First results on the randomized information complexity of elliptic partial differential equations were obtained in [Hei06b]. The results above have some direct consequences for certain instances of this problem. Let $d, m \in \mathbf{N}$, $d \geq 2$, let $Q_1 \subset \mathbf{R}^d$ be a bounded C^∞ domain (see, e.g., [Tri78] for the definition), and let \mathcal{L} be an elliptic differential operator of order $2m$ on Q_1 , that is

$$\mathcal{L}u = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u(x), \tag{27}$$

with boundary operators

$$\mathcal{B}_j u = \sum_{|\alpha| \leq m_j} b_{j\alpha}(x) D^\alpha u(x), \tag{28}$$

where $j = 1, \dots, m$, $m_j \leq 2m - 1$ and $a_\alpha \in C^\infty(Q_1)$ and $b_{j\alpha} \in C^\infty(\partial Q_1)$ are complex-valued infinitely differentiable functions. Consider the homogeneous boundary value problem

$$\mathcal{L}u(x) = f(x) \quad (x \in Q_1^0) \tag{29}$$

$$\mathcal{B}_j u(x) = 0 \quad (x \in \partial Q_1). \tag{30}$$

We assume that $(\mathcal{L}, \{B_j\})$ is regularly elliptic (see [Tri78], 5.2.1/4, for the definition), and that 0 is not in the spectrum of \mathcal{L} , considered as an unbounded operator in $L_q(Q_1)$ with domain of definition $W_{q, \{B_j\}}^{2m}(Q_1)$, where the latter denotes the subspace of $W_q^{2m}(Q_1)$ consisting of those f which satisfy (30). This implies that \mathcal{L} is an isomorphism from $W_{q, \{B_j\}}^{2m}(Q_1)$ to $L_q(Q_1)$ for $1 < q < \infty$, see [Tri78], Theorem 5.5.1(b). Now we put $S = \mathcal{L}^{-1} J_{pq}$, considered as an operator into $W_q^{2m}(Q_1)$, that is,

$$S : W_p^r(Q_1) \xrightarrow{J_{pq}} L_q(Q_1) \xrightarrow{\mathcal{L}^{-1}} W_q^{2m}(Q_1).$$

Hence S is the solution operator for the elliptic problem (29–30), where we consider the problem of approximating the full solution u , the right-hand side f is supposed to belong to $W_p^r(Q_1)$, and the error is measured in the norm of $W_q^{2m}(Q_1)$.

Corollary 2. *Let $r, d \in \mathbf{N}$, $1 \leq p \leq \infty$, $1 < q < \infty$, and $r/d > 1/p - 1/q$. Then*

$$e_n^{\text{ran}}(S, B_{W_p^r(Q_1)}) \asymp n^{-r/d + \max(1/p - 1/q, 0)}. \tag{31}$$

Proof. Using local charts (like, e.g., in [Hei06b]) it is easy to extend Theorem 1 to smooth domains Q_1 in place of $Q = [0, 1]^d$. It is also clear that the case

of complex-valued functions is a direct consequence of the real case. From this and the above-mentioned fact that \mathcal{L}^{-1} is an isomorphic embedding of $L_q(Q_1)$ into $W_q^{2m}(Q_1)$ the upper bound follows.

So does the lower bound if we verify that algorithms with values in $\mathcal{L}^{-1}(L_q(Q_1)) = W_{q, \{\mathcal{B}_j\}}^{2m}(Q_1)$, a subspace of $W_q^{2m}(Q_1)$ containing $S(W_p^r(Q_1))$, cannot be better (up to a constant) than algorithms with values in $W_q^{2m}(Q_1)$. This, however, follows, e.g., from the fact that $W_{q, \{\mathcal{B}_j\}}^{2m}(Q_1)$ is complemented in $W_q^{2m}(Q_1)$, see [Tri78], Theorem 5.5.2(b), completing the proof.

A similar approach (in the sense of using isomorphism properties to reduce approximation of solution operators to approximation of embeddings) was presented in [DNS06a, DNS06b] for the deterministic setting, with $q = 2$. There, however, more general classes of operators and, besides function values, also arbitrary linear functionals are considered.

