# A Near Linear Time Approximation Scheme for Steiner Tree Among Obstacles in the Plane

Matthias Müller-Hannemann and Siamak Tazari*

Darmstadt University of Technology, Dept. of Computer Science
{muellerh,tazari}@algo.informatik.tu-darmstadt.de

**Abstract.** We present a polynomial time approximation scheme (PTAS) for the Steiner tree problem with polygonal obstacles in the plane with running time $O(n \log^2 n)$, where $n$ denotes the number of terminals plus obstacle vertices. To this end, we show how a planar spanner of size $O(n \log n)$ can be constructed that contains a $(1+\epsilon)$-approximation of the optimal tree. Then one can find an approximately optimal Steiner tree in the spanner using the algorithm of Borradaile et al. (2007) for the Steiner tree problem in planar graphs. We prove this result for the Euclidean metric and also for all uniform orientation metrics, i.e. particularly the rectilinear and octilinear metrics.

**Keywords:** Steiner Tree, Obstacles, PTAS, Euclidean Metric, Uniform Orientation Metric, Spanner, Banyan, Planar Graph.

## 1 Introduction

We consider the following network design problem: given a set of points in the plane and a set of disjoint polygonal obstacles, find the shortest network interconnecting the points and avoiding the interior of the obstacles. We refer to the given points as *terminals* and to the obstacle vertices as *corners*. We let $n$ be the total number of terminals *and* corners. The shortest interconnecting network of the terminals will be a tree, a so-called *Steiner tree*, and it might use corners and additional vertices called *Steiner points* (note that we use this term only to refer to points that do not coincide with terminals or corners). This problem is called the *obstacle-avoiding Steiner minimum tree* problem (SMTO) or ESMTO when we are using the Euclidean metric.

Uniform orientation metrics are derived from $\lambda$-*geometries*. In a $\lambda$-geometry, one is allowed to move only along $\lambda \geq 2$ orientations building consecutive angles of $\pi/\lambda$. The *rectilinear* or *Manhattan* metric corresponds to the 2-geometry and the *octilinear* metric to the 4-geometry. We call the corresponding SMT problems $\lambda$-SMT or, when obstacles are to be avoided, $\lambda$-SMTO. In this case, the obstacle edges must obey the restrictions of the given orientations, too.

It has been a long-standing open problem whether these SMT problems *among obstacles* admit a polynomial time approximation scheme (PTAS). With the

* This author was supported by the Deutsche Forschungsgemeinschaft (DFG), grant MU1482/3-1.

F. Dehne, J.-R. Sack, and N. Zeh (Eds.): WADS 2007, LNCS 4619, pp. 151–162, 2007.

recent result of Borradaile et al. [1,2] about Steiner trees in planar graphs, this question can now be answered affirmatively by combining a number of results in the literature (see Section 1.1). However, to obtain a near linear running time, new ideas and more sophisticated techniques are required; this is the main contribution of our work. Our approach is based on constructing a planar graph of size $O(n \log n)$ that contains a $(1+\epsilon)$-approximation of the solution and then find an approximate solution in that graph. The total running time will be $O(n \log^2 n)$. Along the way, we prove a number of spanner results and other properties of SMTOs both for the Euclidean and uniformly oriented case.

The SMT problem and its many variations are of high theoretical (see below) and practical relevance. The applications reach from all kinds of network design to phylogenetic trees. Especially the geometric case with obstacles is very important in VLSI design, since there are usually regions in the plane that may not be crossed by wire. Also, it is often only allowed to route the tree along a rectilinear or octilinear grid and so, SMTs in uniformly oriented metrics are required.

### 1.1 Related Work

The ESMTO problem is clearly $\mathcal{NP}$-hard since it contains the Steiner minimum tree problem without obstacles as a special case [3]. For the SMT problem without obstacles, Arora [4] and Mitchell [5] were the first to present a PTAS. Rao and Smith [6] improved the running time of Arora's algorithm from $O(n(\frac{1}{\epsilon}\log n)^{O(1/\epsilon)})$ to $O(2^{\mathrm{poly}(1/\epsilon)}n + n\log n)$ using a certain spanner graph they call a "banyan" and this is the best running time known so far. However, none of these algorithms are applicable to the case with obstacles since a so-called "patching lemma" that is necessary for these approaches, fails to hold. Provan [7] has shown how to approximate ESMTO by an SMT problem in graphs and derived an FPTAS for the special case when the terminals lie on a constant number of "boundary polygons" and interior points.

