Multi-cumulant Control for Zero-Sum Differential Games: Performance-Measure Statistics and State-Feedback Paradigm

Khanh D. Pham

Space Vehicles Directorate
Air Force Research Laboratory
Kirtland AFB, NM 87117 U.S.A
khanh.pham@kirtland.af.mil

Abstract. This chapter presents an extension of cost-cumulant control theory over a finite horizon for a class of stochastic zero-sum differential games wherein the evolution of the states of the game in response to decision strategies selected by two players from sets of admissible controls is described by a stochastic linear differential equation and a standard integral-quadratic cost. A direct dynamic programming approach for the Mayer optimization problem is used to solve for a multi-cumulant based solution when both players measure the states and minimize the first finite number of cumulants of the standard integral-quadratic cost associated with this special class of differential games. This innovative decision-making paradigm is proposed herein to provide not only a mechanism in which the conflicting interests of noncooperative players can be optimized, but also an analytical tool which is used to provide a complete statistical description of the global performance of the stochastic differential game.

1 Introduction

This chapter considers a closed-loop two-person zero-sum linear-quadratic game wherein the dynamics of the game in response to control variables selected by both players from a class of linear-feedback controllers is described by a stochastic linear differential equation. In seeking optimal control strategies whose respective objectives are minimization and maximization of a finite linear combination of the first k cost cumulants of an integral-quadratic random cost associated with the class of linear stochastic systems over a finite horizon, the recently developed statistical control theory [7]-[17] is extended herein. The extension which is manifested through the resulting cumulant-generating equations, now allows the incorporation of classes of linear feedback controllers to affect and predict more accurately the effects of non-Gaussian perturbations on the accuracy of system performance via a complete statistical description. In other words, using these high-order cost cumulants, it is possible to obtain an approximation of the system performance distribution.

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Since the formulation of multi-cumulant and zero-sum games is parameterized both by the number of cost cumulants and by the scalar coefficients in the linear combination, it may be viewed both as a generalization of linear-quadratic Gaussian control, when the first cost cumulant is optimized and of two-person zero-sum differential games when a certain denumerable linear combination of cost cumulants is optimized. The set of coupled matrix Riccati differential equations is introduced, whose solvability leads to the existence of the closed-loop feedback saddle points for the corresponding multi-cumulant and zero-sum game under some additional mild conditions. It is worth mentioning that the multi-cumulant and zero-sum game is an initial cost problem, in contrast with the more traditional terminal cost class of investigations. One may address an initial cost problem by introducing changes of variables which convert it to a terminal cost problem. However, this modifies the natural context of cost cumulants, which it is preferable to retain. Instead, one may take a more direct dynamic programming approach to the initial cost problem. Such an approach is illustrative of the more general concept of the principle of optimality, an idea tracing its roots back to the 17th century.

2 Problem Formulation

Let's consider a zero-sum stochastic differential game with two noncooperative players, identified as u_1 and u_2. Suppose $(t_0, x_0) \in [t_0, t_f] \times \mathbb{R}^n$ is fixed and a system input noise $w(t) \triangleq w(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^p$ is an p-dimensional stationary Wiener process defined with $\{\mathcal{F}_t\}_{t\geq 0}$ being its natural filtration on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ over $[t_0, t_f]$ with the correlation of increments

$$E\left\{[w(\tau) - w(\xi)][w(\tau) - w(\xi)]^T\right\} = W|\tau - \xi|, \quad W > 0 \ .$$

Also, decision sets $\mathcal{U}_1 \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_1}))$ and $\mathcal{U}_2 \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_2}))$ are assumed to be the subsets of Hilbert space of $\mathbb{R}^{m_1}$-valued and $\mathbb{R}^{m_2}$-valued, square integrable processes on $[t_0, t_f]$ that are adapted to the σ-field $\mathcal{F}_t$ generated by $w(t)$, respectively. Associated with each $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ is a standard finite-horizon integral-quadratic form (IQF) random cost $J : [t_0, t_f] \times \mathbb{R}^n \times \mathcal{U}_1 \times \mathcal{U}_2 \mapsto \mathbb{R}^+$ (for which the first player, u_1 tries to minimize, while the second player, u_2 attempts to maximize it) such that

$$\begin{aligned} J(t_0, x_0; u_1, u_2) &= x^T(t_f) Q_f x(t_f) \\ &+ \int_{t_0}^{t_f} \left[x^T(\tau)Q(\tau)x(\tau) + u_1^T(\tau)R_{11}(\tau)u_1(\tau) - u_2^T(\tau)R_{22}(\tau)u_2(\tau)\right] d\tau \ , \end{aligned} \quad (1)$$

where the system states of the game, $x(t) \triangleq x(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^n$ belong to the Hilbert space $L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^n))$ with $E\left\{\int_{t_0}^{t_f} x^T(\tau)x(\tau)d\tau\right\} < \infty$ and evolve according to the stochastic differential equation

$$dx(t) = (A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t))\,dt + G(t)dw(t) \quad (2)$$
$$x(t_0) = x_0 \ .$$

The system coefficient matrices $A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})$, $B_1 \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times m_1})$, $B_2 \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times m_2})$ and $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p})$ are deterministic bounded matrix-valued functions. The terminal penalty weighting $Q_f \in \mathbb{R}^{n \times n}$, the state weighting $Q \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})$ and control weightings $R_{11} \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_1 \times m_1})$, and $R_{22} \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_2 \times m_2})$ are deterministic bounded matrix-valued functions with properties of symmetry and positive semi-definiteness. In addition, $R_{11}(t)$ and $R_{22}(t)$ are invertible.

To put this stochastic differential game in a class of closed-loop feedback control, it is observed that the system (2) is linear and the performance measure (1) is quadratic. Therefore, it is reasonable to assume that the players choose control actions that are optimal within the class of memoryless perfect-state strategies, $\gamma_1 : [t_0, t_f] \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^n)) \mapsto L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_1}))$ and $\gamma_2 : [t_0, t_f] \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^n)) \mapsto L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_2}))$

$$u_1(t) = \gamma_1(t, x(t)) = K_1(t)x(t) \ , \quad (3)$$
$$u_2(t) = \gamma_2(t, x(t)) = K_2(t)x(t) \ , \quad (4)$$

where the admissible gains $K_1 \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_1 \times n})$ and $K_2 \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_2 \times n})$ are deterministic bounded matrix-valued functions defined in appropriate senses.

For a given initial condition $(t_0, x_0) \in [t_0, t_f] \times \mathbb{R}^n$ and subject to strategies (3)-(4), the dynamics of the game (2) is given by

$$dx(t) = [A(t) + B_1(t)K_1(t) + B_2(t)K_2(t)]\, x(t)dt + G(t)dw(t) \ , \quad (5)$$
$$x(t_0) = x_0 \ ,$$

and its IQF cost in the form of a Chi-square random variable, follows

$$J(t_0, x_0; K_1, K_2) = x^T(t_f)Q_f x(t_f)$$
$$+ \int_{t_0}^{t_f} x^T(\tau) \left[Q(\tau) + K_1^T(\tau)R_{11}(\tau)K_1(\tau) - K_2^T(\tau)R_{22}(\tau)K_2(\tau)\right] x(\tau)d\tau \ . \quad (6)$$

It is necessary to develop a procedure for generating cost cumulants of the two-player zero-sum differential game by adapting the parametric method in [5] to characterize a moment-generating function. These cost cumulants are then used to form performance index in the cost-cumulant control optimization. This approach begins with a replacement of the initial condition (t_0, x_0) by any arbitrary pair (α, x_α). Thus, for the given admissible feedback gains K_1 and K_2, the cost functional (6) is seen as the "cost-to-go", $J(\alpha, x_\alpha)$

$$J(\alpha, x_\alpha) \triangleq x^T(t_f)Q_f x(t_f)$$
$$+ \int_{\alpha}^{t_f} x^T(\tau) \left[Q(\tau) + K_1^T(\tau)R_{11}(\tau)K_1(\tau) - K_2^T(\tau)R_{22}(\tau)K_2(\tau)\right] x(\tau)d\tau \ .$$

The moment-generating function of the vector-valued random process (5) is given by the definition

$$\varphi\left(\alpha, x_\alpha; \theta\right) \triangleq E\left\{\exp\left(\theta J\left(\alpha, x_\alpha\right)\right)\right\} \ , \tag{7}$$

where the scalar $\theta \in \mathbb{R}^+$ is a small parameter. What follows next is the cumulant-generating function

$$\psi\left(\alpha, x_\alpha; \theta\right) \triangleq \ln\left\{\varphi\left(\alpha, x_\alpha; \theta\right)\right\} \ , \tag{8}$$

in which $\ln\{\cdot\}$ denotes the natural logarithmic transformation of an enclosed entity.

