Lower Bounds on Edge Searching

Brian Alspach[1], Danny Dyer[2], Denis Hanson[1], and Boting Yang[3]

[1] Department of Mathematics and Statistics, University of Regina
{alspach,dhanson}@math.uregina.ca
[2] Department of Mathematics and Statistics, Memorial University of Newfoundland
dyer@math.mun.ca
[3] Department of Computer Science, University of Regina
boting@cs.uregina.ca

Abstract. Searching a network for intruders is an interesting and difficult problem. Edge-searching is one such search model, in which intruders may exist anywhere along an edge. Since finding the minimum number of searchers necessary to search a graph is NP–complete, it is natural to look for bounds on the search number. We show lower bounds on the search number using minimum degree, girth, chromatic number, and colouring number.

1 Introduction

Clearing a graph (or network) of an intruder or intruders has a natural division into two classes of problem: those in which intruders may be located only at vertices and those in which intruders may be located at vertices or anywhere along edges. The latter situation is called *edge-searching*. Searching graphs serves as a model for important applied problems (see [3], [4] and [6]). A survey of results can be found in [1].

In this paper, we adopt the convention that multigraphs allow multiple edges, reflexive graphs allow loops, and that graphs allow neither loops nor multiple edges. A reflexive multigraph allows both loops and multiple edges. The specifics of searching a reflexive multigraph G are as follows. Initially, all edges of G are *contaminated*. To search G it is necessary to formulate and carry out a search strategy. A strategy is a sequence of actions performed as consecutive steps designed so that after the final step, all edges of G are *uncontaminated* (or *cleared*). Only three actions are allowed at each step.

1. Place a searcher on a vertex.
2. Move a searcher on a vertex u along an edge uv to v.
3. Remove a searcher from a vertex.

An edge uv in G can be *cleared* in one of two ways. Either at least two searchers are located on vertex u of edge uv, and one of them traverses the edge from u to v while the others remain at u, or at least one searcher is located on vertex u, where all edges incident with u, other than uv, are already cleared. Then the searcher moves from u to v. A cleared edge may become *recontaminated*. This

B. Chen, M. Paterson, and G. Zhang (Eds.): ESCAPE 2007, LNCS 4614, pp. 516–527, 2007.

happens if, at any time, there is a path from an endpoint of the cleared edge to an endpoint of a contaminated edge that does not contain a searcher. We say a vertex is cleared when all edges incident with it are cleared.

Knowing that our goal is a cleared graph, one in which all the edges are cleared, a basic question is: what is the fewest number of searchers for which a search strategy exists? We call this the *search number*, denoted $\mathrm{s}(G)$.

Let $E(i)$ be the set of cleared edges after action i has occurred. A search strategy for a graph G for which $E(i) \subseteq E(i+1)$ for all i is said to be *monotonic*. We may then define the *monotonic search number*, denoted $\mathrm{ms}(G)$. LaPaugh [7] and Bienstock and Seymour [2] proved that for any connected graph G, $\mathrm{s}(G) = \mathrm{ms}(G)$. We will only consider connected graphs throughout this paper.

In general, determining the search number of a graph G is NP-complete [8]. As any successful search strategy gives an upper bound, our goal becomes first to find the "right" way to clear the graph, using as few searchers as possible. Once this strategy is found, we must then prove that no fewer searchers will suffice. Here is where the true difficulty lies: most easily attainable lower bounds are quite poor. We will prove several lower bound results using the graph parameters of minimum degree, girth, and chromatic number.

The following three theorems (see [11]) give lower bounds for the search number of a graph G. The first uses the minimum degree $\delta(G)$, while the second uses the clique number $\omega(G)$ of G, the order of a maximum order complete subgraph.

Theorem 1. *If G is a connected graph then $\mathrm{s}(G) \geqq \delta(G)$. If $\delta(G) \geqq 3$, then $\mathrm{s}(G) \geqq \delta(G) + 1$.*

Theorem 2. *If G is a connected graph and $\omega(G) \geqq 4$, then $\omega(G) \leqq \mathrm{s}(G)$.*

We also recall a well-known theorem on searching. In this paper, if H is minor of G, we write $H \preceq G$, and if H is a subgraph of G, we write $H \subseteqq G$.

Theorem 3. *If $H \preceq G$, then $\mathrm{s}(H) \leqq \mathrm{s}(G)$.*

2 Minimum Degree and Girth

Consider the graph $K_{3,3}$. By Theorem 1, four searchers are necessary. However, with four searchers it is impossible to clear more than two vertices! We introduce the idea of girth to expand our repertoire of lower bounds.

Since the search number of a connected graph is equal to its monotonic search number, we may investigate monotonic search strategies instead. Being able to assume a search is monotonic is very useful. Moreover, Theorem 4 tells us something about how such a search strategy may be formulated. However, we must first introduce the following lemma. A vertex in a graph G is said to be *exposed* if it has edges incident with it that are contaminated as well as edges incident with it that are cleared. Following a search strategy S on G, we define $\mathrm{ex}_S(G, i)$ to be the number of exposed vertices after the i-th step. We also define the *maximum number of exposed vertices* to be $\mathrm{mex}_S(G) = \max_i \mathrm{ex}_S(G, i)$.

