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Predictions in Eternal Inflation

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Abstract. In generic models of cosmological inflation, quantum fluctuations strongly influence the spacetime metric and produce infinitely many regions where the end of inflation (reheating) is delayed until arbitrarily late times. The geometry of the resulting spacetime is highly inhomogeneous on scales of many Hubble sizes. The recently developed string-theoretic picture of the "landscape" presents a similar structure, where an infinite number of de Sitter, flat, and anti-de Sitter universes are nucleated via quantum tunneling. Since observers on the Earth have no information about their location within the eternally inflating universe, the main question in this context is to obtain statistical predictions for quantities observed at a random location. I describe the problems arising within this statistical framework, such as the need for a volume cutoff and the dependence of cutoff schemes on time slicing and on the initial conditions. After reviewing different approaches and mathematical techniques developed in the past two decades for studying these issues, I discuss the existing proposals for extracting predictions and give examples of their applications

5.1 Eternal Inflation

The general idea of eternally inflating spacetime was first introduced and developed in the 1980s [1, 2, 3, 4] in the context of slow-roll inflation. Let us begin by reviewing the main features of eternal inflation, following these early works.

A prototypical model contains a minimally coupled scalar field ϕ (the "inflaton") with an effective potential $V(\phi)$ that is sufficiently flat in some range of ϕ. When the field ϕ has values in this range, the spacetime is approximately de Sitter with the Hubble rate

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3}V(\phi)} \equiv H(\phi)\ . \tag{5.1}$$

(We work in units where $G = c = \hbar = 1$.) The value of H remains approximately constant on timescales of several Hubble times ($\Delta t \gtrsim H^{-1}$), while

S. Winitzki: *Predictions in Eternal Inflation*, Lect. Notes Phys. **738**, 157–191 (2008)
DOI 10.1007/978-3-540-74353-8_5

the field ϕ follows the slow-roll trajectory $\phi_{\mathrm{sr}}(t)$. Quantum fluctuations of the scalar field ϕ in de Sitter background grow linearly with time [5, 6, 7],

$$\langle\hat{\phi}^2(t+\Delta t)\rangle - \langle\hat{\phi}^2(t)\rangle = \frac{H^3}{4\pi^2}\Delta t \,, \tag{5.2}$$

at least for time intervals Δt of order several H^{-1}. Due to the quasi-exponential expansion of spacetime during inflation, Fourier modes of the field ϕ are quickly stretched to super-Hubble length scales. However, quantum fluctuations with super-Hubble wavelengths cannot maintain quantum coherence and become essentially classical [6, 7, 8, 9, 10]; this issue is discussed in more detail in Sect. 5.1.2. The resulting field evolution $\phi(t)$ can be visualized [1, 8, 11] as a Brownian motion with a "random jump" of typical step size $\Delta\phi \sim H/(2\pi)$ during a time interval $\Delta t \sim H^{-1}$, superimposed onto the deterministic slow-roll trajectory $\phi_{\mathrm{sr}}(t)$. A statistical description of this "random walk"-type evolution $\phi(t)$ is reviewed in Sect. 5.2.1.

The "jumps" at points separated in space by many Hubble distances are essentially uncorrelated; this is another manifestation of the well-known "no-hair" property of de Sitter space [12, 13, 14]. Thus the field ϕ becomes extremely inhomogeneous on large (super-horizon) scales after many Hubble times. Moreover, in the semi-classical picture it is assumed [2] that the local expansion rate $\dot{a}/a \equiv H(\phi)$ tracks the local value of the field $\phi(t,\mathbf{x})$ according to the Einstein equation (5.1). Here $a(t,\mathbf{x})$ is the scale factor function which varies with $\mathbf{x}$ only on super-Hubble scales, $a(t,\mathbf{x})\Delta x \gtrsim H^{-1}$. Hence, the spacetime metric can be visualized as having a slowly varying, "locally de Sitter" form (with spatially flat coordinates $\mathbf{x}$),

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - a^2(t,\mathbf{x})\mathrm{d}\mathbf{x}^2 \,. \tag{5.3}$$

The deterministic trajectory $\phi_{\mathrm{sr}}(t)$ eventually reaches a (model-dependent) value ϕ_* signifying the end of the slow-roll inflationary regime and the beginning of the reheating epoch (thermalization). Since the random walk process will lead the value of ϕ away from $\phi = \phi_*$ in some regions, reheating will not begin everywhere at the same time. Moreover, regions where ϕ remains in the inflationary range will typically expand faster than regions near the end of inflation where $V(\phi)$ becomes small. Therefore, a delay of the onset of reheating will be rewarded by additional expansion of the proper 3-volume, thus generating more regions that are still inflating. This feature is called "self-reproduction" of the inflationary spacetime [3]. Since each Hubble-size region evolves independently of other such regions, one may visualize the spacetime as an ensemble of inflating Hubble-size domains (Fig. 5.1).

The process of self-reproduction will never result in a global reheating if the probability of jumping away from $\phi = \phi_*$ and the corresponding additional volume expansion factors are sufficiently large. The corresponding quantitative conditions and their realization in typical models of inflation are reviewed in Sect. 5.3.1. Under these conditions, the process of self-reproduction of inflating regions continues forever. At the same time, every

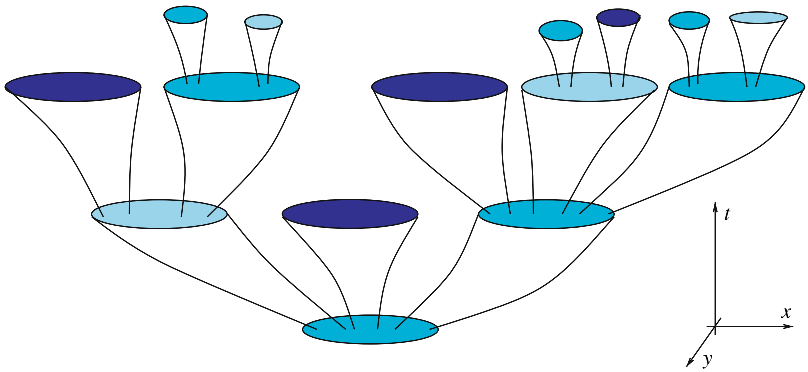

Fig. 5.1. A qualitative diagram of self-reproduction during inflation. *Shaded spacelike domains* represent Hubble-size regions with different values of the inflation field ϕ. The time step is of order H^{-1}. *Dark-colored shades* are regions undergoing reheating ($\phi = \phi_*$); *lighter-colored shades* are regions where inflation continues. On average, the number of inflating regions grows with time

given comoving worldline (except for a set of measure zero; see Sect. 5.3.1) will sooner or later reach the value $\phi = \phi_*$ and enter the reheating epoch. The resulting situation is known as "eternal inflation" [3]. More precisely, the term "eternal inflation" means future-eternal self-reproduction of inflating regions [15].[1] To emphasize the fact that self-reproduction is due to random fluctuations of a field, one refers to this scenario as "eternal inflation of random walk type." Below we use the terms "eternal self-reproduction" and "eternal inflation" interchangeably.

Observers like us may appear only in regions where reheating already took place. Hence, it is useful to consider the locus of all reheating events in the entire spacetime; in the present example, it is the set of spacetime points x there $\phi(x) = \phi_*$. This locus is called the *reheating surface* and is a noncompact, spacelike three-dimensional hypersurface [16, 18]. It is important to realize that a finite, initially inflating 3-volume of space may give rise to a reheating surface having an infinite 3-volume, and even to infinitely many causally disconnected pieces of the reheating surface, each having an infinite 3-volume (see Fig. 5.2). This feature of eternal inflation is at the root of several technical and conceptual difficulties, as will be discussed below.

Everywhere along the reheating surface, the reheating process is expected to provide appropriate initial conditions for the standard "hot big bang" cosmological evolution, including nucleosynthesis and structure formation. In other words, the reheating surface may be visualized as the locus of the "hot big bang" events in the spacetime. It is thus natural to view the reheating

[1] It is worth emphasizing that the term "eternal inflation" refers to future-eternity of inflation in the sense described above, but does not imply past-eternity. In fact, inflationary spacetimes are generically *not* past-eternal [16, 17].

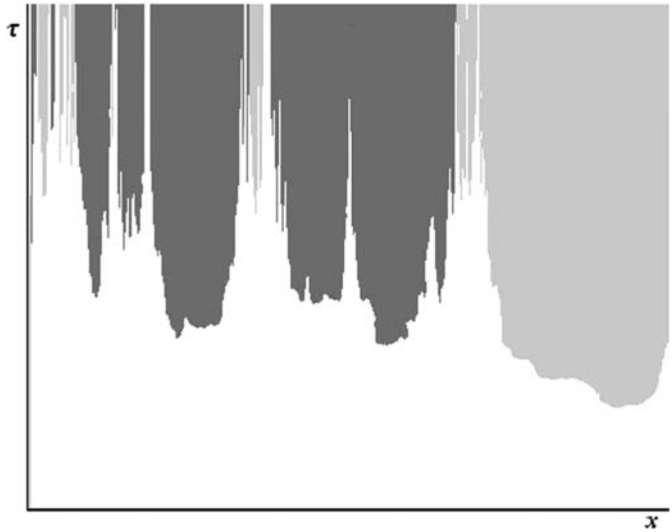

Fig. 5.2. A 1+1-dimensional slice of the spacetime structure in an eternally inflating universe (numerical simulation in [19]). *Shades of different colors* represent different, causally disconnected regions where reheating took place. The reheating surface is the *line* separating the white (inflating) domain and the shaded domains

surface as the initial equal-time surface for astrophysical observations in the post-inflationary epoch. Note that the observationally relevant range of the primordial spectrum of density fluctuations is generated only during the last 60 e-foldings of inflation. Hence, the duration of the inflationary epoch that preceded reheating is not directly measurable beyond the last 60 e-foldings; the total number of e-foldings can vary along the reheating surface and can be in principle arbitrarily large.[2]

The phenomenon of eternal inflation is also found in multi-field models of inflation [22, 23], as well as in scenarios based on Brans–Dicke theory [24, 25, 26], topological inflation [27, 28], braneworld inflation [29], "recycling universe" [30] and the string theory landscape [31]. In some of these models, quantum tunneling processes may generate "bubbles" of a different phase of the vacuum (see Sect. 5.2.5 for more details). Bubbles will be created randomly at various places and times, with a fixed rate per unit 4-volume. In the interior of some bubbles, additional inflation may take place, followed by a new reheating surface. The interior structure of such bubbles is sketched in Fig. 5.3.

[2] For instance, it was shown that holographic considerations do not place any bounds on the total number of e-foldings during inflation [20]. For recent attempts to limit the number of e-foldings using a different approach, see, e.g., [21]. Note also that the effects of "random jumps" are negligible during the last 60 e-foldings of inflation, since the produced perturbations must be of order 10^{-5} according to observations.

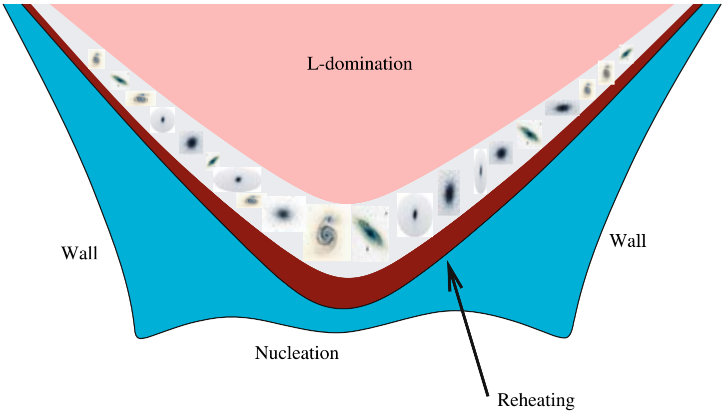

Fig. 5.3. A spacetime diagram of a bubble interior. The infinite, spacelike reheating surface is shown in *darker shade*. Galaxy formation is possible within the spacetime region indicated.

The nucleation event and the formation of bubble walls is followed by a period of additional inflation, which terminates by reheating. Standard cosmological evolution and structure formation eventually give way to a Λ-dominated universe. Infinitely many galaxies and possible civilizations may appear within a thin spacelike slab running along the interior reheating surface. This reheating surface appears to interior observers as an infinite, spacelike hypersurface [32]. For this reason, such bubbles are called "pocket universes," while the spacetime is called a "multiverse." (Generally, the term "pocket universe" refers to a non-compact, connected component of the reheating surface [33].)

In scenarios of this type, each bubble is causally disconnected from most other bubbles.[3] Hence, bubble nucleation events may generate infinitely many statistically inequivalent, causally disconnected patches of the reheating surface, every patch giving rise to a possibly infinite number of galaxies and observers. This feature significantly complicates the task of extracting physical predictions from these models. This class of models is referred to as "eternal inflation of tunneling type."

In the following subsections, I discuss the motivation for studying eternal inflation as well as physical justifications for adopting the effective stochastic picture. Different techniques developed for describing eternal inflation are reviewed in Sect. 5.2. Section 5.3 contains an overview of methods for extracting predictions and a discussion of the accompanying "measure problem."