The PTASs discussed above also apply to $\lambda$-SMTs for all $\lambda \geq 2$. The rectilinear and octilinear case have been shown to be $\mathcal{NP}$-complete in [8,9]. For general fixed $\lambda$ no proof has been published so far, though it is widely believed that these problems are hard, too. Properties of uniformly oriented SMTs have been studied by Brazil et al. [10]. Approximation algorithms for rectilinear SMTO have been proposed by Ganley and Cohoon [11] and for the octilinear case by Müller-Hannemann and Schulze [9,12]. For rectilinear SMTO with a constant number of obstacles, Liu et al. [13] presented a PTAS based on Mitchell's [5] approach. The SMT problem with length restrictions on obstacles has been studied by Müller-Hannemann and Peyer [14] in the rectilinear case, and by Müller-Hannemann and Schulze [12] in the octilinear case, and constant-factor approximation algorithms have been proposed.

The SMT problem in graphs has also been studied widely in the literature. It has been shown to be $\mathcal{APX}$-complete [15] and thus, no PTAS exists unless $\mathcal{P} = \mathcal{NP}$. The best approximation factor known so far is $1.55 + \epsilon$ [16]. The case of planar graphs has very recently been shown to admit a PTAS by Borradaile et al. [1,2]. This results immediately in a PTAS for rectilinear, octilinear, and

Euclidean SMTO using the following results from the literature: the so-called Hanan-grid [11,17] for the rectilinear case, the result of Müller-Hannemann and Schulze [12] for the octilinear case, and Provan's construction [7] together with the planar spanner result of Arikati et al. [18] for the Euclidean case. However, in all these cases, the PTAS of Borradaile et al. has to be run on a graph of size $O(n^2)$ and thus, the total running time will be $O(n^2 \log n)$. In this work, we show alternative constructions with running time $O(n \log^2 n)$.

### 1.2 On Spanners and Banyans

A *t-spanner* of a set of points $P$ is a graph that contains a path between any two points of $P$ that is at most a factor of $t$ longer than the shortest path between them. Spanners have been vastly studied in the literature [19] and have been often used in the design of PTASs [6]. Of particular interest to us are spanners of the *visibility graph* among obstacles in the plane. The visibility graph contains all straight line connections between terminals and corners that do not cross the interior of any obstacle. Clarkson [20] showed how to construct a $(1+\epsilon)$-spanner of linear size of the visibility graph in $O(n \log n)$ time. A linear-sized *planar* spanner for both the rectilinear and Euclidean metric has been shown to exist and also to be computable in $O(n \log n)$ time by Arikati et al. [18,21]. We will show how to extend these ideas to derive sparse planar spanners in the same time bound for all uniform orientation metrics.

Rao and Smith [6] introduced the notion of *banyans*. A banyan is a graph that contains a $(1+\epsilon)$-approximation of the SMT of a given set of points[1] and whose weight is at most a constant factor larger than the SMT. Rao and Smith showed how to construct a banyan of size $O(n)$ in time $O(n \log n)$ in the obstacle-free case.

### 1.3 Our Contribution and Outline

One of the main results of our work is to show how to construct a *planar banyan for SMTO* of size $O(n \log n)$ in $O(n \log^2 n)$ time by building on the framework of Rao and Smith using new ideas and combining other results from the literature, especially the banyan-result for planar graphs contained in [1]. An approximate Steiner tree can then be obtained on this planar graph using [1]. Since the algorithm in [1,2] is exponential in $1/\epsilon$, so is our resulting algorithm.

The main difficulties that arise when obstacles are present are to deal with visibility and the fact that only a subset of corners is included in the SMT, i.e. we do not know which vertices of a spanner will be part of the SMT. In particular, a spanner might include arbitrary short edges between corners that are not part of the SMT and this causes an important proof idea of Rao and Smith to fail. Roughly speaking, they show that in the obstacle-free case, there is always a "long enough" spanner edge near non-negligible SMT edges and so,

[1] Their construction was in fact more powerful as it included an approximate SMT for *any* subset of the terminals.

they introduce a grid of candidate Steiner points in a neighborhood around every spanner edge to capture these SMT vertices. Our main new algorithmic idea is to use $O(\log n)$ layers of candidate Steiner points around each spanner edge, so that we are guaranteed to find such appropriate points even when our spanner edges are short. Another important difference is that we use planar spanners, so that afterwards, we can use the algorithm of [1] instead of building on Arora's approach [4] to obtain our PTAS. We present our algorithm in Section 2 and then present two proofs for the correctness of our algorithm for the Euclidean case in Section 3: one using an analog of the hexagon property [3] and another one using a generalization of the empty ball lemma [6]. Even though our proofs follow the lines of the proofs of Rao and Smith, they differ conceptually at some key points and other techniques have to be used, see Section 3.1.

Afterwards, in Section 4, we turn our attention to uniform orientation metrics and argue how the presented proofs can be modified to work for these cases, too. Along the way, we have to argue that a lemma by Provan [7] about $\delta$-grids among obstacles in the plane still holds true for all uniform orientation metrics. At last, we prove a variation of Arikati et al.'s planar spanner result [18] to apply to uniform orientation metrics.