4 Deterministic Approximation

We already mentioned that for those r, d and p for which $W_p^r(Q)$ is embedded into $C(Q)$, the order of the randomized n -th minimal error coincides with that of the deterministic one. If the embedding does not hold, that is, if (8) is satisfied, the situation is different. First of all, since function values are not well-defined, we replace $B_{W_p^r(Q)}$ by the (dense) subset $B_{W_p^r(Q)} \cap C(Q)$. In this section we show that, although now function values are defined, the Sobolev smoothness $W_p^r(Q)$ does not lead to any rate at all for the deterministic setting. That is, the deterministic n -th minimal error is bounded from below by a positive constant.

Put $F = B_{W_p^r(Q)} \cap C(Q)$, let $n \in \mathbf{N}$ and let $\mathcal{A}_n^{\text{det}}$ be the class of all deterministic algorithms for the approximation of J_{pq} on F , which are of the form

$$A(f) = \varphi(f(x_1), \dots, f(x_n)), \tag{32}$$

where $x_i \in Q$ ($i = 1, \dots, n$) and $\varphi : \mathbf{R}^n \rightarrow L_q(Q)$ is an arbitrary mapping. The error on F is defined as

$$e(J_{pq}, A, F) = \sup_{f \in F} \|S(f) - A(f)\|_{L_q(Q)}.$$

The deterministic n -th minimal error is defined as

$$e_n^{\text{det}}(J_{pq}, F) = \inf_{A \in \mathcal{A}_n^{\text{det}}} e(J_{pq}, A, F).$$

Proposition 2. *Let $1 \leq p, q < \infty$ and $r/d > 1/p - 1/q$. Assume that either*

$$\frac{r}{d} < \frac{1}{p} \tag{33}$$

or

$$\frac{r}{d} = \frac{1}{p} \quad \text{and} \quad 1 < p < \infty. \tag{34}$$

Then

$$e_n^{\det}(J_{pq}, B_{W_p^r(Q)} \cap C(Q)) \asymp 1. \tag{35}$$

For the proof we need the following lemma. Let $B(0, \varrho)$ denote the closed ball in \mathbf{R}^d of radius ϱ around 0.

Lemma 1. *Assume that (33) or (34) holds. Then there is a sequence of functions*

$$(f_m)_{m=1}^\infty \subset W_p^r(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$$

such that for all m

$$f_m(0) = 1, \quad \text{supp } f_m \subseteq B\left(0, \frac{1}{m}\right), \tag{36}$$

and

$$\lim_{m \rightarrow \infty} \|f_m\|_{W_p^r(\mathbf{R}^d)} = 0. \tag{37}$$

Proof. In case of (34) this is a combination of well-known facts from function space theory. Let $\psi \in C^\infty(\mathbf{R}^d)$ be such that $\psi \geq 0$,

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2}, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

and

$$\int_{\mathbf{R}^d} \psi(x) dx = 1.$$

Put

$$g(x) = \begin{cases} \psi(x) \ln \ln \frac{3}{|x|} & \text{if } 0 < |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then

$$\|g\|_{W_p^r(\mathbf{R}^d)} < \infty, \quad \text{supp } g \subseteq B(0, 1), \quad \lim_{x \rightarrow 0} g(x) = +\infty, \tag{38}$$

see [Ada75], Example 5.26. Furthermore, setting

$$h_m = \psi_m * g \quad \text{with} \quad \psi_m(x) = m^d \psi(mx) \quad (x \in \mathbf{R}^d, m \in \mathbf{N}),$$

we get, using Lemma 3.15 of [Ada75] and (38),

$$h_m \in C^\infty(\mathbf{R}^d), \quad \text{supp } h_m \subseteq B(0, 2), \quad h_m(0) > 0 \quad (m \in \mathbf{N}),$$

$$\sup_{m \in \mathbf{N}} \|h_m\|_{W_p^r(\mathbf{R}^d)} < \infty, \quad \lim_{m \rightarrow \infty} h_m(0) = +\infty.$$