[3] Collisions between bubbles are rare [34]; however, effects of bubble collisions are observable in principle [35].

5.1.1 Some Motivation

The hypothesis of cosmological inflation was invoked to explain several outstanding puzzles in observational data [36]. However, some observed quantities (such as the cosmological constant Λ or elementary particle masses) may be expectation values of slowly varying effective fields χ_{a}. Within the phenomenological approach, we are compelled to consider also the fluctuations of the fields χ_{a} during inflation, on the same footing as the fluctuations of the inflaton ϕ. Hence, in a generic scenario of eternal inflation, all the fields χ_{a} arrive at the reheating surface $\phi = \phi_*$ with values that can be determined only statistically. Observers appearing at different points in space may thus measure different values of the cosmological constant, elementary particle masses, spectra of primordial density fluctuations and other cosmological parameters.

It is important to note that inhomogeneities in observable quantities are created on scales far exceeding the Hubble horizon scale. Such inhomogeneities are not directly accessible to astrophysical experiments. Nevertheless, the study of the global structure of eternally inflating spacetime is not merely of academic interest. Fundamental questions regarding the cosmological singularities, the beginning of the universe and of its ultimate fate, as well as the issue of the cosmological initial conditions, all depend on the knowledge of the global structure of the spacetime as predicted by the theory, whether or not this global structure is directly observable (see, e.g., [37, 38]). In other words, the fact that some theories predict eternal inflation influences our assessment of the viability of these theories. In particular, the problem of initial conditions for inflation [39] is significantly alleviated when eternal inflation is present. For instance, it was noted early on that the presence of eternal self-reproduction in the "chaotic" inflationary scenario [40] essentially removes the need for the fine-tuning of the initial conditions [3, 41]. More recently, constraints on initial conditions were studied in the context of self-reproduction in models of quintessence [42] and k-inflation [43].

Since the values of the observable parameters χ_{a} are random, it is natural to ask for the probability distribution of χ_{a} that would be measured by a randomly chosen observer. Understandably, this question has been the main theme of much of the work on eternal inflation. Obtaining an answer to this question promises to establish a more direct contact between scenarios of eternal inflation and experiment. For instance, if the probability distribution for the cosmological constant Λ were peaked near the experimentally observed, puzzlingly small value (see, e.g., [44] for a review of the cosmological constant problem), the smallness of $\mathit{\Lambda}$ would be explained as due to observer selection effects rather than to fundamental physics. Considerations of this sort necessarily involve some anthropic reasoning; however, the relevant assumptions are minimal. The basic goal of theoretical cosmology is to select physical theories of the early universe that are most compatible with astrophysical observations, including the observation of our existence. It appears reasonable to assume that the civilization of Planet Earth evolved near a randomly chosen star compatible with the development of life, within a randomly chosen galaxy where such stars exist. Many models of inflation generically include eternal inflation and hence predict the formation of infinitely many galaxies where civilizations like ours may develop. It is then also reasonable to assume that our civilization is typical among all the civilizations that evolved in galaxies formed at any time in the universe. This assumption is called the "principle of mediocrity" [18].

To use the "principle of mediocrity" for extracting statistical predictions from a model of eternal inflation, one proceeds as follows [18, 45]. In the example with the fields χ_{a} described above, the question is to determine the probability distribution for the values of χ_{a} that a random observer will measure. Presumably, the values of the fields χ_{a} do not directly influence the emergence of intelligent life on planets, although they may affect the efficiency of structure formation or nucleosynthesis. Therefore, we may assume a fixed, χ_{a}-dependent mean number of civilizations $\nu_{\mathrm{civ}}(\chi_{\mathrm{a}})$ per galaxy and proceed to ask for the probability distribution $P_{\mathrm{G}}(\chi_{\mathrm{a}})$ of χ_{a} near a randomly chosen galaxy. The observed probability distribution of χ_{a} will then be

$$P(\chi_{\mathrm{a}}) = P_{\mathrm{G}}(\chi_{\mathrm{a}})\nu_{\mathrm{civ}}(\chi_{\mathrm{a}}) \; . \tag{5.4}$$

One may use the standard "hot big bang" cosmology to determine the average number $\nu_{\mathrm{G}}(\chi_{\mathrm{a}})$ of suitable galaxies per unit volume in a region where reheating occurred with given values of χ_{a}; in any case, this task does not appear to pose difficulties of principle. Then the computation of $P_{\mathrm{G}}(\chi_{\mathrm{a}})$ is reduced to determining the volume-weighted probability distribution $\mathcal{V}(\chi_{\mathrm{a}})$ for the fields χ_{a} within a randomly chosen 3-volume along the reheating surface. The probability distribution of χ_{a} will be expressed as

$$P(\chi_{\mathrm{a}}) = \mathcal{V}(\chi_{\mathrm{a}})\nu_{\mathrm{G}}(\chi_{\mathrm{a}})\nu_{\mathrm{civ}}(\chi_{\mathrm{a}}) \; . \tag{5.5}$$

However, defining $\mathcal{V}(\chi_{\mathrm{a}})$ turns out to be far from straightforward since the reheating surface in eternal inflation is an infinite 3-surface with a complicated geometry and topology. The lack of a natural, unambiguous, unbiased measure on the infinite reheating surface is known as the "measure problem" in eternal inflation. Existing approaches and measure prescriptions are discussed in Sect. 5.3, where two main alternatives (the "volume-based" and "worldline-based" measures) are presented. In Sect. 5.3.2 I give arguments in favor of using the volume-based measure for computing the probability distribution of values χ_{a} measured by a random observer. The volume-based measure has been applied to obtain statistical predictions for the gravitational constant in Brans–Dicke theories [24, 25], cosmological constant (dark energy) [46, 47, 48, 49, 50, 51], particle physics parameters [52, 53, 54] and the amplitude of primordial density perturbations [48, 51, 55, 56].

The issue of statistical predictions has recently come to the fore in conjunction with the discovery of the string theory landscape. According to various estimates, one expects to have between 10^{500} and 10^{1500} possible vacuum states of string theory [31, 57, 58, 59, 60]. The string vacua differ in the geometry of spacetime compactification and have different values of the effective cosmological constant (or "dark energy" density). Transitions between vacua may happen via the well-known Coleman–deLuccia tunneling mechanism [32]. Once the dark energy dominates in a given region, the spacetime becomes locally de Sitter. Then the tunneling process will create infinitely many disconnected "daughter" bubbles of other vacua. Observers like us may appear within any of the habitable bubbles. Since the fundamental theory does not specify a single "preferred" vacuum, the probability distribution of vacua remains to be determined as found by a randomly chosen observer. The "volume-based" and "worldline-based" measures can be extended to scenarios with multiple bubbles, as discussed in more detail in Sect. 5.3.4. Some recent results obtained using these measures are reported in [61, 62].

5.1.2 Physical Justifications of the Semi-classical Picture

The standard framework of inflationary cosmology asserts that vacuum quantum fluctuations with super-horizon wavelengths become classical inhomogeneities of the field ϕ. The calculations of cosmological density perturbations generated during inflation [7, 11, 63, 64, 65, 66, 67] also assume that a "classicalization" of quantum fluctuations takes place via the same mechanism. In the calculations, the statistical average $\left\langle \delta\phi^2 \right\rangle$ of classical fluctuations on super-Hubble scales is simply set equal to the quantum expectation value $\langle 0|\,\hat{\phi}^2\,|0\rangle$ in a suitable vacuum state. While this approach is widely accepted in the cosmology literature, a growing body of research is devoted to the analysis of the quantum-to-classical transition during inflation (see, e.g., [68] for an early review). Since a detailed analysis would be beyond the scope of the present text, I merely outline the main ideas and arguments relevant to this issue.

A standard phenomenological explanation of the "classicalization" of the perturbations is as follows. For simplicity, let us restrict our attention to a slow-roll inflationary scenario with one scalar field ϕ. In the slow-roll regime, one can approximately regard ϕ as a massless scalar field in de Sitter background spacetime [4]. Due to the exponentially fast expansion of de Sitter spacetime, super-horizon Fourier modes of the field ϕ are in squeezed quantum states with exponentially large ($\sim \mathrm{e}^{Ht}$) squeezing parameters [69, 70, 71, 72, 73, 74]. Such highly squeezed states have a macroscopically large uncertainty in the field value ϕ and thus quickly decohere due to interactions with gravity and with other fields. The resulting mixed state is effectively equivalent to a statistical ensemble with a Gaussian-distributed value of ϕ. Therefore one may compute the statistical average $\left\langle \delta\phi^2 \right\rangle$ as the quantum expectation value $\langle 0|\,\hat{\phi}^2\,|0\rangle$ and interpret the fluctuation $\delta\phi$ as a classical "noise." A heuristic description of the "classicalization" [4] is that the quantum commutators of the creation and annihilation operators of the field modes, $[\hat{a}, \hat{a}^\dagger] = 1$, are much smaller than the expectation values $\left\langle a^\dagger a \right\rangle \gg 1$ and are thus negligible.

A related issue is the backreaction of fluctuations of the scalar field ϕ on the metric.[4] According to the standard theory (see, e.g., [67, 83] for reviews), the perturbations of the metric arising due to fluctuations of ϕ are described by an auxiliary scalar field (sometimes called the "Sasaki–Mukhanov variable") in a fixed de Sitter background. Thus, the "classicalization" effect should apply equally to the fluctuations of ϕ and to the induced metric perturbations. At the same time, these metric perturbations can be viewed, in an appropriate coordinate system, as fluctuations of the local expansion rate $H(\phi)$ due to local fluctuations of ϕ [4, 7, 84]. Thus one arrives at the picture of a "locally de Sitter" spacetime with the metric (5.3), where the Hubble rate $\dot{a}/a = H(\phi)$ fluctuates on super-horizon length scales and locally follows the value of ϕ via the classical Einstein equation (5.1).

The picture as outlined is phenomenological and does not provide a description of the quantum-to-classical transition in the metric perturbations at the level of field theory. For instance, a fluctuation of ϕ leading to a local increase of $H(\phi)$ necessarily violates the null energy condition [85, 86, 87]. The cosmological implications of such "semi-classical" fluctuations (see, e.g., the scenario of "island cosmology" [88, 89, 90]) cannot be understood in detail within the framework of the phenomenological picture.

A more fundamental approach to describing the quantum-to-classical transition of perturbations was developed using non-equilibrium quantum field theory and the influence functional formalism [91, 92, 93, 94]. In this approach, decoherence of a pure quantum state of ϕ into a mixed state is entirely due to the self-interaction of the field ϕ. In particular, it is predicted that no decoherence would occur for a free field with $V(\phi) = m^2\phi^2/2$. This result is at variance with the accepted paradigm of "classicalization" as outlined above. If the source of the "noise" is the coupling between different perturbation modes of ϕ, the typical amplitude of the "noise" will be second order in the perturbation. This is several orders of magnitude smaller than the amplitude of "noise" found in the standard approach. Accordingly, it is claimed [95, 96] that the magnitude of cosmological perturbations generated by inflation is several orders of magnitude smaller than the results currently accepted as standard, and that the shape of the perturbation spectrum depends on the details of the process of "classicalization" [97]. Thus, the results obtained via the influence functional techniques do not appear to reproduce the phenomenological picture of "classicalization" as outlined above. This mismatch emphasizes the need for a deeper understanding of the nature of the quantum-to-classical transition for cosmological perturbations.

Finally, let us mention a different line of work which supports the "classicalization" picture. In [8, 98, 99, 100, 101, 102], calculations of (renormalized) expectation values such as $\langle\hat{\phi}^2\rangle$, $\langle\hat{\phi}^4\rangle$, etc., were performed for field operators $\hat{\phi}$

[4] The backreaction effects of the long-wavelength fluctuations of a scalar field during inflation have been investigated extensively (see, e.g., [75, 76, 77, 78, 79, 80, 81, 82]).

in a fixed de Sitter background. The results were compared with the statistical averages

$$\int P(\phi,t)\phi^2 \mathrm{d}\phi, \quad \int P(\phi,t)\phi^4 \mathrm{d}\phi, \quad \text{etc.}\,, \tag{5.6}$$

where the distribution $P(\phi, t)$ describes the "random walk" of the field ϕ in the Fokker–Planck approach (see Sect. 5.2.1). It was shown that the leading late-time asymptotics of the quantum expectation values coincide with the corresponding statistical averages (5.6). These results appear to validate the "random walk" approach, albeit in a limited context (in the absence of backreaction).

5.2 Stochastic Approach to Inflation

The *stochastic approach to inflation* is a semi-classical, statistical description of the spacetime resulting from quantum fluctuations of the inflation field(s) and their backreaction on the metric [1, 2, 103, 104, 105, 106, 107, 108, 109, 110, 111]. In this description, the spacetime remains everywhere classical but its geometry is determined by a stochastic process. In the next subsections I review the main tools used in the stochastic approach for calculations in the context of random walk type, slow-roll inflation. Models involving tunneling-type eternal inflation are considered in Sect. 5.2.5.

