A Congruence Between a Siegel and an Elliptic Modular Form

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Preface

The winter semester 2002/2003 was the last semester before my retirement from the university. It also happened that I was the chairman of the Colloquium and the speaker foreseen for February 7 had to cancel his visit.

At about the same time I found some numerical support for a very general conjecture relating divisibilities of certain special values of L-functions to congruences between modular forms. I have been thinking about this kind of relationship for many years, but I never had any idea how one could find experimental evidence. But in the early 2003 C. Faber and G. van der Geer had written a program that produced lists of eigenvalues of Hecke operators on some special Siegel modular forms. After a few days of suspense we could compare their list with my list of eigenvalues of elliptic modular forms and verify the congruence in our examples.

I was very exited about this and spontaneously invited myself to give the Colloquium lecture, which is documented in the text below. (Bonn Spring 2007)

1 Elliptic and Siegel Modular Forms

I have to recall some well known facts from the classical theory of modular forms. We have the upper half plane

$$\mathbb{H} = \{z \mid x + iy \text{ with } y > 0\} .$$

On this upper half plane we have an action of $Sl_2(\mathbb{R})$, which is given by

$$\begin{gathered} Sl_2(\mathbb{R}) \times \mathbb{H} \longrightarrow \mathbb{H} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \mapsto \frac{az+b}{cz+d} . \end{gathered}$$

The stabilizer of $i \in \mathbb{H}$ is the maximal compact subgroup $SO(2)$ and we can identify $\mathbb{H} = Sl_2(\mathbb{R})/SO(2)$. Let k be a positive (even) integer. A *holomorphic modular form of weight k with respect to $Sl_2(\mathbb{Z})$* is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z}) ,$$

and which satisfies a growth condition. To formulate this growth condition we restrict f to a "neighborhood of infinity" $\mathbb{H}(c) = \{z | \Im(z) > c\}$. On this neighborhood the group $\Gamma_\infty = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ with $n \in \mathbb{Z}$ acts and the map $z \mapsto e^{2\pi i z}$ identifies $\Gamma_\infty \backslash \mathbb{H}(c)$ to a punctured disk. Since f satisfies $f(z) = f(z+1)$ we can view its restriction to $\Gamma_\infty \backslash \mathbb{H}(c)$ as a function in the variable q. The growth condition requires that f has a (Fourier or Laurent) expansion

$$f(q) = a_0 + a_1 q + a_2 q^2 \dots ,$$

i.e. it extends to a holomorphic function on the disk. If $a_0 = 0$, then f is called a *cusp form.*

Remark: The quotient $Sl_2(\mathbb{Z}) \backslash \mathbb{H}$ has the structure of a Riemann surface, which can be compactified to a compact Riemann surface $\overline{Sl_2(\mathbb{Z}) \backslash \mathbb{H}}$ by adding one point at ∞. We write the maximal compact subgroup

$$SO(2) = U(1) = K = \left\{ e(\phi) \mid e(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \right\} .$$

Since $\mathbb{H} = Sl_2(\mathbb{R})/SO(2)$, the representation $\rho_k : SO(2) \to \mathbb{C}^\times$, which is given by $e(\phi) \mapsto e(\phi)^k$ defines a $Sl_2(\mathbb{R})$-invariant holomorphic line bundle $\mathcal{L}_k$ on $\mathbb{H}$, this gives us a line bundle, also called $\mathcal{L}_k$, on $Sl_2(\mathbb{Z}) \backslash \mathbb{H}$. This line bundle can be extended in a specific way to a line bundle on the compactification. Then the space of modular forms of weight k can be canonically identified with the space of sections $H^0(\overline{Sl_2(\mathbb{Z}) \backslash \mathbb{H}}, \mathcal{L}_k)$.

We have the two modular forms of weight 4 and 6

$$E_4(z) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(cz+d)^4} ,$$

$$E_6(z) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(cz+d)^6} ,$$

and then we have the q-expansions

$$E_4(q) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 \dots$$
$$E_6(q) = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 - 1575504q^5 + \dots .$$

The space of cusp forms has dimension 1 for the values $k = 12, 16, 18, 20, 22, 26$. The modular form

$$\Delta(z) = \frac{E_4(q)^3 - E_6(q)^2}{12^3} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \dots$$

is the generator of the space of cusp forms of weight 12.

The space of cusp forms of weight 22 is generated by

$$\begin{aligned} f(q) =& \frac{E_6(q)E_4(q)^4 - E_6(q)^3 \cdot E_4(q)}{12^3} = q - 288q^2 - 128844q^3 - 2014208q^4 \\ &+ 21640950q^5 + 37107072q^6 - 768078808q^7 + 1184071680q^8 \\ &+ 6140423133q^9 - 6232593600q^{10} - 94724929188q^{11} \pm \dots . \end{aligned}$$

Now I have to say a few words on Siegel modular forms. We start from a lattice

$$L = \mathbb{Z}^4 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}f_2 \oplus \mathbb{Z}f_1$$

on which we have an alternating pairing which on the basis vectors is given by

$$\langle e_1, f_1 \rangle = \langle e_2, f_2 \rangle = -\langle f_1, e_1 \rangle = -\langle f_2, e_2 \rangle = 1 ,$$

and all other values of the pairing are zero. The group of automorphisms of this symplectic form is a semi-simple group scheme $Sp_2/\operatorname{Spec}(\mathbb{Z})$. This is the symplectic group of genus 2.