Lemma 1. *If G is a connected reflexive multigraph, then for any monotonic search strategy S using* $\mathrm{ms}(G)$ *searchers,* $\mathrm{mex}_S(G) \leqq \mathrm{ms}(G) \leqq \mathrm{mex}_S(G) + 1$.

Proof. The first inequality is straightforward; every exposed vertex must contain a searcher, so there cannot be more exposed vertices than searchers.

For the second inequality, it suffices to show that if there is a monotonic search strategy S which clears G using k searchers, and if $\mathrm{mex}_S(G) \leqq k-2$, then we can formulate another monotonic search strategy T which clears G using only $k-1$ searchers, and $\mathrm{mex}_T(G) = \mathrm{mex}_S(G)$.

Let $\mathbb{S}$ denote the set of all search strategies for G that use k searchers. For a given search strategy $S \in \mathbb{S}$, label the searchers in the order that they are first placed in G. The first unlabelled searcher placed will be labelled γ_1, the second unlabelled searcher placed will be labelled γ_2, and so on. Certainly, if any searcher is not placed on the graph, then it may be removed from S without affecting the search strategy. Thus, we may assume that every searcher is labelled, and every searcher is at some point placed on the graph. We also shall index each action using successive positive integers starting with 1, 2, ...

Whenever a vertex v is exposed, then there must be at least one searcher on v in order to prevent recontamination of the edges incident with v. If there is more than one searcher located at v, then we arbitrarily designate one of the searchers as the *guard* for v. Of course, if there is only one searcher located at v, then that searcher automatically is designated as the guard. We shall call a searcher *important* if the searcher at some point either clears an edge or becomes the guard on an exposed vertex.

Now consider the searcher γ_k. We first want to show that there is a strategy in $\mathbb{S}$ for which γ_k is not important. To this end, we assume that $\mathbb{S}$ contains at least one strategy for which γ_k is important. For any strategy S for which γ_k is important, let $L(S)$ denote the index of the last action in S and let $k(S)$ denote the index of the action that results in γ_k becoming important. In other words, either action $k(S)$ is the first time γ_k clears an edge, or action $k(S)$ results in γ_k first being a guard at some vertex.

Over all strategies $S \in \mathbb{S}$ for which γ_k is important, let $L(S) - k(S)$ be minimum. Suppose $L(S) - k(S) > 0$.

Consider the case that γ_k becomes important because action $k(S)$ consists of γ_k clearing an edge uv by traversing the edge from u to v. Define a new strategy S' as follows. We let S' coincide with S for actions with indices $1, 2, \ldots, k(S)-1$. At this point, γ_k is located at vertex u and in strategy S should move along uv from u to v in order to clear it. Because $\mathrm{mex}_S(G) \leqq k-2$, there is at least one searcher $\gamma_i \neq \gamma_k$ who is not being used to protect an exposed vertex. If γ_i also is located on vertex u, then let γ_i traverse the edge uv from u to v as the action indexed with $k(S)$ in S'. From this point on, S' is the same as S except that the roles of γ_i and γ_k are interchanged. If γ_i is not located on vertex u, then remove γ_i from its current vertex as the action indexed by $k(S)$ in S'. Now place γ_i on u as the action of S' indexed $k(S)+1$. Remove γ_k from u as the next action of S' and replace γ_k on the vertex from which γ_i just came. These four actions have interchanged the locations of γ_i and γ_k. From this point on, the action indexed

t in S' is the same as the action indexed $t-4$ in S with the roles of γ_i and γ_k interchanged.

In both subcases for which γ_k became important, it is easy to see that $L(S')-k(S') < L(S)-k(S)$. Now consider the remaining case that γ_k becomes important because following action $k(S)$, γ_k is a guard on an exposed vertex u. If there are two or more searchers on u following action $k(S)$, designate a searcher γ_i other than γ_k to be the guard. Then interchange the roles of γ_i and γ_k from that point on. The resulting strategy S' certainly satisfies $L(S') - k(S') < L(S) - k(S)$. Hence, just before the action $k(S)$ is carried out, there is just one searcher on u, in addition to γ_k, and this searcher leaves u on action $k(S)$. Because $\text{mex}_S(G) \leqq k-2$, there is another non-guard searcher γ_j located at some vertex $v \neq u$. Take four steps to interchange γ_k and γ_j and then define S' to be the initial part of S, the four actions just concluded, and the completion of S with the roles of γ_j and γ_k interchanged. Again it is easy to see that $L(S')-k(S') < L(S)-k(S)$.

Therefore, if there exists a search strategy S with γ_k being important and $L(S) - k(S) > 0$, there must be such a search strategy with $L(S) = k(S)$. Suppose S is such a strategy with $L(S) = k(S)$. This means that γ_k becomes important only on the last action of S, and this action must complete clearing the graph. Hence, on the last action $L(S)$, γ_k traverses an edge uv from u to v clearing the last contaminated edge of G. Because γ_k is not important and u is incident with a contaminated edge, there must be another searcher on u acting as the guard. Have this searcher clear the edge uv instead of γ_k. This results in a strategy for which γ_k is not important.

