Variational Decomposition Model in Besov Spaces and Negative Hilbert-Sobolev Spaces

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Abstract. In this paper, we propose a new variational decomposition model which splits an image into two components: a first one containing the structure and a second one the texture or noise. Our decomposition model relies on the use of two semi-norms: the Besov semi-norm for the geometrical component, the negative Hilbert-Sobolev norms for the texture or noise. And the proposed model can be understood as generalizations of Daubechies-Teschke's model and have been motivated also by Lorenz's idea. And we illustrate our study with numerical examples for image decomposition and denoising.

1 Introduction

Image decomposition is of important interest in mathematical image processing. In principle, it can be understood as an inverse problem. Consequently, it can be done by regularization techniques and minimization of related variational functionals.

One classical model of such functionals is the total variation minimizing process introduced by Rudin-Osher-Fatemi [1]. However, since ROF model will remove the texture when tuning parameter is small enough, Meyer proposes that the oscillating components (texture or noise) should be modeled using a different space of functions that is in some sense dual to BV space. So, this leads to a new image decomposition model in theory [2]. Meyer's model cannot be solved directly, due to the existence of the weaker norm. Thus, a lot of people begin to study regarding practical methods of Meyer's model. For example, Vese-Osher proposed to solve Meyer's model using three Euler-Lagrange equations based on L^p norm [3]. Osher-Sole-Vese put forward the method combing total variation minimization with the H^{-1} norm based on VO model [4]. But, it is a pity that the PDEs based these variational models is usually numerically intensive. Thus, in [5], Daubechies-Teschke suggested a special variational model for image decomposition:

$$\inf_{u,v} F(u,v) = 2\alpha |u|_{B^1_{1,1}(\Omega)} + \|f-(u+v)\|^2_{L^2(\Omega)} + \gamma \|v\|^2_{H^{-1}(\Omega)} . \tag{1}$$

Since function spaces of interest in problem (1) can be characterized by means of wavelet coefficients, they propose a wavelet based scheme of (1) instead of solving PDE systems. Later in [8], Linh Lieu successfully generalized Osher-Sole-Vese's

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model for image restoration and decomposition in a total variation minimization framework. She proposed that the oscillating component can be modeled by tempered distributions belonging to the negative Hilbert-Sobolev spaces H^{-s} ($s>0$).

Here inspired from Linh Lieu's idea, it occurred to us that the textured (or noisy) component v in (1) can be characterized via negative Hilbert-Sobolev spaces H^{-s}. In addition, since Besov spaces $B^{\beta}_{p,q}(\Omega)$ ($\beta>0$, $0<p,q\le\infty$) cover a wide range of classical smoothness spaces and the Besov semi-norms can be expressed through equivalent norms of wavelet coefficients[6], we propose to generalize the first term $|u|_{B^{1}_{1,1}(\Omega)}$ to $|u|_{B^{\beta}_{p,q}(\Omega)}$ in (1). But we are only interested in the especially simple case $p=q$. Therefore, the new variational model for image decomposition is

$$\inf_{u,v} E(u,v)=2\alpha|u|_{B^{\beta}_{p,p}(\Omega)}+\|f-(u+v)\|^2_{L^2(\Omega)}+\gamma\|v\|^2_{H^{-s}(\Omega)}, \tag{2}$$

where α and γ are tuning parameters, $1\le p\le\infty$.

The outline of the paper is as follows. In section 2 we give minimization process of the new variational model (2). It can be understood as generalizations of [5, 6]. In section 3 we discuss some examples of the new variational problem. Section 4 shows numerical results of image decomposition and denoising examples using (2). Finally, we give the conclusions in section 5.

2 Minimization of the New Model

In this section, we consider the minimization of the new variational problem (2). Since $B^{\beta}_{2,2}(\Omega)=H^{\beta}(\Omega)$ [5], we consider only the spaces $B^{\beta}_{p,p}(\Omega)$ 、 $L^2(\Omega)=B^{0}_{2,2}(\Omega)$ and $H^{-s}(\Omega)=B^{-s}_{2,2}(\Omega)$ in (2).