5.2.1 Random Walk-Type Eternal Inflation

The important features of random walk-type eternal inflation can be understood by considering a simple slow-roll inflationary model with a single scalar field ϕ and a potential $V(\phi)$. The slow-roll evolution equation is

$$\dot{\phi} = -\frac{1}{3H}\frac{\mathrm{d}V}{\mathrm{d}\phi} = -\frac{1}{4\pi}\frac{\mathrm{d}H}{\mathrm{d}\phi} \equiv v(\phi)\,, \tag{5.7}$$

where $H(\phi)$ is defined by (5.1) and $v(\phi)$ is a model-dependent function describing the "velocity" $\dot{\phi}$ of the deterministic evolution of the field ϕ. The slow-roll trajectory $\phi_{\mathrm{sr}}(t)$, which is a solution of (5.7), is an attractor [112, 113] for trajectories starting with a wide range of initial conditions.[5]

As discussed in Sect. 5.1.2, the super-horizon modes of the field ϕ are assumed to undergo a rapid quantum-to-classical transition. Therefore one regards the spatial average of ϕ on scales of several H^{-1} as a *classical* field variable. The spatial averaging can be described with help of a suitable window function,

$$\langle\phi(\mathbf{x})\rangle \equiv \int W(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\mathrm{d}^3\mathbf{y}\,. \tag{5.8}$$

[5] See [43] for a precise definition of an attractor trajectory in the context of inflation.

It is implied that the window function $W(\mathbf{x})$ decays quickly on physical distances $a|\mathbf{x}|$ of order several H^{-1}. From now on, let us denote the volume-averaged field simply by ϕ (no other field ϕ will be used).

As discussed above, the influence of quantum fluctuations leads to random "jumps" superimposed on top of the deterministic evolution of the volume-averaged field $\phi(t, \mathbf{x})$. This may be described by a Langevin equation of the form [2]

$$\dot{\phi}(t, \mathbf{x}) = v(\phi) + N(t, \mathbf{x}) \,, \tag{5.9}$$

where $N(t, \mathbf{x})$ stands for "noise" and is assumed to be a Gaussian random function with known correlator [2, 114, 115, 116]

$$\left\langle N(t, \mathbf{x}) N(\tilde{t}, \tilde{\mathbf{x}}) \right\rangle = C(t, \tilde{t}, |\mathbf{x} - \tilde{\mathbf{x}}|\,; \phi) \,. \tag{5.10}$$

An explicit form of the correlator C depends on the specific window function W used for averaging the field ϕ on Hubble scales [116]. However, the window function W is merely a phenomenological device used in lieu of a complete *ab initio* derivation of the stochastic inflation picture. One expects, therefore, that results of calculations should be robust with respect to the choice of W. In other words, any uncertainty due to the choice of the window function must be regarded as an imprecision inherent in the method. For instance, a robust result in this sense is an exponentially fast decay of correlations on timescales $\Delta t \gtrsim H^{-1}$,

$$C(t, \tilde{t}, |\mathbf{x} - \tilde{\mathbf{x}}|\,; \phi) \propto \exp\left(-2H(\phi)\left|t - \tilde{t}\right|\right) \,, \tag{5.11}$$

which holds for a wide class of window functions [116].

For the purposes of the present consideration, we only need to track the evolution of $\phi(t, \mathbf{x})$ along a single comoving worldline $\mathbf{x} = \text{const}$. Thus, we will not need an explicit form of $C(t, \tilde{t}, |\mathbf{x} - \tilde{\mathbf{x}}|\,; \phi)$ but merely the value at coincident points $t = \tilde{t}$, $\mathbf{x} = \tilde{\mathbf{x}}$, which is computed in the slow-roll inflationary scenario as [2]

$$C(t, t, 0; \phi) = \frac{H^2(\phi)}{4\pi^2} \,. \tag{5.12}$$

[This represents the fluctuation (5.2) accumulated during one Hubble time, $\Delta t = H^{-1}$.] Due to the property (5.11), one may neglect correlations on time scales $\Delta t \gtrsim H^{-1}$ in the "noise" field.[6] Thus, the evolution of ϕ on time scales $\Delta t \gtrsim H^{-1}$ can be described by a finite-difference form of the Langevin equation (5.9),

$$\phi(t + \Delta t) - \phi(t) = v(\phi)\Delta t + \sqrt{2D(\phi)\Delta t}\,\xi(t) \,, \tag{5.13}$$

where

$$D(\phi) \equiv \frac{H^3(\phi)}{8\pi^2} \tag{5.14}$$

[6] Taking these correlations into account leads to a picture of "color noise" [117, 118]. In what follows, we only consider the simpler picture of "white noise" as an approximation adequate for the issues at hand.

and ξ is a normalized random variable representing "white noise,"

$$\langle \xi \rangle = 0, \quad \left\langle \xi^2 \right\rangle = 1, \tag{5.15}$$

$$\langle \xi(t)\xi(t+\Delta t) \rangle = 0 \quad \text{for} \quad \Delta t \gtrsim H^{-1}. \tag{5.16}$$

Equation (5.13) is interpreted as describing a Brownian motion $\phi(t)$ with the systematic "drift" $v(\phi)$ and the "diffusion coefficient" $D(\phi)$. In a typical slow-roll inflationary scenario, there will be a range of ϕ where the noise dominates over the deterministic drift,

$$v(\phi)\Delta t \ll \sqrt{2D(\phi)\Delta t}, \quad \Delta t \equiv H^{-1} \, . \tag{5.17}$$

Such a range of ϕ is called the "diffusion-dominated regime." For ϕ near the end of inflation, the amplitude of the noise is very small, and so the opposite inequality holds. This is the "deterministic regime" where the random jumps can be neglected and the field ϕ follows the slow-roll trajectory.

5.2.2 Fokker–Planck Equations

A useful description of the statistical properties of $\phi(t)$ is furnished by the probability density $P(\phi, t)\mathrm{d}\phi$ of having a value ϕ at time t. As in the case of the Langevin equation, the values $\phi(t)$ are measured along a single, randomly chosen comoving worldline $\mathbf{x}$ = const. The probability distribution $P(\phi, t)$ satisfies the Fokker–Planck (FP) equation whose standard derivation we omit [119, 120],

$$\partial_t P = \partial_\phi \left[-v(\phi)P + \partial_\phi \left(D(\phi)P \right) \right] \, . \tag{5.18}$$

The coefficients $v(\phi)$ and $D(\phi)$ are in general model-dependent and need to be calculated in each particular scenario. These calculations require only the knowledge of the slow-roll trajectory and the mode functions of the quantized scalar perturbations. For ordinary slow-roll inflation with an effective potential $V(\phi)$, the results are well-known expressions (5.7) and (5.14). The corresponding expressions for models of k-inflation were derived in [43] using the relevant quantum theory of perturbations [121].

It is well known that there exists a "factor ordering" ambiguity in translating the Langevin equation into the FP equation if the amplitude of the "noise" depends on the position. Specifically, the factor $D(\phi)$ in (5.13) may be replaced by $D(\phi + \theta\Delta t)$, where $0 < \theta < 1$ is an arbitrary constant. With $\theta \neq 0$, the term $\partial_{\phi\phi} (DP)$ in (5.18) will be replaced by a different ordering of the factors,

$$\partial_{\phi\phi} \left(DP \right) \to \partial_\phi \left[D^\theta \partial_\phi \left(D^{1-\theta} P \right) \right] \, . \tag{5.19}$$

Popular choices $\theta = 0$ and $\theta = 1/2$ are called the Ito and the Stratonovich factor ordering, respectively. Motivated by the considerations of [122], we choose $\theta = 0$ as shown in (5.13) and (5.18). Given the phenomenological nature of the Langevin equation (5.13), one expects that any ambiguity due to the choice of θ represents an imprecision inherent in the stochastic approach. This imprecision is typically of order $H^2 \ll 1$ [123].

The quantity $P(\phi, t)$ may be also interpreted as the fraction of the *comoving* volume (i.e., coordinate volume $\mathrm{d}^3\mathbf{x}$) occupied by the field value ϕ at time t. Another important characteristic is the volume-weighted distribution $P_V(\phi, t)\mathrm{d}\phi$, which is defined as the *proper* 3-volume (as opposed to the comoving volume) of regions having the value ϕ at time t. (To avoid considering infinite volumes, one may restrict one's attention to a finite comoving domain in the universe and normalize $P_V(\phi, t)$ to unit volume at some initial time $t = t_0$.) The volume distribution satisfies a modified FP equation [4, 106, 108],

$$\partial_t P_V = \partial_\phi \left[-v(\phi) P_V + \partial_\phi \left(D(\phi) P_V \right) \right] + 3H(\phi) P_V \ , \tag{5.20}$$

which differs from (5.18) by the term $3HP_V$ that describes the exponential growth of 3-volume in inflating regions.[7]

Presently we consider scenarios with a single scalar field; however, the formalism of FP equations can be straightforwardly extended to multi-field models (see, e.g., [122]). For instance, the FP equation for a two-field model is

$$\partial_t P = \partial_{\phi\phi} \left(DP \right) + \partial_{\chi\chi} \left(DP \right) - \partial_\phi \left(v_\phi P \right) - \partial_\chi \left(v_\chi P \right) \ , \tag{5.21}$$

where $D(\phi, \chi)$, $v_\phi(\phi, \chi)$ and $v_\chi(\phi, \chi)$ are appropriate coefficients.

5.2.3 Methods of Solution

In principle, one can solve the FP equations forward in time by a numerical method, starting from a given initial distribution at $t = t_0$. To specify the solution uniquely, the FP equations must be supplemented by boundary conditions at both ends of the inflating range of ϕ [110, 111]. At the reheating boundary ($\phi = \phi_*$), one imposes the "exit" boundary conditions,

$$\partial_\phi \left[D(\phi) P \right]_{\phi=\phi_*} = 0, \quad \partial_\phi \left[D(\phi) P_V \right]_{\phi=\phi_*} = 0 \ . \tag{5.22}$$

These boundary conditions express the fact that random jumps are very small at the end of inflation and cannot move the value of ϕ away from $\phi = \phi_*$. If the potential $V(\phi)$ reaches Planck energy scales at some $\phi = \phi_{\max}$ (this happens generally in "chaotic"-type inflationary scenarios with unbounded potentials), the semi-classical picture of spacetime breaks down for regions with $\phi \sim \phi_{\max}$. Hence, a boundary condition must be imposed also at $\phi = \phi_{\max}$. For instance, one can use the absorbing boundary condition,

$$P(\phi_{\max}) = 0 \ , \tag{5.23}$$

[7] A more formal derivation of (5.20) as well as details of the interpretation of the distributions P and P_V in terms of ensembles of worldlines can be found in [124].

which means that Planck-energy regions with $\phi = \phi_{\max}$ disappear from consideration [110, 111].

Once the boundary conditions are specified, one may write the general solution of the FP equation (5.18) as

$$P(\phi, t) = \sum_{\lambda} C_{\lambda} P^{(\lambda)}(\phi)\, \mathrm{e}^{\lambda t} , \tag{5.24}$$

where the sum is performed over all the eigenvalues λ of the differential operator

$$\hat{L}P \equiv \partial_\phi \left\{ -v(\phi)P + \partial_\phi \left[D(\phi)P \right] \right\} , \tag{5.25}$$

and the corresponding eigenfunctions $P^{(\lambda)}$ are defined by

$$\hat{L}P^{(\lambda)}(\phi) = \lambda P^{(\lambda)}(\phi) . \tag{5.26}$$

The constants C_{λ} can be expressed through the initial distribution $P(\phi, t_0)$.

By an appropriate change of variables $\phi \to z$, $P(\phi) \to F(z)$, the operator $\hat{L}$ may be brought into a manifestly self-adjoint form [104, 106, 107, 108, 123, 125],

$$\hat{L} \to \frac{\mathrm{d}^2}{\mathrm{d}z^2} + U(z) . \tag{5.27}$$

Then one can show that all the eigenvalues λ of $\hat{L}$ are non-positive; in particular, the (algebraically) largest eigenvalue $\lambda_{\max} \equiv -\gamma < 0$ is non-degenerate and the corresponding eigenfunction $P^{(\lambda_{\max})}(\phi)$ is positive everywhere [43, 123]. Hence, this eigenfunction describes the late-time asymptotic of the distribution $P(\phi, t)$,

$$P(\phi, t) \propto P^{(\lambda_{\max})}(\phi)\, \mathrm{e}^{-\gamma t} . \tag{5.28}$$

The distribution $P^{(\lambda_{\max})}(\phi)$ is the "stationary" distribution of ϕ per comoving volume at late times. The exponential decay of the distribution $P(\phi, t)$ means that at late times most of the comoving volume (except for an exponentially small fraction) has finished inflation and entered reheating.