From this strategy, form a new strategy T which is exactly the same, but with all of γ_k's actions are removed. Since γ_k is unimportant, every edge is still cleared, and the maximum number of exposed vertices is the same, but only $k-1$ searchers are used.

Theorem 4. *If G is a connected reflexive graph with no vertices of degree* 2*, then there exists a monotonic search S with* $\text{ms}(G)$ *searchers such that* $\text{ms}(G) = \text{mex}_S(G) + 1$.

Proof. Let G be a connected reflexive graph G with no vertices of degree 2. Assume that for every monotonic search strategy S on G, $\text{mex}_S(G) = \text{ms}(G) = k$. Since S is a search strategy, there is a moment when the number of exposed vertices becomes $\text{mex}_S(G)$ for the last time. Let S' be a monotonic search strategy which has the minimum number of instances where the number of exposed vertices goes from being less than k to being k and has the minimum number of edge clearings after the last time the number of exposed vertices becomes k. The only action which can increase the number of exposed vertices is clearing an edge, which can expose at most two additional vertices. Let xy be the last edge cleared before the number of exposed vertices becomes $\text{mex}_S(G)$ for the last time. We consider four cases as to how xy can be cleared.

Case 1: The edge xy is a loop, with $x = y$. Since clearing xy can expose at most one additional vertex, the number of exposed vertices must be $k-1$. If x was already exposed, clearing xy would not increase the number of exposed vertices. Thus, x must not have been an exposed vertex. But since there must be a searcher on each of the $k-1$ exposed vertices, this leaves only one searcher to clear the loop xy. But a single searcher cannot clear a loop, a contradiction. Thus, xy cannot be a loop.

Case 2: The number of exposed vertices just before xy is cleared is $k-2$, and at this time neither x nor y is exposed. Label the $k-2$ exposed vertices as $v_1, v_2, \dots v_{k-2}$, and assume that searcher γ_i rests on vertex v_i, $1 \leqq i \leqq k-2$. The edge xy must be such that neither x nor y is some v_i. Assume that after xy is cleared, searcher γ_{k-1} is on x and γ_k is on y.

If there are any pendant edges or loops attached to some v_i that are not cleared, we can use searcher γ_k to clear these edges first. If this reduces the number of exposed vertices, then at some later action k vertices must be exposed because the number of exposed vertices increasing to k occurs a minimum number of times in S'. This later point must have more cleared edges, contradicting the minimality of S'. Thus, clearing such an edge cannot reduce the number of exposed vertices. But then, clearing xy next would produce a search strategy with fewer edges to be cleared after the number of exposed vertices becomes k for the last time, again contradicting the minimality of S'. Similarly, if there are any contaminated edges between v_i and v_j, γ_k may clear these edges first, and then xy, again contradicting the minimality of S'. So we may assume that all edges between the v_i have already been cleared, as have all pendant edges and loops incident with them.

If some vertex v_i is incident with only one contaminated edge, then γ_i may clear that edge first, and then γ_k may clear xy, again contradicting the minimality of S'. Thus, each v_i must have at least two contaminated edges incident with it, and the γ_i, $1 \leqq i \leqq k-2$, must remain where they are as blockers.

Neither x nor y are exposed before xy is cleared so that all edges incident with x and y are contaminated. After xy is cleared, both x and y are exposed. Thus, each of them is incident with a contaminated edge. Since G has no vertices of degree 2, both x and y must have at least two contaminated edges incident with them, and thus neither γ_{k-1} nor γ_k may move, contradicting that S' is a search strategy.

Case 3a: The number of exposed vertices just before xy is cleared is $k-1$ and one of the vertices of xy already is an exposed vertex. Label the exposed vertices v_1, v_2, $\dots$, v_{k-1}, and assume that they have searchers on them, with searcher γ_i on vertex v_i, $1 \leqq i \leqq k-1$. Without loss of generality, assume that $x = v_{k-1}$. Since the vertex v_{k-1} is still exposed, we may assume that γ_{k-1} stays on v_{k-1}, that the remaining searcher γ_k clears $v_{k-1}y$ by traversing the edge from v_{k-1} to y, and that there is another contaminated edge $v_{k-1}z$ incident with v_{k-1}.

If there are any pendant edges or loops attached to some v_i that are not cleared, we use the remaining searcher γ_k to clear these edges first, and then $v_{k-1}y$, contradicting the minimality of S'. In particular, $v_{k-1}z$ is not pendant so that z must have degree at least 3. Similarly, if there are any contaminated edges between v_i and v_j, γ_k may clear these edges first, then $v_{k-1}y$, again contradicting the minimality of S'. So we may assume that all edges between the v_i already have been cleared, as have all pendant edges and loops incident with them.