Its group of real points

$$Sp_2(\mathbb{R}) = \{g \in GL_4(\mathbb{R}) \mid \langle gx, gy \rangle = \langle x, y \rangle\}$$

contains $U(2)$ as a maximal compact subgroup and we can form the quotient space

$$\mathbb{H}_2 = Sp_2(\mathbb{R})/U(2) .$$

This is the space of symmetric 2×2 matrices

$$Z = X + iY$$

with complex entries whose imaginary part Y is positive definite. Hence we have a complex structure on this space. This complex structure can also be seen in the following way: let $P(\mathbb{C})$ in $Sp_2(\mathbb{C})$ be the stabilizer of the isotropic plane $\{e_1 - if_1, e_2 - if_2\} \subset \mathbb{C}^4$, then we have an open embedding

$$\mathbb{H}_2 = Sp_2(\mathbb{R})/U(2) \hookrightarrow Sp_2(\mathbb{C})/P(\mathbb{C}) ,$$

the group $SU(2)$ is the group of real points of $P(\mathbb{C})$ intersected with its complex conjugate $\bar{P}(\mathbb{C})$. The object on the right is the Grassmann variety of isotropic complex planes in $(\mathbb{C}^4, \langle\ ,\ \rangle)$. It is projective and of dimension 3. The group $\Gamma = Sp_2(\mathbb{Z})$ acts upon $\mathbb{H}_2$ and the quotient $\Gamma\backslash\mathbb{H}_2$ is a quasiprojective algebraic variety over $\mathbb{C}$. We have a homomorphism

$P(\mathbb{C}) \to GL_2(\mathbb{C})$. For any pair of integers $i \geq 0, j$ the holomorphic representation

$$\rho : GL_2(\mathbb{C}) \longrightarrow \mathrm{Sym}^i(\mathbb{C}^2) \otimes \det^j$$

defines a holomorphic vector bundle $\mathcal{E}_{ij}$ on the flag variety $Sp_2(\mathbb{C})/P(\mathbb{C})$ which is $Sp_2(\mathbb{C})$-equivariant. Hence its restriction – also called $\mathcal{E}_{ij}$ – to $\mathbb{H}_2$ is a $Sp_2(\mathbb{R})$ equivariant holomorphic bundle on $\mathbb{H}_2$ and hence descends to a holomorphic bundle on $\Gamma\backslash\mathbb{H}_2$. We can consider the space of holomorphic sections

$$H^0(\Gamma\backslash\mathbb{H}_2, \mathcal{E}_{ij}) ,$$

and define the subspace of modular forms M_{ij} (which satisfy some growth condition) and the subspace S_{ij} of cusp forms; these are rapidly decreasing at infinity. These spaces are called the spaces of modular forms (cusp forms) of weight i, j. (See remark above.)

There are formulas by R. Tsushima for the dimensions of these spaces S_{ij} (Riemann–Roch–Hirzebruch or the trace formula), and for small values i, j the dimensions are zero. We say that i, j is a regular pair if $i > 0, j > 3$. We have 29 cases of regular pairs i, j where S_{ij} is of dimension one.

2 The Hecke Algebra and a Congruence

Whenever we have such a space of modular forms we have an action of the algebra of Hecke operators on it. This is an algebra generated by operators T_p (for $Sl_2(\mathbb{Z})$) and $T_p^{(\nu)}$, $\nu = 1, 2$ (for $Sp_2(\mathbb{Z})$), which are attached to a prime p and which induce endomorphisms $T_p^{(\nu)} : S_{ij} \to S_{ij}$, and which commute with each other. If we pick a prime, then we can consider the matrix

$$\begin{pmatrix} p & & & 0 \\ & p & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

which is in $GSp_2(\mathbb{Q})$, and if $f(Z) \in H^0(\Gamma\backslash\mathbb{H}_2, \mathcal{E}_{ij})$, then $f\left(\begin{pmatrix} p & & & 0 \\ & p & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} Z\right)$ is not invariant under Γ; it is only a section in

$$H^0(\Gamma_0(p)\backslash\mathbb{H}_2, \mathcal{E}_{ij}) ,$$

where $\Gamma_0(p) \subset \Gamma$ is a subgroup of finite index. We can form a trace by summing over $\Gamma_0(p)\backslash\Gamma$ and up to a normalizing factor this will be our operator

$$T_p^{(1)} : S_{ij} \longrightarrow S_{ij} .$$

If now $\dim S_{ij} = 1$, then the operator $T_p^{(1)} : S_{ij} \to S_{ij}$ induces the multiplication by a number $\lambda(p)$ on S_{ij} and if $j \geq 3$, then we get a sequence of integers

$$\{\lambda(p)\}_{p \in \mathrm{Primes}}$$

which, of course, depends on i, j.

We also have the Hecke operators for classical modular forms, and our cusp form f of weight 22 is also an eigenform for the operators T_p. In this case the situation is simple. Because f is normalized, i.e. $a_1 = 1$, we have

$$T_p f = a_p f$$

where a_p is the p-th Fourier coefficient. We have $\dim S_{4,10} = 1$ and formulate the conjecture

Conjecture: *For $S_{4,10}$ we have a congruence*

$$\lambda(p) \equiv p^8 + a_p + p^{13} \mod 41 \quad \textit{for all primes } p\ .$$

Prop. *The conjecture holds for* $2 \leq p \leq 11$.