Similarly, one can represent the general solution of (5.20) by

$$P_V(\phi, t) = \sum_{\tilde{\lambda}} C_{\tilde{\lambda}} P_V^{(\tilde{\lambda})}(\phi) \mathrm{e}^{\tilde{\lambda} t} , \tag{5.29}$$

where

$$[\hat{L} + 3H(\phi)] P^{(\tilde{\lambda})}(\phi) = \tilde{\lambda} P^{(\tilde{\lambda})}(\phi) . \tag{5.30}$$

By the same method as for the operator $\hat{L}$, it is possible to show that the spectrum of eigenvalues $\tilde{\lambda}$ of the operator $\hat{L} + 3H(\phi)$ is bounded from above and that the largest eigenvalue $\tilde{\lambda}_{\max} \equiv \tilde{\gamma}$ admits a non-degenerate, everywhere positive eigenfunction $P^{(\tilde{\gamma})}(\phi)$. However, the largest eigenvalue $\tilde{\gamma}$ may be either positive or negative. If $\tilde{\gamma} > 0$, the late-time behavior of $P_V(\phi, t)$ is

$$P_V(\phi, t) \propto P^{(\tilde{\gamma})}(\phi) e^{\tilde{\gamma} t} , \tag{5.31}$$

which means that the total proper volume of all the inflating regions grows with time. This is the behavior expected in eternal inflation: the number of independently inflating domains increases without limit. Thus, the condition $\tilde{\gamma} > 0$ is the criterion for the presence of eternal self-reproduction of inflating domains. The corresponding distribution $P^{(\tilde{\gamma})}(\phi)$ is called the "stationary" distribution [26, 110, 111, 126].

If $\tilde{\gamma} \leq 0$, eternal inflation does not occur and the entire space almost surely (i.e., with probability 1) enters the reheating epoch at a finite time.

If the potential $V(\phi)$ is of "new" inflationary type [6, 127, 128, 129, 130] and has a global maximum at say $\phi = \phi_0$, the eigenvalues γ and $\tilde{\gamma}$ can be estimated (under the usual slow-roll assumptions on V) as [123]

$$\gamma \approx \frac{V''(\phi_0)}{8\pi V(\phi_0)} H(\phi_0) < 0, \quad \tilde{\gamma} \approx 3H(\phi_0) > 0 . \tag{5.32}$$

Therefore, eternal inflation is generic in the "new" inflationary scenario.

Let us comment on the possibility of obtaining solutions $P(\phi, t)$ in practice. With the potential $V(\phi) = \lambda\phi^4$, the full time-dependent FP equation (5.18) can be solved analytically via a non-linear change of variable $\phi \to \phi^{-2}$ [125, 131, 132]. This exact solution, as well as an approximate solution $P(\phi, t)$ for a general potential, can also be obtained using the saddle-point evaluation of a path-integral expression for $P(\phi, t)$ [133]. In some cases the eigenvalue equation $\hat{L}P^{(\lambda)} = \lambda P^{(\lambda)}$ may be reduced to an exactly solvable Schrödinger equation. These cases include potentials of the form $V(\phi) = \lambda e^{\mu\phi}$, $V(\phi) = \lambda\phi^{-2}$, $V(\phi) = \lambda \cosh^{-2}(\mu\phi)$; see, e.g., [123] for other examples.

A general approximate method for determining $P(\phi, t)$ for arbitrary potentials [134, 135, 136] consists of a perturbative expansion,

$$\phi(t) = \phi_0(t) + \delta\phi_1(t) + \delta\phi_2(t) + \cdots , \tag{5.33}$$

applied directly to the Langevin equation. The result is (at the lowest order) a Gaussian approximation with a time-dependent mean and variance [134],

$$P(\phi, t) \approx \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left\{ -\frac{[\phi - \phi_0(t)]^2}{2\sigma^2(t)} \right\} , \tag{5.34}$$

$$\sigma^2(t) \equiv \frac{H'^2(\phi_{\mathrm{sr}})}{\pi} \int_{\phi_{\mathrm{sr}}}^{\phi_{\mathrm{in}}} \frac{H^3}{H'^3} d\phi, \tag{5.35}$$

$$\phi_0(t) \equiv \phi_{\mathrm{sr}}(t) + \frac{H''}{2H'}\sigma^2(t) + \frac{H'}{4\pi}\left(\frac{H_{\mathrm{in}}^3}{H_{\mathrm{in}}'^2} - \frac{H^3}{H'^2} \right), \tag{5.36}$$

where $\phi_{\mathrm{sr}}(t)$ is the slow-roll trajectory and ϕ_{in} is the initial value of ϕ. While methods based on the Langevin equation do not take into account boundary conditions or volume weighting effects, the formula (5.34) provides an adequate approximation to the distribution $P(\phi, t)$ in a useful range of ϕ and t [136].

5.2.4 Gauge Dependence Issues

An important feature of the FP equations is their dependence on the choice of the time variable. One can consider a replacement of the form

$$t \to \tau, \quad \mathrm{d}\tau \equiv T(\phi)\mathrm{d}t \,, \tag{5.37}$$

understood in the sense of integrating along comoving worldlines $\mathbf{x} = \text{const}$, where $T(\phi) > 0$ is an arbitrary function of the field. For instance, a possible choice is $T(\phi) \equiv H(\phi)$, which makes the new time variable dimensionless,

$$\tau = \int H \mathrm{d}t = \ln a \,. \tag{5.38}$$

This time variable is called "scale factor time" or "e-folding time" since it measures the number of e-foldings along a comoving worldline.

The distributions $P(\phi, \tau)$ and $P_V(\phi, \tau)$ are defined as before, except for considering the 3-volumes along hypersurfaces of equal τ. These distributions satisfy FP equations similar to (5.18)–(5.20). With the replacement (5.37), the coefficients of the new FP equations are modified as follows [123],

$$D(\phi) \to \frac{D(\phi)}{T(\phi)}, \quad v(\phi) \to \frac{v(\phi)}{T(\phi)} \,, \tag{5.39}$$

while the "growth" term $3HP_V$ in (5.20) is replaced by $3HT^{-1}P_V$. The change in the coefficients may significantly alter the qualitative behavior of the solutions of the FP equations. For instance, stationary distributions defined through the proper time t and the e-folding time $\tau = \ln a$ were found to have radically different behaviors [18, 111, 126]. This sensitivity to the choice of the "time gauge" τ is unavoidable since hypersurfaces of equal τ may preferentially select regions with certain properties. For instance, most of the proper volume in equal-t hypersurfaces is filled with regions that have gained expansion by remaining near the top of the potential $V(\phi)$, while hypersurfaces of equal scale factor will under-represent those regions. Thus, a statement such as "most of the volume in the universe has values of ϕ with high $V(\phi)$" is largely gauge dependent.

In the early works on eternal inflation [25, 110, 111, 126], the late-time asymptotic distribution of volume $P_V^{(\tilde{\gamma})}(\phi)$ along hypersurfaces of equal proper time [see (5.31)] was interpreted as the stationary distribution of field values in the universe. However, the high sensitivity of this distribution to the choice of the time variable makes this interpretation unsatisfactory. Also, it was noted [137] that equal proper-time volume distributions predict an unacceptably small probability for the currently observed CMB temperature. The reason for this result is the extreme bias of the proper-time gauge toward over-representing regions where reheating occurred very recently [18, 138]. One might ask whether hypersurfaces of equal scale factor or some other choice of time gauge would provide less biased answers. However, it turns out [124] that there exists no *a priori* choice of the time gauge τ that provides unbiased equal-τ probability distributions for all potentials $V(\phi)$ in models of slow-roll inflation (see Sect. 5.3.3 for details).

Although the FP equations necessarily involve a dependence on gauge, they do provide a useful statistical picture of the distribution of fields in the universe. The FP techniques can also be used for deriving several gauge-independent results. For instance, the presence of eternal inflation is a gauge-independent statement (see also Sect. 5.3.1): if the largest eigenvalue $\tilde{\gamma}$ is positive in one gauge of the form (5.37), then $\tilde{\gamma} > 0$ in every other gauge [139]. Using the FP approach, one can also compute the fractal dimension of the inflating domain [139, 140] and the probability of exiting inflation through a particular point ϕ_* of the reheating boundary in the configuration space (in case there exists more than one such point).

The exit probability can be determined as follows [43, 122]. Let us assume for simplicity that there are two possible exit points ϕ_* and ϕ_{E}, and that the initial distribution is concentrated at $\phi = \phi_0$, i.e.,

$$P(\phi, t = 0) = \delta(\phi - \phi_0) \ , \tag{5.40}$$

where $\phi_{\mathrm{E}} < \phi_0 < \phi_*$. The probability of exiting inflation through $\phi = \phi_{\mathrm{E}}$ during a time interval $[t, t + \mathrm{d}t]$ is

$$\mathrm{d}p_{\mathrm{exit}}(\phi_{\mathrm{E}}) = -v(\phi_{\mathrm{E}})P(\phi_{\mathrm{E}}, t)\mathrm{d}t \tag{5.41}$$

[note that $v(\phi_{\mathrm{E}}) < 0$]. Hence, the total probability of exiting through $\phi = \phi_{\mathrm{E}}$ at any time is

$$p_{\mathrm{exit}}(\phi_{\mathrm{E}}) = \int_0^\infty \mathrm{d}p_{\mathrm{exit}}(\phi_{\mathrm{E}}) = -v(\phi_{\mathrm{E}}) \int_0^\infty P(\phi_{\mathrm{E}}, t)\mathrm{d}t \ . \tag{5.42}$$

Introducing an auxiliary function $F(\phi)$ as

$$F(\phi) \equiv -v(\phi) \int_0^\infty P(\phi, t)\mathrm{d}t \ , \tag{5.43}$$

one can show that $F(\phi)$ satisfies the gauge-invariant equation,

$$\partial_\phi \left[\partial_\phi \left(\frac{D}{v} F \right) - F \right] = \delta(\phi - \phi_0) \ . \tag{5.44}$$

This is in accord with the fact that $p_{\mathrm{exit}}(\phi_{\mathrm{E}}) = F(\phi_{\mathrm{E}})$ is a gauge-invariant quantity. Equation (5.44) with the boundary conditions

$$F(\phi_*) = 0, \quad \partial_\phi \left(\frac{D}{v} F \right) \bigg|_{\phi = \phi_{\mathrm{E}}} = 0 \ , \tag{5.45}$$

can be straightforwardly integrated and yields explicit expressions for the exit probability $p_{\mathrm{exit}}(\phi_{\mathrm{E}})$ as a function of the initial value ϕ_0 [43]. The exit probability $p_{\mathrm{exit}}(\phi_*)$ can be determined similarly.

5.2.5 Self-Reproduction of Tunneling Type

Until now, we considered eternal self-reproduction due to random walk of a scalar field. Another important class of models includes self-reproduction due to bubble nucleation.[8] Such scenarios of eternal inflation were studied in [34, 141, 142, 143, 144, 145].

In a locally de Sitter universe dominated by dark energy, nucleation of bubbles of false vacuum may occur due to tunneling [14, 32, 146, 147]. Since the bubble nucleation rate κ per unit 4-volume is very small [32, 148],

$$\kappa = \mathrm{O}(1) H^{-4} \exp\left(-S_{\mathrm{I}} - \frac{\pi}{H^2}\right) , \tag{5.46}$$

where S_{I} is the instanton action and H is the Hubble constant of the de Sitter background, bubbles will generically not merge into a single false-vacuum domain [34]. Hence, infinitely many bubbles will be nucleated at different places and times. The resulting "daughter" bubbles may again contain an asymptotically de Sitter, infinite universe, which again gives rise to infinitely many "grand-daughter" bubbles. This picture of eternal self-reproduction was called the "recycling universe" [30]. Some (or all) of the created bubbles may support a period of additional inflation followed by reheating, as shown in Fig. 5.3.

In the model of [30], there were only two vacua which could tunnel into each other. A more recently developed paradigm of "string theory landscape" [31] involves a very large number of metastable vacua, corresponding to local minima of an effective potential in field space. The value of the potential at each minimum is the effective value of the cosmological constant Λ in the corresponding vacuum. Figure 5.4 shows a phenomenologist's view of the "landscape." Vacua with $\Lambda \leq 0$ do not allow any further tunneling[9] and are called "terminal" vacua [153], while vacua with $\Lambda > 0$ are called "recyclable" since they can tunnel to other vacua with $\Lambda > 0$ or $\Lambda \leq 0$. Bubbles of recyclable vacua will give rise to infinitely many nested "daughter" bubbles. A conformal diagram of the resulting spacetime is outlined in Fig. 5.5. Of course, only a finite number of bubbles can be drawn; the bubbles actually form a fractal structure in a conformal diagram [154].

A statistical description of the "recycling" spacetime can be obtained [30, 153] by considering a single comoving worldline $\mathbf{x} = \mathrm{const}$ that passes through different bubbles at different times. (It is implied that the worldline is randomly chosen from an ensemble of infinitely many such worldlines passing through different points $\mathbf{x}$.) Let the index $\alpha = 1, ..., N$ label all the available types of bubbles. For calculations, it is convenient to use the e-folding time $\tau \equiv \ln a$. We are interested in the probability $f_\alpha(\tau)$ of passing through

[8] Both processes may be combined in a single scenario [30], but we shall consider them separately for clarity.

