Drawing Colored Graphs on Colored Points*

Melanie Badent[1], Emilio Di Giacomo[2], and Giuseppe Liotta[2]

[1] Department of Computer and Information Science, University of Konstanz
melanie.badent@uni-konstanz.de
[2] Dip. di Ingegneria Elettronica e dell'Informazione, Università degli Studi di Perugia
{digiacomo, liotta}@diei.unipg.it

Abstract. Let G be a planar graph with n vertices whose vertex set is partitioned into subsets $V_0, \ldots, V_{k-1}$ for a positive integer $1 \leq k \leq n$ and let S be a set of n distinct points in the plane partitioned into subsets $S_0, \ldots, S_{k-1}$ with $|V_i| = |S_i|$ $(0 \leq i \leq k-1)$. This paper studies the problem of computing a crossing-free drawing of G such that each vertex of V_i is mapped to a distinct point of S_i. Lower and upper bounds on the number of bends per edge are proved for any $3 \leq k \leq n$. As a special case, we improve the upper and lower bounds presented in a paper by Pach and Wenger for $k = n$ [*Graphs and Combinatorics* (2001), 17:717–728].

1 Introduction and Overview

Let G be a planar graph with n vertices whose vertex set is partitioned into subsets $V_0, \ldots, V_{k-1}$ for some positive integer $1 \leq k \leq n$ and let S be a set of n distinct points in the plane partitioned into subsets $S_0, \ldots, S_{k-1}$ with $|V_i| = |S_i|$ $(0 \leq i \leq k-1)$. Each index i is a *color*, G is a *k-colored planar graph*, and S is a *k-colored set of points compatible with* G. This paper studies the problem of computing a *k-colored point-set embedding of G on S*, i.e. a crossing-free drawing of G such that each vertex of V_i is mapped to a distinct point of S_i.

Computing k-colored point-set embeddings of k-colored planar graphs has applications in graph drawing, where the *semantic constraints* for the vertices of a graph G define the placement that these vertices must have in a readable visualization of G (see, e.g., [7]). For example, in the context of data base systems design some particularly relevant entities of an ER schema may be required to be drawn in the center and/or along the boundary of the diagram (see, e.g., [18]); in social network analysis, a typical technique to visualize and navigate large networks is to group the vertices into clusters and to draw the vertices of the same cluster close to each other and relatively far from those of other clusters (see, e.g., [6]). A natural way of modelling these types of semantic constraints is to color a (sub)set of the vertices of the input graph and to specify a set of locations having the same color for their placement in the drawing.

The problem of computing k-colored point-set embeddings of k-colored planar graphs has therefore attracted considerable interest in the graph drawing

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and computational geometry communities, where particular attention has been devoted to the *curve complexity* of the computed drawings, i.e. the maximum number of bends along each edge. Namely, reducing the number of bends along the edges is a fundamental optimization goal when computing aesthetically pleasing drawings of graphs (see, e.g., [7]). Before presenting our results, we briefly review the literature on the subject. Since there is not a unified terminology, we slightly rephrase some of the known results; in what follows, n denotes both the number of vertices of a k-colored planar graph and the number of points of a k-colored set of points compatible with the graph.

Kaufmann and Wiese [16] study the "mono-chromatic version" of the problem, that is they focus on 1-colored point-set embeddings. Given a 1-colored planar graph G (i.e. a planar graph G) and a (1-colored) set S of points in the plane they show how to compute a 1-colored point-set embedding of G on S such that the curve complexity is at most two, which is proved to be worst case optimal. Further studies on 1-chromatic point-set embeddings can be found in [4,5,11]; these papers are devoted to characterizing which 1-colored planar graphs with n vertices admit 1-colored point-set embeddings of curve complexity zero on any set of n points and to presenting efficient algorithms for the computation of such drawings.