For an orthogonal wavelet ψ, which is in $B^{s}_{2,2}(\Omega)$ ($s>\beta$), we have the following norm equivalence [5]:

$$\begin{aligned}
\|f-(u+v)\|^2_{L^2(\Omega)} &\approx \sum_{\lambda\in J}|f_\lambda-(u_\lambda+v_\lambda)|^2\\
\|v\|^2_{H^{-s}(\Omega)} &\approx \sum_{\lambda\in J}2^{-2s|\lambda|}|v_\lambda|^2 \qquad ,\\
|u|_{B^{\beta}_{p,p}(\Omega)} &\approx \left(\sum_{\lambda\in J}2^{\beta|\lambda|p}2^{|\lambda|(p-2)}|u_\lambda|^p\right)^{\frac{1}{p}}
\end{aligned} \tag{3}$$

where $J=\{\lambda=(i,j,k):k\in J_j, j\in Z, i=1,2,3\}$, $|\lambda|=j$ if $\lambda\in J_j$, f_λ , u_λ , v_λ denote the λ-th wavelet coefficients.

Replacing the norms in (2) by (3), we obtain the equivalent sequence in wavelet framework

$$W_f(u,v)=2\alpha\left(\sum_{\lambda\in J}2^{\beta|\lambda|p}2^{|\lambda|(p-2)}|u_\lambda|^p\right)^{\frac{1}{p}}+\sum_{\lambda\in J}\left(|f_\lambda-(u_\lambda+v_\lambda)|^2+\gamma 2^{-2s|\lambda|}|v_\lambda|^2\right). \tag{4}$$

Let u_λ be fixed in (4), then the derivative of $W_f(u,v)$ with respect to v_λ can be expressed by

$$D_{v_\lambda}\left(W_f(u,v)\right) = -2(f_\lambda - u_\lambda) + 2(1+\gamma 2^{-2s|\lambda|})v_\lambda .$$

Set $D_{v_\lambda}\left(W_f(u,v)\right)=0$, one has

$$v_\lambda = (1+\gamma 2^{-2s|\lambda|})^{-1}(f_\lambda - u_\lambda) . \tag{5}$$

Replacing v_λ by (5) in (4), we have

$$W_f(u,v) = 2\alpha\left(\sum_{\lambda\in J} 2^{\beta|\lambda|p}2^{|\lambda|(p-2)}\left|u_\lambda\right|^p\right)^{\frac{1}{p}} + \sum_{\lambda\in J}\frac{\gamma 2^{-2s|\lambda|}}{1+\gamma 2^{-2s|\lambda|}}(f_\lambda - u_\lambda)^2 . \tag{6}$$

Set $\mu_\lambda = \dfrac{\gamma 2^{-2s|\lambda|}}{1+\gamma 2^{-2s|\lambda|}}$ and $\phi\left((u_\lambda)\right) = \left(\sum_{\lambda\in J} 2^{\beta|\lambda|p}2^{|\lambda|(p-2)}\left|u_\lambda\right|^p\right)^{\frac{1}{p}}$ in (6), then one has

$$Q_{f_\lambda}(u_\lambda) = 2\alpha\phi\left((u_\lambda)\right) + \sum_{\lambda\in J}\mu_\lambda(f_\lambda - u_\lambda)^2 . \tag{7}$$

Note that here ϕ is positive homogeneous of degree one. Since the duality between positive homogeneous functions and convex sets holds for convex functions, we consider only the case $1\le p\le\infty$ in this paper.

In the following, we, inspired from [6, 9], minimize (7) using duality result from convex analysis.