If some vertex v_i is incident with only one contaminated edge, then γ_i may clear that edge first, then γ_k may clear $v_{k-1}y$, again contradicting the minimality of S'. Thus, each v_i must have at least two contaminated edges incident with it, and all the γ_i, $1 \leqq i \leqq k-2$, must remain where they are as blockers. Note that $\deg(y) > 1$, as otherwise searching $v_{k-1}y$ does not expose a new vertex. Since $\deg(y) \geqq 3$, we know that once γ_k clears $v_{k-1}y$, γ_k must remain on y. After $v_{k-1}y$ is cleared, if v_{k-1} has two or more contaminated edges incident with it, then γ_{k-1} must remain at v_{k-1}. Then no searchers may move, contradicting that S' is a search strategy. Thus, the only contaminated edge remaining incident with v_{k-1} must be $v_{k-1}z$. Thus, the next action in S' must be that γ_{k-1} clears $v_{k-1}z$. Since $\deg(z) \geqq 3$, z must have at least two contaminated edges incident with it, and thus γ_{k-1} also cannot move, contradicting that S' is a search strategy.

Case 3b: The number of exposed vertices is $k-1$, and neither of the vertices of xy is already exposed. Since the number of exposed vertices increases by 1 after xy is cleared, exactly one of x and y must have degree 1. (If both were degree 1, the graph would be disconnected.) Without loss of generality, assume that $\deg(x) = 1$. Then $\deg(y) \geqq 3$. Assume that the $k-1$ exposed vertices are labelled v_i and that the searcher γ_i is on v_i, $1 \leqq i \leqq k-1$. Then the searcher γ_k must clear xy.

As in the previous case, all edges between v_i must be cleared, as must all pendant edges and loops incident with them. Also, each v_i must have at least two contaminated edges incident with it. Thus, none of the γ_i, $1 \leqq i \leqq k-1$, may move. Similarly, since $\deg(y) \geqq 3$, y must have at least two contaminated edges incident with it, meaning that γ_k cannot move. This contradicts that S' is a search strategy.

This theorem tells us that there exist search strategies for some graphs that "save" searchers, in the sense we may keep a searcher in reserve, to never be stationed at an exposed vertex, but instead to clear edges between stationed searchers. If we consider the analogy of a graph filled with gas, we may always keep a searcher from being exposed, or by rotating searchers reduce the amount of "exposure" to a toxic substance.

The use of graph instead of multigraph in Theorem 4 is intentional. While it is possible that the result may be extended to some multigraphs, this proof does not suffice.

We first introduce a lemma from [10] to be used in the proof of Theorem 5.

Lemma 2. *If G is a graph and $\delta(G) \geqq 3$, then the number of cycles with pairwise distinct vertex sets is greater than $2^{\frac{\delta}{2}}$.*

Theorem 5. *If G is a connected graph with $\delta(G) \geqq 3$, then $\mathrm{s}(G) \geqq \delta(G) + g(G) - 2$.*

Proof. From Lemma 2, we know that the girth of G is finite, that $g = g(G) \geqq 3$, and that G has at least 3 cycles. Since $\delta = \delta(G) \geqq 3$, it follows from Theorem 4 that there exists a monotonic search S with $\text{ms}(G) = \text{s}(G)$ searchers such that $\text{mex}_S(G) = \text{ms}(G) - 1$. Let $E_0, E_1, ..., E_m$ be the sequence of cleared edge sets corresponding to S. Let G_i be the graph induced by the cleared edges in E_i.

Case 1. $\delta \geqq g = 3$. Consider the smallest i such that G has one cleared vertex u at step i. Since $\deg(u) \geqq \delta$, G must have at least δ exposed vertices adjacent to u. Since S exposes at most $\text{ms}(G) - 1$ vertices, $\delta \leqq \text{s}(G) - 1$, and thus $\text{s}(G) \geqq \delta + 1 = \delta + g - 2$.

Case 2. $\delta \geqq g = 4$. Let i be the least number such that G has at least two cleared vertices u and v at step i. If u and v are adjacent, they can have no common neighbours, and since $\deg(u) \geqq \delta$ and $\deg(v) \geqq \delta$, they must both be adjacent to at least $\delta - 1$ exposed vertices each. This accounts for $2\delta - 2$ searchers, and $2\delta - 2 \geqq \delta + g - 2$, as required. If u and v are not adjacent, then they may share common neighbours. At worst, all their neighbours are common. Consider the graph G_{i-1}. Since u and v are not adjacent, only one of them can become cleared by the next move. Assume that v is already cleared at step $i - 1$, and u becomes clear at step i. Then v has at least δ exposed vertices adjacent to it, and certainly u itself is exposed at this point. Thus G must have at least $\delta + 1$ different exposed vertices at step $i - 1$. Since S exposes at most $\text{ms}(G) - 1$ vertices, $\delta + 1 \leqq \text{ms}(G) - 1$, and thus $\text{ms}(G) \geqq \delta + 2 = \delta + g - 2$.