One might say that this is really not so much evidence for the conjecture. But here are certain numbers, namely, $4, 10, 22, 8, 13$ and 41, which seem to be somewhat arbitrary. I did not play with these numbers until I found a congruence. I picked all these numbers in advance and only then I checked the congruence, which I expected to be true for this specific choice.

The congruence is a generalization of a classical congruence. If we write the Δ-function

$$\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 \pm \ldots = \sum_{n=1}^{\infty} \tau(n) q^n\ ,$$

then we have the famous Ramanujan congruence

$$\tau(p) \equiv p^{11} + 1 \mod 691 \quad \text{for all primes } p\ .$$

But there is a difference: Usually people interpret this last congruence as a congruence between the q-expansions of two modular forms, namely the Δ-function and the Eisenstein series $E_{12}(z)$. Since the Fourier coefficients are the same as the eigenvalues of the Hecke operators we also get the congruence between the eigenvalues. For the congruence between the Siegel modular form and the elliptic modular form we only have a congruence between Hecke eigenvalues. I do not see a congruence between Fourier coefficients.

I want to say something about the numbers, how I get them and I want to say a few words about the meaning of this congruence.

3 The Special Values of the *L*-function

We start from our modular cusp form of weight 22

$$f(q) = q - 288q^2 - 128844q^3 - 2014208q^4 + 21640950q^5 + \ldots = \sum_{n=1}^{\infty} a_n q^n ,$$

we have its associated L-function $L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, and because f is an eigenform for the Hecke algebra this L-function has an Euler product expansion

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{1 - a_p p^{-s} + p^{21-2s}} .$$

Actually it is better to consider the Mellin transform

$$\int_0^{\infty} f(iy) y^s \frac{dy}{y} = \frac{\Gamma(s)}{(2\pi)^s} \cdot L(f,s) = \Lambda(f,s) .$$

From this integral representation we easily get the functional equation

$$\Lambda(f, 22-s) = -\Lambda(f,s) .$$

Now we consider the "special" values $\Lambda(f,21), \Lambda(f,20), \ldots, \Lambda(f,11)$. It follows from the theory of modular symbols (Manin–Vishik) that there exist two real numbers $\Omega_-, \Omega_+ \neq 0$ (the *periods*) such that

$$\frac{\Lambda(f,21)}{\Omega_-}, \quad \frac{\Lambda(f,20)}{\Omega_+}, \quad \frac{\Lambda(f,19)}{\Omega_-}, \quad \ldots \in \mathbb{Q}$$

These periods are only defined up to elements in $\mathbb{Q}^\times$, but a closer look allows us to pin them down up to a factor in $\mathbb{Z}^\times = \{\pm 1\}$. In this case we can simply try to normalize them such that

$$\left\{ \frac{\Lambda(f,21)}{\Omega_-}, \quad \frac{\Lambda(f,19)}{\Omega_-}, \quad \ldots, \quad \frac{\Lambda(f,11)}{\Omega_-} \right\}$$

and

$$\left\{ \frac{\Lambda(f,20)}{\Omega_+}, \quad \ldots, \quad \frac{\Lambda(f,14)}{\Omega_+}, \quad \frac{\Lambda(f,12)}{\Omega_+} \right\}$$

are sets of co-prime integers. Of course, it is not so difficult to produce these lists of integers. (From this list we conclude that the normalization of Ω_- was not the right one. This is related to the fact that

$$131 \cdot 593 \mid \zeta(-21) ,$$

and this produces a congruence between f and an Eisenstein series

$$a_p \equiv p^{21} + 1 \mod 131 \cdot 593 .$$

This forces us to replace Ω_- by $131 \cdot 593 \cdot \Omega_-$.) With this modification the list for the odd case is

$$\{2^5{\cdot}3^3{\cdot}5^6{\cdot}7{\cdot}13{\cdot}17{\cdot}19/(131{\cdot}593),\ 2^5{\cdot}3{\cdot}5^2{\cdot}13{\cdot}17\,,\ 2{\cdot}3{\cdot}5^3{\cdot}7{\cdot}13,\ 2{\cdot}5^2{\cdot}13{\cdot}17,\ 5^37,\ 0\}$$

and for the even case

$$\{2^5 \cdot 3^3 \cdot 5 \cdot 19,\ 2^3 \cdot 7 \cdot 13^2,\ 3 \cdot 5 \cdot 7 \cdot 13,\ 2 \cdot 3 \cdot 41,\ 2 \cdot 3 \cdot 7\}\ .$$

We have exactly one "large" prime dividing a value. This is

$$41 \mid \frac{\Lambda(f,14)}{\Omega_+}\ ,$$

and this divisibility is the source for the congruence above.