Proposition 1. Let $\{f_\lambda\}\in \ell^2(J)$ and $1\le p\le\infty$. Then the wavelet coefficients of the minimizer of problem (7) is

$$u_\lambda = \left(Id - \Pi_{\theta_\lambda C}\right)(f_\lambda) . \tag{8}$$

where $\theta_\lambda = {}^{\alpha}\!/_{\mu_\lambda}$ and Π_C is the orthogonal projection onto the convex set

$$C = \left\{ x\in \ell^2(J) \,\middle|\, \sum_{\lambda\in J} x_\lambda y_\lambda \le \phi\left((y_\lambda)\right), \forall y\in \ell^2(J)\right\}. \tag{9}$$

Proof. Since ϕ is homogeneous of degree one, it is standard [7] that the Legendre-Fenchel transform of ϕ

$$\phi^*(w_\lambda) = \sup\left(\langle u_\lambda, w_\lambda\rangle_{\ell^2(J)} - \phi(u_\lambda)\right) = \sup\left(\left(\sum_{\lambda\in J} u_\lambda w_\lambda\right) - \phi(u_\lambda)\right)$$

is the indicator function of a convex set C:

$$\phi^*(w_\lambda) = \begin{cases} 0 & if\ w_\lambda\in C \\ +\infty & otherwise \end{cases} . \tag{10}$$

Since ϕ is convex and l.s.c., $\phi^{**}=\phi$. Hence $\phi(u_\lambda) = \sup_{w_\lambda\in C}\langle u_\lambda, w_\lambda\rangle_{\ell^2(J)} = \sup_{w_\lambda\in C}\left(\left(\sum_{\lambda\in J} u_\lambda w_\lambda\right)\right)$.

If u_λ is a minimizer of (7), then necessary condition is

$$0 \in \partial Q_{f_\lambda}\left((u_\lambda)\right). \tag{11}$$

Since the subgradient of the second term of (7) with respect to u_λ is $\{-2\mu_\lambda(f_\lambda - u_\lambda)\}$, one has

$$\partial Q_{f_\lambda}(u_\lambda) = 2\alpha\partial\phi\left((u_\lambda)\right) - 2\mu_\lambda(f_\lambda - u_\lambda).$$

Hence

$$\frac{f_\lambda - u_\lambda}{\theta_\lambda} \in \partial\phi\left((u_\lambda)\right), \tag{12}$$

where $\theta_\lambda = \alpha/\mu_\lambda$. From the inversion rules for subgradients ([7] prop. 11.3), we know that (12) is equivalent to:

$$0 \in \frac{f_\lambda - u_\lambda}{\theta_\lambda} - \frac{f_\lambda}{\theta_\lambda} + \frac{1}{\theta_\lambda}\partial\phi^*\left(\frac{f_\lambda - u_\lambda}{\theta_\lambda}\right). \tag{13}$$

So $w = \dfrac{f_\lambda - u_\lambda}{\theta_\lambda}$ is the minimizer of $\dfrac{\left\| w - f_\lambda/\theta_\lambda \right\|}{2} + \dfrac{1}{\theta_\lambda}\phi^*(w)$.

Being ϕ^* given by (10), w is given by the orthogonal projection of $\dfrac{f_\lambda}{\theta_\lambda}$ on the convex set C. Indeed, from (13), one has

$$\frac{f_\lambda}{\theta_\lambda} \in \left(Id + \frac{1}{\theta_\lambda}\partial\phi^*\right)\left(\frac{f_\lambda - u_\lambda}{\theta_\lambda}\right) \Rightarrow w \in \left(Id + \frac{1}{\theta_\lambda}\partial\phi^*\right)^{-1}\left(\frac{f_\lambda}{\theta_\lambda}\right). \tag{14}$$

Set $\Pi_C\left(\dfrac{f_\lambda}{\theta_\lambda}\right) = \left(Id + \dfrac{1}{\theta_\lambda}\partial\phi^*\right)^{-1}\left(\dfrac{f_\lambda}{\theta_\lambda}\right)$, then $\Pi_{\theta_\lambda C}(f_\lambda) = \theta_\lambda\left(Id + \dfrac{1}{\theta_\lambda}\partial\phi^*\right)^{-1}(f_\lambda)$. Thus

$$\left(\frac{1}{\theta_\lambda}\right)\Pi_{\theta_\lambda C}(f_\lambda) = \Pi_C\left(\frac{f_\lambda}{\theta_\lambda}\right) = \frac{f_\lambda - u_\lambda}{\theta_\lambda} \Rightarrow u_\lambda = \left(Id - \Pi_{\theta_\lambda C}\right)(f_\lambda).$$