Case 3. $\delta \geqq g \geqq 5$. Let i be the least number such that G has at least two cleared vertices u and v at step i. If these two vertices are adjacent, then one must have $\delta - 1$ exposed vertices adjacent to it, and the other must have at least $\delta - 2$ exposed vertices adjacent to it (it may be adjacent to a third cleared vertex). Thus $2\delta - 3 \leqq \text{ms}(G) - 1$, and $\text{ms}(G) \geqq 2\delta - 2 \geqq \delta + g - 2$. If u and v are not adjacent, they have at most one neighbour in common, and hence again must have at least $2\delta - 3$ exposed vertices between them. Thus, as above, $\text{ms}(G) \geqq \delta + g - 2$.

Case 4. $g > \delta = 3$. Consider the smallest i such that G_i contains exactly one cycle C. Then each vertex of this cycle is either exposed or cleared. (Since only one edge was cleared, if G_i contained more than one cycle, then G_{i-1} must have contained a cycle.) Let u be a cleared vertex in C. Consider the graph H obtained when the edges of C are removed from G_i. Certainly, H is a forest, as G_i contained exactly one cycle. Then u is certainly in one of the non-trivial component trees that make up H. Since there are no vertices of degree 1 in G, any vertices of degree 1 in H must be exposed. Thus, there is an exposed vertex in the tree containing u. Further, this exposed vertex cannot be an exposed vertex in C, as this would mean that G_i contains two cycles. Thus, for every cleared vertex in C, there is an exposed vertex in G. Certainly, for every exposed vertex in C there is a corresponding exposed vertex (itself), and the number of exposed vertices is at least g. Since the monotonic search strategy S exposes at most $\text{ms}(G) - 1$ vertices, $g \leqq \text{ms}(G) - 1$, and thus $\text{ms}(G) \geqq g + 1 \geqq \delta + g - 2$.

Case 5. $g > \delta \geqq 4$. Let i_1 be the smallest i such that G_{i_1} has two or more cycles. Accordingly, we know G_i has at most one cycle for any $i < i_1$. If C_1 and C_2 are two of the cycles formed and are vertex-disjoint, then as before, there is an exposed vertex that corresponds to each vertex in each cycle. But at most one exposed vertex may correspond to a vertex in both cycles. Thus the number of exposed vertices is at least $2g-1$, and so $\text{ms}(G) \geqq 2g \geqq \delta+g-2$. If C_1 and C_2 share exactly one common vertex, then there are at least $2g-2$ exposed vertices at step i_2. Again, $\text{ms}(G) \geqq 2g-1 \geqq \delta+g-2$. If C_1 and C_2 share more than one vertex, then G_{i_2} contains exactly three cycles. In this case, we consider step i_2, the first moment that the graph G_i contains four or more cycles.

Let C be the subgraph of G formed by $V(C) = V(C_1) \cup V(C_2)$ and $E(C) = E(C_1) \cup E(C_2)$, as shown in Figure 1(i). Let one of the new cycles formed be C_3. If C_3 is vertex-disjoint from C, then G_{i_2} contains two vertex-disjoint cycles, and as before, the number of exposed vertices is at least $2g-1$. Thus, $\text{ms}(G) \geqq 2g \geqq \delta+g-2$. If C_3 and C share exactly one vertex, then there are at least $2g-2$ exposed vertices at step i_2. Again, $\text{ms}(G) \geqq 2g-1 \geqq \delta+g-2$. Otherwise, C and C_3 share two or more vertices. We consider some subcases (see Figure 1).

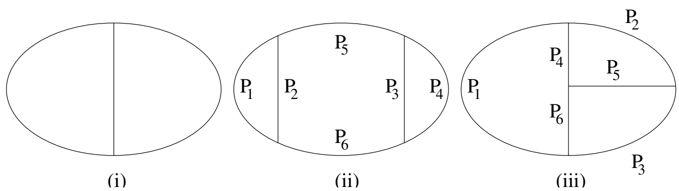

Fig. 1. (i) The graph C; (ii) Case 5(a); (iii) Case 5(b)

Case 5(a). In this case, we consider four cycles: the cycle induced by the paths P_1 and P_2; the cycle induced by P_2, P_3, P_5, and P_6; the cycle induced by P_3 and P_4; and finally the cycle induced by P_1, P_4, P_5, and P_6. These cycles all have length at least g. We note that either or both of P_5 and P_6 may be paths of length zero. Summing the lengths of the cycles, we see that we count each path, and hence each edge, exactly twice. Thus, in this subgraph G', $E' = E(G') \geqq 2g$. We next consider how many vertices are in $V' = V(G')$. If neither P_5 nor P_6 are paths of length zero, then summing vertex degrees over V' shows that $2(|V'|-4)+3\cdot 4 = 2|E|$, or that $|V'| = |E'|-2 \geqq 2g-2$. In this case, every vertex corresponds to an exposed vertex, and so $\text{ms}(G) \geqq 2g-1 \geqq \delta+g-2$. If exactly one of P_5 or P_6 is a path of length zero, then summing vertex degrees over V' shows that $2(|V'|-3)+2\cdot 3+4 = 2|E'|$, or that $|V'| = |E'|-2 \geqq 2g-2$. All but one of these vertices must correspond to an exposed vertex, so $\text{ms}(G) \geqq 2g-2 \geqq \delta+g-2$. Finally, if both P_5 and P_6 are paths of length zero, then summing vertex degrees over V' shows that $2(|V'|-2)+2\cdot 4 = 2|E'|$, or that $|V'| = |E'|-2$. In this case, however, all but two vertices must correspond to an exposed vertex, so the number of exposed vertices is at least $|E'|-4 \geqq 2g-4 \geqq \delta+g-3$, since $g \geqq \delta+1$. Thus, $\text{ms}(G) \geqq \delta+g-2$.