2-colored point-set embeddings are studied in [10] where it is proved that subclasses of outerplanar graphs, including paths, cycles, caterpillars, and wreaths all admit a 2-colored point-set embedding on any 2-colored set of points such that the resulting drawing has constant curve complexity. It is also shown in [10] that there exists a 3-connected 2-colored planar graph G and a 2-colored set of points S such that every 2-colored point-set embedding of G on S has at least one edge requiring $\Omega(n)$ bends. These results are extended in [8], where an $O(n \log n)$-time algorithm is described to compute a 2-colored point-set embedding with constant curve complexity for every 2-colored outerplanar graph and it is proved that for any positive integer h there exists a 3-colored outerplanar graph G and a 3-colored set of points such that any 3-colored point-set embedding of G on S has at least one edge having more than h bends. Characterizations of families of 2-colored planar graphs which admit a 2-colored point-set embedding having curve complexity zero on any compatible 2-colored set of points can be found in [1,2,13,14,15].

Key references for the "n-chromatic version" of the problem are the works by Halton [12] and by Pach and Wenger [17]. Halton [12] proves that an n-colored planar graph always admits an n-colored point-set embedding on any n-colored set of points; however, he does not address the problem of optimizing the curve complexity of the computed drawing. About ten years later, Pach and Wenger [17] re-visit the question and show that an n-colored planar graph G always has an n-colored point-set embedding on any n-colored set of points such that each edge of the drawing has at most $120n$ bends; they also give a probabilistic argument to prove that, asymptotically, the upper bound on the curve complexity is tight for a linear number of edges. More precisely, let G be an n-colored planar graph with m independent edges and let S be a set of n

points in convex position such that each point is colored at random with one of n distinct colors. Pach and Wenger prove that, almost surely, at least $\frac{m}{20}$ edges of G have at least $\frac{m}{40^3}$ bends on any n-colored point-set embedding of G on S.

The present paper describes a unified approach to the problem of computing k-colored point-set embeddings for $3 \leq k \leq n$. The research is motivated by the following observations: (i) The literature has either focused on very few colors or on the n colors case; in spite of the practical relevance of the problem, little seems to be known about how to draw graphs where the vertices are grouped into $3 \leq k \leq n$ clusters and there are semantic constraints for the placement of these vertices. (ii) The $\Omega(n)$ lower bound on the curve complexity for 2-colored point-set embeddings described in [10] implies that for any $2 \leq k \leq n$ there can be k-colored point-set embeddings which require a linear number of bends per edge. This could lead to the conclusion that in order to compute k-colored point-set embeddings that are optimal in terms of curve complexity one can arbitrarily n-color the input graph, consistently color the input set of points, and then use the drawing algorithm by Pach and Wenger [17]. However, the lower bound of [10] shows $\Omega(n)$ curve complexity for a *constant number* of edges, whereas the drawing technique of Pach and Wenger gives rise to a *linear number* of edges each having a linear number of bends. Hence, the total number of bends in a drawing obtained by the technique of [17] is $O(n^2)$ and it is not known whether there are small values of k for which $o(n^2)$ bends would always be possible. (iii) There is a large gap between the multiplicative constant factors that define the upper and the lower bound of the curve complexity of n-colored point-set embeddings [17]. Since the readability of a drawing of a graph is strongly affected by the number of bends along the edges, it is natural to study whether there exists an algorithm that guarantees curve complexity less than $120n$. Our main results are as follows.

- A lower bound on the curve complexity of k-colored point-set embeddings is presented which establishes that $\Omega(n^2)$ bends may be necessary even for small values of k. Namely, it is shown that for any k such that $3 \leq k \leq n$ there exists a k-colored planar graph G and a k-colored set of points S compatible with G such that any k-colored point-set embedding of G on S has at least $\frac{n}{6} - 1$ edges each having at least $\frac{n}{6} - 1$ bends. This lower bound generalizes and improves the one in [17] for $k = n$.
- An $O(n^2 \log n)$-time algorithm is described that receives as input a k-colored planar graph G ($3 \leq k \leq n$), a k-colored set of points S compatible with G, and computes a k-colored point-set embedding of G on S with curve complexity at most $3n+2$. This reduces by about forty times the previously known upper bound for $k = n$ [17].
- Motivated by the previously described lower bound, special colorings of the input graph are studied which can guarantee a curve complexity that does not depend on n. Namely, it is shown that if the k-colored planar graph G has $k-1$ vertices each having a distinct color and $n-k+1$ vertices of the same color, it is always possible to compute a k-colored point-set embedding whose curve complexity is at most $9k-1$.

For proofs omitted in this abstract refer to the full version of this paper [3].