Due to space limitations, several details and proofs had to be shortened or omitted. We refer the interested reader to the full version of this paper.

## 2 The Algorithm

The main result of our work is the following theorem:

**Theorem 2.1.** *The Steiner minimum tree problem among disjoint polygonal obstacles in the plane admits a PTAS in the Euclidean metric and in all uniform orientation metrics. The running time is $O(n \log^2 n)$, where $n$ is the total number of terminals and obstacle corners.*

Our algorithm is summarized in Alg. 1. We are given a set of terminals $Z$ and a set of disjoint polygonal obstacles $O$ as described in the introduction. In the first step, we find a $(1+\epsilon_1)$-spanner $G_1$ of the visibility graph of $Z \cup O$. In the Euclidean case, this can be done using the algorithm of Clarkson [20] or Arikati et al. [18,21] in $O(n \log n)$ time. Arikati et al. also provide an algorithm for the rectilinear case and we will show in Section 4.2 how to extend this result to other uniform orientation metrics. Note that these algorithms construct a spanner without having to build the full visibility graph. The graph $G_1$ will have $O(n)$ edges. Around each such edge, we place $\lceil \log_2 n \rceil$ circles with doubling radii and place a grid of constant size inside each of them. Here, we make use of a constant $\kappa$ that depends on the metric being used. For $\epsilon_1 \leq 1$, in the Euclidean metric, $\kappa$ can be chosen to be $\leq 226$, and in the rectilinear metric $\leq 50$. This introduces a set $P_0$ of $O(n \log n)$ "candidate Steiner points" from which we remove the ones that lie inside obstacles. This can be done using a sweep-line algorithm in $O(k \log k) = O(n \log^2 n)$-time with $k = n + |P_0| = O(n \log n)$. Let $G_2$ be the visibility graph of $Z \cup P_0 \cup O$ and let $G_3$ be a *planar* $(1+\epsilon_2)$-spanner of $G_2$. $G_3$ can

**Algorithm 1.** A PTAS for SMTO

**Input :** a set of terminals $Z$ and a set of disjoint polygonal obstacles $O$ in the plane and the desired accuracy $0 < \epsilon \leq 1$.
**Output :** a $(1+\epsilon)$-approximation of the obstacle-avoiding Steiner minimum tree of the terminals.
**Note :** $\kappa$ is a constant and can be $\leq 226$ in the Euclidean case and $\leq 50$ in the rectilinear case, $\epsilon_1$ and $\epsilon_2$ have to be chosen appropriately, e.g. $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{22}$.

**begin**
  find a $(1+\epsilon_1)$-spanner $G_1$ of the visibility graph of $Z \cup O$;
  let $P_0 = \emptyset$;
  **for** *each edge* $e$ *in* $G_1$ **do** `// let` $\ell$ `be the length of` $e$
    **for** $i = 0$ *to* $\lceil \log_2 n \rceil$ **do**
      let $r = \kappa 2^i \ell / \epsilon_1$;
      let $C$ be a circle of radius $r$ around the midpoint of $e$;
      place a grid with spacing $\delta(r) = r\epsilon_1^3/\kappa^2$ inside $C$;
      `// the grid has` $\leq 4\kappa^4/\epsilon_1^6 = O(1)$ `points`
      add these points to $P_0$;
  remove all the points from $P_0$ that lie inside obstacles;
  `// let` $G_2$ `be the visibility graph of` $Z \cup P_0 \cup O$
  find a planar $(1+\epsilon_2)$-spanner $G_3$ of $G_2$;
  find a $(1+\epsilon/3)$-approximate SMT $T$ of $Z$ in $G_3$;
  `// using the PTAS of Borradaile et al. [1]`
  **return** $T$;
**end**

be found using Arikati et al.'s algorithm or our extension of it for other uniform orientation metrics. Let $k = O(n \log n)$ be the number of vertices of $G_2$. The spanner algorithms need $O(k \log k)$ time and introduce $O(k)$ additional Steiner points to achieve planarity. Thus, $G_3$ can be constructed in $O(n \log^2 n)$ time and has $O(n \log n)$ vertices (note that $G_2$ is not constructed explicitly). Now we find a $(1+\epsilon/3)$-approximate Steiner minimum tree of the terminals $Z$ in $G_3$ using the PTAS of Borradaile et al. [1,2] for the Steiner tree problem in planar graphs. The time needed for this step is $O(k \log k)$ and hence, the total runtime of our algorithm is $O(n \log^2 n)$.