Theorem 1. *Cost Cumulant Generating Function.*
For all $\alpha \in [t_0, t_f]$ and the small parameter $\theta \in \mathbb{R}^+$, define

$$\varphi\left(\alpha, x_\alpha; \theta\right) \triangleq \varrho\left(\alpha; \theta\right) \exp\left(x_\alpha^T \Upsilon(\alpha; \theta) x_\alpha\right) \ , \tag{9}$$

$$\upsilon\left(\alpha; \theta\right) \triangleq \ln\{\varrho\left(\alpha; \theta\right)\} \ . \tag{10}$$

Then, the cost-cumulant generating function is expressed by

$$\psi\left(\alpha, x_\alpha; \theta\right) = x_\alpha^T \Upsilon(\alpha; \theta) x_\alpha + \upsilon\left(\alpha; \theta\right) \ , \tag{11}$$

where the scalar solution $\upsilon\left(\alpha; \theta\right)$ solves the time-backward differential equation with the terminal boundary condition $\upsilon\left(t_f; \theta\right) = 0$

$$\frac{d}{d\alpha} \upsilon\left(\alpha; \theta\right) = -\mathrm{Tr}\left\{\Upsilon(\alpha; \theta) G\left(\alpha\right) W G^T\left(\alpha\right)\right\} \ , \tag{12}$$

and the matrix-valued solution $\Upsilon(\alpha; \theta)$ satisfies the time-backward differential equation together with its terminal-valued condition $\Upsilon(t_f; \theta) = \theta Q_f$

$$\begin{aligned} \frac{d}{d\alpha} \Upsilon(\alpha; \theta) = &-[A(\alpha) + B_1(\alpha) K_1(\alpha) + B_2(\alpha) K_2(\alpha)]^T \Upsilon(\alpha; \theta) \\ &- \Upsilon(\alpha; \theta)[A(\alpha) + B_1(\alpha) K_1(\alpha) + B_2(\alpha) K_2(\alpha)] \\ &- 2\Upsilon(\alpha; \theta) G(\alpha) W G^T(\alpha) \Upsilon(\alpha; \theta) \\ &- \theta\left[Q(\alpha) + K_1^T(\alpha) R_{11}(\alpha) K_1(\alpha) - K_2^T(\alpha) R_{22}(\alpha) K_2(\alpha)\right] \ . \end{aligned} \tag{13}$$

In addition, the auxiliary solution $\varrho(\alpha; \theta)$ is satisfying the time-backward differential equation with the terminal boundary condition $\varrho\left(t_f; \theta\right) = 1$

$$\frac{d}{d\alpha} \varrho\left(\alpha; \theta\right) = -\varrho\left(\alpha; \theta\right) \mathrm{Tr}\left\{\Upsilon(\alpha; \theta) G\left(\alpha\right) W G^T\left(\alpha\right)\right\} \ . \tag{14}$$

Proof. For any given θ, let $\varpi\left(\alpha, x_\alpha; \theta\right) \triangleq \exp\left(\theta J\left(\alpha, x_\alpha\right)\right)$. The moment-generating function becomes

$$\varphi\left(\alpha, x_\alpha; \theta\right) = E\left\{\varpi\left(\alpha, x_\alpha; \theta\right)\right\} \ ,$$

with time derivative of

$$\frac{d}{d\alpha}\varphi\left(\alpha,x_\alpha;\theta\right)=-\varphi\left(\alpha,x_\alpha;\theta\right)\theta x_\alpha^T\big[Q(\alpha)+K_1^T(\alpha)R_{11}(\alpha)K_1(\alpha)\\ -K_2^T(\alpha)R_{22}(\alpha)K_2(\alpha)\big]x_\alpha\ .$$

Using the standard Ito's formula in [1], one gets

$$\begin{aligned}d\varphi\left(\alpha,x_\alpha;\theta\right)&=E\left\{d\varpi\left(\alpha,x_\alpha;\theta\right)\right\}\ ,\\ &=E\Big\{\varpi_\alpha\left(\alpha,x_\alpha;\theta\right)d\alpha+\varpi_{x_\alpha}\left(\alpha,x_\alpha;\theta\right)dx_\alpha\\ &\quad+\frac{1}{2}\mathrm{Tr}\left\{\varpi_{x_\alpha x_\alpha}(\alpha,x_\alpha;\theta)G(\alpha)WG^T(\alpha)\right\}d\alpha\Big\}\ ,\\ &=\varphi_\alpha\left(\alpha,x_\alpha;\theta\right)d\alpha\\ &\quad+\varphi_{x_\alpha}\left(\alpha,x_\alpha;\theta\right)\big[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\big]x_\alpha d\alpha\\ &\quad+\frac{1}{2}\mathrm{Tr}\left\{\varphi_{x_\alpha x_\alpha}\left(\alpha,x_\alpha;\theta\right)G\left(\alpha\right)WG^T\left(\alpha\right)\right\}d\alpha\ ,\end{aligned}$$

when combined with (9) leads to

$$\begin{aligned}&-\varphi\left(\alpha,x_\alpha;\theta\right)\theta x_\alpha^T\big[Q(\alpha)+K_1^T(\alpha)R_{11}(\alpha)K_1(\alpha)-K_2^T(\alpha)R_{22}(\alpha)K_2(\alpha)\big]x_\alpha\\ &=\frac{\frac{d}{d\alpha}\varrho\left(\alpha;\theta\right)}{\varrho\left(\alpha;\theta\right)}\varphi\left(\alpha,x_a;\theta\right)+\varphi\left(\alpha,x_\alpha;\theta\right)x_\alpha^T\frac{d}{d\alpha}\Upsilon(\alpha;\theta)x_\alpha+\varphi\left(\alpha,x_\alpha;\theta\right)\Big\{x_\alpha^T\big[A(\alpha)\\ &+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\big]^T\Upsilon(\alpha;\theta)x_\alpha\\ &\qquad+x_\alpha^T\Upsilon_a(\alpha;\theta)\left[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\right]x_\alpha\Big\}\\ &+\varphi\left(\alpha,x_\alpha;\theta\right)\Big\{2x_\alpha^T\Upsilon(\alpha;\theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha;\theta)x_\alpha+\mathrm{Tr}\left\{\Upsilon(\alpha;\theta)G(\alpha)WG^T(\alpha)\right\}\Big\}.\end{aligned}$$

To have constant and quadratic terms independent of x_α, it is required that

$$\begin{aligned}\frac{d}{d\alpha}\Upsilon(\alpha;\theta)&=-[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)]^T\Upsilon(\alpha;\theta)\\ &\quad-\Upsilon(\alpha;\theta)[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)]\\ &\quad-2\Upsilon(\alpha;\theta)G\left(\alpha\right)WG^T\left(\alpha\right)\Upsilon(\alpha;\theta)\\ &\quad-\theta\big[Q(\alpha)+K_1^T(\alpha)R_{11}(\alpha)K_1(\alpha)-K_2^T(\alpha)R_{22}(\alpha)K_2(\alpha)\big]\ ,\\ \frac{d}{d\alpha}\varrho\left(\alpha;\theta\right)&=-\varrho\left(\alpha;\theta\right)\mathrm{Tr}\left\{\Upsilon(\alpha;\theta)G\left(\alpha\right)WG^T\left(\alpha\right)\right\}\ ,\end{aligned}$$

with the terminal conditions $\Upsilon(t_f;\theta)=\theta Q_f$ and $\varrho\left(t_f;\theta\right)=1$. Finally, the remaining time-backward differential equation satisfied by $\upsilon\left(\alpha;\theta\right)$ is given by

$$\frac{d}{d\alpha}\upsilon\left(\alpha;\theta\right)=-\mathrm{Tr}\left\{\Upsilon(\alpha;\theta)G\left(\alpha\right)WG^T\left(\alpha\right)\right\}\ ,\quad \upsilon\left(t_f;\theta\right)=0\ .$$