2 Preliminaries

A *drawing* of a graph G is a geometric representation of G such that each vertex is a distinct point of the Euclidean plane and each edge is a simple Jordan curve connecting the points which represent its end-vertices. A drawing is *planar* if any two edges can only share the points that represent common end-vertices. A graph is *planar* if it admits a planar drawing.

Let $G = (V, E)$ be a graph. A *k-coloring* of G is a partition $\{V_0, V_1, \dots, V_{k-1}\}$ of V where the integers $0, 1, \dots, k-1$ are called *colors*. In the rest of this section the index i is $0 \le i \le k-1$ if not differently specified. For each vertex $v \in V_i$ we denote by $col(v)$ the color i of v. A graph G with a k-coloring is called a *k-colored graph*. Let S be a set of distinct points in the plane. We always assume that the points of S have distinct x-coordinates (this condition can always be satisfied by means of a suitable rotation of the plane). For any point $p \in S$ we denote by $x(p)$ and $y(p)$ the x- and y-coordinates of p, respectively. A *k-coloring* of S is a partition $\{S_0, S_1, \dots, S_{k-1}\}$ of S. A set S of distinct points in the plane with a k-coloring is called a *k-colored set of points*. For each point $p \in S_i$ $col(p)$ denotes the color i of p. A k-colored set of points S is *compatible with* a k-colored graph G if $|V_i| = |S_i|$ for every i; if G is planar, we say that G has a *k-colored point-set embedding* on S if there exists a planar drawing of G such that: (i) every vertex v is mapped to a distinct point p of S with $col(p) = col(v)$, (ii) each edge e of G is drawn as a polyline λ; a point shared by any two consecutive segments of λ is called a *bend* of e. The *curve complexity* of a drawing is the maximum number of bends per edge. Throughout the paper n denotes the number of vertices of graph and m the number of its edges.

3 Lower Bound on the Curve Complexity

A *diamond graph* is a 3-colored planar graph as the one depicted in Figure 1(a). More formally, let $n \ge 12$, let $n'' = (n \mod 12)$ and let $n' = n - n'' = 12h$ for some $h > 0$; a *diamond graph* $G_n = (V, E)$ is defined as follows: $V = V_0 \cup V_1 \cup V_2$; $V_0 = \{v_i \mid 0 \le i \le \frac{n'}{3} + \lceil \frac{n''}{2} \rceil\}$; $V_1 = \{u_i \mid 0 \le i \le \frac{n'}{3} + \lfloor \frac{n''}{2} \rfloor\}$; $V_2 = \{w_i \mid 0 \le i \le \frac{n'}{3}\}$; $E = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4$; $E_0 = \{(v_i, v_{i+1}) \mid 0 \le i \le \frac{n'}{3} + \lceil \frac{n''}{2} \rceil - 1\}$; $E_1 = \{(u_i, u_{i+1}) \mid 0 \le i \le \frac{n'}{3} + \lfloor \frac{n''}{2} \rfloor - 1\}$; $E_2 = \{(w_i, w_{i+1}),\ (w_{i+1}, w_{i+2}),\ (w_{i+2}, w_{i+3}),\ (w_{i+3}, w_i) \mid 0 \le i \le 4h-1, i \mod 4 = 0\}$; $E_3 = \{(w_{i+1}, w_{i+4}),\ (w_{i+3}, w_{i+4}),\ (w_{i+1}, w_{i+6}),\ (w_{i+3}, w_{i+6}) \mid 0 \le i \le 4h-5, i \mod 4 = 0\}$; $E_4 = \{(w_{4h-1}, v_{\frac{n'}{3} + \lceil \frac{n''}{2} \rceil}),\ (w_{4h-3}, v_0),\ (w_0, u_0),\ (w_2, u_{\frac{n'}{3} + \lfloor \frac{n''}{2} \rfloor})\}$.

Let $S' = S_0 \cup S_1$ be a 2-colored set of points all belonging to a horizontal straight line ℓ; S' is a *bi-colored sequence* if $|S_0| = |S_1|$ or $|S_0| = |S_1|+1$ and given two points p and q of S' such that there is no point r with $x(p) < x(r) < x(q)$, then $col(p) \ne col(q)$. A 3-*colored set of points with an alternating bi-colored sequence* is a 3-colored set of points $S = S_0 \cup S_1 \cup S_2$ such that $S' = S_0 \cup S_1$ is an alternating bi-colored sequence and no point of S_2 is on ℓ. A 3-colored set of points with an alternating bi-colored sequence is shown in Figure 1(b).