Here replacing u_λ by (8) in (5), one obtain the expression of v_λ. Therefore, minimizers of (2) can be expressed as:

$$v = \sum_{\lambda \in J}(1 + \gamma 2^{-2s|\lambda|})^{-1}\left(\Pi_{\theta_\lambda C}(f_\lambda)\right)\psi_\lambda, \tag{15}$$

and

$$u = \langle f, 1\rangle + \sum_{\lambda \in J}\left(\left(Id - \Pi_{\theta_\lambda C}\right)(f_\lambda)\right)\psi_\lambda, \tag{16}$$

where the scale function is equal to one and ψ is orthogonal wavelet.

3 Some Examples of the New Model

In order to illustrate concretely the minimization of the new model, we consider the three cases $p=1$, $p=2$ and $p=\infty$ separately in this section. Here what is important to us is that one can obtain the convex sets that are related to three examples. In terms of the description of section 2 and Lorenz's work [6, 9], one has

$$C=\left\{x_\lambda \in l^2(J)\left|\left(\sum_{\lambda\in J} 2^{-2|\lambda|\beta}\left|x_\lambda\right|^2\right)^{\frac{1}{2}} \le 1\right.\right\},\quad (p=2) \tag{17}$$

$$C=\left\{x_\lambda \in l^2(J)\left|\sup_{\lambda\in J} 2^{-|\lambda|(\beta-1)}\left|x_\lambda\right| \le 1\right.\right\},\quad (p=1) \tag{18}$$

and

$$C=\left\{x_\lambda \in l^2(J)\left|\sum_{\lambda\in J} 2^{-|\lambda|(\beta+1)}\left|x_\lambda\right| \le 1\right.\right\},\quad (p=\infty)\,. \tag{19}$$

3.1 The Penalty $|\cdot|_{B^{\beta}_{1,1}(\Omega)}$

From (18), one obtains the convex set which is located by the projection:

$$\theta_\lambda C=\left\{x\in \ell^2(J)\left|\sup_{\lambda\in} 2^{-|\lambda|(\beta-1)}\left|x_\lambda\right| \le \theta_\lambda\right.\right\}. \tag{20}$$

Then this projection is performed by the following clipping function [6], i.e.

$$\Pi_{\theta_\lambda C}(f_\lambda)=\left(C_{2^{|\lambda|(\beta-1)}\theta_\lambda}(f_\lambda)\right)=\begin{cases} 2^{|\lambda|(\beta-1)}\theta_\lambda & f_\lambda \ge \theta_\lambda \\ f_\lambda & |f_\lambda|<\theta_\lambda \\ -2^{|\lambda|(\beta-1)}\theta_\lambda & f_\lambda \le -\theta_\lambda \end{cases}. \tag{21}$$

Clearly, (8) is a soft shrinkage function:

$$u_\lambda = S_{2^{|\lambda|(\beta-1)}\theta_\lambda}\left(f_\lambda\right). \tag{22}$$

Replacing u_λ by (22) in (5), one has

$$v_\lambda=\left(1+\gamma 2^{-2s|\lambda|}\right)^{-1} C_{2^{|\lambda|(\beta-1)}\theta_\lambda}\left(f_\lambda\right). \tag{23}$$

If set $\beta=1$ and $s=1$, (22) and (23) reduce to Daubechies-Teschke's results [5].

3.2 The Penalty $|\cdot|_{B^{\beta}_{2,2}(\Omega)}$

In this case, it can be seen as the example for $1<p<\infty$. From (17), we know that the projection which one must calculate is the orthogonal projection onto the convex set:

$$\theta_\lambda C=\left\{x\in \ell^2(J)\left|\sum_{\lambda\in J} 2^{-2|\lambda|\beta}\left|x_\lambda\right|^2 \le \theta_\lambda{}^2\right.\right\}. \tag{24}$$

Then this projection is characterized by the constrained minimization problem

$$\min \sum_{\lambda\in J}(x_\lambda - f_\lambda)^2 \ s.t. \sum_{\lambda\in J} 2^{-2|\lambda|\beta}|x_\lambda|^2 \le \theta_\lambda^2 . \tag{25}$$