4 Cohomology with Coefficients

To explain this connection I have to recall some other facts from the theory of Siegel modular varieties. The space

$$\Gamma \backslash \mathbb{H}_2$$

can be interpreted as the parameter space of principally polarized abelian surface over $\mathbb{C}$. Roughly, we can attach to a point in $\mathbb{H}_2$ a triple

$$\langle L, \langle\ ,\rangle\ , I\rangle = \mathcal{A}_I$$

where I is a complex structure on $L \otimes \mathbb{R}$, which is an isometry for the pairing and s.t. the associated hermitian form is positive definite. (I personally prefer to view $\mathbb{H}_2$ as the space of such complex structures on $L \otimes \mathbb{R}$.) This $\mathcal{A}_I$ is an abelian surface and $\mathcal{A}_I \cong \mathcal{A}_{I'}$ if there is a $\gamma \in \Gamma$ such that $\gamma I = I'$, and this γ provides an isomorphism $\gamma^* : \mathcal{A}_I \cong \mathcal{A}_{I'}$. Here we encounter a minor difficulty, because γ is not unique, and γ^* depends on the choice of γ. Therefore we can not attach an abelian variety to a point $\tilde{I} \in \Gamma \backslash \mathbb{H}_2$. But if we pass to a suitably small normal congruence subgroup $\Gamma' \subset \Gamma$ then it is clear that we have a family $\pi : \mathcal{A} \to \Gamma' \backslash \mathbb{H}_2$ of principally polarized abelian varieties over $\Gamma' \backslash \mathbb{H}_2$. Then the family of cohomology groups $H^1(\mathcal{A}_{\tilde{I}}, \mathbb{Z})$ defines a local system of free $\mathbb{Z}$-modules of rank 4 over $\Gamma' \backslash \mathbb{H}_2$. This local system descends to a sheaf on $\Gamma \backslash \mathbb{H}_2$.

This sheaf is also obtained from the standard representation

$$\rho_{10} : \Gamma \longrightarrow Gl(L) = Gl(\mathcal{M}_{1,0})\ .$$

We define a representation $\rho_{01} : \Gamma \to Gl(\mathcal{M}_{0,1})$ where the module $\mathcal{M}_{0,1}$ is defined by

$$\Lambda^2 \mathcal{M}_{1,0} = \mathcal{M}_{0,1} \oplus \mathbb{Z}\ .$$

We can form the modules $\mathrm{Sym}^m(\mathcal{M}_{1,0}) \otimes \mathrm{Sym}^n(\mathcal{M}_{0,1})$ and these modules have a unique submodule (or quotient)

$$\rho_{m,n} : \varGamma \to Gl(\mathcal{M}_{m,n}) ,$$

which is defined be the requirement that it has the largest dominant weight amoung all highest weights of submodules.

(The representations ρ_{10} (resp. ρ_{01}) have highest weight γ_β (resp. γ_α), which are the two fundamental dominant weights. Then $\mathcal{M}_{m,n}$ is the unique irreducible submodule with highest weight $\lambda = m\gamma_\beta + n\gamma_\alpha$. At this point is an ambiguity: Instead of taking the submodule we could take the unique irreducible quotient having this fundamental weight (actually this ambiguity already occurs when we form the symmetric products). Then the submodule will map injectively into the quotient and the image is a submodule of finite index. This index will be only divisible by "small" primes $\leq m, n$, they do not play a role in our considerations, in other words it does not matter whether we take the submodule or the quotient.)

These representations yield sheaves $\tilde{\mathcal{M}}_{m,n}$ of $\mathbb{Z}$-modules. For an open set $U \subset \varGamma\backslash\mathbb{H}_2$ and its inverse image $\tilde{U} \subset \mathbb{H}_2$ we have

$$\begin{aligned}&\tilde{\mathcal{M}}_{m,n}(U) = \{f : \tilde{U} \to \mathcal{M}_{m,n} | f \quad \text{locally constant and}\\ &f(\gamma u) = \rho_{n,m}(\gamma) f(u) \quad \text{for all} \quad \gamma \in \varGamma\} .\end{aligned}$$

For m even, these modules give us sheaves $\tilde{\mathcal{M}}_{m,n}$ on the space $\varGamma\backslash\mathbb{H}_2$ which are almost local systems. We can consider the cohomology groups

$$H^i_c(\varGamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n}), H^i(\varGamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n})$$

where $H^\bullet_c$ denotes the cohomology with compact support. These cohomology groups sit in an exact sequence

$$\begin{aligned}&\to H^{i-1}(\partial\overline{(\varGamma\backslash\mathbb{H}_2)}, \tilde{\mathcal{M}}_{m,n}) \to H^i_c(\varGamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n}) \to H^i(\varGamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n})\\ &\to H^i(\partial\overline{(\varGamma\backslash\mathbb{H}_2)}, \tilde{\mathcal{M}}_{m,n}) \to ,\end{aligned}$$

where $\partial\overline{(\varGamma\backslash\mathbb{H}_2)}$ is the boundary of the Borel–Serre compactification.

We denote by $H^i_!(\varGamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n})$ the image of the cohomology with compact supports in the cohomology. The coefficient system $\tilde{\mathcal{M}}_{m,n}$ is called regular if $n, m > 0$. In this case all cohomology groups $H^i_!(\varGamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n} \otimes \mathbb{Q})$ vanish for $i \neq 3$. We have a Hodge filtration on $H^3_!(\varGamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n} \otimes \mathbb{C})$, and the lowest step of this filtration is given by (Faltings)

$$S_{m,n+3} \hookrightarrow H^3_!(\varGamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n} \otimes \mathbb{C}) .$$

To get the connection to the conjecture we choose $m = 4$, $n = 7$.

Now I formulate a second conjecture. We invert some small primes (say ≤ 22), and we denote the resulting ring by $R = \mathbb{Z}[\frac{1}{2}, \dots, \frac{1}{19}]$.