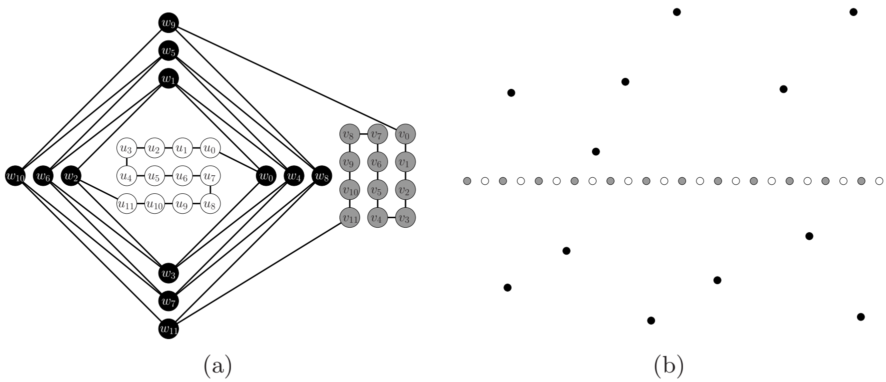

Fig. 1. (a) A diamond graph for $h = 3$. (b) A 3-colored set of points with an alternating bi-colored sequence.

Let G_n ($n \geq 12$) be the diamond graph with n vertices and let S be a 3-colored set of points with an alternating bi-colored sequence and compatible with G_n. Let Γ_n be a 3-colored point-set embedding of G_n on S. Let $p_0, p_1, \ldots, p_{8h+n''-1}$ be the points of the bi-colored sequence of S ordered according to their x-coordinates. Denote with z_i the vertex of G_n which is mapped to p_i. Notice that z_i and z_{i+1} are not adjacent in Γ_n because one of them belongs to V_0 and the other one belongs to V_1 in G_n. Connect in Γ_n z_i and z_{i+1} with a straight-line segment ($i = 0, \ldots, 8h+n''-2$); the obtained path is called *bi-colored path on Γ_n*.

Lemma 1. *Let G_n ($n \geq 12$) be a diamond graph and let S be a 3-colored set of points with an alternating bi-colored sequence such that S is compatible with G_n. Let Γ_n be a 3-colored point-set embedding of G_n on S, let e be an edge of Γ_n, and let Π be the bi-colored path on Γ_n. If Π crosses e b times, then e has at least $b-1$ bends.*

Lemma 2. *Let G_n ($n \geq 12$) be a diamond graph and let S be a 3-colored set of points with an alternating bi-colored sequence such that S is compatible with G_n. Let Γ_n be a 3-colored point-set embedding of G_n on S and let Π be the bi-colored path on Γ_n. Π crosses at least $\frac{n'}{6} - 1$ edges of Γ_n, where $n' = n - (n \mod 12)$; also, Π crosses each of these edges at least $\frac{n'}{6}$ times.*

Proof. For a planar drawing of G_n and a cycle $C \in G_n$ we say that C *separates* a subset $V' \subset V$ from a subset $V'' \subset V$ if all vertices of V' lie in the interior of the region bounded by C and all vertices of V'' are in the exterior of this region. In every planar drawing of G_n each of the h cycles defined by the edges in the set E_2 separates all vertices in V_0 from all vertices in V_1. Thus, every edge of Π must cross these h cycles. Analogously, in every planar drawing of G_n, each of the $h-1$ cycles defined by the edges in the set E_3 separates all vertices in V_0 from all vertices in V_1. Therefore, every edge of Π must also cross these $h-1$ cycles. The number of edges in Π is $\frac{2n'}{3} + n'' - 1$, where $n'' = n - n' = n \mod 12$, and hence each cycle is crossed $\frac{2n'}{3} + n'' - 1$ times. Since each cycle has four edges, we have that at least $2h - 1 = \frac{n'}{6} - 1$ edges (one per cycle) are crossed at least $\lceil \frac{n'}{6} + \frac{n''}{4} - \frac{1}{4} \rceil \geq \lceil \frac{12h}{6} - \frac{1}{4} \rceil = \lceil 2h - \frac{1}{4} \rceil = 2h = \frac{n'}{6}$ times. □