Case 5(b). In this case, we again consider four cycles: the cycle induced by the paths P_1, P_4, and P_6; the cycle induced by the paths P_2, P_4, and P_5; the cycle induced by the paths P_3, P_5, and P_6; and the cycle induced by the paths P_1, P_2, and P_3. Each cycle has length at least g. Consider the sum of the lengths of the cycles. Each path is counted twice, as is each edge. Thus, in this subgraph G', the total number of edges $|E'| \geqq 2g$. We sum the degrees of the vertices, and find that $2(|V'|-4)+4\cdot 3 = 2|E'|$, or that $|V'| = |E'|-2 \geqq 2g-2$. Since each vertex in G' corresponds to an exposed vertex, we see that $\mathrm{ms}(G) \geqq 2g-1 \geqq \delta+g-2$.

In fact, this result is best possible. Recall that the *complete bipartite graph* $K_{a,b}$ on $a+b$ distinct vertices, where $1 \leqq a \leqq b$, is the graph with vertex set $V(K_{a,b}) = \{v_1, v_2, \ldots, v_a\} \cup \{u_1, u_2, \ldots, u_b\}$ and edge set $E(K_{a,b}) = \{u_i v_j | 1 \leqq i \leqq b, 1 \leqq j \leqq a\}$. We now have sufficient tools to calculate the search number of the complete bipartite graph for all possible values of a and b.

Corollary 1. *Let* $1 \leqq a \leqq b$.

1. *If* $a = 1$ *and* $1 \leqq b \leqq 2$, *then* $\mathrm{s}(K_{a,b}) = 1$.
2. *If* $a = 1$ *and* $b \geqq 3$, *then* $\mathrm{s}(K_{a,b}) = 2$.
3. *If* $a = b = 2$, *then* $\mathrm{s}(K_{a,b}) = 2$.
4. *If* $a = 2$ *and* $b \geqq 3$, *then* $\mathrm{s}(K_{a,b}) = 3$.
5. *If* $3 \leqq a \leqq b$, *then* $\mathrm{s}(K_{a,b}) = a+2$.

A similar result can be shown for a complete multipartite graph.

Theorem 6. *For a complete multipartite graph* $K_{m_1,\ldots,m_k}$, *where* $m_1 \leqq \ldots \leqq m_k$, *if* $m_k \geqq 3$ *and* $k \geqq 3$, *then* $\mathrm{s}(G) = \sum_{i=1}^{k-1} m_i + 2$.

Proof. Let $\sum_{i=1}^{k-1} m_i = x$. It is easy to see that $\mathrm{s}(K_{m_1,\ldots,m_k}) \leqq x+2$. Suppose $K_{m_1,\ldots,m_k}$ can be cleared by $x+1$ searchers. By Theorem 4, there exists a monotonic search strategy S with $\mathrm{ms}(G)$ searchers such that $\mathrm{mex}_S(G) = \mathrm{ms}(G)-1 \leqq x$. Let $V_1, \ldots, V_k$ be the k parts of the vertex set with $|V_j| = m_j$, and $v \in V_i$ be the first cleared vertex using strategy S. Thus, all neighbours of v must be exposed vertices. If $m_i < m_k$, v has at least $x+1$ neighbours. This contradicts that $\mathrm{mex}_S(G) \leqq x$. If $m_i = m_k$, the x neighbours of v must be exposed vertices. Since $\mathrm{mex}_S(G) \leqq x$, each of other vertices in V_i must be contaminated. Since $m_i = m_k \geqq 3$, each of these exposed vertices has at least 2 contaminated edges incident on it. When we use the only free searcher to clear any edge incident on a vertex in $V_i - \{v\}$, we have $x+1$ exposed vertices, each of which is incident with at least two contaminated edges. Thus, no searcher can move, a contradiction.

The Petersen graph P is a cubic graph with girth 5. Thus, $\mathrm{s}(P) \geqq 6$. In fact, 6 searchers are sufficient. To see this, place a searcher on each of the vertices of a 5-cycle in P. Use a sixth searcher to clear the 5-cycle induced by these vertices. Move each searcher from the vertex it is on along the single remaining contaminated edge incident with it. This leaves searchers on every remaining uncleared vertex, and the sixth searcher can then clear the 5-cycle induced by these vertices, clearing the graph. In the same fashion, Theorem 5 implies that the Heawood graph and the McGee graph, which have girths 6 and 7, respectively, must have search numbers at least 7 and 8. In fact, it can be shown that these numbers are also sufficient to clear these graphs. The search strategies are similar to those for the Petersen graph.