[9] Asymptotically flat $\Lambda = 0$ vacua cannot support tunneling [149, 150, 151]; vacua with $\Lambda < 0$ will quickly collapse to a "big crunch" singularity [32, 152].

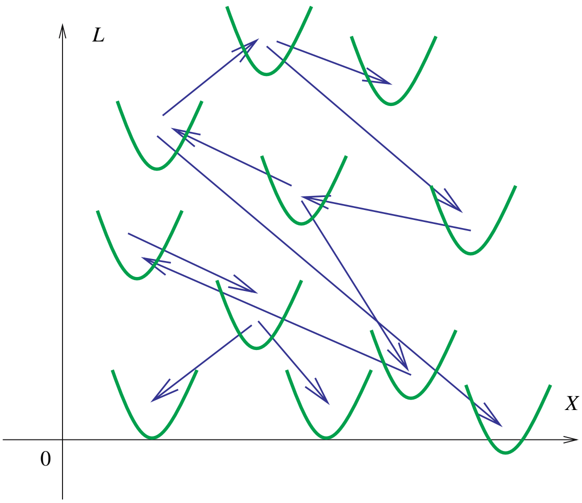

Fig. 5.4. A schematic representation of the "landscape of string theory," consisting of a large number of local minima of an effective potential. The variable X collectively denotes various fields and Λ is the effective cosmological constant. *Arrows* show possible tunneling transitions between vacua

a bubble of type α at time τ. This probability distribution is normalized by $\sum_\alpha f_\alpha = 1$; the quantity $f_\alpha(\tau)$ can be also visualized as the fraction of the comoving volume occupied by bubbles of type α at time τ. Denoting by $\kappa_{\alpha\beta}$ the nucleation rate for bubbles of type α within bubbles of type β [computed according to (5.46)], we write the "master equation" describing the evolution of $f_\alpha(\tau)$,

$$\frac{\mathrm{d}f_\beta}{\mathrm{d}\tau} = \sum_\alpha \left(-\kappa_{\alpha\beta} f_\beta + \kappa_{\beta\alpha} f_\alpha\right) \equiv \sum_\alpha M_{\beta\alpha} f_\alpha, \tag{5.47}$$

where we introduced the auxiliary matrix $M_{\alpha\beta}$. Given a set of initial conditions $f_\alpha(0)$, one can evolve $f_\alpha(\tau)$ according to (5.47).

To proceed further, one may now distinguish the following two cases: either terminal vacua exist (some β such that $\kappa_{\alpha\beta} = 0$ for all α) or all the vacua

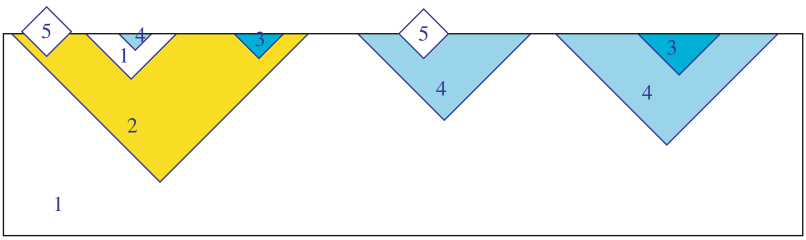

Fig. 5.5. A conformal diagram of the spacetime where self-reproduction occurs via bubble nucleation. Regions labeled "*5*" are asymptotically flat ($\Lambda = 0$)

are recyclable. (Theory suggests that the former case is more probable [59].) If terminal vacua exist, then the late-time asymptotic solution can be written as [153]

$$f_\alpha(\tau) \approx f_\alpha^{(0)} + s_\alpha \mathrm{e}^{-q\tau}, \tag{5.48}$$

where $f_\alpha^{(0)}$ is a constant vector that depends on the initial conditions and has non-zero components only in terminal vacua, and s_α does not depend on initial conditions and is an eigenvector of $M_{\alpha\beta}$ such that $\sum_\alpha M_{\beta\alpha} s_\alpha = -q s_\beta$, $q > 0$. This solution shows that all comoving volume reaches terminal vacua exponentially quickly. (As in the case of random walk inflation, there are infinitely many "eternally recycling" points $\mathbf{x}$ that never enter any terminal vacua, but these points form a set of measure zero.)

If there are no terminal vacua, the solution $f_\alpha(\tau)$ approaches a constant distribution [155],

$$\lim_{\tau\to\infty} f_\alpha(\tau) \approx f_\alpha^{(0)}, \quad \sum_\beta M_{\beta\alpha} f_\alpha^{(0)} = 0 \, , \tag{5.49}$$

$$f_\alpha^{(0)} = H_\alpha^4 \exp\left(\frac{\pi}{H_\alpha^2}\right) . \tag{5.50}$$

In this case, the quantities $f_\alpha^{(0)}$ are independent of initial conditions and are interpreted as the fractions of time spent by the comoving worldline in bubbles of type α.

One may adopt another approach and ignore the duration of time spent by the worldline within each bubble. Thus, one describes only the *sequence* of the bubbles encountered along a randomly chosen worldline [62, 155, 156]. If the worldline is initially in a bubble of type α, then the probability $\mu_{\beta\alpha}$ of entering the bubble of type β as the next bubble in the sequence after α is

$$\mu_{\beta\alpha} = \frac{\kappa_{\beta\alpha}}{\sum_\gamma \kappa_{\gamma\alpha}} \, . \tag{5.51}$$

(For terminal vacua α, we have $\kappa_{\gamma\alpha} = 0$ and so we may define $\mu_{\beta\alpha} = 0$ for convenience.) Once again we consider landscapes without terminal vacua separately from terminal landscapes. If there are no terminal vacua, then the matrix $\mu_{\alpha\beta}$ is normalized, $\sum_\beta \mu_{\beta\alpha} = 1$, and is thus a stochastic matrix [157] describing a Markov process of choosing the next visited vacuum. The sequence of visited vacua is infinite, so one can define the mean frequency $f_\alpha^{(\mathrm{mean})}$ of visiting bubbles of type α. If the probability distribution for the first element in the sequence is $f_{(0)\alpha}$, then the distribution of vacua after k steps is given (in the matrix notation) by the vector

$$\mathbf{f}_{(k)} = \boldsymbol{\mu}^k \mathbf{f}_{(0)}, \tag{5.52}$$

where $\boldsymbol{\mu}^k$ means the kth power of the matrix $\boldsymbol{\mu} \equiv \mu_{\alpha\beta}$. Therefore, the mean frequency of visiting a vacuum α is computed as an average of $f_{(k)\alpha}$ over n consecutive steps in the limit of large n:

$$\mathbf{f}^{(\text{mean})} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{f}_{(k)} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{\mu}^k \mathbf{f}_{(0)}. \tag{5.53}$$

[It is proved in the theory of Markov processes that the limit $f_\alpha^{(\text{mean})}$ given by (5.53) almost surely coincides with the mean frequency of visiting the state α; see, e.g., [158, Chap. 5, Theorem 2.1], and [159, Theorem 3.5.9.].] It turns out that the distribution $\mathbf{f}^{(\text{mean})}$ is independent of the initial state $\mathbf{f}_{(0)}$ and coincides with the distribution (5.49) found in the continuous-time description [155].

If there exist terminal vacua, then almost all sequences will have a finite length. The distribution of vacua in a randomly chosen sequence is still well-defined and can be computed using (5.53) without the normalizing factor $1/n$,

$$\mathbf{f}^{(\text{mean})} = (\mathbf{1} - \boldsymbol{\mu})^{-1} \boldsymbol{\mu} \mathbf{f}_{(0)}, \tag{5.54}$$

but now the resulting distribution depends on the initial state $\mathbf{f}_{(0)}$ [62, 156].

5.3 Predictions and Measure Issues

As discussed in Sect. 5.1.1, a compelling question in the context of eternal inflation is how to make statistical predictions of observed parameters. One begins by determining whether eternal inflation is present in a given model.

5.3.1 Presence of Eternal Inflation

Since the presence of eternal self-reproduction in models of tunneling type is generically certain (unless the nucleation rate for terminal vacua is unusually high), in this section we restrict our attention to eternal inflation of the random walk type.

The hallmark of eternal inflation is the unbounded growth the total number of independent inflating regions. The total proper 3-volume of the inflating domain also grows without bound at late times, at least when computed along hypersurfaces of equal proper time or equal scale factor. However, the proper 3-volume is a gauge-dependent quantity, and one may construct time gauges where the 3-volume decreases with time even in an everywhere expanding universe [124]. The 3-volume of an *arbitrary* family of equal-time hypersurfaces cannot be used as a criterion for the presence of eternal inflation. However, a weaker criterion is sufficient: eternal inflation is present if (and only if) there *exists* a choice of time slicing with an unbounded growth of the 3-volume of inflating domains [124]. Equivalently, eternal inflation is present if a finite comoving volume gives rise to infinite physical volume [137]. Thus, the presence or absence of eternal inflation is a gauge-independent statement. One may, of course, use a particular gauge (such as the proper time or e-folding time) for calculations, as long as the result is known to be gauge-independent. It can be shown that eternal inflation is present if and only if the 3-volume grows in the e-folding time slicing [124].

The presence of eternal inflation has been analyzed in many specific scenarios. For instance, eternal inflation is generic in "chaotic" [3, 41, 160] and "new" [8] inflationary models. It is normally sufficient to establish the existence of a "diffusion-dominated" regime, that is, a range of ϕ where the typical amplitude $\delta\phi \sim H$ of "jumps" is larger than the typical change of the field, $\dot{\phi}\Delta t$, during one Hubble time $\Delta t = H^{-1}$. For models of scalar field inflation, the condition is

$$H^2 \gg H' . \tag{5.55}$$

Such a range of ϕ is present in most slow-roll models of inflation. (For an example of an inflationary scenario where eternal inflation is generically *not* present, see [161].) A strict formal criterion for the presence of eternal inflation is the positivity of the largest eigenvalue $\tilde{\gamma}$ of the operator $\hat{L}+3H(\phi)$, as defined in Sect. 5.2.3.

The causal structure of the eternally inflating spacetime and the topology of the reheating surface can be visualized using the construction of "eternal comoving points" [139]. These are comoving worldlines $\mathbf{x} = \text{const}$ that forever remain within the inflating domain and never enter the reheating epoch. These worldlines correspond to places where the reheating surface reaches $t = \infty$ in a spacetime diagram (see Fig. 5.2). It was shown in [139] using topological arguments that the presence of inflating domains at arbitrarily late times entails the existence of infinitely many such "eternal points." The set of all eternal points within a given three-dimensional spacelike slice is a measure of zero fractal set. The fractal dimension of this set can be understood as the fractal dimension of the inflating domain [124, 139, 140] and is invariant under any *smooth* coordinate transformations in the spacetime.

The existence of eternal points can be used as another invariant criterion for the presence of eternal inflation. The probability $X(\phi)$ of having an eternal point in an initial Hubble-size region with field value ϕ can be found as the solution of a gauge-invariant, non-linear diffusion equation [139]

$$D\partial_{\phi\phi}X + v\partial_{\phi}X - 3H\left(1 - X\right)\ln\left(1 - X\right) = 0 , \tag{5.56}$$

with zero boundary conditions. Eternal inflation is present if there exists a non-trivial solution $X(\phi) \not\equiv 0$ of this equation.

5.3.2 Observer-Based Measure in Eternal Inflation

In theories where observable parameters χ_{a} are distributed randomly, one would like to predict the values of χ_{a} most likely to be observed by a random (or "typical") observer. More generally, one looks for the probability distribution $P(\chi_{\mathrm{a}})$ of observing the values χ_{a}. As discussed in Sect. 5.1.1, considerations of this type necessarily involve some form of the "principle of mediocrity" [18]. On a more formal level, one needs to construct an ensemble of the "possible observers" and to define a probability measure on this ensemble. In inflationary cosmology, observers appear only along the reheating surface. If eternal inflation is present, the reheating surface contains infinitely many causally disconnected and (possibly) statistically inequivalent domains. The principal difficulties in the probabilistic approach are due to a lack of a natural definition of measure on such surfaces.[10]

Existing proposals for an observer-based measure fall in two major classes, which may be designated as "volume-based" vs. "worldline-based." The difference between these classes is in the approach taken to construct the ensemble of observers. In the "volume" approach [18, 19, 45, 111, 153, 163], the ensemble contains every observer appearing in the universe, at any time or place. In the "worldline" approach [156, 164, 165, 166], the ensemble consists of observers appearing near a single, randomly selected comoving worldline $\mathbf{x} = \text{const}$ (a timelike geodesic could also be used). If the ensemble contains infinitely many observers (this is typically the case for volume-based ensembles), a regularization is needed to obtain specific probability distributions. Finding and applying suitable regularization procedures is a separate technical issue explored in Sect. 5.3.3 and 5.3.4. I begin with a general discussion of these measure prescriptions.