Then we get an exact sequence (*assumption*)

$$0 \to H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \to H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R)$$
$$\to H^3\left(\partial\overline{(\Gamma\backslash\mathbb{H}_2)}, \tilde{\mathcal{M}}_{4,7} \otimes R\right) \to 0 \,,$$

I know that this is true if I replace R by $\mathbb{Q}$.

Now we can show that we have an action of the Hecke operators on these modules and we have

$$H^3(\partial\overline{(\Gamma\backslash\mathbb{H}_2)}, \tilde{\mathcal{M}}_{4,7} \otimes R) = R$$

where $T^{(1)}_{(p)}$ acts on R with the eigenvalue

$$p^8 + a_p + p^{13} \,.$$

(For this assertion I refer to my lecture notes volume or [Modsym].)

Now we formulate another *assumption*

$$H^3_!(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \cong R^4 \,.$$

Then we know that $T_p^{(1)}$ acts as a scalar by multiplication by $\lambda(p)$ on this cohomology group. We have $\lambda(2) \neq 2^8 + a_2 + 2^{13}$ and hence we can decompose

$$H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes \mathbb{Q}) = H^3_!(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes \mathbb{Q}) \oplus H^3_{\mathrm{Eis}}(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes \mathbb{Q}) \,.$$

Now the main assertion of the second conjecture is:

If we intersect this decomposition with the integral cohomology, then

$$H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \supset H^3_!(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \oplus H^3_{\mathrm{Eis}}(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \,.$$

and the index of the direct sum in $H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R)$ is divisible by 41. *(The denominator of the Eisenstein class is divisible by* 41.*)*

The point with this second conjecture is that it implies the first conjecture and it can be verified on a computer. To do this we have to find a way to compute the cohomology groups. This can be done by using a suitable acyclic covering of $\Gamma\backslash\mathbb{H}_2$, and then the cohomology is computed from the Čech complex of this covering. We could also try to use a cell decomposition. This method will allow us to check the two assumptions. I think the problem will be that the number of cells will not be so big, but we have nontrivial coefficient systems, its dimension in a general point is 1820. It will be still more difficult to implement the action of the Hecke operator, because one has to pass to a finer cell decomposition, which also computes the cohomology and where the Hecke operators can be implemented as a homomorphism between the two complexes.

Of course, we could also compute $\bmod 41$, then we find $\lambda(2) \equiv 2^8 + a_2 + 2^{13} \bmod 41$, and our conjecture would say that $T_2^{(1)} \bmod 41$ is not diagonalizable.

Why didn't I do this earlier? In my lecture notes volume (Chap III, 3.1) I discuss the above conjecture in greater generality and I raise the question whether computer experiments should be made. There I say that these computations would "... einen beträchtlichen Aufwand erfordern, aber die Frage entscheiden, ob es sich lohnt, das Problem zu behandeln".

Recently I got some kind of unexpected help from C. Faber und G. van der Geer. They produced some tables of eigenvalues $\lambda(p)$ for certain local systems $\mathcal{M}_{m,n}$ with some small values n, m.

They make use of the fact that $\Gamma\backslash\mathbb{H}_2$ is actually the set of the complex points of a quasiprojective scheme $\mathcal{A}_2/\operatorname{Spec}(\mathbb{Z})$, and that our local systems $\tilde{\mathcal{M}}_{m,n}$ have an algebraic-geometric meaning. They are "motivic" sheaves, and it is not quite clear what that means. But in any case we can pick a prime ℓ and then $\tilde{\mathcal{M}}_{m,n}\otimes\mathbb{Z}_\ell$ will be an ℓ-adic sheaf on $\mathcal{A}_2$. Then we have the Grothendieck fixed point formula

$$\operatorname{tr}\,(\Phi_p \mid H_c^\bullet(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{F}}_p, \tilde{\mathcal{M}}_{m,n} \otimes \mathbb{Z}_\ell)) = \sum_{x\in\mathcal{A}_2(\mathbb{F}_p)} \operatorname{tr}\,(\Phi_p \mid \tilde{\mathcal{M}}_{m,n,x})\ ,$$

where Φ_p is the Frobenius at p. The right hand side can be computed because we have the modular interpretation.

The left hand side consists of several pieces (Eisenstein cohomology, endoscopic contributions, if $m = 0$ there may be some Saito–Kurokawa lifts), and the trace of Φ_p on these pieces can be computed explicitly (for small n, m) and can be expressed in terms of modular forms for $Sl_2(\mathbb{Z})$ and in terms of algebraic Hecke characters. This can be brought to the right hand side, and the resulting expression can be computed explicitly for small values n, m.

Then we are left with the "genuine" part in $H_c^3(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{F}}_p, \tilde{\mathcal{M}}_{m,n} \otimes \mathbb{Z}_\ell)$, and this part will be of rank $4 \cdot \dim S_{m,n+3}$. If now $\dim S_{m,n+3} = 1$, then this "genuine" part will be of rank 4 and we have

$$\operatorname{tr}\,(\Phi_p \mid H^3_{\text{genuine}}(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{F}}_p, \tilde{\mathcal{M}}_{m,n} \otimes \mathbb{Z}_\ell)) = \lambda(p)\ .$$

But now the $\lambda(p)$ can be computed from the right hand side, if we take the effort to compute the sum over $\mathcal{A}_2(\mathbb{F}_p)$ and the non "genuine" traces.