□

Now cost cumulants can be generated for the zero-sum stochastic differential game by looking at a MacLaurin series expansion of the cumulant-generating function

$$\psi(\alpha, x_\alpha; \theta) = \sum_{i=1}^{\infty} \kappa_i(\alpha, x_\alpha) \frac{\theta^i}{i!} = \sum_{i=1}^{\infty} \frac{\partial^{(i)}}{\partial \theta^{(i)}} \psi(\alpha, x_\alpha; \theta) \bigg|_{\theta=0} \frac{\theta^i}{i!} \ , \tag{15}$$

in which $\kappa_i(\alpha, x_\alpha)$'s are called the cost cumulants. Note that the series coefficients can be computed using (11)

$$\frac{\partial^{(i)}}{\partial \theta^{(i)}} \psi(\alpha, x_\alpha; \theta) \bigg|_{\theta=0} = x_\alpha^T \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha; \theta) \bigg|_{\theta=0} x_\alpha + \frac{\partial^{(i)}}{\partial \theta^{(i)}} \upsilon(\alpha; \theta) \bigg|_{\theta=0} . \tag{16}$$

In view of results (15) and (16), cost cumulants for the stochastic differential game problem can be obtained as

$$\kappa_i(\alpha, x_\alpha) = x_\alpha^T \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha; \theta) \bigg|_{\theta=0} x_\alpha + \frac{\partial^{(i)}}{\partial \theta^{(i)}} \upsilon(\alpha; \theta) \bigg|_{\theta=0} \ , \tag{17}$$

for any finite $1 \leq i < \infty$. For notational convenience, the following definitions are introduced:

$$H(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha; \theta) \bigg|_{\theta=0} \text{ and } D(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^{(i)}} \upsilon(\alpha; \theta) \bigg|_{\theta=0} . \tag{18}$$

The next theorem yields an attractive method of generating cost cumulants in time domain. This computational method is preferred to that of (16) in the formulation of cost-cumulant control problems.

Theorem 2. *Cost-Cumulants in Zero-Sum Stochastic Differential Games. Suppose that (A, B_1) and (A, B_2) are uniformly stabilizable. The players choose control strategies $(u_1(t), u_2(t)) = (K_1(t)x(t), K_2(t)x(t))$ for the zero-sum differential game characterized by (5) and (6). For $k \in \mathbb{Z}^+$ fixed and $1 \leq i \leq k$, the kth cost cumulant in the zero-sum stochastic game is given by*

$$\kappa_k(t_0, x_0; K_1, K_2) = x_0^T H(t_0, k) x_0 + D(t_0, k) \ , \tag{19}$$

in which the cumulant variables $\{H(\alpha, i)\}_{i=1}^k$ and $\{D(\alpha, i)\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the following differential equations (with the dependence of $H(\alpha, i)$ and $D(\alpha, i)$ upon the admissible gains K_1 and K_2 suppressed)

$$\begin{aligned} \frac{d}{d\alpha} H(\alpha, 1) = & - \left[A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha)\right]^T H(\alpha, 1) \\ & - H(\alpha, 1) \left[A(\alpha) + B_1(\alpha)K_1(\alpha) + B_2(\alpha)K_2(\alpha)\right] \\ & - Q(\alpha) - K_1^T(\alpha) R_{11}(\alpha) K_1(\alpha) + K_2^T(\alpha) R_{22}(\alpha) K_2(\alpha) \ , \end{aligned} \tag{20}$$

and, for $2 \leq i \leq k$

$$\frac{d}{d\alpha}H(\alpha,i) = -\left[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\right]^T H(\alpha,i) - H(\alpha,i)\left[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\right] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}H(\alpha,j)G(\alpha)WG^T(\alpha)H(\alpha,i-j)\ , \tag{21}$$

together with $1 \leq i \leq k$

$$\frac{d}{d\alpha}D(\alpha,i) = -\mathrm{Tr}\left\{H(\alpha,i)G(\alpha)WG^T(\alpha)\right\}\ , \tag{22}$$

where the terminal conditions $H(t_f,1) = Q_f$, $H(t_f,i) = 0$ *for* $2 \leq i \leq k$ *and* $D(t_f,i) = 0$ *for* $1 \leq i \leq k$.

Proof. The cost-cumulant expression in (19) is readily justified by using the result (17) and the definitions (18). What remains is to show that the solutions $H(\alpha,i)$ and $D(\alpha,i)$ for $1 \leq i \leq k$ indeed satisfy (20)-(22). Note that the equations (20)-(22) are satisfied by the solutions $H(\alpha,i)$ and $D(\alpha,i)$ and can be obtained by repeatedly taking the derivative with respect to θ of (12)-(13) together with the assumption $A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)$ is stable for all $\alpha \in [t_0,t_f]$. □

In the subsequent development, the subset of symmetric matrices of the vector space of all $n \times n$ matrices with real elements is denoted by $\mathbb{S}^n$. Now, let the k-tuple variables $\mathcal{H}$ and $\mathcal{D}$ be defined as follows

$$\mathcal{H}(\cdot) \triangleq (\mathcal{H}_1(\cdot),\ldots,\mathcal{H}_k(\cdot)) \,\text{and}\, \mathcal{D}(\cdot) \triangleq (\mathcal{D}_1(\cdot),\ldots,\mathcal{D}_k(\cdot))\ ,$$

for each element $\mathcal{H}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{S}^n)$ of $\mathcal{H}$ and $\mathcal{D}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R})$ of $\mathcal{D}$ having the representations

$$\mathcal{H}_i(\cdot) \triangleq H(\cdot,i) \,\text{and}\, \mathcal{D}_i(\cdot) \triangleq D(\cdot,i)$$

with the right members satisfying the dynamic equations (20)-(22) on the horizon $[t_0,t_f]$. For ease of presentation, the following mappings are introduced:

$$\mathcal{F}_i : [t_0,t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^{m_1 \times n} \times \mathbb{R}^{m_2 \times n} \mapsto \mathbb{S}^n$$
$$\mathcal{G}_i : [t_0,t_f] \times (\mathbb{S}^n)^k \mapsto \mathbb{R}$$

where the actions are given by

$$\begin{aligned}\mathcal{F}_1(\alpha,\mathcal{H},K_1,K_2) &\triangleq -\left[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\right]^T\mathcal{H}_1(\alpha)\\&\quad-\mathcal{H}_1(\alpha)\left[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\right]\\&\quad-Q(\alpha)-K_1^T(\alpha)R_{11}(\alpha)K_1(\alpha)+K_2^T(\alpha)R_{22}(\alpha)K_2(\alpha)\\\mathcal{F}_i(\alpha,\mathcal{H},K_1,K_2) &\triangleq -\left[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\right]^T\mathcal{H}_i(\alpha)\\&\quad-\mathcal{H}_i(\alpha)\left[A(\alpha)+B_1(\alpha)K_1(\alpha)+B_2(\alpha)K_2(\alpha)\right]\\&\quad-\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{i-j}(\alpha)\ ,\quad 2\le i\le k\\\mathcal{G}_i(\alpha,\mathcal{H}) &\triangleq -\mathrm{Tr}\left\{\mathcal{H}_i(\alpha)G(\alpha)WG^T(\alpha)\right\}\ ,\quad 1\le i\le k\ .\end{aligned}$$