Note that the first step of the PTAS of Borradaile et al. is to determine a subgraph $G_4$ of $G_3$ that contains a $(1+\epsilon/3)$-approximation of the SMT of $G_3$ and has weight at most a constant times the weight of the SMT of $G_3$. Hence, $G_4$ is a planar banyan of the terminal set $Z$ and so, our algorithm also delivers a planar banyan of a set of terminals among obstacles in the plane.

*A note on the running time.* Of course, the constants hidden in the $O$-notations above all depend on $1/\epsilon$. Our algorithm builds the planar graph $G_3$ in time $O(\frac{\kappa^4}{\epsilon^{11}} n \log^2 n)$ and its size is more precisely $O(\frac{\kappa^4}{\epsilon^{11}} n \log n)$. The PTAS of Borradaile et al. takes time singly exponential in $1/\epsilon$ [2].

# 3 Correctness

We present two proofs for the correctness of Alg. 1. The first one results in better constants but does not work in the rectilinear case. The other one is more general and can even be partly extended to give us some structural information about SMTOs in higher dimensions but uses much larger constants. The proof technique and the generalization of the empty ball lemma used in the second proof might also be interesting in their own right. In the next section, we discuss uniform orientation metrics where we include a simpler proof for the rectilinear case that results in small constants.

## 3.1 Key Differences

The main new idea in our algorithm compared to that of Rao and Smith's is the use of $O(\log n)$ layers of grids around each edge of the spanner $G_1$. We had to do this because in our case, we do not have the so-called spanner path property (Lemma 34 in [6]), that essentially says that two vertices that are connected in the SMT by an edge of length $L$, can not be connected in a spanner by a chain of "tiny" edges of length $< L$. In our case, two terminals and/or corners *can* be connected by a path consisting entirely of "tiny" edges, finding their way among obstacles. But we know that any two vertices in the spanner are connected by a short path with at most $n$ edges and one of them can be made "long enough" by multiplying it with a power of 2 if necessary.

Also, we have the additional problem that many vertices of our spanner need not be part of the optimal Steiner tree: we do not know which corners will be included in the SMT. And there is the issue of visibility. To deal with these problems, we formulate and prove Lemma 3.1, which is a technical lemma that is very important in both of our proofs. The reason it works well is that since we do not make use of the spanner path property, it is not necessary for us that the vertices that we find close to a Steiner point $A$ be in fact close to $A$ *in the SMT* or even be part of the SMT at all. This enabled us to prove our generalizations of the hexagon property and the empty ball lemma and use them to prove our main theorem.

## 3.2 First Proof

We use the notation $d(A, B)$ for the length of the shortest obstacle-avoiding path between two points $A$ and $B$ and the notation $d_G(A, B)$ for the shortest path between $A$ and $B$ in a graph $G$. We start by mentioning some well known facts about Euclidean Steiner minimum trees [22,7,23] (recall that we use the term Steiner point for vertices of the tree that do not coincide with terminals or corners):

**Fact 0:** The SMT is a tree that includes all terminals as vertices. It might include corners or Steiner points as additional internal vertices.
**Fact 1:** Every Steiner point has 3 incident edges making angles of 120°.

**Fact 2:** A Steiner point may not occur on the boundary of some obstacle.
**Fact 3:** Every terminal and corner has degree at most 3 in the SMT.
**Fact 4:** If there are $k$ terminals, there are at most $k-2$ Steiner points.
**Fact 5:** Two edges of the SMT meet only at a common endpoint, i.e. the SMT is not self-intersecting.
**Fact 6:** ($120°$ wedge property) If $s$ is a Steiner point of the SMT, then in any closed $120°$ wedge with apex $s$, there exists a terminal or corner $v$ and an SMT path $sv$ that lies entirely inside the wedge.

The following lemma is of central importance for our work:

**Lemma 3.1.** *Let $S$ be a closed convex region of the plane and let $A \in S$ be a point that is not contained in the interior of any obstacle. Then, we have*

*(i) a terminal or corner in $S$ that is visible to $A$; or*
*(ii) the maximal visible area to $A$ in $S$ is a closed convex region $S' \subseteq S$ that contains no terminal or corner and that shares its border with $S$ except for finitely many straight line segments, where its border may consist of obstacle edges. Furthermore, any obstacle-avoiding path contained in $S$ and connected to $A$ is contained in $S'$.*

Consider an SMT edge $AB$ and some fixed distance $D$. Let $H_A$ be the regular hexagon of side length $D$ that has $A$ as a corner, does not contain $AB$, and builds two $120°$ angles with $AB$, i.e. if we extend $AB$, it would cut $H_A$ in half. Furthermore, we define an SMT edge $AB$ to be *locally $D$-bounded* if when walking from $A$ or $B$ for at most 3 SMT edges or until we encounter a terminal or corner (whichever comes first), all edges we pass have length at most $D$. We have the following property:

**Lemma 3.2 (Generalization of the hexagon property).** *Let $AB$ be a locally $D$-bounded SMT edge. Then the regular hexagon $H_A$ of side length $D$ defined above contains a terminal or a corner that is visible to $A$ (this terminal or corner could be $A$ itself).*

Let $AB$ be an SMT edge of length $L$. For given constants $c \geq 1$ and $\epsilon_1 > 0$, we define $AB$ to be *locally long* if it is locally $cL/\epsilon_1$-bounded. Otherwise we call it *locally short*. The next lemma builds the heart of our first proof of Theorem 2.1. It assures that near every locally long SMT edge $AB$, we find a spanner edge of $G_1$ that is long enough, so that a layer of grids around it will enclose the points $A$ and $B$; and short enough, so that the grid spacing does not introduce too large an error.

**Lemma 3.3.** *Let $AB$ be a locally long SMT edge of length $L$ as defined above with some constants $c \geq 1$ and $0 < \epsilon_1 \leq 1$ to be specified. Consider Alg. 1 with a constant $\kappa \geq 8c+2$. Then there exists an edge $e$ of length $\ell$ in the $(1+\epsilon_1)$-spanner $G_1$ and an integer $0 \leq i \leq \lceil \log_2 n \rceil$, so that $L \leq 2^i \ell \leq \kappa L/\epsilon_1$ and so that $A$ and $B$ are included in a circle of radius $\kappa 2^i \ell/\epsilon_1$ around the midpoint of $e$.*

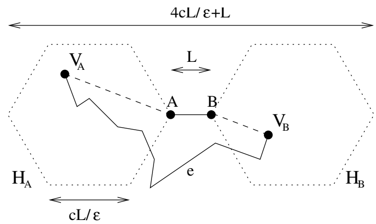


**Fig. 1.** Proof of Lemma 3.3; $L \le d(V_A, V_B) \le 4cL/\epsilon_1 + L$

*Proof.* By the hexagon property above with $D = cL/\epsilon_1$, we know that there exists a terminal or corner $V_A$ inside $H_A$ and a terminal or corner $V_B$ inside $H_B$, so that $V_A$ is visible to $A$ and $V_B$ is visible to $B$ (note that $V_A$ resp. $V_B$ could be equal to $A$ resp. $B$). Then we know that $L \le d(V_A, V_B) \le L + 4cL/\epsilon_1 =: M$ (see Fig. 1). Now consider the shortest path between $V_A$ and $V_B$ in the spanner $G_1$. It consists of at most $n$ edges and its length is at least $L$ and at most $(1+\epsilon_1)M = ((4c+1)\epsilon_1 + 4c + \epsilon_1^2)L/\epsilon_1 \le \kappa L/\epsilon_1$ if we choose $\kappa \ge 8c+2$. Also, this path lies entirely inside a circle $Q$ of radius $R := (1+\epsilon_1/2)M$ around the midpoint of the edge $AB$, since otherwise it would be too long for $G_1$ to be a $(1+\epsilon_1)$-spanner. Hence, there exists an edge $e$ of length $\ell$ on this path inside $Q$, so that $L/n \le \ell \le \kappa L/\epsilon_1$. If $\ell < L$, one can choose an integer $0 \le i \le \lceil \log_2 n \rceil$ so that $L \le 2^i \ell \le 2L \le \kappa L/\epsilon_1$ otherwise choose $i = 0$. Also, since $e$ is inside $Q$, the distance between the midpoint of $e$ to $A$ and $B$ is at most $R + L/2 = ((2c+1.5)\epsilon_1 + 4c + \epsilon_1^2/2)L/\epsilon_1 \le \kappa 2^i \ell/\epsilon_1$.

*Proof (First proof of Theorem 2.1 for the Euclidean metric).* Let us denote the length of a tree $T$ by $\ell(T)$. Let $T^\star$ be an optimal obstacle-avoiding Steiner tree of the terminal set and let $T$ be the tree returned by Alg. 2.1. We show that the graph $G_3$ contains a Steiner tree $T'$ with $\ell(T') \le (1+\epsilon/2)\ell(T^\star)$. Then we know that $\ell(T) \le (1+\epsilon/3)\ell(T') \le (1+\epsilon)\ell(T^\star)$ and we are done.

We partition the edges of $T^\star$ into locally long and locally short edges as defined above and construct the tree $T'$ as follows: for every locally long edge in $T^\star$, we find appropriate endpoints and a short path in $G_3$ to add to $T'$; then we get a number of connected components and interconnect them with a minimum forest in $G_3$. We now analyze the length of $T'$.