Finally we define f_m by

$$f_m(x) = h_m(0)^{-1}h_m(2mx) \quad (x \in \mathbf{R}^d, m \in \mathbf{N}).$$

Relation (34) implies

$$\|f_m\|_{W_p^r(\mathbf{R}^d)} \leq h_m(0)^{-1}\|h_m(2m\cdot)\|_{W_p^r(\mathbf{R}^d)} \leq ch_m(0)^{-1} \rightarrow 0,$$

while (36) is obviously fulfilled by definition. This completes the proof in the case of (34).

If (33) holds, we choose ψ as above and put

$$f_m(x) = \psi(mx) \quad (x \in \mathbf{R}^d, m \in \mathbf{N}).$$

Clearly, (36) is satisfied, and it follows from (33) that

$$\|f_m\|_{W_p^r(\mathbf{R}^d)} \leq cm^{r-d/p} \rightarrow 0 \quad (m \rightarrow \infty).$$

Proof of Proposition 2. The upper bound follows from the fact that J_{pq} is bounded. To prove the lower bound, let x_1, \dots, x_n be any fixed distinct points in Q . For $m \in \mathbf{N}$ consider the function $v_m \in C(Q)$ given by

$$v_m(x) = \left(1 + n\|f_m\|_{W_p^r(\mathbf{R}^d)}\right)^{-1} \left(1 - \sum_{i=1}^n f_m(x - x_i)\right) \quad (x \in Q).$$

We have

$$\|v_m\|_{W_p^r(Q)} \leq 1 \tag{39}$$

and, using (37),

$$\begin{aligned} \|v_m\|_{L_q(Q)} &\geq \int_Q v_m(x)dx \\ &= \left(1 + n\|f_m\|_{W_p^r(\mathbf{R}^d)}\right)^{-1} \left(1 - \sum_{i=1}^n \int_Q f_m(x - x_i)dx\right) \\ &\geq \left(1 + n\|f_m\|_{W_p^r(\mathbf{R}^d)}\right)^{-1} \left(1 - n\|f_m\|_{W_p^r(\mathbf{R}^d)}\right) \rightarrow 1 \end{aligned} \tag{40}$$

as $m \rightarrow \infty$. Finally, by (36), for m large enough,

$$v_m(x_i) = 0 \quad (i = 1, \dots, n). \tag{41}$$

Now (39–41) combined with standard results from information-based complexity theory [TWW88], Ch. 3.1, prove Proposition 2.

Proposition 2 was independently obtained by Novak and Woźniakowski (unpublished notes).

5 Comments

Comparing the rates in Theorem 1 and in Proposition 2 for the case (8) of non-embedding into $C(Q)$, we see that on $B_{W_p^r(Q)} \cap C(Q)$ randomization gives a speedup over the deterministic setting of

$$\frac{e_n^{\text{ran}}(J_{pq}, B_{W_p^r(Q)} \cap C(Q))}{e_n^{\text{det}}(J_{pq}, B_{W_p^r(Q)} \cap C(Q))} \asymp n^{-r/d + \max(1/p - 1/q, 0)},$$

which is non-trivial in all cases, since $r/d > \max(1/p - 1/q, 0)$ by assumption. If $p = 1$, the maximal exponent of the speedup is r/d , reached for $q = 1$, and r/d can be arbitrarily close to 1. If $1 < p < \infty$, the maximal exponent of the speedup is again r/d and can now be as large as $1/p$, reached for $q \leq p$.

Let us also mention that the lower bounds presented here for the randomized setting hold true for the more general case of adaptive (standard) information, as introduced e.g. in [Hei06a]. Similarly for the deterministic case. The latter follows from general results [TWW88].

As is readily seen from its proof, Proposition 2 remains true if J_{pq} is replaced by I of Corollary 1, the parameter q and the condition $r/d > 1/p - 1/q$ being omitted. Furthermore, the argument used in the proof of Corollary 2 shows that we can also replace J_{pq} in Proposition 2 by S .

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