For a compact formulation, the product mappings are established as such

$$\begin{aligned}&\mathcal{F}_1\times\cdots\times\mathcal{F}_k:[t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^{m_1\times n}\times\mathbb{R}^{m_2\times n}\mapsto(\mathbb{S}^n)^k\\&\mathcal{G}_1\times\cdots\times\mathcal{G}_k:[t_0,t_f]\times(\mathbb{S}^n)^k\mapsto\mathbb{R}^k\end{aligned}$$

along with the corresponding notations $\mathcal{F}\triangleq\mathcal{F}_1\times\cdots\times\mathcal{F}_k$ and $\mathcal{G}\triangleq\mathcal{G}_1\times\cdots\times\mathcal{G}_k$. Thus, the dynamic equations of motion (20)-(22) can be rewritten as

$$\frac{d}{d\alpha}\mathcal{H}(\alpha)=\mathcal{F}(\alpha,\mathcal{H}(\alpha),K_1(\alpha),K_2(\alpha))\ ,\quad \mathcal{H}(t_f)=\mathcal{H}_f \tag{23}$$

$$\frac{d}{d\alpha}\mathcal{D}(\alpha)=\mathcal{G}(\alpha,\mathcal{H}(\alpha))\ ,\qquad \mathcal{D}(t_f)=\mathcal{D}_f\ , \tag{24}$$

where the terminal values $\mathcal{H}_f=(Q_f,0,\dots,0)$ and $\mathcal{D}_f=(0,\dots,0)$.

Note that the product system uniquely determines $\mathcal{H}$ and $\mathcal{D}$ once the admissible feedback gains K_1 and K_2 are specified. Hence, $\mathcal{H}$ and $\mathcal{D}$ are considered as $\mathcal{H}(\cdot,K_1,K_2)$ and $\mathcal{D}(\cdot,K_1,K_2)$, respectively. The performance index in cost-cumulant control can now be formulated in the admissible feedback gains K_1 and K_2.

Definition 1. *Performance Index in Cost-Cumulant Control.*
Fix $k\in\mathbb{Z}^+$ and the sequence $\mu=\{\mu_i\ge 0\}_{i=1}^k$ with $\mu_1>0$. Then for the given initial condition (t_0,x_0), the performance index $\phi_0:[t_0,t_f]\times(\mathbb{S}^n)^k\times\mathbb{R}^k\mapsto\mathbb{R}^+$ of the finite-horizon cost-cumulant control is defined by

$$\begin{aligned}\phi_0\left(t_0,\mathcal{H}(t_0,K_1,K_2),\mathcal{D}(t_0,K_1,K_2)\right)&\triangleq\sum_{i=1}^k\mu_i\kappa_i(K_1,K_2)\\&=\sum_{i=1}^k\mu_i\left[x_0^T\mathcal{H}_i(t_0,K_1,K_2)x_0+\mathcal{D}_i(t_0,K_1,K_2)\right]\ ,\end{aligned} \tag{25}$$

where additional parametric design freedom μ_i mutually chosen by players represent different levels of influence as they deem important to the overall cost distribution. Symmetric solutions $\{\mathcal{H}_i(t_0,K_1,K_2)\ge 0\}_{i=1}^k$ and $\{\mathcal{D}_i(t_0,K_1,K_2)\ge 0\}_{i=1}^k$ evaluated at $\alpha=t_0$ satisfy (23)-(24).

For the given terminal data $(t_f, \mathcal{H}_f, \mathcal{D}_f)$, the classes $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ and $\mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ of admissible feedback gains may be defined as follows.

Definition 2. *Admissible Feedback Gain Strategies.*
Let the compact subsets $\overline{K}_1 \subset \mathbb{R}^{m_1 \times n}$ and $\overline{K}_2 \subset \mathbb{R}^{m_2 \times n}$ be the sets of allowable gain values. For the given $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, the sets of admissible control strategies $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ and $\mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ are assumed to be the classes of $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m_1 \times n})$ and $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m_2 \times n})$ with values $K_1(\cdot) \in \overline{K}_1$ and $K_2(\cdot) \in \overline{K}_2$ for which solutions to the dynamic equations of motion (23)-(24) exist on the finite horizon $[t_0, t_f]$.

Then one may state the cost-cumulant control optimization problem for the zero-sum stochastic differential game.

Definition 3. *Optimization Problem.*
Suppose that $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then, the cost-cumualnt control optimization problem over $[t_0, t_f]$ is given by

$$\min_{K_1(\cdot) \in \mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}} \max_{K_2(\cdot) \in \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}} \phi_0\left(t_0, \mathcal{H}(t_0, K_1, K_2), \mathcal{D}(t_0, K_1, K_2)\right) \tag{26}$$

subject to the dynamic equations (23)-(24) for $\alpha \in [t_0, t_f]$.

Next, the fundamental theorem of calculus and stochastic differential rules is utilized to derive the existence of a saddle point.

Theorem 3. *Existence of a Saddle Point.*
Consider the linear-quadratic zero-sum stochastic differential game

$$dx(t) = [A(t) + B_1(t)K_1(t) + B_2(t)K_2(t)]\, x(t)dt + G(t)dw(t) \ ,$$
$$x(t_0) = x_0 \ ,$$

which in turn, is associated with the finite-horizon IQF cost

$$J(t_0, x_0; K_1, K_2) = x^T(t_f)Q_f x(t_f)$$
$$+ \int_{t_0}^{t_f} x^T(\tau)\left[Q(\tau) + K_1^T(\tau)R_{11}(\tau)K_1(\tau) - K_2^T(\tau)R_{22}(\tau)K_2(\tau)\right] x(\tau)d\tau \ .$$

For any given $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, there exists a saddle point $(K_1^*, K_2^*) \in \mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ such that there hold

$$\phi_0\left(t_0, \mathcal{H}(t_0, K_1^*, K_2), \mathcal{D}(t_0, K_1^*, K_2)\right) \leq \phi_0\left(t_0, \mathcal{H}(t_0, K_1^*, K_2^*), \mathcal{D}(t_0, K_1^*, K_2^*)\right)$$
$$\phi_0\left(t_0, \mathcal{H}(t_0, K_1^*, K_2^*), \mathcal{D}(t_0, K_1^*, K_2^*)\right) \leq \phi_0\left(t_0, \mathcal{H}(t_0, K_1, K_2^*), \mathcal{D}(t_0, K_1, K_2^*)\right) \ .$$

It is now concluded that the existence of a saddle point yields both necessary and sufficient conditions for the minimax problem to be equivalent to the corresponding maximin problem. In other words, the Issacs condition holds according to [3]. The value function, $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ for the game starting at the time-states triple $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is defined as follows.

Definition 4. *Value Function.*
The value function $\mathcal{V} : [t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^k \mapsto \mathbb{R}^+ \cup \{+\infty\}$ *associated with the Mayer problem is defined by*

$$\begin{aligned}\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) &\triangleq \min_{K_1(\cdot)\in\mathcal{K}^1_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}} \max_{K_2(\cdot)\in\mathcal{K}^2_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}} \phi_0\left(t_0, \mathcal{H}(t_0, K_1, K_2), \mathcal{D}(t_0, K_1, K_2)\right) \\ &= \max_{K_2(\cdot)\in\mathcal{K}^2_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}} \min_{K_1(\cdot)\in\mathcal{K}^1_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}} \phi_0\left(t_0, \mathcal{H}(t_0, K_1, K_2), \mathcal{D}(t_0, K_1, K_2)\right) \ ,\end{aligned}$$

for any $(\varepsilon, \mathcal{Y}, \mathcal{Z}) \in [t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^k$.

Conventionally, set $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \infty$ when either $\mathcal{K}^1_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}$ or $\mathcal{K}^2_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}$ is empty. The development in the sequel is motivated by the excellent treatment in [4], and is intended to follow it closely. Unless otherwise specified, the dependence of trajectory solutions $\mathcal{H}$ and $\mathcal{D}$ on the admissible gains K_1 and K_2 is omitted for notational clarity.

Theorem 4. *Necessary Conditions.*
The value function evaluated along any trajectory corresponding to a pair of control strategy gains feasible for its terminal states is a non-increasing function of time. The value function evaluated along any optimal trajectory is constant.