A number of previously considered "volume-based" measure proposals were found to be lacking in one aspect or another [18, 111, 123, 164, 167, 168], the most vexing problem being the dependence on the choice of the time gauge [110, 111, 137]. The requirement of time gauge independence is sufficiently important to reject any measure proposal that suffers from the gauge ambiguity. A prescription manifestly free from gauge dependence is the "spherical cutoff" measure [45]. This prescription provides unambiguous predictions for models of random walk-type eternal inflation if the reheating condition $\phi = \phi_*$ corresponds to a topologically compact and connected locus in the field space $\{\phi, \chi_\mathrm{a}\}$ (see Sect. 5.3.3). For models where the reheating condition is met at several disconnected loci in field space (tunneling-type eternal inflation belongs to this class), one can use the recently proposed prescription of "comoving cutoff" [153, 163]. Since no other volume-based prescriptions are currently considered viable, we refer to the mentioned spherical cutoff/comoving cutoff prescriptions simply as the "volume-based measure."

Similarly, existing measure proposals of the "worldline" type appear to converge essentially to a single prescription [62, 156] (however, see [169, 170] for the most recent developments). We refer to this prescription as the "worldline-based measure."

The main difference between the worldline-based and volume-based measures is in their dependence on the initial state. When considering the volume-based measure, one starts from a finite initial spacelike 3-volume. (Final results

[10] To avoid confusion, let us note that the recent work [162] proposes a measure in the phase space of trajectories rather than an observer-based measure in the sense discussed here.

are insensitive to the choice of this 3-surface in spacetime or to its geometry.) The initial state consists of the initial values of the fields χ_a within the initial volume, and possibly a label α corresponding to the type of the initial bubble. When considering the worldline-based measure, one assumes knowledge of these data at the initial point of the worldline.[11] It turns out that the volume-based measure always yields results that are independent of the initial conditions. This agrees with the concept of the "stationarity" of a self-reproducing universe [110, 111, 126]; the universe forgets the initial state in the course of eternal self-reproduction. In contrast, probabilities obtained using the worldline-based measure always depend on the initial state (except for the case of a "non-terminal" landscape, i.e., a landscape scenario without terminal vacua). A theory of initial conditions is necessary to obtain a specific prediction from the worldline-based measure.

At this time, there is no consensus as to which of the two measures is the physically relevant one. The present author is inclined to regard the two measures as reasonable answers to two differently posed questions. The first question is to determine the probability distribution for observed values of χ_a, given that the observer is randomly chosen from all the observers present in the entire spacetime. Since we have no knowledge as to the total duration of inflation in our past or the total number of bubble nucleations preceding the most recent one, it appears reasonable to include in the ensemble all the observers that will ever appear anywhere in the spacetime. The answer to the first question is thus provided by the volume-based measure.

The second question is posed in a rather different manner. In the context of tunneling-type eternal inflation, upon discovering the type of our bubble we may wish to leave a message to a future civilization that may arise in our future after an unspecified number of nested bubble nucleations. The analogous situation in the context of random walk-type inflation is a hypothetical observer located within an inflating region of spacetime who wishes to communicate with future civilizations that will eventually appear when inflation is over. The only available means of communication is leaving information on paper in a sealed box. The message might contain the probability distribution $P(\chi_a)$ for parameters χ_a that we expect the future civilization to observe. In this case, the initial state is known at the time of writing the message. It is clear that the box can be discovered only by future observers near its worldline. It is then natural to choose the ensemble of observers appearing along this worldline. Starting from the known initial state, one would then compute $P(\chi_a)$ according to the worldline-based measure.

Calculations using the worldline-based measure usually do not require regularization (except for the case of non-terminal landscape) because the

[11] Naturally, it is assumed that the initial state is in the self-reproduction regime. For random walk-type models, the initial 3-volume is undergoing inflation rather than reheating and for tunneling models, the initial volume is not situated within a terminal bubble.

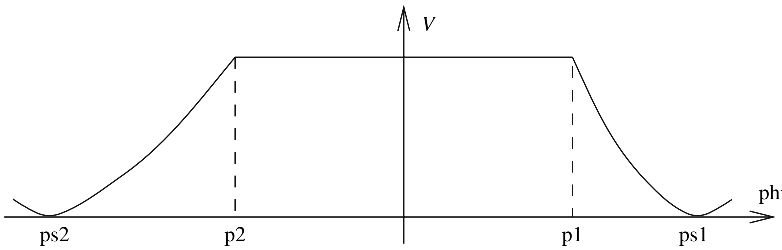

Fig. 5.6. Illustrative inflationary potential with a flat self-reproduction regime $\phi_1 < \phi < \phi_2$ and deterministic regimes $\phi_*^{(1)} < \phi < \phi_1$ and $\phi_2 < \phi < \phi_*^{(2)}$

worldline-based ensemble of observers is almost surely finite [156]. For instance, in random walk-type models the worldline-based measure predicts the exit probability distribution p_{exit}, which can be computed by solving a suitable differential equation [see (5.44)]. However, the ensemble used in the volume-based measure is infinite and requires a regularization. Known regularization methods are reviewed in Sect. 5.3.3 and 5.3.4.

A simple toy model [18] where the predictions of the volume-based measure can be obtained analytically is a slow-roll scenario with a potential shown in Fig. 5.6. The potential is flat in the range $\phi_1 < \phi < \phi_2$ where the evolution is diffusion-dominated, while the evolution of regions with $\phi > \phi_2$ or $\phi < \phi_1$ is completely deterministic (fluctuation-free). It is assumed that the diffusion-dominated range $\phi_1 < \phi < \phi_2$ is sufficiently wide to cause eternal self-reproduction of inflating regions. There are two thermalization points, $\phi = \phi_*^{(1)}$ and $\phi = \phi_*^{(2)}$, which may be associated to different types of true vacuum and thus to different observed values of cosmological parameters. The question is to compare the volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ of regions thermalized into these two vacua. Since there is an infinite volume thermalized into either vacuum, one looks for the volume ratio $\mathcal{V}_1/\mathcal{V}_2$.

The potential is symmetric in the range $\phi_1 < \phi < \phi_2$, so it is natural to assume that Hubble-size regions exiting the self-reproduction regime at $\phi = \phi_1$ and at $\phi = \phi_2$ are equally abundant. Since the evolution within the ranges $\phi_*^{(1)} < \phi < \phi_1$ and $\phi_2 < \phi < \phi_*^{(2)}$ is deterministic, the regions exiting the self-reproduction regime at $\phi = \phi_1$ or $\phi = \phi_2$ will be expanded by fixed amounts of e-foldings, which we may denote N_1 and N_2, respectively,

$$N_j = -4\pi \int_{\phi_j}^{\phi_*^{(j)}} \frac{H(\phi)}{H'(\phi)} \mathrm{d}\phi \,. \tag{5.57}$$

Therefore the volume of regions thermalized at $\phi = \phi_*^{(j)}$, where $j = 1, 2$, will be increased by the factors $\exp(3N_j)$. Hence, the volume ratio is

$$\frac{\mathcal{V}_1}{\mathcal{V}_2} = \exp\left(3N_1 - 3N_2\right). \tag{5.58}$$

5.3.3 Regularization for a Single Reheating Surface

The task at hand is to define a measure that ascribes equal weight to each observer ever appearing anywhere in the universe. As discussed in Sect. 5.1.1, it is sufficient to construct a measure $\mathcal{V}(\chi_\mathrm{a})$ of the 3-volume along the reheating surface. The volume-based measure $P(\chi_\mathrm{a})$ will then be given by (5.5). In the presence of eternal inflation, the proper 3-volume of the reheating surface diverges even when we limit the spacetime domain under consideration to the comoving future of a finite initial spacelike 3-volume. Therefore, the reheating surface needs to be regularized.

In this section we consider the case when the reheating condition is met at a topologically compact and connected locus in the configuration space $\{\phi, \chi_\mathrm{a}\}$. In this case, every connected component of the reheating surface in spacetime will contain all the possible values of the fields χ_a, and all such connected pieces are statistically equivalent. Hence, it suffices to consider a single connected piece of the reheating surface. A situation of this type is illustrated in Fig. 5.7. A sketch of the random walk in configuration space is shown in Fig. 5.8.

A simple regularization scheme is known as the "equal-time cutoff." One considers the part of the reheating surface formed before a fixed time $t_{\max}$; that part is finite as long as $t_{\max}$ is finite. Then one can compute the distribution of the quantities of interest within that part of the reheating surface. Subsequently, one takes the limit $t_{\max} \to \infty$. The resulting

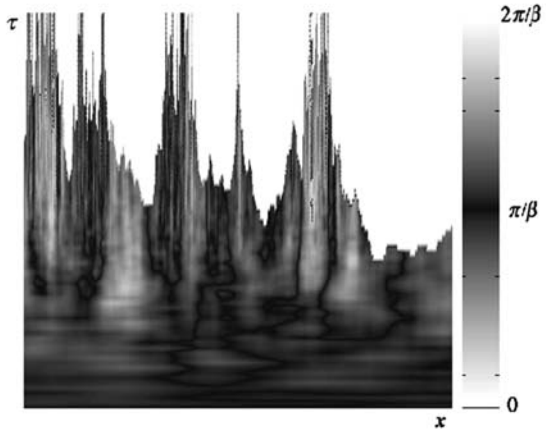

Fig. 5.7. A 1+1-dimensional slice of spacetime in a two-field inflationary model (numerical simulation in [19]). *Shades* denote different values of the field χ, which takes values in the periodically identified interval $[0, 2\pi/\beta]$. The *white region* represents the thermalized domain. The boundary of the thermalized domain is the reheating surface (cf. Fig. 5.2), which contains all the possible values of the field χ

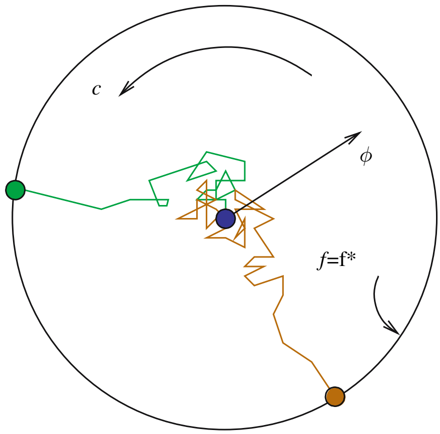

Fig. 5.8. A random walk in configuration space for a two-field inflationary model considered in [19]. The center of the field space is a diffusion-dominated regime. The reheating condition, $\phi = \phi_*$, selects a compact and connected region (a circle) in configuration space. The problem is to determine the volume-weighted probability distribution for the values of χ at reheating

distribution can be found from the solution $P_V^{(\tilde{\gamma})}(\phi)$ of the "stationary" FP equation,

$$[\hat{L} + 3H(\phi)]P_V^{(\tilde{\gamma})} = \tilde{\gamma} P_V^{(\tilde{\gamma})} , \tag{5.59}$$

with the largest eigenvalue $\tilde{\gamma}$ (see Sect. 5.2.3). However, both the eigenvalue $\tilde{\gamma}$ and the distribution $P_V^{(\tilde{\gamma})}(\phi)$ depend rather sensitively on the choice of the equal-time hypersurfaces. Since there appears to be no preferred choice of the cutoff hypersurfaces in spacetime, the equal-time cutoff cannot serve as an unbiased measure. Also, it was shown in [124] that the unbiased result (5.58) cannot be obtained via an equal-time cutoff with any choice of the time gauge.

The "spherical cutoff" measure prescription [45] regularizes the reheating surface in a different way. A finite region within the reheating surface is selected as a spherical region of radius R around a randomly chosen center point. (Since the reheating 3-surface is spacelike, the distance between points can be calculated as the length of the shortest path within the reheating 3-surface.) Then the distribution of the quantities of interest is computed within the spherical region. Subsequently, the limit $R \to \infty$ is evaluated. Since every portion of the reheating surface is statistically the same, the results are independent of the choice of the center point. The spherical cutoff is gauge-invariant since it is formulated entirely in terms of the intrinsic properties of the reheating surface.

While the spherical cutoff prescription successfully solves the problem of regularization, there is no universally applicable analytic formula for the resulting distribution. Application of the spherical cutoff to general models of random walk inflation requires a direct numerical simulation of the stochastic field dynamics in the inflationary spacetime. Such simulations were reported in [19, 111, 140, 171] and used the Langevin equation (5.9) with a specific stochastic ansatz for the noise field $N(t, \mathbf{x})$.

Apart from numerical simulations, the results of the spherical cutoff method may be obtained analytically in a certain class of models [19]. One such case is a multi-field model where the potential $V(\phi, \chi_\mathrm{a})$ is independent of χ_a within the range of ϕ where the "diffusion" in ϕ dominates. Then the distribution in χ_a is flat when field ϕ exits the regime of self-reproduction and resumes the deterministic slow-roll evolution. One can derive a gauge-invariant Fokker–Planck equation for the volume distribution $P_V(\phi, \chi_\mathrm{a})$, using ϕ as the time variable [19],

$$\partial_\phi P_V = \partial_{\chi\chi}\left(\frac{D}{v_\phi}P_V\right) - \partial_\chi\left(\frac{v_\chi}{v_\phi}P_V\right) + \frac{3H}{v_\phi}P_V, \tag{5.60}$$

where D, v_ϕ and v_χ are the coefficients of the FP equation (5.21). This equation is valid for the range of ϕ where the evolution of ϕ is free of fluctuations. By solving (5.60), one can calculate the volume-based distribution of χ_a predicted by the spherical cutoff method as $P_V(\phi_*, \chi)$. Note that the mentioned restriction on the potential $V(\phi, \chi_\mathrm{a})$ is important. In general, the field ϕ cannot be used as the time variable since the surfaces of constant ϕ are not everywhere spacelike due to large fluctuations of ϕ in the diffusion-dominated regime.