Using Lagrange multipliers $\mu > 0$, this problem can be rewritten as

$$\min_{x_\lambda}\left\{ F(x_\lambda) = \sum_{\lambda\in J}(f_\lambda - x_\lambda)^2 + \mu 2^{-2|\lambda|\beta}|x_\lambda|^2 \right\} .$$

Set $F'(x_\lambda) = 0$, one has

$$x_\lambda = \frac{f_\lambda}{1+\mu 2^{-2|\lambda|\beta}} . \tag{26}$$

Replacing x_λ by (26) in (24) yields

$$\theta_\lambda^2 = \sum_{\lambda\in J}\frac{2^{-2|\lambda|\beta}}{\left(1+\mu 2^{-2|\lambda|\beta}\right)^2}|f_\lambda|^2 . \tag{27}$$

Here we discover that the right side of (27) is monotonically decreasing and continuous in μ. If μ increases from 0 to ∞, (27) decreases from $2^{-2|\lambda|\beta}|f_\lambda|^2$ to 0. Thus, this indicates that there is a Lagrange multipliers $\mu > 0$ such that (26) is the projection. Replacing $\prod_{\theta_\lambda C}$ by (26) in (8), one has

$$u_\lambda = \frac{1}{1+2^{2|\lambda|\beta+1}\left(\frac{1}{2\mu}\right)} f_\lambda . \tag{28}$$

This is a linear shrinkage operator which depends on the scale $|\lambda|$ and Besov smooth order β, where $\mu = \frac{1}{2\theta_\lambda}$.

Replacing u_λ by (28) in (5), we have

$$v_\lambda = \left(1+\gamma 2^{-2s|\lambda|}\right)^{-1}\frac{2^{2|\lambda|\beta}}{2^{2|\lambda|\beta}+\mu}(f_\lambda) . \tag{29}$$

3.3 The Penalty $|\cdot|_{B^{\beta}_{\infty,\infty}(\Omega)}$

In this section, (19) shows that the convex set which we concern is

$$\theta_\lambda C = \left\{ x \in \ell^2(J) \,\middle|\, \sum_{\lambda\in J} 2^{-|\lambda|(\beta+1)}|x_\lambda| \le \theta_\lambda \right\} . \tag{30}$$

Similar to the case $p = 2$, we have

$$x_\lambda = f_\lambda - \frac{\mu}{2}2^{-|\lambda|(\beta+1)}\operatorname{sign}(f_\lambda) . \tag{31}$$

From section 3.2., we know that here the projection is the soft shrinkage, i.e. $x_\lambda = S_{\frac{\mu}{2}2^{-|\lambda|(\beta+1)}}(f_\lambda)$. Therefore, replacing $\Pi_{\theta_\lambda C}$ by $S_{\frac{\mu}{2}2^{-|\lambda|(\beta+1)}}$ in (8) yields the clipping function:

$$u_\lambda = C_{\frac{\mu}{2}2^{-|\lambda|(\beta+1)}}(f_\lambda). \quad (32)$$

Replacing u_λ by (32) in (5), one obtains

$$v_\lambda = \left(1+\gamma 2^{-2s|\lambda|}\right)^{-1} S_{\frac{\mu}{2}2^{-|\lambda|(\beta+1)}}(f_\lambda). \quad (33)$$

Finally, replacing v_λ and u_λ separately by (23)、(29)、(33) and (22)、(28)、(32) in (15) and (16), we obtain the associated minimizers of the new model in three cases.

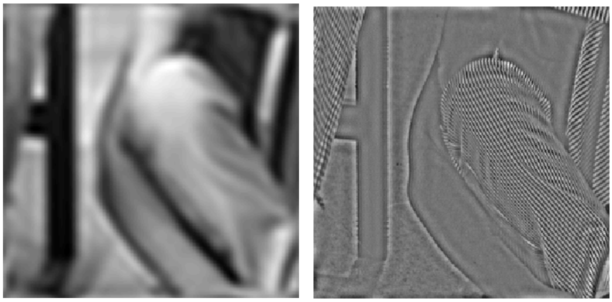
4 Numerical Examples

In this section we present numerical results obtained by applying our proposed new model to image decomposition and denoising in the case $p=1$, $p=2$ and $p=\infty$. In our implementation, the stationary wavelet transform is used. We will show numerical results obtained with various values of β and s. For denoising, the peak-signal-to-noise (PSNR) are used to evaluate the denoising performance.