It is important to note that these properties are necessary conditions for optimality. The next theorem shows that these conditions are also sufficient for optimality.

Theorem 5. *Sufficient Condition.*
Let $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ *be an extended real-valued function defined on*

$$[t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^k$$

such that $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \phi_0\big(\varepsilon, \mathcal{Y}, \mathcal{Z}\big)$.

Let t_f, $\mathcal{H}_f$, $\mathcal{D}_f$ *be given terminal conditions, and suppose that, for each trajectory pair* $(\mathcal{H}, \mathcal{D})$ *corresponding to a control strategy pair* (K_1, K_2) *in* $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$, $\mathcal{W}(\alpha, \mathcal{H}(\alpha), \mathcal{D}(\alpha))$ *is finite and non-increasing on* $[t_0, t_f]$.

If (K_1^*, K_2^*) *is a control strategy pair in* $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ *such that for the corresponding trajectory pair* $(\mathcal{H}^*, \mathcal{D}^*)$, $\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha))$ *is constant then the pair* (K^*, K_2^*) *is a saddle point and* $\mathcal{W}(t_f, \mathcal{H}_f, \mathcal{D}_f) = \mathcal{V}(t_f, \mathcal{H}_f, \mathcal{D}_f)$.

Corollary 1. *Restriction of Strategy Gains.*
Let (K_1^*, K_2^*) *be an optimal control strategy pair in* $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ *and* $(\mathcal{H}^*, \mathcal{D}^*)$ *the corresponding trajectory pair of dynamic equations*

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}(\alpha) &= \mathcal{F}(\alpha, \mathcal{H}(\alpha), K_1(\alpha), K_2(\alpha)) \ , \quad \mathcal{H}(t_f) = \mathcal{H}_f \\ \frac{d}{d\alpha}\mathcal{D}(\alpha) &= \mathcal{G}(\alpha, \mathcal{H}(\alpha)) \ , \quad \mathcal{D}(t_f) = \mathcal{D}_f \ .\end{aligned}$$

Then, the restriction of the pair (K_1^*, K_2^*) *to* $[t_0, \alpha]$ *is an optimal control strategy pair for the control problem with the terminal-valued condition* $(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha))$ *when* $t_0 \le \alpha \le t_f$.

Both necessary and sufficient conditions implied by these properties for a control gain to be optimal give hints that one may find a function $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) : [t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^k \mapsto \mathbb{R}^+$ such that $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \phi_0\big(\varepsilon, \mathcal{Y}, \mathcal{Z}\big)$, $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is constant on the corresponding trajectory pair, and $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is non-increasing on other trajectories.

Note that the value function $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is supposed to be continuously differentiable in $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ which then results in the uniqueness of a saddle point (K_1^*, K_2^*). Formally speaking, the result regarding the differentiability of the value function, which is adapted from [4], is stated as follows.

Theorem 6. *Differentiability of Value Function.*
Let admissible feedback gains $K_1^*(\alpha, \mathcal{H}, \mathcal{D})$ and $K_2^*(\alpha, \mathcal{H}, \mathcal{D})$ constitute a saddle point. Further, let $t_0(\varepsilon, \mathcal{Y}, \mathcal{Z})$ and $(\mathcal{H}(t_0(\varepsilon, \mathcal{Y}, \mathcal{Z}); \varepsilon, \mathcal{Y}), \mathcal{D}(t_0(\varepsilon, \mathcal{Y}, \mathcal{Z}); \varepsilon, \mathcal{Z}))$ be the initial time and initial states for the trajectories of

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}, K_1^*(\alpha, \mathcal{H}, \mathcal{D}), K_2^*(\alpha, \mathcal{H}, \mathcal{D})) \ ,$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha, \mathcal{H}) \ ,$$

with the terminal-valued condition $(\varepsilon, \mathcal{Y}, \mathcal{Z})$. Then, the value function $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is differentiable at each point at which $t_0(\varepsilon, \mathcal{Y}, \mathcal{Z})$ and $\mathcal{H}(t_0(\varepsilon, \mathcal{Y}, \mathcal{Z}); \varepsilon, \mathcal{Y})$ and $\mathcal{D}(t_0(\varepsilon, \mathcal{Y}, \mathcal{Z}); \varepsilon, \mathcal{Z})$ are differentiable with respect to $(\varepsilon, \mathcal{Y}, \mathcal{Z})$.

As a tenet of transition from the principle of optimality, a family of games based on different starting points is now considered. Let's begin with an interlude of time, ε in mid-play. At its commencement, the path has reached some definitive points. Consider all possible $(\mathcal{H}, \mathcal{D})$ which may be reached at the end of the interlude for all possible choices of (K_1, K_2). Suppose that for each endpoint, the game beginning there has already been solved. Then the value function $\mathcal{V}(\varepsilon, \mathcal{H}, \mathcal{D})$ resulting from each choice of (K_1, K_2) is known, and they are to be so chosen as to render it minimax. As the duration of the interlude approaches t_f, this leads to a sufficient condition to Hamilton-Jacobi-Isaacs (HJI) equation. By adapting to the initial-cost problem and the terminologies present in the cost-cumulant control, the HJI equation satisfied by the value function $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is then given.

Definition 5. *Playable Set.*
Let the playable set $\mathcal{Q}$ *be defined as*

$$\mathcal{Q} \triangleq \big\{(\varepsilon, \mathcal{Y}, \mathcal{Z}) \in [t_0, t_f] \times (\mathbb{S}^n)^k \times \mathbb{R}^k \, \text{such that} \, \mathcal{K}^1_{\varepsilon, \mathcal{Y}, \mathcal{Z}; \mu} \times \mathcal{K}^2_{\varepsilon, \mathcal{Y}, \mathcal{Z}; \mu} \neq 0\big\} \ .$$

Theorem 7. *HJI Equation-Mayer Problem.*
Let $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ be any interior point of the playable set $\mathcal{Q}$ at which the value function $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is differentiable. Then $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ satisfies the partial differential inequality

$$0 \geq \frac{\partial}{\partial \varepsilon}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) \cdot \operatorname{vec}(\mathcal{F}(\varepsilon,\mathcal{Y},K_1,K_2)) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) \cdot \operatorname{vec}(\mathcal{G}(\varepsilon,\mathcal{Y})) \ ,$$

for all $(K_1, K_2) \in \overline{K}_1 \times \overline{K}_2$.

If there exists a saddle point $(K_1^*, K_2^*) \in \mathcal{K}^1_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu} \times \mathcal{K}^2_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}$*, then the partial differential equation of differential games*

$$0 = \min_{K_1 \in \overline{K}_1} \max_{K_2 \in \overline{K}_2} \left\{ \frac{\partial}{\partial \varepsilon}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) \cdot \operatorname{vec}(\mathcal{F}(\varepsilon,\mathcal{Y},K_1,K_2)) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) \cdot \operatorname{vec}(\mathcal{G}(\varepsilon,\mathcal{Y})) \right\} \quad (27)$$

is satisfied together with $\mathcal{V}(t_0, \mathcal{H}_0, \mathcal{D}_0) = \phi_0(t_0, \mathcal{H}_0, \mathcal{D}_0)$ *and vec*$(\cdot)$ *the vectorizing operator of enclosed entities. The optimum in (27) is achieved by the left limit* $(K_1^*(\varepsilon)^-, K_2^*(\varepsilon)^-)$ *of the optimal strategy pair at* ε.

The construction of a scalar-valued function which is a candidate for the value function is discussed in the following theorem.

Theorem 8. *Verification Theorem.*
Fix $k \in \mathbb{Z}^+$. *Let* $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z})$ *be a continuously differentiable solution of the HJI equation*

$$0 = \min_{K_1 \in \overline{K}_1} \max_{K_2 \in \overline{K}_2} \left\{ \frac{\partial}{\partial \varepsilon}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) \cdot \operatorname{vec}(\mathcal{F}(\varepsilon,\mathcal{Y},K_1,K_2)) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) \cdot \operatorname{vec}(\mathcal{G}(\varepsilon,\mathcal{Y})) \right\}$$

and satisfy the boundary condition

$$\mathcal{W}(t_0, \mathcal{H}_0, \mathcal{D}_0) = \phi_0\left(t_0, \mathcal{H}_0, \mathcal{D}_0\right) \ , \qquad \text{for } (t_0, \mathcal{H}_0, \mathcal{D}_0) \in \mathcal{M} \ , \quad (28)$$

where $\mathcal{M} = \{t_0\} \times (\mathbb{S}^n)^k \times \mathbb{R}^k$.