Theorem 1. *For every $n \geq 12$ and for every $3 \leq k \leq n$ there exists a k-colored planar graph G with n vertices and a k-colored set of points S compatible with G such that any k-colored point-set embedding of G on S has at least $\frac{n'}{6} - 1$ edges each having at least $\frac{n'}{6} - 1$ bends, where $n' = n - (n \mod 12)$.*

We conclude this section comparing the result of Theorem 1 with the known lower bound for $k = n$ [17]. Let G be an n-colored graph with m independent edges and let S be a set of n points in convex position such that each point is colored at random with one of n distinct colors. In [17] it is proved that, almost surely, at least $\frac{m}{20}$ edges of G have at least $\frac{m}{40^3}$ bends on any possible n-colored point-set embedding of G on S. A comparison with the result in Theorem 1 can be easily done by observing that the maximum number of independent edges in a graph with n vertices is at most $n/2$.

4 Upper Bound on the Curve Complexity

Theorem 1 shows that in terms of curve complexity the problem of computing a k-colored point-set embedding for any $k \geq 3$ is as difficult as computing an n-colored point-set embedding. Therefore, a drawing algorithm that is asymptotically optimal in terms of curve complexity for all values of k such that $1 \leq k \leq n$ could be designed as follows: (1) Randomly assign each vertex of color i of the input graph to a distinct point of color i of the input set of points. (2) Apply the drawing algorithm of Pach and Wenger [17], which constructs an n-colored point-set embedding whose curve complexity is at most $120n$. However, since optimizing the number of bends per edge is an important requirement that guarantees the readability of a drawing of a graph [7], we present in this section a new approach to the computation of n-colored point-set embeddings which reduces the maximum number of bends per edge from at most $120n$ to at most $3n + 2$.

The key idea is to translate the geometric problem into an equivalent topological problem, namely that of suitably augmenting a planar graph by adding dummy edges that do not cross the real edges too many times. The main ingredients for this approach are: (i) The notion of *augmenting k-colored Hamiltonian path* for a k-colored planar graph G. (ii) A theorem that proves that the number of crossings between the edges of an augmenting k-colored Hamiltonian path and the edges of a k-colored planar graph give an upper bound on the curve complexity of a k-colored point-set embedding of G. (iii) An augmentation algorithm that, for any linear ordering of the vertices of G, computes an augmenting k-colored Hamiltonian path which visits the vertices according to this ordering and that crosses each edge of G at most $3n - 1$ times.

A *k-colored sequence* σ is a linear sequence of (possibly repeated) colors $c_0, c_1, \ldots, c_{n-1}$ such that $0 \leq c_j \leq k - 1$ $(0 \leq j \leq n - 1)$. We say that σ is *compatible* *with* a k-colored graph G if, for every $0 \leq i \leq k-1$, color i occurs $|V_i|$ times in σ. Let S be a k-colored set of points and let $p_0, p_1, \ldots, p_{n-1}$ be the points of S ordered according to their x-coordinates. We say that S *induces* the k-colored sequence $\sigma = col(p_0), col(p_1), \ldots, col(p_{n-1})$.

A graph G has a *Hamiltonian path* if it has a simple path that contains all the vertices of G. If G is a k-colored graph and $\sigma = c_0, c_1, \ldots, c_{n-1}$ is a k-colored sequence compatible with G, a *k-colored Hamiltonian path of G consistent with σ* is a Hamiltonian path $v_0, v_1, \ldots, v_{n-1}$ such that $col(v_i) = c_i$ $(0 \leq i \leq n-1)$. Suppose that G is a k-colored planar graph and that G does not have a k-colored Hamiltonian path consistent with σ. One can augment G to a (not necessarily planar) k-colored graph G' by adding to G a suitable number of dummy edges and such that G' has a k-colored Hamiltonian path $\mathcal{H}'$ consistent with σ and that includes all dummy edges.