5.3.4 Regularization for Multiple Types of Reheating Surfaces

Let us begin by considering a simpler example: an inflationary scenario with an *asymmetric* slow-roll potential $V(\phi)$ having two minima $\phi_*^{(1)}$, $\phi_*^{(2)}$. This scenario has two possibilities for thermalization, possibly differing in the observable parameters χ_a. More generally, one may consider a scenario with n different minima of the potential, possibly representing n distinct reheating scenarios. It is important that the minima $\phi_*^{(j)}$, $j = 1, ..., n$ are topologically disconnected *in the configuration space*. This precludes the possibility that different minima are reached within one connected component of the reheating hypersurface in spacetime. Additionally, the fields χ_a may fluctuate across each connected component in a way that depends on the minimum j. Thus, the distribution $P(\chi_\mathrm{a}; j)$ of the fields χ_a at each connected component of the reheating surface may depend on j. In other words, the different components of the reheating surface may be statistically inequivalent with respect to the distribution of χ_a on them. To use the volume-based measure for making predictions in such models, one needs a regularization method that is applicable to situations with a large number of disconnected and statistically inequivalent components of the reheating hypersurface.

In such situations, the spherical cutoff prescription (see Sect. 5.3.3) yields only the distribution of χ_a across one connected component, since the sphere of a finite radius R will never reach any other components of the reheating surface. Therefore, the spherical cutoff needs to be supplemented by a "weighting" prescription, which would assign weights p_j to the minimum labeled j. In scenarios of tunneling type, such as the string theory landscape, observers may find themselves in bubbles of types $\alpha = 1, ..., N$. Again, a weighting prescription is needed to determine the probabilities p_α of being in a bubble of type α.

Two different weighting prescriptions have been formulated, using the volume-based [153, 163] and the worldline-based approach [156], respectively. The first prescription is called the "comoving cutoff" while the second the "worldline" or the "holographic" cutoff. For clarity, we illustrate these weighting prescriptions on models of tunneling type where infinitely many nested bubbles of types $\alpha = 1, ...N$ are created and where some bubbles are "terminal," i.e., contain no "daughter" bubbles. In the volume-based approach, each bubble receives equal weight in the ensemble; in the worldline-based approach, only bubbles intersected by a selected worldline are counted and given equal weight. Let us now examine these two prescriptions in more detail.

Since the set of all bubbles is infinite, one needs to perform a "regularization," that is, one needs to select a very large but finite subset of bubbles. The weight p_α will be calculated as the fraction of bubbles of type α within the selected subset; then the number of bubbles in the subset will be taken to infinity. Technically, the two prescriptions differ in the details of the regularization. The difference between the two prescriptions can be understood pictorially (Fig. 5.9). In a spacetime diagram, one draws a finite number of timelike comoving geodesic worldlines emitted from an initial 3-surface toward the future. (It can be shown that the results are independent of the choice of these lines, as long as that choice is uncorrelated with the bubble nucleation process [163].) Each of these lines will intersect only a finite number of bubbles, since the final state of any worldline is (almost surely) a terminal bubble. The subset of bubbles needed for the regularization procedure is defined as the set of all bubbles intersected by at least one line. At this point, the volume-based approach assigns equal weight to each bubble in the subset, while the worldline-based approach assigns equal weight to each bubble along each worldline. As a result, the volume-based measure counts each bubble in the subset only once, while the worldline-based measure counts each bubble as many times as it is intersected by some worldlines. After determining the weights p_α by counting the bubbles as described, one increases the number of worldlines to infinity and evaluates the limit values of p_α.

It is clear that the volume-based measure represents the counting of bubbles in the entire universe, and it is appropriate that each bubble is being counted only once. On the other hand, the worldline-based measure counts bubbles occurring along a single worldline, ignoring the bubbles produced in other parts of the universe and introducing an unavoidable bias due to the initial conditions at the starting point of the worldline.

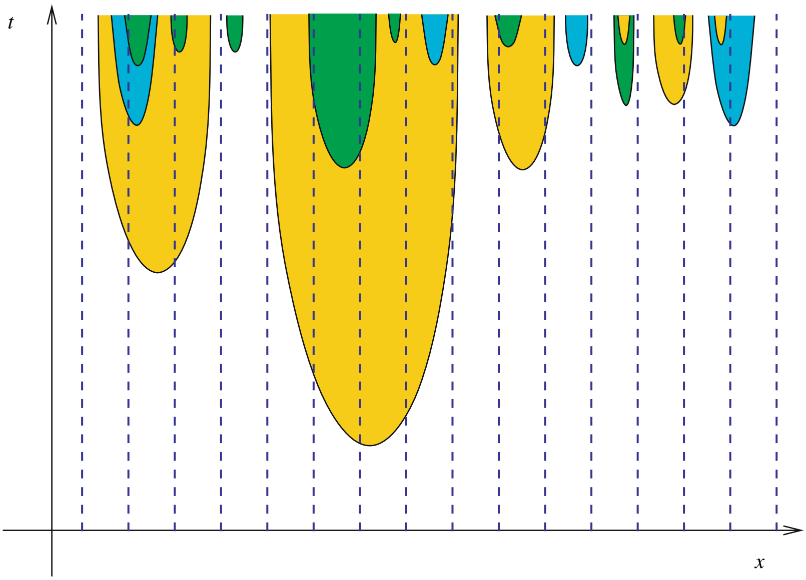

Fig. 5.9. Weighting prescriptions for models of tunneling type. *Shaded regions* are bubbles of different types. *Dashed vertical lines* represent randomly chosen comoving geodesics used to define a finite subset of bubbles

Explicit formulas for p_α were derived for a tunneling-type scenario (with terminal vacua) in the volume-based approach [153],

$$p_\alpha = \sum_\beta H_\beta^q \kappa_{\alpha\beta} s_\beta \ , \tag{5.61}$$

where $\kappa_{\alpha\beta}$ is the matrix of nucleation rates, q and s_α are the quantities defined by (5.48), and H_β is the Hubble parameter in bubbles of type β. The expressions for p_α obtained from the worldline-based measure are given by (5.54). As we have noted before, the volume-based measure assigns weights p_α that are independent of initial conditions, while the weights obtained from the worldline-based measure depend sensitively on the type of bubble where the counting begins.

In the case of a non-terminal landscape, both the volume-based and the worldline-based measures give identical results for p_α, which coincide with the mean frequency (5.49) of visiting a bubble of type α [155, 156].

With the weighting prescriptions just described, the volume-based and the worldline-based measure proposals can be considered complete. In other words, we have two alternative prescriptions that can be applied (in principle) to arbitrary models of random walk or tunneling-type eternal inflation. Further research is needed to reach a definite conclusion concerning the viability of these measure prescriptions.

Acknowledgments

The author is grateful to Gabriel Lopes Cardoso, Matthew Johnson, and Andrei Linde for useful discussions, and Andrei Barvinsky and Alex Vilenkin for comments on the manuscript.

References

1. A. Vilenkin, Phys. Rev. **D27**, 2848 (1983). 157, 158, 166
2. A. A. Starobinsky, in *Current Topics in Field Theory, Quantum Gravity and Strings, Lecture Notes in Physics* 206, eds H. J. de Vega and N. Sanchez (Springer Verlag), p. 107. 157, 158, 166, 167
3. A. D. Linde, Phys. Lett. **B175**, 395–400 (1986). 157, 158, 159, 162, 178
4. A. S. Goncharov, A. D. Linde and V. F. Mukhanov, Int. J. Mod. Phys. **A2**, 561–591 (1987). 157, 164, 165, 169
5. A. Vilenkin and L. H. Ford, Phys. Rev. **D26**, 1231 (1982). 158
6. A. D. Linde, Phys. Lett. **B116**, 335 (1982). 158, 171
7. A. A. Starobinsky, Phys. Lett. **B117**, 175–178 (1982). 158, 164, 165
8. A. Vilenkin, Nucl. Phys. **B226**, 527 (1983). 158, 165, 178
9. R. H. Brandenberger, Nucl. Phys. **B245**, 328 (1984). 158
10. A. H. Guth and S.-Y. Pi, Phys. Rev. **D32**, 1899–1920 (1985). 158
11. J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. **D28**, 679 (1983). 158, 164
12. G. W. Gibbons and S. W. Hawking, Phys. Rev. **D15**, 2738–2751 (1977). 158
13. S. W. Hawking and I. G. Moss, Phys. Lett. **B110**, 35 (1982). 158
14. S. W. Hawking and I. G. Moss, Nucl. Phys. **B224**, 180 (1983). 158, 174
15. A. Vilenkin, Eternal inflation and chaotic terminology, `gr-qc/0409055`. 159
16. A. Borde and A. Vilenkin, Phys. Rev. Lett. **72**, 3305–3309 (1994). 159
17. A. Borde, A. H. Guth and A. Vilenkin, Phys. Rev. Lett. **90**, 151301 (2003). 159
18. A. Vilenkin, Phys. Rev. **D52**, 3365–3374 (1995). 159, 163, 172, 178, 179, 181
19. V. Vanchurin, A. Vilenkin and S. Winitzki, Phys. Rev. **D61**, 083507 (2000). 160, 179, 182,
20. D. A. Lowe and D. Marolf, Phys. Rev. **D70**, 026001 (2004). 160
21. C.-H. Wu, K.-W. Ng and L. H. Ford, Constraints on the duration of inflationary expansion from quantum stress tensor fluctuations, `gr-qc/0608002`. 160
22. A. D. Linde, Phys. Lett. **B259**, 38–47 (1991). 160
23. A. D. Linde, Phys. Rev. **D49**, 748–754 (1994). 160
24. J. Garcia-Bellido, A. D. Linde and D. A. Linde, Phys. Rev. **D50**, 730–750 (1994). 160, 163
25. J. Garcia-Bellido, Nucl. Phys. **B423**, 221–242 (1994). 160, 163, 172
26. J. Garcia-Bellido and D. Wands, Phys. Rev. **D52**, 5636–5642 (1995). 160, 171
27. A. D. Linde, Phys. Lett. **B327**, 208–213 (1994). 160
28. A. Vilenkin, Phys. Rev. Lett. **72**, 3137–3140 (1994). 160
29. K. E. Kunze, Phys. Lett. **B587**, 1–6 (2004). 160
30. J. Garriga and A. Vilenkin, Phys. Rev. **D57**, 2230–2244 (1998). 160, 174
31. L. Susskind, The anthropic landscape of string theory, `hep-th/0302219`. 160, 164, 174
32. S. R. Coleman and F. De Luccia, Phys. Rev. **D21**, 3305 (1980). 161, 164, 174
33. A. H. Guth, Eternal inflation, Ann. N.Y. Acad. Sci. **950**, 66–82 (2001). 161
34. A. H. Guth and E. J. Weinberg, Nucl. Phys. **B212**, 321 (1983). 161, 174