Let $(t_f, \mathcal{H}_f, \mathcal{D}_f)$ *be a point of* $\mathcal{Q}$, (K_1, K_2) *a control strategy pair in* $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ *and* $\mathcal{H}$ *and* $\mathcal{D}$ *the corresponding solutions of the equations*

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}(\alpha), K_1(\alpha), K_2(\alpha)) \ , \quad \mathcal{H}(t_f) = \mathcal{H}_f$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha, \mathcal{H}(\alpha)) \ , \quad \mathcal{D}(t_f) = \mathcal{D}_f \ .$$

Then, $\mathcal{W}(\alpha, \mathcal{H}(\alpha), \mathcal{D}(\alpha))$ is a non-increasing function of α. If (K_1^*, K_2^*) is a control strategy pair in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ defined on $[t_0, t_f]$ with corresponding solution, $\mathcal{H}^*$ and $\mathcal{D}^*$ of the above equations such that for $\alpha \in [t_0, t_f]$

$$\begin{aligned} 0 &= \frac{\partial}{\partial \varepsilon}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) \\ &+ \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y})}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) \cdot \operatorname{vec}(\mathcal{F}(\alpha, \mathcal{H}^*(\alpha), K_1^*(\alpha), K_2^*(\alpha))) \\ &+ \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z})}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) \cdot \operatorname{vec}(\mathcal{G}(\alpha, \mathcal{H}^*(\alpha))) \ , \end{aligned} \tag{29}$$

then (K_1^*, K_2^*) is a saddle-point strategy pair in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ and

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \ , \tag{30}$$

where $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is the value function.

It is observed that to have a saddle-point solution (K_1^*, K_2^*) in $\mathcal{K}^1_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{K}^2_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ defined and continuous for all $\alpha \in [t_0, t_f]$, the solution $\mathcal{H}(\alpha)$ to (23) when evaluated at $\alpha = t_0$ must also exist. Therefore, it is necessary that $\mathcal{H}(\alpha)$ is finite for all $\alpha \in [t_0, t_f)$. Moreover, the solution of (23) exists and is continuously differentiable in a neighborhood of t_f. Applying the results from [2], these solutions can further be extended to the left of t_f as long as $\mathcal{H}(\alpha)$ remains finite. Hence, the existence of unique and continuously differentiable solutions to (23) are certain if $\mathcal{H}(\alpha)$ are bounded for all $\alpha \in [t_0, t_f)$. As the result, the candidate value functions $\mathcal{V}(\alpha, \mathcal{H}, \mathcal{D})$ are continuously differentiable as well.

Theorem 9. *Necessary and Sufficient Conditions for a Saddle-Point Solution. (K_1^*, K_2^*) is a saddle-point strategy if and only if $\mathcal{H}(\alpha)$ is bounded for all $\alpha \in [t_0, t_f)$.*

3 Multi-cumulant Saddle-Point Solution

Recall that the optimization problem being considered herein is in "Mayer form" and can be solved by applying an adaptation of the Mayer form verification theorem of dynamic programming given in [4]. In the framework of dynamic programming, it is often required to denote the terminal time and states of a family of optimization problems as $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ rather than $(t_f, \mathcal{H}_f, \mathcal{D}_f)$. That is, for $\varepsilon \in [t_0, t_f]$ and $1 \le i \le k$, the states of the system (23)-(24) defined on the interval $[t_0, \varepsilon]$ have terminal values denoted by $\mathcal{H}(\varepsilon) \equiv \mathcal{Y}$ and $\mathcal{D}(\varepsilon) \equiv \mathcal{Z}$. Since the cumulant-based performance index (25) is quadratic affine in terms of arbitrarily fixed x_0, this observation then suggests a solution to (27) may be sought in the form

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = x_0^T \sum_{i=1}^{k} \mu_i(\mathcal{Y}_i + \mathcal{E}_i(\varepsilon))\, x_0 + \sum_{i=1}^{k} \mu_i(\mathcal{Z}_i + \mathcal{T}_i(\varepsilon)) \ , \tag{31}$$

where the parametric functions of time $\mathcal{E}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{S}^n)$ and $\mathcal{T}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ are yet to be determined. The next theorem shows how the partial differential equation in the notation of $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ looks like using inverse vectorizing transformation.

Corollary 2. *Time Derivative of a Candidate Function.*
Fix $k \in \mathbb{Z}^+$ and let $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ be any interior point of the reachable set $\mathcal{Q}$ at which the real-valued function (31) is differentiable. Then, the time derivative of $\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is found to be

$$\frac{d}{d\varepsilon}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \sum_{i=1}^{k} \mu_i \left(\mathcal{G}_i(\varepsilon, \mathcal{Y}) + \frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) \right) + x_0^T \sum_{i=1}^{k} \mu_i \left(\mathcal{F}_i(\varepsilon, \mathcal{Y}, K_1, K_2) + \frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon) \right) x_0 \ . \tag{32}$$

The substitution of this hypothesized solution (31) into (27) and making use of the result (32) yield

$$\begin{aligned} 0 &= \min_{K_1 \in \overline{K}_1} \max_{K_2 \in \overline{K}_2} \left\{ \frac{\partial}{\partial\varepsilon}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y})}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \cdot \operatorname{vec}(\mathcal{F}_i(\varepsilon, \mathcal{Y}, K_1, K_2)) \right. \\ &\qquad \left. + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z})}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \cdot \operatorname{vec}(\mathcal{G}_i(\varepsilon, \mathcal{Y})) \right\} \\ &= \min_{K_1 \in \overline{K}_1} \max_{K_2 \in \overline{K}_2} \left\{ x_0^T \left(\sum_{i=1}^{k} \mu_i \frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon) \right) x_0 + \sum_{i=1}^{k} \mu_i \frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) \right. \\ &\qquad \left. + x_0^T \left(\sum_{i=1}^{k} \mu_i \mathcal{F}_i(\varepsilon, \mathcal{Y}, K_1, K_2) \right) x_0 + \sum_{i=1}^{k} \mu_i \mathcal{G}_i(\varepsilon, \mathcal{Y}) \right\} \ . \end{aligned} \tag{33}$$

It is important to observe that

$$\begin{aligned} \sum_{i=1}^{k} \mu_i \mathcal{F}_i(\varepsilon, \mathcal{Y}, K_1, K_2) &= -\left[A(\varepsilon) + B_1(\varepsilon)K_1 + B_2(\varepsilon)K_2\right]^T \sum_{i=1}^{k} \mu_i \mathcal{Y}_i \\ &\quad - \sum_{i=1}^{k} \mu_i \mathcal{Y}_i \left[A(\varepsilon) + B_1(\varepsilon)K_1 + B_2(\varepsilon)K_2\right] \\ &\quad - \mu_1 Q(\varepsilon) - \mu_1 K_1^T R_{11}(\varepsilon) K_1 + \mu_1 K_2^T R_{22}(\varepsilon) K_2 \\ &\quad - \sum_{i=2}^{k} \mu_i \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{Y}_j G(\varepsilon) W G^T(\varepsilon) \mathcal{Y}_{i-j} \ , \\ \sum_{i=1}^{k} \mu_i \mathcal{G}_i(\varepsilon, \mathcal{Y}) &= -\sum_{i=1}^{k} \mu_i \operatorname{Tr}\left\{\mathcal{Y}_i G(\varepsilon) W G^T(\varepsilon)\right\} \ . \end{aligned}$$