If G' is not planar, we can apply a planarization algorithm (see, e.g., [7]) to G' with the constraint that only crossings between dummy edges and edges of $G - \mathcal{H}'$ are allowed. Such a planarization algorithm constructs an embedded planar graph G'' where each edge crossing is replaced with a dummy vertex, called *division vertex*. By this procedure an edge e of $\mathcal{H}'$ can be transformed into a path whose internal vertices are division vertices. The subdivision of $\mathcal{H}'$ obtained this way is called an *augmenting k-colored Hamiltonian path of G consistent with* σ and is denoted as $\mathcal{H}''$. If every edge e of G is crossed at most d times in G' (i.e. e is split by at most d division vertices in G''), $\mathcal{H}''$ is said to be an *augmenting k-colored Hamiltonian path of G consistent with σ and inducing at most d division vertices per edge*. If G' is planar, then $\mathcal{H}''$ coincides with $\mathcal{H}'$. If both end-vertices of $\mathcal{H}''$ are on the external face of the augmented Hamiltonian form of G, then $\mathcal{H}''$ is said to be *external*.

Let v_d be a division vertex for an edge e of G. Since a division vertex corresponds to a crossing between e and an edge of $\mathcal{H}'$, there are four edges incident on v_d in G''; two of them are dummy edges that belong to $\mathcal{H}''$, the other two are two "pieces" of edge e obtained by splitting e with v_d. Let (u, v_d) and (v, v_d) be the latter two edges. We say that v_d is a *flat division vertex* if it is encountered after u and before v while walking along $\mathcal{H}''$; v_d is a *pointy division vertex* otherwise. The following theorem refines and improves a result presented in [8].

Theorem 2. *Let G be a k-colored planar graph, let σ be a k-colored sequence compatible with G, and let $\mathcal{H}$ be an augmenting k-colored Hamiltonian path of G consistent with σ having at most d_f flat and d_p pointy division vertices per edge. If $\mathcal{H}$ is external then G admits a k-colored point-set embedding on any set of points that induces σ such that the maximum number of bends along each edge is $d_f + 2d_p + 1$.*

Based on Theorem 2, we show our upper bound by proving that for *any* n-colored sequence σ an n-colored planar graph G always admits an augmenting k-colored Hamiltonian path of G consistent with σ such that $d_f \leq 3n-3$ and $d_p \leq 2$, which implies a curve complexity of $3n+2$. The algorithm to compute an augmenting k-colored Hamiltonian path of G consistent with σ relies on a morphing technique that starts with a special type of planar drawing where all vertices are aligned and transforms it into a drawing with aligned vertices that respects the given linear ordering.

Let $G = (V, E)$ be a planar graph. A *topological book embedding* of G is a planar drawing such that all vertices of G are represented as points of a horizontal straight line ℓ called *spine* and each edge intersects the spine a finite number of times. The straight line ℓ defines two half-planes one above and one below ℓ which are called the *top page* and the *bottom page*, respectively. In a topological book embedding each edge can be either completely contained in the top page, or completely contained in the bottom page, or can cross the spine. A crossing between an edge and the spine is called a *spine crossing*. In order to simplify the description of our results, we assume that a topological book embedding is such that every edge is a sequence of circular arcs; each circular arc of an edge e is called an *arc* of e. It is also assumed that if an edge e crosses the spine at a point p, the two arcs of e sharing p belong to opposite pages.

A *monotone topological book embedding* is a topological book embedding such that each edge crosses the spine at most once. Also, let $e = (u, v)$ be an edge of a monotone topological book embedding that crosses the spine at a point p; e is such that if u precedes v in the left-to-right order along the spine then p is between u and v, the arc with endpoints u and p is in the bottom page, and the arc with endpoints u and v is in the top page.

Theorem 3. [9] *Every planar graph admits a monotone topological book embedding. Also, a monotone topological book embedding can be computed in $O(n)$ time, where n is the number of the vertices in the graph.*

Given a monotone topological book embedding Γ of a planar graph G, we transform Γ into a new topological book embedding Γ' such that the linear ordering of the vertices along the spine coincides with an arbitrary given linear ordering λ of the vertices of G. Every vertex v of G has a *source position* $s(v)$ defined by the point representing v in Γ and a target position $t(v)$ in Γ' defined by the point representing v in Γ'. The linear ordering of the target positions of the vertices of G in Γ' coincides with λ. The transformation from Γ to Γ' moves each vertex of G from its source to its target position by processing the vertices in Γ from left to right. The *trajectory* of vertex v is the straight-line segment $\overline{s(v)t(v)}$. When v is moved to its target position the shape of those edges that are incident to v and of those edges that are intersected by the trajectory of v is changed in order to guarantee the planarity of the drawing.