Let $AB$ be a locally long edge of $T^\star$ of length $L$. By Lemma 3.3, we find an edge $e$ of length $\ell$ in $G_1$ and a circle $C$ of radius $r = \kappa 2^i \ell/\epsilon_1$ around the midpoint of $e$ for some integer $0 \le i \le \lceil \log_2 n \rceil$, so that $A$ and $B$ are included in $C$. The grid inside $C$ has spacing[2] $\delta = r\epsilon_1^3/\kappa^2 = 2^i \ell \epsilon_1^2/\kappa \le L\epsilon_1$ since $2^i \ell \le \kappa L/\epsilon_1$ by Lemma 3.3. A technical lemma of Provan (Lemma 3.2 in [7]) says that for a given Steiner point $A$ in a $\delta$-grid among obstacles, we can always find a terminal, corner or grid point $A'$ that is visible to $A$, so that $d(A, A') \le 2\delta$. Let $A'$ and $B'$ be vertices of $G_3$ that are visible to and closest to $A$ and $B$, respectively. Add the shortest path between $A'$ and $B'$ in $G_3$ to $T'$. We have

[2] The grid spacing in [6] is $r\epsilon_1^2/\kappa^2$ but we believe that the exponent of $\epsilon_1$ should be 3.

$$d(A', B') \le d(A', A) + d(A, B) + d(B, B') \\ \le L + 4\delta \le L + 4L\epsilon_1 = (1 + 4\epsilon_1)L \tag{1}$$

and thus, $d_{G_3}(A', B') \le (1 + \epsilon_2)(1 + 4\epsilon_1)L$.

We leave the detailed analysis of locally short edges for the full paper; one can show that the overhead caused by ignoring locally short edges is at most equal to the total length of all locally short edges and can be upper bounded[3] by $(1 + \epsilon_2)\epsilon_1\ell(T^\star)$. So, we get that

$$\ell(T') \le (1 + \epsilon_2)(1 + 4\epsilon_1)\ell(T^\star) + (1 + \epsilon_2)\epsilon_1\ell(T^\star) \le (1 + \epsilon/2)\ell(T^\star) \tag{2}$$

if $\epsilon_1$ and $\epsilon_2$ are chosen appropriately, e.g. $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{22}$.

*Second Proof.* Due to space limitations, we leave our second proof for the full paper. Here, we just state our generalization of the empty ball lemma:

**Lemma 3.4 (Generalization of the empty ball lemma).** *Let $S_1$ and $S_2$ be closed convex regions in the plane whose interiors are free of terminals and obstacle edges but whose borders may partly consist of obstacle-edges. Denote the parts of their borders that are not obstacle-edges as the* free border. *Assume that $S_2$ encloses $S_1$ and that the distance between every point on the free border of $S_1$ to any point on the free border of $S_2$ is at least $\gamma > 0$. Then, for any obstacle-avoiding SMT, the number of Steiner points inside $S_1$ is bounded by a constant $s_0 \le (96e)^8$ (where $e$ is the base of the natural logarithm).*

## 4 Uniform Orientation Metrics

We briefly discuss how our proofs adapt to uniform orientation metrics and provide a somewhat different proof for the rectilinear case that results in much better constants. Afterwards, we prove our generalization of Arikati et al.'s [18] planar spanner result to the cases with $\lambda \ge 3$.

### 4.1 Adapting the Proofs

*The Cases $\lambda \ge 3$.* Brazil et al. [10] showed that for $\lambda \ge 3$, there always exists an SMT, such that the minimum angle at a Steiner point is $90° \le \alpha_{\min} \le 120°$ and the maximum angle is $120° \le \alpha_{\max} \le 150°$. For these cases, we get an $\alpha_{\max}$-wedge property like Fact 6 of the Euclidean case and we can use it to prove an analog of the Hexagon property (Lemma 3.2). Also, using this $\alpha_{\max}$-wedge property one can generalize Provan's lemma [7] to ensure that in a $\delta$-grid around a Steiner point $A$, one can always find a grid point, terminal, or corner $A'$ that is visible to $A$, so that $d(A, A') \le \delta/cos\frac{\alpha_{\max}}{2}$. Using these two results, one can generalize both of our proofs from Section 3 straightforwardly to all $\lambda$-geometries with $\lambda \ge 3$, where again, the first proof results in much better constants (but possibly different ones from the Euclidean case).

[3] To achieve this, we have to set $c = 28$ and thus, we can choose $\kappa = 8c + 2 = 226$.