Differentiating the expression within the bracket of (33) with respect to K_1 and K_2 yield the necessary conditions for an extremum of the performance index (25) on $[t_0, \varepsilon]$,

$$-2B_1^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{Y}_i M_0 - 2\mu_1 R_{11}(\varepsilon)K_1 M_0 = 0 \ ,$$
$$-2B_2^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{Y}_i M_0 + 2\mu_1 R_{22}(\varepsilon)K_2 M_0 = 0 \ .$$

Because M_0 is an arbitrary rank-one matrix, it must be true that

$$K_1(\varepsilon, \mathcal{Y}, \mathcal{Z}) = -R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r \ , \tag{34}$$

$$K_2(\varepsilon, \mathcal{Y}, \mathcal{Z}) = R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r \ , \tag{35}$$

where $\widehat{\mu}_r \triangleq \mu_i/\mu_1$ for $\mu_1 > 0$. Substituting the gain expressions (34) and (35) into the right member of the HJI equation (33) yields the value of the minimax

$$\begin{aligned}
x_0^T\Bigg[&\sum_{i=1}^{k}\mu_i\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon) - A^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{Y}_i - \sum_{i=1}^{k}\mu_i\mathcal{Y}_i A(\varepsilon)\\
&- \mu_1 Q(\varepsilon) + \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{Y}_i\\
&+\sum_{i=1}^{k}\mu_i\mathcal{Y}_i B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{Y}_s - \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{Y}_i\\
&-\sum_{i=1}^{k}\mu_i\mathcal{Y}_i B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{Y}_s - \mu_1\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{Y}_s\\
&+ \mu_1\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{Y}_s\\
&- \sum_{i=2}^{k}\mu_i\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{Y}_j G(\varepsilon)WG^T(\varepsilon)\mathcal{Y}_{i-j}\Bigg]x_0\\
&+\sum_{i=1}^{k}\mu_i\frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) - \sum_{i=1}^{k}\mu_i\mathrm{Tr}\left\{\mathcal{Y}_i G(\varepsilon)WG^T(\varepsilon)\right\} \ .
\end{aligned} \tag{36}$$

It is now necessary to exhibit time-dependent functions $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$ and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$ which will render the left side of (36) equal to zero for $\varepsilon \in [t_0, t_f]$, when $\{\mathcal{Y}_i\}_{i=1}^k$ are evaluated along solution trajectories of the cumulant-generating equations.

Studying the expression (36) reveals that $\mathcal{E}_i(\cdot)$ and $\mathcal{T}_i(\cdot)$ for $1 \le i \le k$ satisfying the time-backward differential equations

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{E}_1(\varepsilon) = {} & A^T(\varepsilon)\mathcal{H}_1(\varepsilon) + \mathcal{H}_1(\varepsilon)A(\varepsilon) + Q(\varepsilon) \\ & - \mathcal{H}_1(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\ & - \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\mathcal{H}_1(\varepsilon) \\ & + \mathcal{H}_1(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\ & + \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\mathcal{H}_1(\varepsilon) \\ & + \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\ & - \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\ , \end{aligned} \tag{37}$$

and, for $2 \le i \le k$

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon) = {} & A^T(\varepsilon)\mathcal{H}_i(\varepsilon) + \mathcal{H}_i(\varepsilon)A(\varepsilon) \\ & - \mathcal{H}_i(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\ & - \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\mathcal{H}_i(\varepsilon) \\ & + \mathcal{H}_i(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\ & + \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\mathcal{H}_i(\varepsilon) \\ & + \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{i-j}(\varepsilon)\ , \end{aligned} \tag{38}$$

together with

$$\frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) = \mathrm{Tr}\left\{\mathcal{H}_i(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\}\ , \qquad 1 \le i \le k\ , \tag{39}$$

will work. Furthermore, at the boundary condition, it is necessary to have $\mathcal{W}(t_0, \mathcal{H}_0, \mathcal{D}_0) = \phi_0(t_0, \mathcal{H}_0, \mathcal{D}_0)$. Or, equivalently, $x_0^T \sum_{i=1}^{k} \mu_i(\mathcal{H}_{i0} + \mathcal{E}_i(t_0))\, x_0 + \sum_{i=1}^{k} \mu_i(\mathcal{D}_{i0} + \mathcal{T}_i(t_0)) = x_0^T \sum_{i=1}^{k} \mu_i \mathcal{H}_{i0} x_0 + \sum_{i=1}^{k} \mu_i \mathcal{D}_{i0}$. Thus, matching the boundary condition yields the corresponding initial value conditions $\mathcal{E}_i(t_0) = 0$ and $\mathcal{T}_i(t_0) = 0$ for (37)-(39). Applying the feedback gains specified in (34) and (35) along the solution trajectories of (23)-(24), these equations become Riccati-type

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{H}_1(\varepsilon) = & -A^T(\varepsilon)\mathcal{H}_1(\varepsilon) - \mathcal{H}_1(\varepsilon)A(\varepsilon) - Q(\varepsilon) \\
& + \mathcal{H}_1(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\
& + \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\mathcal{H}_1(\varepsilon) \\
& - \mathcal{H}_1(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\
& - \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\mathcal{H}_1(\varepsilon) \\
& - \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\
& + \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\ ,
\end{aligned} \tag{40}$$

and, for $2 \le i \le k$

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{H}_i(\varepsilon) = & -A^T(\varepsilon)\mathcal{H}_i(\varepsilon) - \mathcal{H}_i(\varepsilon)A(\varepsilon) \\
& + \mathcal{H}_i(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\
& + \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_1(\varepsilon)R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\mathcal{H}_i(\varepsilon) \\
& - \mathcal{H}_i(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon) \\
& - \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B_2(\varepsilon)R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\mathcal{H}_i(\varepsilon) \\
& - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{i-j}(\varepsilon)\ ,
\end{aligned} \tag{41}$$

together, for $1 \le i \le k$

$$\frac{d}{d\varepsilon}\mathcal{D}_i(\varepsilon) = -\mathrm{Tr}\left\{\mathcal{H}_i(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\} \tag{42}$$

where the terminal-valued conditions $\mathcal{H}_1(t_f) = Q_f$, $\mathcal{H}_i(t_f) = 0$ for $2 \le i \le k$ and $\mathcal{D}_i(t_f) = 0$ for $1 \le i \le k$. Thus, whenever these equations (40)-(42) admit solutions $\{\mathcal{H}_i(\cdot)\}_{i=1}^k$ and $\{\mathcal{D}_i(\cdot)\}_{i=1}^k$, then the existence of $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$ and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$ satisfying (37)-(39) are assured. By comparing the equations (37)-(39) to those of (40)-(42), one may recognize that these sets of equations are related to one another by

$$\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon) = -\frac{d}{d\varepsilon}\mathcal{H}_i(\varepsilon) \text{ and } \frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) = -\frac{d}{d\varepsilon}\mathcal{D}_i(\varepsilon)$$

for $1 \le i \le k$. Enforcing the initial value conditions of $\mathcal{E}_i(t_0) = 0$ and $\mathcal{T}_i(t_0) = 0$ uniquely implies that

$$\mathcal{E}_i(\varepsilon) = \mathcal{H}_i(t_0) - \mathcal{H}_i(\varepsilon) \text{ and } \mathcal{T}_i(\varepsilon) = \mathcal{D}_i(t_0) - \mathcal{D}_i(\varepsilon)$$

for all $\varepsilon \in [t_0, t_f]$ and yields a value function

$$\begin{aligned}\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) &= \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}) \\ &= x_0^T \sum_{i=1}^k \mu_i \mathcal{H}_i(t_0) x_0 + \sum_{i=1}^k \mu_i \mathcal{D}_i(t_0) \ ,\end{aligned}$$

for which the sufficient condition (29) of the verification theorem is satisfied. Therefore, the respective feedback gains (34) and (35) for Player 1 and Player 2 optimizing the performance index (25), become optimal

$$K_1^*(\varepsilon) = -R_{11}^{-1}(\varepsilon)B_1^T(\varepsilon)\sum_{r=1}^k \widehat{\mu}_r \mathcal{H}_r^*(\varepsilon) \ , \tag{43}$$

$$K_2^*(\varepsilon) = R_{22}^{-1}(\varepsilon)B_2^T(\varepsilon)\sum_{r=1}^k \widehat{\mu}_r \mathcal{H}_r^*(\varepsilon) \ . \tag{44}$$