To better explain the various steps of this morphing technique from Γ to Γ', we introduce the notion of 2-*spine drawing* of a planar graph G which generalizes the definition of topological book embedding. A 2-spine drawing Γ^* of G is a planar drawing such that each vertex is represented as a point of one among two parallel horizontal lines called *spines* of Γ^*. Each edge $e = (u, v)$ of G can have both end-vertices represented in Γ^* as points both in the same spine or in different spines. If both u and v are in the same spine, edge e is drawn in Γ^* as a sequence of arcs; if u is in the upper spine and v is in the lower spine, then when going from u to v along e in Γ^* we find a (possibly empty) sequence of arcs whose endpoints are in the upper spine, a straight-line segment between the two spines, and a (possibly empty) sequence of arcs whose endpoints are in the lower spine. A sequence of arcs of an edge e whose endpoints are in the upper (respectively, lower) spine is called an *upper sequence* of e (respectively, a *lower sequence* of e). The straight-line segment of an edge e between the two spines is called the *inter-spine segment* of e. Note that a 2-spine drawing such that all vertices are points of one of the spines is a topological book embedding.

Lemma 3. *Let G be a planar graph and let λ be a given linear ordering of the vertices of G. G admits a topological book embedding such that the left-to-right order of the vertices along the spine is λ.*

Sketch of Proof. Based on Theorem 3, G has a monotone topological book embedding that we call Γ. Let ℓ be the spine of Γ and let $v_0, \ldots, v_{n-1}$ be the vertices of G in the left-to-right order they have along ℓ. Let ℓ' be a horizontal line below ℓ. For each vertex v of G we define a target position on ℓ' such that the left-to-right order of these target positions corresponds to λ.

We process each vertex of Γ in the left-to-right order along ℓ. At each step a vertex is moved to its target position on ℓ' and a 2-spine drawing with spines ℓ and ℓ' of G is computed. Indeed, in order to compute a topological book embedding Γ' of G such that Γ' satisfies the statement, we compute a sequence $\Gamma_0, \ldots, \Gamma_n$ of 2-spine drawings with spines ℓ and ℓ' such that Γ_0 coincides with Γ and Γ_n coincides with Γ'. At Step i $(0 \leq i \leq n-1)$ the 2-spine drawing Γ_i is transformed into Γ_{i+1} by moving v_i to its target position on ℓ' and by changing the shape of the edges accordingly.

When vertex v_i is moved to its target position, we maintain the planar embedding and only change the shape of the edges incident on v_i and the shape of any edge that is intersected by the trajectory of v_i. In the remainder, we assume that the target positions along ℓ' are such that a trajectory of a vertex intersects an arc with end-points p and q only if one of the end-points of the trajectory is in the closed interval defined by p and q. (This assumption can be satisfied by suitably choosing the radii of the arcs of the edges and the distance between spines ℓ and ℓ'.)

Transformation of the shape of the edges intersected by the trajectory of v_i: The trajectory τ of v_i can intersect both inter-spine segments of some edges or arcs belonging to the the lower sequence of some edges. Notice that if τ intersects both inter-spine segments and arcs, then the inter-spine segments are encountered before the arcs when going from $s(v_i)$ to $t(v_i)$; see also Figure 2. Let $s_0, s_1, \ldots, s_{h-1}$ be the segments crossed by τ in the order they are encountered when going from $s(v_i)$ to $t(v_i)$ along τ; denote by x_j the endpoint of s_j that is on ℓ and by x'_j the endpoint of s_j that is on ℓ' $(0 \leq j \leq h-1)$. Two cases are possible: **Case a:** x'_j is to the left of x'_{j+1} along ℓ', and therefore x_j is to the left of x_{j+1} along ℓ (see Figure 2(a)); **Case b:** x'_j is to the right of x'_{j+1} along ℓ', and therefore x_j is to the right of x_{j+1} along ℓ (see Figure 2(b)). Let $c_0, c_1, \ldots, c_{l-1}$ be the arcs crossed by τ in the order they are encountered going from $s(v_i)$ to $t(v_i)$ along τ; denote by y_j and z_j the endpoints of c_j, with y_j to the left of z_j $(0 \leq j \leq l-1)$. Notice that y_j is to the left of y_{j+1} and z_j is to the