After I saw the preprint by C. Faber and G. van der Geer I realized that I might be able to check the first conjecture in a special case. I had to go through the values $\frac{L(f,k)}{\Omega_{\varepsilon(k)}}$ for the modular cusp form f of weight ≤ 22. (For higher weights except 26 the dimension of these space are ≥ 2. The eigenvalues of the Hecke operators are algebraic integers and also the normalized L-values will be algebraic integers, and the computations will be much more complicated.) I had to find a "large" prime dividing one of the values, and I found for our form of weight 22

$$41 \mid \frac{L(f,14)}{\Omega_+}\ .$$

I computed the numbers $4, 7$ and $7 + 3 = 10$ from these data and wrote an e-mail to G. van der Geer inquiring the dimension of $S_{4,10}$. Several answers were possible. The dimension could be zero. This would be devastating. The dimension could be >1, this would mean a horrible additional computational effort. But the answer was

Re: Kohomologie lokaler Systemen
Lieber Guenter,
die Dimension ist dann 1. Die ersten Eigenwerte sind wie folgt:

$$-2^4 \cdot 3 \cdot 5 \cdot 7$$

$$2^3 \cdot 3^4 \cdot 5 \cdot 17$$

$$-2^2 \cdot 3 \cdot 5^2 \cdot 17 \cdot 1439$$

$$2^4 \cdot 5^2 \cdot 7^2 \cdot 17 \cdot 31 \cdot 59$$

$$2^3 \cdot 3 \cdot 11 \cdot 17 \cdot 5650223$$

d.h. fuer die Primzahlen 2, 3, 5, 7, 11.
Mit bestem Gruss, Gerard

I read this message in my office in the Beringstrasse and I had the values of the a_p at home on my laptop. After two oral examinations of computer science students I went home and checked the numbers. I was extremely pleased when I found that the congruences hold.

(Actually van der Geer was also pleased because he considered it as confirmation of his computations with Faber. (I have multiplied the values in his table by -1, probably this has to be done because the trace occurs in odd degree))

5 Why the Denominator?

We stick to the case $\mathcal{M}_{4,7}$, and f is still our modular cusp form of weight 22. If we had a splitting under the Hecke-algebra

$$H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) = H^3_!(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \oplus H^3(\partial(\overline{\Gamma\backslash\mathbb{H}_2}), \tilde{\mathcal{M}}_{4,7} \otimes R) ,$$

then we could construct a mixed Tate motive $\mathcal{X}(f)$ which sits in an exact sequence

$$0 \to R(-8) \to \mathcal{X}(f) \to R(-13) \to 0$$

and hence defines an element in the extension group

$$[\mathcal{X}(f)] \in \mathrm{Ext}^1_{\mathcal{MM}}(R(-13), R(-8)) = \mathrm{Ext}^1_{\mathcal{MM}}(R(-5), R(0)) .$$

(For more details see [Mixmot].) This Ext^1 group is some kind of undefined object, but we can attach to our object $\mathcal{X}(f)$ elements in two other extension groups, namely:

(i) an extension class in the category of mixed Hodge structures

$$[\mathcal{X}(f)]_{BdRh} \in \mathrm{Ext}^1_{BdRh}(R(-13), R(-8)) = \mathrm{Ext}^1_{BdRh}(R(-5), R(0)) = \mathbb{R}$$

(See MixMot 1.5.2). It is some kind of general belief that those elements in the extension group of mixed Hodge structures, which come from mixed motives $\mathcal{X}$ over $\mathbb{Z}$, are in fact elements of the form

$$[\mathcal{X}_{BdRh}] = a(\mathcal{X})\zeta'(-4) \text{ with } a(\mathcal{X}) \in \mathbb{Q}\,.$$

This last conjecture can be verified in our particular case we have the formula

$$[\mathcal{X}_{BdRh}(f)] = c \cdot \frac{\frac{\Lambda(f,13)}{\Omega_-}}{\frac{\Lambda(f,14)}{\Omega_+}}\zeta'(-4)$$

where c is a rational number containing only small primes.

(ii) For any prime ℓ we can attach an ℓ-adic extension class

$$[\mathcal{X}(f)]_\ell \in \mathrm{Ext}^1_{Gal}(R_\ell(-13), R_\ell(-8)) = H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q},)R_\ell(5))$$

and this cohomology group contains certain specific elements $c_\ell(5)$, these are the Soulé elements. These elements should also be generators of the image of

$$\mathrm{Ext}^1_{\mathcal{MM}}(R_\ell(-13), R_\ell(-8)) \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), R_\ell(5))$$

if we tensor by $\mathbb{Q}$. We write the Galois cohomology group multiplicatively and now it is general belief that we must have

$$[\mathcal{X}(f)]_\ell = c_\ell(5)^{c\cdot\frac{\frac{\Lambda(f,13)}{\Omega_-}}{\frac{\Lambda(f,14)}{\Omega_+}}}\,.$$

From now on we choose $\ell = 41$ (of course we could replace ℓ by 41 in the following considerations but this causes some confusion), then the value of the ℓ-adic ζ-function $\zeta_\ell(5) \not\equiv 0 \mod \ell$ and this implies that $c_\ell(5)$ is a primitive element in the Galois cohomology group. But the rational exponent has ℓ in its denominator, this contradicts the existence of our mixed motive and this motive has been constructed under the assumption that ℓ does not divide the denominator of the Eisenstein class.