*The Rectilinear Case.* In the rectilinear case, we do not have an $\alpha$-wedge property for an $\alpha < 180°$; in fact, $180°$-angles can occur at any Steiner point. But instead, the structure of rectilinear Steiner trees is well-studied. Particularly, one can derive the following lemma using results from [24,25]:

**Lemma 4.1.** *For a given set of terminals and disjoint rectilinear obstacles in the plane, there exists an obstacle-avoiding RSMT that has the following properties: (i) for any two Steiner points $A$ and $B$ that are connected by a horizontal line-segment, $B$ is not connected to a third Steiner point by a vertical line segment; (ii) if there is a grid with spacing $\delta$ around a Steiner point $A$, then there exists a grid point, terminal, or corner $A'$ that is visible to $A$, so that $d(A, A') \leq 2\delta$.*

Using this lemma, both of our proofs from the last section adapt straightforwardly to the rectilinear case. Furthermore, one can choose $c = 12$ and also $\kappa = 4c + 2 = 50$.

### 4.2 Planar Spanners

Consider a $\lambda$-geometry and let $\omega = \pi/\lambda$ be the smallest allowed angle. Before we start with the construction of our spanner, we need the following technical lemma:

**Lemma 4.2.** *Consider a $\lambda$-geometry with smallest allowed angle $\omega$ and let a set of disjoint polygonal obstacles be given whose edges are parallel to the allowed directions. Consider two points $A$ and $B$ in the plane. Then there exists a shortest path (with respect to the metric of the $\lambda$-geometry) between $A$ and $B$ that passes through $A = v_0, v_1, v_2, \ldots, v_k = B$, so that each $v_i$ with $0 < i < k$ is a corner and so that the path between each $v_i$ and $v_{i+1}$ is either a straight line in an allowed direction or is comprised of two straight lines in allowed directions that build an angle of $\pi - \omega$ with each other.*

Arikati et al. [18,21] showed how to find a planar rectilinear $(1+\epsilon)$-spanner of the visibility graph among disjoint polygonal obstacles in the plane that uses at most $O(n)$ Steiner points in time $O(n \log n)$. This spanner might include obstacle-edges that are not rectilinear but their length is measured in the rectilinear metric. We first show that one can rotate the axes of the coordinate system to build an arbitrary angle and still obtain such a spanner:

**Lemma 4.3.** *Given a set of terminals and a set of disjoint polygonal obstacles in the plane, one can find a planar $(1 + \epsilon)$-spanner of the visibility graph of size $O(n)$ in $O(n \log n)$ time that uses only edges in two directions $d_1$ and $d_2$ building an angle $\omega$ with each other (except for parts of the spanner that coincide with obstacle edges).*[4]

Now we can use a similar trick to the one used by Arikati et al. to obtain their Euclidean spanner: let $d_1, d_2, \ldots, d_\lambda$ be the allowed directions, so that two

[4] The spanning property is with respect to the metric induced by $d_1$ and $d_2$.

consecutive ones build an angle of $\omega = \pi/\lambda$ with each other. Find $(1+\epsilon)$-spanners $G_1, \ldots, G_\lambda$, so that $G_i$ uses only edges parallel to $d_i$ and $d_{i+1}$ (or $d_\lambda$ and $d_1$) using Lemma 4.3. Let $G$ be the graph obtained by superimposing all these graphs on each other, i.e. putting them on each other and adding all intersection points as new vertices to the graph. A straightforward adaption of the proof of Arikati et al. for the Euclidean case (published in the thesis of Zeh [21]) shows that $G$ will still have $O(n)$ vertices[5] and can be obtained in $O(n \log n)$ time. Also, by Lemma 4.2, $G$ is indeed a $(1+\epsilon)$-spanner of the visibility graph (an approximate shortest path between each $v_i$ and $v_{i+1}$ of the lemma lies entirely in a spanner $G_j$), i.e. we have

**Theorem 4.4.** *Consider a $\lambda$-geometry and let a set of terminals and a set of disjoint polygonal obstacles whose edges are in the allowed directions be given, so that the total number of terminals and corners is $n$. Then one can find a planar $(1+\epsilon)$-spanner (with respect to the metric of the $\lambda$-geometry) of the visibility graph of size $O(n)$ in $O(n \log n)$ time that uses only edges in the allowed directions.*