Theorem 10. *Multi-Cumulant Saddle-Point Solution.*
Consider the linear-quadratic zero-sum stochastic differential game (5)-(6) in which the pairs (A, B_1) and (A, B_2) are uniformly stabilizable on $[t_0, t_f]$. Let $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \ge 0\}_{i=1}^k$ with $\mu_1 > 0$. Then, the optimal cost-cumulant control via state-feedback is achieved by the saddle-point gains

$$K_1^*(\alpha) = -R_{11}^{-1}(\alpha)B_1^T(\alpha)\sum_{r=1}^k \widehat{\mu}_r \mathcal{H}_r^*(\alpha) \ , \tag{45}$$

$$K_2^*(\alpha) = R_{22}^{-1}(\alpha)B_2^T(\alpha)\sum_{r=1}^k \widehat{\mu}_r \mathcal{H}_r^*(\alpha) \ , \tag{46}$$

where additional parametric design freedom $\widehat{\mu}_r \triangleq \mu_i/\mu_1$ *mutually selected by Players 1 and 2 represent different levels of influence as they deem important to the global performance of the game and* $\{\mathcal{H}_r^*(\alpha) \geq 0\}_{r=1}^k$ *are the optimal solutions of the time-backward differential equations*

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}_1^*(\alpha) = &- [A(\alpha) + B_1(\alpha)K_1^*(\alpha) + B_2(\alpha)K_2^*(\alpha)]^T \mathcal{H}_1^*(\alpha) \\ &- \mathcal{H}_1^*(\alpha)[A(\alpha) + B_1(\alpha)K_1^*(\alpha) + B_2(\alpha)K_2^*(\alpha)] \\ &- Q(\alpha) - K_1^{*T}(\alpha)R_{11}(\alpha)K_1^*(\alpha) + K_2^{*T}(\alpha)R_{22}(\alpha)K_2^*(\alpha) \ , \end{aligned} \quad (47)$$

and, for $2 \leq r \leq k$

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}_r^*(\alpha) = &- [A(\alpha) + B_1(\alpha)K_1^*(\alpha) + B_2(\alpha)K_2^*(\alpha)]^T \mathcal{H}_r^*(\alpha) \\ &- \mathcal{H}_r^*(\alpha)[A(\alpha) + B_1(\alpha)K_1^*(\alpha) + B_2(\alpha)K_2^*(\alpha)] \\ &- \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{r-s}^*(\alpha) \ , \end{aligned} \quad (48)$$

with the terminal-boundary conditions $\mathcal{H}_1^*(t_f) = Q_f$*, and* $\mathcal{H}_r^*(t_f) = 0$ *when* $2 \leq r \leq k$.

4 Conclusions

This paper dealt with a class of two-player zero-sum differential games modeled in a stochastic environment for realistic conditions. Both players were assumed to have exact knowledge of the state, the payoff functional and the control capabilities of each. Matrix differential equations for generating statistics of the IQF random cost used in this game were derived. A more direct dynamic programming approach was used to solve for a saddle-point solution that can address both control strategy selection and performance analysis aspects. This saddle-point solution was computed by two multi-cumulant control gains within the class of linear memoryless-feedback strategies which then minimized a linear combination of first k cumulants of the IQF random cost of the game. Hopefully, these results will make some new theoretical contributions and performance analysis tools to differential game communities. Finally, this theoretical development provides framework and analyses to applications of boost phase missile interception whose the solution offers two optimal conflicting guidance laws: (1) a hit-to-kill homing guidance law for intercepting boosting ballistic missiles in minimum time and divert fuel and (2) an evasion strategy for a ballistic missile to achieve burnout before the kill vehicle arrives, and force the kill vehicle use maximum divert fuel. Future work will address the efficacy of the theoretical work herein via numerical simulation results.

References

1. Davis, M. H. A.: Linear Estimation and Stochastic Control. A Halsted Press. John Wiley & Sons. New York (1977)

2. Dieudonne, J.: Foundations of Modern Analysis. Academic Press. New York and London (1960)
3. Elliot, R. J.: The existence of value in stochastic differential games. SIAM Journal on Control and Optimization **14** (1976) 85–94
4. Fleming, W. H., Rishel, R. W.: Deterministic and Stochastic Optimal Control. New York: Springer-Verlag (1975)
5. Jacobson, D. H.: Optimal Stochastic Linear Systems with Exponential Performance Criteria and Their Relation to Deterministic Games. IEEE Transactions on Automatic Control **AC-18** (1973) 124–131
6. Pham, K. D., Liberty, S. R., Sain, M. K., Spencer, B. F., Jr.: Generalized Risk Sensitive Building Control: Protecting Civil Structures with Multiple Cost Cumulants. Proceedings of American Control Conference (1999) 500–504
7. Pham, K. D., Liberty, S. R., Sain, M. K.: Evaluating Cumulant Controllers on a Benchmark Structure Protection Problem in the Presence of Classic Earthquakes. Proceedings of 37th Annual Allerton Conference on Communication, Control, and Computing (1999) 617–626
8. Pham, K. D., Sain, M. K., Liberty, S. R., Spencer, B. F., Jr.: The Role and Use of Optimal Cost Cumulants for Protection of Civil Structures. The 14th ASCE Engineering Mechanics Conference. Proceeding CD-ROM (2000)
9. Pham, K. D., Liberty, S. R., Sain, M. K., Spencer, B. F., Jr.: First Generation Seismic-AMD Benchmark: Robust Structural Protection by the Cost Cumulant Control Paradigm. Proceedings of American Control Conference (2000) 1–5
10. Pham, K. D., Sain, M. K., Liberty, S. R., Spencer, B. F., Jr.: Optimum Multiple Cost Cumulants for Protection of Civil Structures. The 8th ASCE Specialty Conference on Probabilistic Mechanics and Structural Reliability Conference. Proceeding of CD-ROM (2000)
11. Pham, K. D., Sain, M. K., Liberty, S. R.: Robust Cost-Cumulants Based Algorithm for Second and Third Generation Structural Control Benchmarks. Proceedings of American Control Conference (2002) 3070–3075
12. Pham, K. D., Sain, M. K., Liberty, S. R.: Finite Horizon Full-State Feedback kCC Control in Civil Structures Protection. Stochastic Theory and Adaptive Control, Lecture Notes in Control and Information Sciences. Proceedings of a Workshop held in Lawrence, Kansas. Edited by B. Pasik-Duncan. Springer-Verlag, Berlin Heidelberg, Germany **280** (2002) 369–383
13. Pham, K. D., Sain, M. K., Liberty, S. R.: Cost Cumulant Control: State-Feedback, Finite-Horizon Paradigm with Application to Seismic Protection. Special Issue of Journal of Optimization Theory and Applications. Edited by A. Miele. Kluwer Academic/Plenum Publishers. New York **115** (2002) 685–710
14. Pham, K. D., Jin, G., Sain, M. K., Spencer, B. F., Jr., Liberty, S. R.: Generalized LQG Techniques for the Wind Benchmark Problem. Special Issue of ASCE Journal of Engineering Mechanics on the Structural Control Benchmark Problem **130** (2004) 466–470
15. Pham, K. D., Sain, M. K., Liberty, S. R.: Infinite Horizon Robustly Stable Seismic Protection of Cable-Stayed Bridges Using Cost Cumulants. Proceedings of American Control Conference (2004) 691–696
16. Pham, K. D., Sain, M. K., Liberty, S. R.: Statistical Control for Smart Base-Isolated Buildings via Cost Cumulants and Output Feedback Paradigm. Proceedings of American Control Conference (2005) 3090–3095
17. Pham, K. D.: Minimax Design of Statistics-Based Control with Noise Uncertainty for Highway Bridges. Proceedings of DETC 2005/2005 ASME 20th Biennial Conference on Mechanical Vibration and Noise (2005)