3 Chromatic Number

If a graph G has a clique of order k, then at least k colours are required for a proper colouring. Thus, for any graph G, $\omega(G) \leqq \chi(G)$. Since we know that the clique number is a lower bound on the search number, it is reasonable to wonder whether Theorem 2 can be extended to the chromatic number.

Recall that the least number k such that the vertices of G can be ordered from v_1 to v_n in which each vertex is preceded by less than k of its neighbours is called the *colouring number* $\mathrm{col}(G)$ of G. Theorem 7 comes from [5]. Corollary 2 then follows directly from Theorems 7, 1, and 3.

Theorem 7. *For every connected graph G, $\chi(G) \leqq \mathrm{col}(G) \leqq \max\{\delta(H) | H \subseteqq G\} + 1$.*

Corollary 2. *For every connected graph G, $\chi(G) - 1 \leqq \mathrm{s}(G)$.*

We also offer a constructive proof for Corollary 2 which gives a proper colouring of G using at most $\mathrm{s}(G) + 1$ colours.

We begin by introducing the homeomorphic reduction of a reflexive multigraph X. Let $V' = \{u \in V(X) : \deg(u) \neq 2\}$. A *suspended path* in X is a path of length at least 2 joining two vertices of V' such that all internal vertices of the path have degree 2. A *suspended cycle* in X is a cycle of length at least 2 such that exactly one vertex of the cycle is in V' and all other vertices have degree 2. Let $V' = \{u \in V(X) : \deg(u) \neq 2\}$. The *homeomorphic reduction* of X is the reflexive multigraph X' obtained from X with vertex set V' and the following edges. Any loop of X incident with a vertex of V' is a loop of X' incident with the same vertex. Any edge of X joining two vertices of V' is an edge of X' joining the same two vertices. Any suspended path of X joining two vertices of V' is replaced by a single edge in X' joining the same two vertices. Any suspended cycle of X containing a vertex u of V' is replaced by a loop in X' incident with u. In the special case that X has connected components that are cycles, these cycles are replaced by loops on a single vertex.

Lemma 3. *If X is a connected reflexive multigraph and Y is its homeomorphic reduction, then $\mathrm{s}(X) = \mathrm{s}(Y)$.*

To obtain a bound on the search number involving chromatic number, we return to the idea of the maximum number of exposed vertices in a search.

Theorem 8. *If G is a connected reflexive multigraph with homeomorphic reduction G' and a monotonic search strategy S for G' such that $\mathrm{mex}_S(G') \geqq 3$, then $\chi(G) \leqq \mathrm{mex}_S(G') + 1$.*

Proof. Let $\mathrm{mex}_S(G') = k$. We will show that G is $(k+1)$-colourable. We first show that G' is $(k+1)$-colourable. Following the monotonic search strategy S that exposes at most k vertices in G', we can design a colouring such that it can colour G' using at most $k+1$ colours.

Initially, searchers are placed on G'. When a vertex first becomes exposed (or in the case of vertices of degree 1, becomes cleared), the vertex is coloured. This colour cannot be changed or erased in the following searching process. We now consider how to colour a vertex v in the moment it becomes exposed (or cleared, in the case of vertices of degree 1). Before this moment, v cannot be adjacent to any cleared vertex. Thus, each coloured vertex that is adjacent to v must be an exposed vertex. Since the number of exposed vertices is less then or equal to k, we can always assign v a colour that is different from the colours of the adjacent vertices of v. Thus, while S clears G', we can assign a colour to each vertex of G' such that any pair of adjacent vertices has different colours. Thus, G' is $(k+1)$-colourable.

We now show that G is $(k+1)$-colourable. For each vertex u in G', assign the colour of u in G' to the corresponding vertex u in G. Any uncoloured vertex in G must be on a suspended path or a suspended cycle. If it is on a suspended cycle, one vertex in this cycle has already been coloured. At most two more colours are needed to colour the remaining vertices of this cycle, but since $k \geqq 3$, we have a sufficient number of colours to do so. Similarly, if the vertex is in a suspended path, the ends of the suspended path have already been coloured. Now at most one more colour is needed to colour the remaining vertices of this path, but again, we have sufficient colours to do so. Hence, G is $(k+1)$-colourable. Therefore, $\chi(G) \leqq k+1$.

Combining Theorem 8 with Lemma 1, we obtain the following corollary, an improvement on Corollary 2.

Corollary 3. *If G is a connected reflexive multigraph and $\mathrm{s}(G) \geqq 3$, then $\chi(G) - 1 \leqq \mathrm{s}(G)$.*

Of course, we can do better. As demonstrated in Theorem 4, there are graphs where the maximum number of exposed vertices is one less than the search number.