## References


1. Borradaile, G., Kenyon-Mathieu, C., Klein, P.N.: A polynomial-time approximation scheme for Steiner tree in planar graphs. In: SODA '07: Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 1285–1294 (2007)
2. Borradaile, G., Kenyon-Mathieu, C., Klein, P.N.: Steiner tree in planar graphs: An $O(n \log n)$ approximation scheme with singly exponential dependence on epsilon. In: WADS '07: Proceedings of the 10th Workshop on Algorithms and Data Structures. LNCS, Springer, Heidelberg (2007)
3. Garey, M., Graham, R., Johnson, D.: The complexity of computing Steiner minimal trees. SIAM Journal on Applied Mathematics 32, 835–859 (1977)
4. Arora, S.: Polynomial time approximation schemes for the Euclidean traveling salesman and other geometric problems. Journal of the ACM 45, 753–782 (1998)
5. Mitchell, J.S.B.: Guillotine subdivisions approximate polygonal subdivisions: A simple polynomial-time approximation scheme for geometric TSP, $k$-MST, and related problems. SIAM Journal on Computing 28, 1298–1309 (1999)
6. Rao, S.B., Smith, W.D.: Approximating geometrical graphs via "spanners" and "banyans". In: STOC '98: Proceedings of the 30th annual ACM Symposium on Theory of Computing, pp. 540–550. ACM Press, New York (1998)
7. Provan, J.S.: An approximation scheme for finding steiner trees with obstacles. SIAM Journal on Computing 17, 920–934 (1988)
8. Garey, M., Johnson, D.: The rectilinear Steiner tree problem is $\mathcal{NP}$-complete. SIAM Journal on Applied Mathematics 32, 826–834 (1977)
9. Müller-Hannemann, M., Schulze, A.: Hardness and approximation of octilinear Steiner trees. In: Deng, X., Du, D.-Z. (eds.) ISAAC 2005. LNCS, vol. 3827, pp. 256–265. Springer, Heidelberg (2005)


[5] This is done by utilizing the special structure of the spanners; each spanner is based on a division of the plane into $O(n)$ regions, each containing a grid of constant size; it is shown that each region of each spanner can overlap with at most a constant number of regions of every other spanner.

10. Brazil, M., Thomas, D., Winter, P.: Minimum networks in uniform orientation metrics. SIAM Journal on Computing 30, 1579–1593 (2000)
11. Ganley, J.L., Cohoon, J.P.: Routing a multi-terminal critical net: Steiner tree construction in the presence of obstacles. In: Proceedings of the International Symposium on Circuits and Systems, pp. 113–116 (1994)
12. Müller-Hannemann, M., Schulze, A.: Approximation of octilinear Steiner trees constrained by hard and soft obstacles. In: Arge, L., Freivalds, R. (eds.) SWAT 2006. LNCS, vol. 4059, pp. 242–254. Springer, Heidelberg (2006)
13. Liu, J., Zhao, Y., Shragowitz, E., Karypis, G.: A polynomial time approximation scheme for rectilinear Steiner minimum tree construction in the presence of obstacles. In: 9th IEEE International Conference on Electronics, Circuits and Systems. vol. 2, pp. 781–784 (2002)
14. Müller-Hannemann, M., Peyer, S.: Approximation of rectilinear Steiner trees with length restrictions on obstacles. In: Dehne, F., Sack, J.-R., Smid, M. (eds.) WADS 2003. LNCS, vol. 2748, pp. 207–218. Springer, Heidelberg (2003)
15. Bern, M., Plassmann, P.: The Steiner problem with edge lengths 1 and 2. Information Processing Letters 32, 171–176 (1989)
16. Robins, G., Zelikovsky, A.: Improved Steiner tree approximation in graphs. In: SODA '00: Proceedings of the 11th Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 770–779 (2000)
17. Hanan, M.: On Steiner's problem with rectilinear distance. SIAM Journal on Applied Mathematics 14, 255–265 (1966)
18. Arikati, S.R., Chen, D.Z., Chew, L.P., Das, G., Smid, M.H.M., Zaroliagis, C.D.: Planar spanners and approximate shortest path queries among obstacles in the plane. In: Díaz, J. (ed.) ESA 1996. LNCS, vol. 1136, pp. 514–528. Springer, Heidelberg (1996)
19. Eppstein, D.: Spanning trees and spanners. In: Sack, J.R., Urrutia, J. (eds.) Handbook of Computational Geometry, pp. 425–461. Elsevier, Amsterdam (2000)
20. Clarkson, K.: Approximation algorithms for shortest path motion planning. In: STOC '87: Proceedings of the 19th Annual ACM Symposium on Theory of Computing, pp. 56–65. ACM Press, New York (1987)
21. Zeh, N.: I/O-Efficient Algorithms for Shortest Path Related Problems. PhD thesis, Carleton University, Ottawa, Canada (2002)
22. Gilbert, E.N., Pollak, H.O.: Steiner minimal trees. SIAM Journal on Applied Mathematics 16, 1–29 (1968)
23. Zachariasen, M., Winter, P.: Obstacle-avoiding Euclidean Steiner trees in the plane: An exact approach. In: Goodrich, M.T., McGeoch, C.C. (eds.) ALENEX 1999. LNCS, vol. 1619, pp. 282–295. Springer, Heidelberg (1999)
24. Hwang, F.: On Steiner minimal trees with rectilinear distance. SIAM Journal on Applied Mathematics 30, 104–114 (1976)
25. Prömel, H., Steger, A.: The Steiner Tree Problem: A Tour through Graphs, Algorithms, and Complexity. Advanced Lectures in Mathematics, Vieweg (2002)