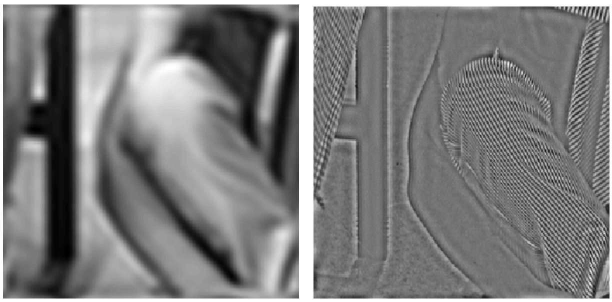
Firstly, we try texture removal with an intercepting part of Barbara image (shown in Figure 1). The results are shown in Figure 2. We can see that the new model (2) can separates better the textured details v from non-textured image kept in u.

Secondly, we show the denoising results obtained from the proposed new model (2). We add Gaussian white noise of $\sigma=10$ to the clean Lena image (shown in Figure 3). Table 1 gives PSNR for the denosing results. In Figure 4, we show denoisng results from our proposed model using $\beta=1$, $s=1$ and $\beta=2$, $s=2$, respectively, for the $B^\beta_{p,q}$ semi-norm and H^{-s} norm. These show that the proposed new model (2) can denoise effectively.

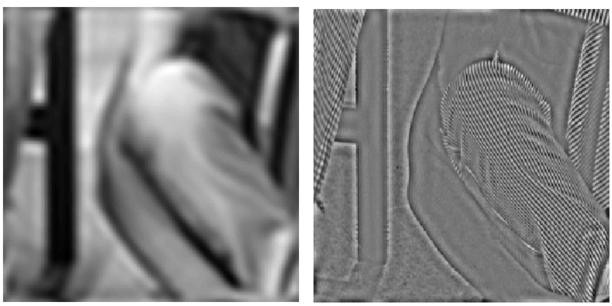
Fig. 1. Original image



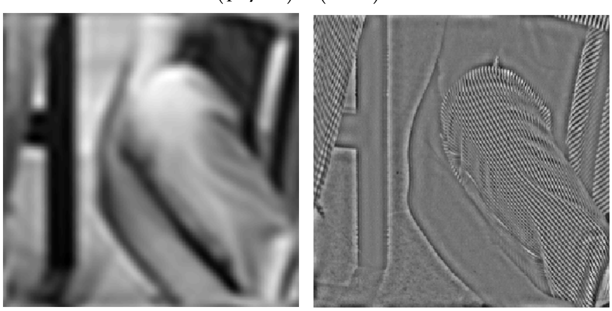
$(p, \beta, s) = (1,1,1)$ [5]



$(p, \beta, s) = (1,2,2)$

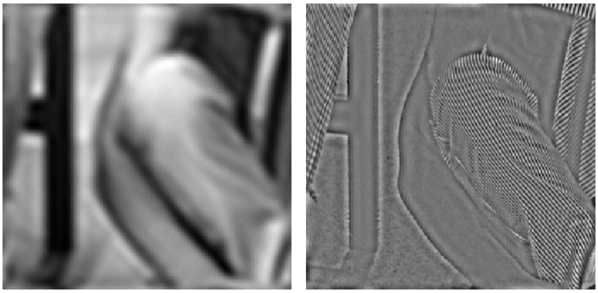


$(p, \beta, s) = (2,1,1)$

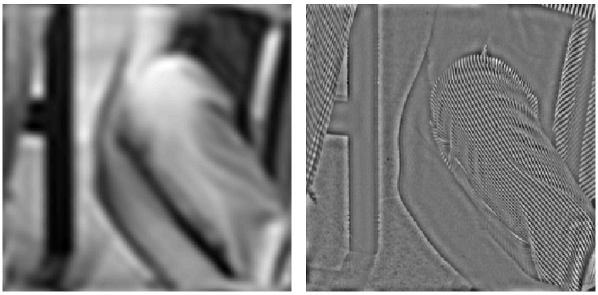


$(p, \beta, s) = (2,2,2)$

Fig. 2. Decomposition results of a natural textured image from the new model (2) based on the different parameter choice (p, β, s)