Corollary 4. *If G is a connected reflexive graph with $\mathrm{s}(G) \geqq 3$ and the property that its homeomorphic reduction is not a multigraph, then $\chi(G) \leqq \mathrm{s}(G)$.*

Proof. Since G is not a multigraph, the homeomorphic reduction can only have multiple edges if two or more suspended paths have the same end points. Forbidding this, the homeomorphic reduction must be a graph with no vertices of degree 2, as required by Theorem 4. The result follows.

We now demonstrate an infinite family of graphs for which Corollary 2 provides a better bound than any of the others demonstrated here. Let P be the graph with vertex set $V(P) = \{v_i\}_{i=1}^{p+1}$, and edge set $E(P) = \{v_i v_j | 1 \leqq i < j \leqq p\} \cup \{v_1 v_{p+1}\}$. Thus, the graph P is a complete graph on p vertices with an extra edge incident with a vertex of degree 1.

We will employ the Mycielski construction [9]. Given a graph G, we form the graph $M(G)$, with vertex set $V(M(G)) = V(G) \cup V'(G) \cup \{u\}$, where $V'(G)$ contains the "twins" of $V(G)$. That is, $V'(G) = \{x' | x \in V(G)\}$. The edge set

$E(V(M)) = E(G) \cup \{x'y | xy \in E(G)\} \cup \{x'u | x' \in V'(G)\}$. That is, for each vertex $v \in V$, we introduce a new vertex v' adjacent to the neighbours of v. Finally, we add a new "super vertex" u which is adjacent to each new vertex v'. Similarly, we may define an infinite family of graphs by repeatedly applying a Mycielski construction. Define $M^0(G) = G$, and $M^t(G) = M(M^{t-1}(G))$ for $t \geqq 1$.

The Mycielski construction based on the 5-cycle C_5 was introduced in [9] to create an infinite family of triangle-free graphs with arbitrarily large chromatic number. In fact, $\chi(M^t(C_5)) = t + 3$ for $t \geqq 0$. More generally, for any graph G, it can be shown that $\omega(M^t(G)) = \omega(G)$, $\delta(M^t(G)) = \delta(G) + t$, and $\chi(M^t(G)) = \chi(G) + t$ for $t \geqq 0$.

Taking the graph P as defined above, it is clear that $\delta(P) = 1$, $\omega(P) = p$, and $\chi(P) = p$. Applying the Mycielski construction, we see that $\delta(M^t(P)) = 1 + t$, $\omega(M^t(P)) = p$, and $\chi(M^t(P)) = p + t$. As well, since P is a subgraph of $M^t(P)$, we know that $g(M^t(P)) = 3$ so long as $p \geqq 3$. So for large p and t, Theorem 5 tells us that $\delta(M^t(P)) + 1 \leqq t + 2 \leqq \mathrm{s}(M^t(P))$. Similarly, Theorem 2 tells us that $\omega(M^t(P)) = p \leqq \mathrm{s}(M^t(P))$. But Corollary 2 tells us that $\chi(M^t(P)) - 1 = p + t - 1 \leqq \mathrm{s}(M^t(P))$, a clear improvement.

References

1. Alspach, B.: Searching and sweeping graphs: A brief survey. Combinatorics 04 (Catania, 2004) Matematiche (Catania) 59 (2004), Fasc. I–II, pp. 5–37 (2004)
2. Bienstock, D., Seymour, P.: Monotonicity in graph searching. Journal of Algorithms 12, 239–245 (1991)
3. Fellows, M., Langston, M.: On search, decision and the efficiency of polynomial time algorithm. In: 21st ACM Symp. on Theory of Computing, pp. 501–512 (1989)
4. Frankling, M., Galil, Z., Yung, M.: Eavesdropping games: A graph-theoretic approach to privacy in distributed systems. Journal of ACM 47, 225–243 (2000)
5. Halin, R.: Unterteilungen vollständiger Graphen in Graphen mit unendlicher chromatischer Zahl. Abh. Math. Sem. Univ. Hamburg 31, 156–165 (1967)
6. Kirousis, L.M., Papadimitriou, C.H.: Searching and pebbling. Theoret. Comput. Sci. 47(2), 205–218 (1986)
7. LaPaugh, A.S.: Recontamination does not help to search a graph. Journal of ACM 40, 224–245 (1993)
8. Megiddo, N., Hakimi, S.L., Garey, M., Johnson, D., Papadimitriou, C.H.: The complexity of searching a graph. Journal of ACM 35, 18–44 (1988)
9. Mycielski, J.: Sur le coloriage des graphes. Coll. Math. 3, 161–162 (1955)
10. Tusa, Z.: Exponentially many distinguishable cycles in graphs. Graphs, designs and combinatorial geometries (Catania, 1989). J. Combin. Inform. System Sci. 15, 281–285 (1990)
11. Yang, B., Dyer, D., Alspach, B.: Sweeping graphs with large clique number (extended abstract). In: Fleischer, R., Trippen, G. (eds.) ISAAC 2004. LNCS, vol. 3341, pp. 880–892. Springer, Heidelberg (2004)