6 Arithmetic Implications

We get a diagram (still $\ell = 41$) of ℓ-adic Galois modules

$$\begin{array}{ccccccc} 0 \to H^3_!(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{4,7} \otimes R_\ell) \to & H^3(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{4,7} \otimes R_\ell) & \to R_\ell(-13) \to 0 \\ & \cup & r \nearrow \\ & H^3_!(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{4,7} \otimes R_\ell) \oplus R_\ell(-13) & \end{array}$$

where the image of the homomorphism r is contained in $\ell R_\ell(-13)$. This gives us an injective homomorphism

$$\psi : \mathbb{Z}/(\ell)(-13) \hookrightarrow H^3_!(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{4,7} \otimes \mathbb{Z}/(\ell)) \,.$$

The module $H^3_!(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{4,7} \otimes \mathbb{Z}/(\ell))$ is of dimension 4 over $\mathbb{F}_\ell$ and the cup product provides a non -degenerated pairing of this module with itself into $\mathbb{Z}/(\ell)(-21)$. The orthogonal complement Y of the image $\psi(\mathbb{Z}/(\ell)(-13))$ is of dimension 3 over $\mathbb{F}_\ell$ and we get two exact sequences

$$0 \to \mathbb{Z}/(\ell)(-13) \to Y \to X \to 0$$

and

$$0 \to X \to H^3_!(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{4,7} \otimes \mathbb{Z}/(\ell))/\psi(\mathbb{Z}/(\ell)(-13)) \to \mathbb{Z}/(\ell)(-8) \to 0 \,.$$

The module X is actually the reduction of the ℓ-adic representation attached to f mod ℓ. It also has a non degenerate pairing with itself with values in $\mathbb{Z}/(\ell)(-21)$ and the two sequences are dual to each other. The sequences give us two extension classes, the first one a class

$$\begin{aligned}[Y] \in \mathrm{Ext}^1_{Gal}(X, \mathbb{Z}/(\ell)(-13)) = & H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Hom}(X, \mathbb{Z}/(\ell)(-13))) \xrightarrow{\sim} \\ & H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), X \otimes \mathbb{Z}/(\ell)(8))\end{aligned}$$

and under the isomorphism $[Y]$ is mapped to the extension class of the second sequence.

Now we can hope that this extension class is actually an element in the Selmer group of the Scholl-Deligne motive $M(f)$ attached to f, and that it is in fact an element of order ℓ. If this turns out to be the case, then we have produced an element in the Selmer group whose existence is predicted by the general philosophy of the Bloch–Kato–Birch–Swinnerton Dyer conjecture.

References

Fa-vdG Faber, Carel; van der Geer, Gerard: Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes.I, C. R. Math. Acad. Sci. Paris 338 (2004), no. 5, 381–384. II, no. 6, 467–470

Ha-Eis G. Harder, Eisensteinkohomologie und die Konstruktion gemischter Motive, SLN 1562
The following two manuscripts on mixed motives and modular symbols can be found on my home page
www.math.uni-bonn.de/people/harder/
in my ftp-directory folder Eisenstein.

Mixmot Modular construction of mixed motives II

Modsym Modular Symbols and Special Values of Automorphic L-Functions

Appendix

In the meanwhile C. Faber, G. van der Geer and I did some further computations. We have still another one dimensional space of modular cusp forms, this is spanned by the modular form

$$g(q) = \Delta(q)E_6(q)E_4(q)^2 =$$

$$q - 48q^2 - 195804q^3 - 33552128q^4 - 741989850q^5 + 9398592q^6 + \dots$$

of weight 26. We have the following divisibilities by "large" primes

$$43|\frac{L(g,23)}{\Omega_-},\ 97|\frac{L(g,21)}{\Omega_-},\ 29|\frac{L(g,19)}{\Omega_-}\ .$$

The corresponding spaces of modular forms $S_{18,5}, S_{14,7}, S_{10,9}$ have dimension 1.

Now let ℓ be one of the primes 41, 43, 97, 29. Let

$$f(q) = q + a_2q^2 + a_3q^3 \dots$$

be the corresponding modular form of weight 22 or 26. Let $S_{i,j}$ be the corresponding one dimensional space of Siegel modular forms and let $\tilde{\mathcal{M}}_{i,j-3} = \tilde{\mathcal{M}}_{m,n}$ be the corresponding local system. The Hecke algebra acts on the cohomology

$$H^3_!(\mathcal{A}_2 \times_{\mathbb{Z}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{m,n} \otimes R_\ell)$$

and we should find an isotypical submodule of rank 4 on which the Hecke operators act by the scalar by which they acts on $S_{i,j}$. The local Hecke algebra at a prime p is generated by two Hecke operators $T_{p,\alpha}, T_{p,\beta}$ which correspond to the double classes

$$Sp_2(\mathbb{Z}_p)\begin{pmatrix} p & & & 0 \\ & p & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} Sp_2(\mathbb{Z}_p) \text{ and } Sp_2(\mathbb{Z}_p)\begin{pmatrix} p^2 & & & 0 \\ & p & & \\ & & p & \\ 0 & & & 1 \end{pmatrix} Sp_2(\mathbb{Z}_p)\ .$$

(See [Ha-Eis] 3.1.2.1) So we get sequences of eigenvalues

$$\{\lambda_\alpha(p), \lambda_\beta(p)\}_{p\in \text{Primes}}$$

Now we state the conjecture that in all four cases we have congruences

$$\lambda_\alpha(p) \equiv p^{n+1} + a_p + p^{n+m+2} \mod \ell$$

and

$$\lambda_\beta(p) \equiv a_p(1 + p^{m+1}) + (p^2 - 1)p^{n_\alpha + n_\beta} \mod \ell$$

for all primes p.