$(p, \beta, s) = (\infty, 1, 1)$



$(p, \beta, s) = (\infty, 2, 2)$

Fig. 2. (*continued*)

Table 1. PSNR for the denoising results

The values of p, β and s			PSNR
Noisy image			28.1058
$p=1$	$\beta=1$	$s=1$ [5]	31.0034
	$\beta=2$	$s=2$	29.5262
$p=2$	$\beta=1$	$s=1$	31.1634
	$\beta=2$	$s=2$	29.9701
$p=\infty$	$\beta=1$	$s=1$	31.3668
	$\beta=2$	$s=2$	31.3687

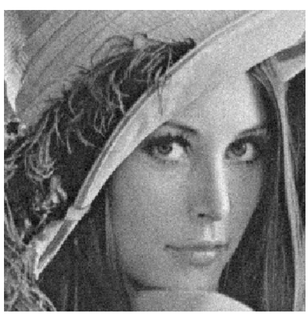
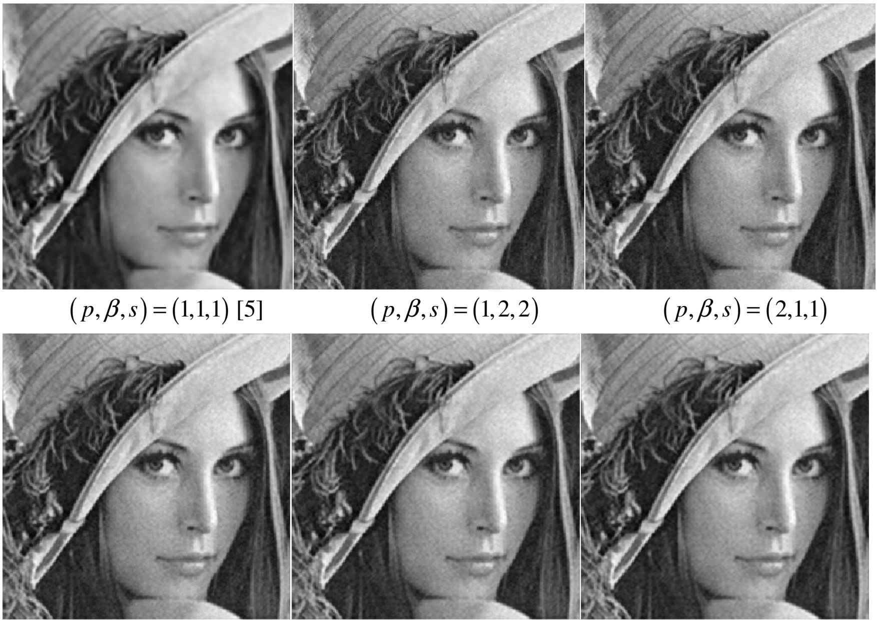


Fig. 3. Noisy image



$(p,\beta,s)=(1,1,1)$ [5] $(p,\beta,s)=(1,2,2)$ $(p,\beta,s)=(2,1,1)$

$(p,\beta,s)=(2,2,2)$ $(p,\beta,s)=(\infty,1,1)$ $(p,\beta,s)=(\infty,2,2)$

Fig. 4. Denoising results from the new model (2) for different parameter choice (p,β,s)

5 Conclusion

In this paper, we have presented a new variational model for image decomposition, which is based on Besov spaces and negative Hilbert-Sobolev spaces. And we, inspired by Lorenz, give proof for the general characterization of the solution of the proposed model based on the orthogonal projections onto the convex set, as well as some material examples. But the optimal choice of tuning parameters in new model is still a remaining problem.

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