35. J. Garriga, A. H. Guth and A. Vilenkin, Eternal inflation, bubble collisions, and the persistence of memory, hep-th/0612242. 161
36. A. H. Guth, Phys. Rev. **D23**, 347–356 (1981). 162
37. J. Garriga, V. F. Mukhanov, K. D. Olum and A. Vilenkin, Int. J. Theor. Phys. **39**, 1887–1900 (2000). 162
38. J. Garriga and A. Vilenkin, Phys. Rev. **D64**, 043511 (2001). 162
39. D. S. Goldwirth and T. Piran, Phys. Rept. **214**, 223–291 (1992). 162
40. A. D. Linde, Phys. Lett. **B129**, 177–181 (1983). 162
41. A. D. Linde, Mod. Phys. Lett. **A1**, 81 (1986). 162, 178
42. J. Martin and M. A. Musso, Phys. Rev. **D71**, 063514 (2005). 162
43. F. Helmer and S. Winitzki, Phys. Rev. **D74**, 063528 (2006). 162, 166, 168, 170, 173
44. S. M. Carroll, Living Rev. Rel. **4**, 1 (2001). 162
45. A. Vilenkin, Phys. Rev. Lett. **81**, 5501–5504 (1998). 163, 179, 183
46. J. Garriga, T. Tanaka and A. Vilenkin, Phys. Rev. **D60**, 023501 (1999). 163
47. J. Garriga and A. Vilenkin, Phys. Rev. **D61**, 083502 (2000). 163
48. J. Garriga, M. Livio and A. Vilenkin, Phys. Rev. **D61**, 023503 (2000). 163
49. J. Garriga and A. Vilenkin, Phys. Rev. **D67**, 043503 (2003). 163
50. J. Garriga, A. Linde and A. Vilenkin, Phys. Rev. **D69**, 063521 (2004). 163
51. J. Garriga and A. Vilenkin, Prog. Theor. Phys. Suppl. **163**, 245–257 (2006). 163
52. M. Tegmark, A. Vilenkin and L. Pogosian, Phys. Rev. **D71**, 103523 (2005). 163
53. M. Tegmark, A. Aguirre, M. Rees and F. Wilczek, Phys. Rev. **D73**, 023505 (2006). 163
54. L. J. Hall, T. Watari and T. T. Yanagida, Phys. Rev. **D73**, 103502 (2006). 163
55. M. Susperregi, Phys. Rev. **D55**, 560–572 (1997). 163
56. B. Feldstein, L. J. Hall and T. Watari, Phys. Rev. **D72**, 123506 (2005). 163
57. W. Lerche, D. Lust and A. N. Schellekens, Nucl. Phys. **B287**, 477 (1987). 164
58. R. Bousso and J. Polchinski, JHEP, **06**, 006 (2000). 164
59. S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, Phys. Rev. **D68**, 046005 (2003). 164, 176
60. F. Denef and M. R. Douglas, JHEP **05**, 072 (2004). 164
61. D. Schwartz-Perlov and A. Vilenkin, JCAP **0606**, 010 (2006). 164
62. A. Aguirre, S. Gratton and M. C. Johnson, Hurdles for recent measures in eternal inflation, hep-th/0611221. 164, 176, 177, 179
63. V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33**, 532–535 (1981). 164
64. V. F. Mukhanov and G. V. Chibisov, Sov. Phys. JETP **56**, 258–265 (1982). 164
65. S. W. Hawking, Phys. Lett. **B115**, 295 (1982). 164
66. A. H. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110–1113 (1982). 164
67. V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215**, 203–333 (1992). 164, 165
68. R. H. Brandenberger, Rev. Mod. Phys. **57**, 1 (1985). 164
69. D. Polarski and A. A. Starobinsky, Class. Quant. Grav. **13**, 377–392 (1996). 164
70. C. Kiefer and D. Polarski, Annalen Phys. **7**, 137–158 (1998). 164
71. C. Kiefer, J. Lesgourgues, D. Polarski and A. A. Starobinsky, Class. Quant. Grav. **15**, L67–L72 (1998). 164
72. C. Kiefer, D. Polarski and A. A. Starobinsky, Int. J. Mod. Phys. **D7**, 455–462 (1998). 164
73. C. Kiefer, D. Polarski and A. A. Starobinsky, Phys. Rev. **D62**, 043518 (2000). 164
74. C. Kiefer, I. Lohmar, D. Polarski and A. A. Starobinsky, Class. Quant. Grav. **24**, 1699 (2007). 164

75. L. Raul W. Abramo, R. H. Brandenberger and V. F. Mukhanov, Phys. Rev. **D56**, 3248–3257 (1997). 165
76. N. Afshordi and R. H. Brandenberger, Phys. Rev. **D63**, 123505 (2001). 165
77. L. R. Abramo and R. P. Woodard, Phys. Rev. **D65**, 043507 (2002). 165
78. L. R. Abramo and R. P. Woodard, Phys. Rev. **D65**, 063515 (2002). 165
79. L. R. Abramo and R. P. Woodard, Phys. Rev. **D65**, 063516 (2002). 165
80. G. Geshnizjani and R. Brandenberger, JCAP **0504**, 006 (2005). 165
81. G. Geshnizjani and N. Afshordi, JCAP **0501**, 011 (2005). 165
82. E. O. Kahya and V. K. Onemli, Quantum stability of a w < -1 phase of cosmic acceleration, `gr-qc/0612026`. 165
83. H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1–166 (1984). 165
84. A. V. Frolov and L. Kofman, JCAP **0305**, 009 (2003). 165
85. A. Borde and A. Vilenkin, Phys. Rev. **D56**, 717–723 (1997). 165
86. S. Winitzki, Null energy condition violations in eternal inflation, `gr-qc/0111109`. 165
87. T. Vachaspati, Eternal inflation and energy conditions in de sitter spacetime, `astro-ph/0305439`. 165
88. S. Dutta and T. Vachaspati, Phys. Rev. **D71**, 083507 (2005). 165
89. Y.-S. Piao, Phys. Rev. **D72**, 103513 (2005). 165
90. S. Dutta, Phys. Rev. **D73**, 063524 (2006). 165
91. M. Morikawa, Prog. Theor. Phys. **77**, 1163–1177 (1987). 165
92. M. Morikawa, Phys. Rev. **D42**, 1027–1034 (1990). 165
93. B. L. Hu and A. Matacz, Phys. Rev. **D51**, 1577–1586 (1995). 165
94. E. Calzetta and B. L. Hu, Phys. Rev. **D52**, 6770–6788 (1995). 165
95. A. Matacz, Phys. Rev. **D55**, 1860–1874 (1997). 165
96. A. Matacz, Phys. Rev. **D56**, 1836–1840 (1997). 165
97. H. Kubotani, T. Uesugi, M. Morikawa and A. Sugamoto, Prog. Theor. Phys. **98**, 1063–1080 (1997). 165
98. R. P. Woodard, Nucl. Phys. Proc. Suppl. **148**, 108–119 (2005). 165
99. N. C. Tsamis and R. P. Woodard, Nucl. Phys. **B724**, 295–328 (2005). 165
100. R. P. Woodard, Generalizing starobinskii's formalism to yukawa theory and to scalar QED, `gr-qc/0608037`. 165
101. S.-P. Miao and R. P. Woodard, Phys. Rev. **D74**, 044019 (2006). 165
102. T. Prokopec, N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. **24**, 201–230 (2007). 165
103. S.-J. Rey, Nucl. Phys. **B284**, 706 (1987). 166
104. K. Nakao, Y. Nambu and M. Sasaki, Prog. Theor. Phys. **80**, 1041 (1988). 166, 170
105. H. E. Kandrup, Phys. Rev. **D39**, 2245 (1989). 166
106. Y. Nambu and M. Sasaki, Phys. Lett. **B219**, 240 (1989). 166, 169, 170
107. Y. Nambu, Prog. Theor. Phys. **81**, 1037 (1989). 166, 170
108. M. Mijic, Phys. Rev. **D42**, 2469–2482 (1990). 166, 169, 170
109. D. S. Salopek and J. R. Bond, Phys. Rev. **D43**, 1005–1031 (1991). 166
110. A. D. Linde and A. Mezhlumian, Phys. Lett. **B307**, 25–33 (1993). 166, 169, 170, 171, 172,
111. A. D. Linde, D. A. Linde and A. Mezhlumian, Phys. Rev. **D49**, 1783–1826 (1994). 166, 169, 170, 171, 172, 179, 180, 184
112. D. S. Salopek and J. R. Bond, Phys. Rev. **D42**, 3936–3962 (1990). 166
113. A. R. Liddle, P. Parsons and J. D. Barrow, Phys. Rev. **D50**, 7222–7232 (1994). 166
114. M. Bellini, H. Casini, R. Montemayor and P. Sisterna, Phys. Rev. **D54**, 7172–7180 (1996). 167

115. H. Casini, R. Montemayor and P. Sisterna, Phys. Rev. **D59**, 063512 (1999). 167
116. S. Winitzki and A. Vilenkin, Phys. Rev. **D61**, 084008 (2000). 167
117. S. Matarrese, M. A. Musso and A. Riotto, JCAP **0405**, 008 (2004). 167
118. M. Liguori, S. Matarrese, M. Musso and A. Riotto, JCAP **0408**, 011 (2004). 167
119. N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981). 168
120. H. Risken, *The Fokker–Planck Equation* (Springer-Verlag, 1989). 168
121. J. Garriga and V. F. Mukhanov, Phys. Lett. **B458**, 219–225 (1999). 168
122. A. Vilenkin, Phys. Rev. **D59**, 123506 (1999). 168, 169, 173
123. S. Winitzki and A. Vilenkin, Phys. Rev. **D53**, 4298–4310 (1996). 169, 170, 171, 172, 179
124. S. Winitzki, Phys. Rev. **D71**, 123507 (2005). 169, 173, 177, 178, 183
125. S. Matarrese, A. Ortolan and F. Lucchin, Phys. Rev. **D40**, 290 (1989). 170, 171
126. J. Garcia-Bellido and A. D. Linde, Phys. Rev. **D51**, 429–443 (1995). 171, 172, 180
127. A. D. Linde, Phys. Lett. **B108**, 389–393 (1982). 171
128. A. D. Linde, Phys. Lett. **B114**, 431 (1982). 171
129. A. D. Linde, Phys. Lett. **B116**, 340 (1982). 171
130. A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220–1223 (1982). 171
131. H. M. Hodges, Phys. Rev. **D39**, 3568–3570 (1989). 171
132. I. Yi, E. T. Vishniac and S. Mineshige, Phys. Rev. **D43**, 362–368 (1991). 171
133. Y. V. Shtanov, Phys. Rev. **D52**, 4287–4294 (1995). 171
134. J. Martin and M. Musso, Phys. Rev. **D73**, 043516 (2006). 171
135. S. Gratton and N. Turok, Phys. Rev. **D72**, 043507 (2005). 171
136. J. Martin and M. Musso, Phys. Rev. **D73**, 043517 (2006). 171
137. M. Tegmark, JCAP **0504**, 001 (2005). 172, 177, 179
138. A. D. Linde, D. A. Linde and A. Mezhlumian, Phys. Lett. **B345**, 203–210 (1995). 172
139. S. Winitzki, Phys. Rev. **D65**, 083506 (2002). 173, 178
140. M. Aryal and A. Vilenkin, Phys. Lett. **B199**, 351 (1987). 173, 178, 184
141. J. R. Gott, Nature **295**, 304–307 (1982). 174
142. J. R. Gott and T. S. Statler, Phys. Lett. **B136**, 157–161 (1984). 174
143. M. Bucher, A. S. Goldhaber and N. Turok, Phys. Rev. **D52**, 3314–3337 (1995). 174
144. M. Bucher, A. S. Goldhaber and N. Turok, Nucl. Phys. Proc. Suppl. **43**, 173–176 (1995). 174
145. K. Yamamoto, M. Sasaki and T. Tanaka, Astrophys. J. **455**, 412–418 (1995). 174
146. S. R. Coleman, Phys. Rev. **D15**, 2929–2936 (1977). 174
147. K.-M. Lee and E. J. Weinberg, Phys. Rev. **D36**, 1088 (1987). 174
148. J. Garriga, Phys. Rev. **D49**, 6327–6342 (1994). 174
149. E. Farhi and A. H. Guth, Phys. Lett. **B183**, 149 (1987). 174
150. E. Farhi, A. H. Guth and J. Guven, Nucl. Phys. **B339**, 417–490 (1990). 174
151. R. Bousso, Phys. Rev. **D71**, 064024 (2005). 174
152. L. F. Abbott and S. R. Coleman, Nucl. Phys. **B259**, 170 (1985). 174
153. J. Garriga, D. Schwartz-Perlov, A. Vilenkin and S. Winitzki, JCAP **0601**, 017 (2006). 174, 176, 179, 185, 186
154. S. Winitzki, Phys. Rev. **D71**, 123523 (2005). 174
155. V. Vanchurin and A. Vilenkin, Phys. Rev. **D74**, 043520 (2006). 176, 177, 186
156. R. Bousso, Phys. Rev. Lett. **97**, 191302 (2006). 176, 177, 179, 181, 185, 186
157. P. Lancaster, *Theory of Matrices* (Academic Press, New York, 1969). 176

158. J. L. Doob, *Stochastic Processes* (Wiley, New York, 1953). 177
159. D. Kannan, *Introduction to Stochastic Processes* (North-Holland, New York, 1979). 177
160. A. D. Linde, JETP Lett. **38**, 176–179 (1983). 178
161. X. Chen, S. Sarangi, S. H. Henry Tye and J. Xu, JCAP **0611**, 015 (2006). 178
162. G. W. Gibbons and N. Turok, The measure problem in cosmology, `hep-th/0609095`. 179
163. R. Easther, E. A. Lim and M. R. Martin, JCAP **0603**, 016 (2006). 179, 185
164. J. Garriga and A. Vilenkin, Phys. Rev. **D64**, 023507 (2001). 179
165. R. Bousso, B. Freivogel and I-S. Yang, Phys. Rev. **D74**, 103516 (2006). 179
166. R. Bousso, Precision cosmology and the landscape, `hep-th/0610211`. 179
167. A. D. Linde and A. Mezhlumian, Phys. Rev. **D53**, 4267–4274 (1996). 179
168. A. Vilenkin and S. Winitzki, Phys. Rev. **D55**, 548–559 (1997). 179
169. A. Aguirre, S. Gratton and M. C. Johnson, Measures on transitions for cosmology from eternal inflation `hep-th/0612195`. 179
170. V. Vanchurin, Phys. Rev. **D75**, 023524 (2007). 179
171. A. D. Linde and D. A. Linde, Phys. Rev. **D50**, 2456–2468 (1994). 184