For our four primes ℓ above and the corresponding modular forms the conjecture for $\lambda_\alpha(p)$ has been checked for all $p \le 37$.

The general rule is: If k is even and f an eigenform of weight k. Let $K = \mathbb{Q}(f)$ be its field of definition. Let us assume that a large prime $\mathfrak{l}$ divides $L(f,\nu)/\Omega_{\epsilon(\nu)}$. Then we solve the equations

$$k = 2n + m + 4, \nu = n + m + 3 .$$

Then we can construct an Eisenstein class in $H^3_!(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{m,n} \otimes K)$ whose denominator is divisible by $\mathfrak{l}$.

Added on April 3, 2003 (the day when the first Abel-Prize was given to J.-P. Serre):
I also checked congruences for the modular cusp forms of weight 24. In this case we have two eigenforms

$$f(q) = \sum_n^\infty a_n q^n = q - (540 - 12\sqrt{144169})q^2 + (169740 + 576\sqrt{144169})q^3 \ldots$$

where we take the positive root-, and we have the conjugate eigenform

$$f'(q) = \sum_n^\infty a'_n q^n .$$

We put $\omega = \frac{1+\sqrt{144169}}{2}$. In this case we find periods $\Omega_\pm, \Omega'_\pm$ such that

$$\frac{L(f,k)}{\Omega_{\epsilon(k)}} \in \mathbb{Z}[\omega], \frac{L(f',k)}{\Omega'_{\epsilon(k)}} \in \mathbb{Z}[\omega] .$$

We normalize the periods such that these numbers for a fixed choice of the sign $\epsilon(k)$ are coprime and such that $\frac{L(f,k)}{\Omega_{\epsilon(k)}}$ and $\frac{L(f',k)}{\Omega'_{\epsilon(k)}}$ are conjugate.

The primes 73 and 179 split in $\mathbb{Q}(\sqrt{144169})$ and for 73 the decomposition is

$$\mathfrak{l} = (73, 53 + 36\sqrt{144169}), \mathfrak{l}' = (73, 53 - 36\sqrt{144169})$$

$$(73) = \mathfrak{l}\mathfrak{l}' .$$

We find $\frac{L(f,19)}{\Omega_-} \in \mathfrak{l}$. The corresponding space $S_{12,7}$ has dimension 1, if $\lambda(p)$ is the sequence of eigenvalues the congruence

$$\lambda(p) \equiv p^5 + a_p + p^{18} \mod \mathfrak{l}$$

has been checked for all primes $p \le 19$. Of course we get a second congruence if we conjugate it.

For 179 we have a splitting

$$(179) = \mathfrak{l}\mathfrak{l}' ,$$

with $\mathfrak{l} = (179, 54+61\sqrt{144169}), \mathfrak{l}' = (179, 54-61\sqrt{144169})$. We find $\frac{L(f,17)}{\Omega_-} \in \mathfrak{l}$, again the corresponding space $S_{8,9}$ has dimension 1. If $\lambda(p)$ is the sequence of eigenvalues the congruences

$$\lambda(p) \equiv p^7 + a_p + p^{16} \mod \mathfrak{l}$$

and of course its conjugates have been checked for the same set of primes p. (There is a slight risk that I mixed up the two primes $\mathfrak{l}, \mathfrak{l}'$.)

Added on March 25, 2005:
When lecturing on this subject, I had sometimes difficulties to get the numbers right. Therefore I formulate the rules:

We start from an elliptic modular form f for $Sl_2(\mathbb{Z})$ which is of (even) weight k, it should be an eigenform for the Hecke-algebra. Then its eigenvalues generate a field $\mathbb{Q}(f)$.

Then we look at the values

$$\left\{\frac{\Lambda(f,k-1)}{\Omega_+}, \frac{\Lambda(f,k-3)}{\Omega_+}, \quad \dots, \quad \frac{\Lambda(f,k-\nu)}{\Omega_+}, \quad \dots, \quad \frac{\Lambda(f,k-\mu(k))}{\Omega_+}\right\}$$

and

$$\left\{\frac{\Lambda(f,k-2)}{\Omega_+}, \frac{\Lambda(f,k-4)}{\Omega_+}, \quad \dots, \quad \frac{\Lambda(f,k-\nu)}{\Omega_+}, \quad \dots, \quad \frac{\Lambda(f,k-\mu'(k))}{\Omega_+}\right\},$$

where in the first row the ν are odd and in the second row they are even. The last value is the one which is nearest to the central point $\frac{k}{2}$ from above.

Then we look for large primes

$$\ell | \frac{\Lambda(f,k-\nu)}{\Omega_{\epsilon(\nu)}}.$$

Now we choose the highest weight $\lambda = m\gamma_\beta + n\gamma_\alpha$. The numbers m, n must satisfy

$$2n + m + 3 = n + 1 + n + m + 2 = k - 1 \quad \text{and} \quad n + m + 3 = k - \nu$$

hence we get

$$n = \nu - 1 \quad \text{and} \quad m = k - 2\nu - 2$$