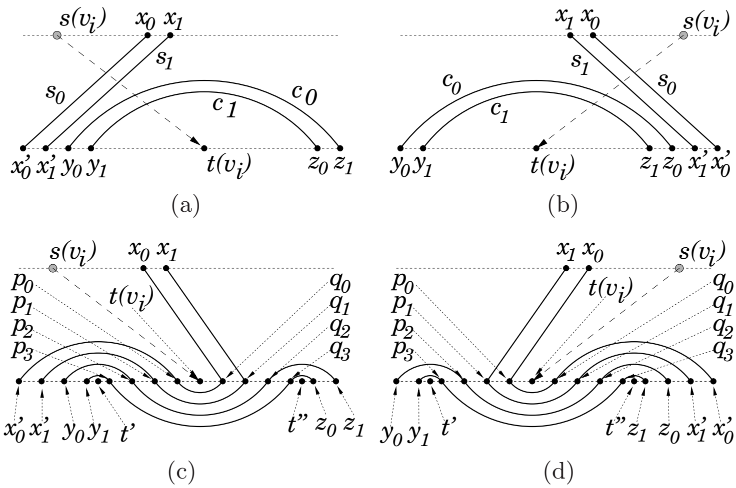

Fig. 2. Transformation of the shape of the edges intersected by the trajectory of v_i

right of z_{j+1}. Refer to Figures 2(c) and 2(d). Let t' and t'' be two points of ℓ' such that t', $t(v_i)$ and t'' appear in this left-to-right order along ℓ' and no vertex or spine crossing is between t' and $t(v_i)$ and between $t(v_i)$ and t'' on ℓ'. Choose $h+l$ points $p_0, p_1, \ldots, p_{h-1}, p_h, \ldots, p_{h+l-1}$ such that each p_j $(0 \leq j \leq h+l-1)$ is between t' and $t(v_i)$ on ℓ' and p_j is to the right of p_{j+1} on ℓ'. Choose $h+l$ points $q_0, q_1, \ldots, q_{h-1}, q_h, \ldots, q_{h+l-1}$ such that each q_j $(0 \leq j \leq h+l-1)$ is between $t(v_i)$ and t'' on ℓ' and q_j is to the left of q_{j+1} on ℓ'. If **Case a** holds (see Figures 2(c)), replace each segment $s_j = \overline{x_j x'_j}$ $(0 \leq j \leq h-1)$ with: (i) an arc with endpoints x'_j and p_j; (ii) an arc with endpoints p_j and q_j; (iii) a straight-line segment $\overline{q_j x_j}$. If **Case b** holds (see Figures 2(d)), replace each segment $s_j = \overline{x_j x'_j}$ $(0 \leq j \leq h-1)$ with: (i) an arc with endpoints x'_j and q_j; (ii) an arc with endpoints q_j and p_j; (iii) a straight-line segment $\overline{p_j x_j}$. Replace each arc c_j $(0 \leq j \leq l-1)$ whose endpoints are y_j and z_j with: (i) an arc with endpoints y_j and p_{h+j}; (ii) an arc with endpoints p_{h+j} and q_{h+j}; (ii) an arc with endpoints q_{h+j} and z_j.

Transformation of the shape of the edges incident on v_i: We partition the edges incident on v_i into four sets. The set $E_{t,l}$ (respectively, $E_{b,l}$) contains the edges $e = (v_j, v_i)$ such that $j < i$ and the arc of e incident on v_i is in the top (respectively, bottom) page of Γ. Analogously we can define sets $E_{t,r}$ and $E_{b,r}$ for the edges (v_i, v_j) with $i < j$.

Let $e = (v_j, v_i)$ be an edge of $E_{t,l}$ or $E_{b,l}$. When we move v_i, v_j has already been moved to ℓ' (because $j < i$) and therefore when going from v_j to v_i along e in Γ_i we find the (possibly empty) lower sequence σ_l of e, the inter-spine segment s_e of e, and the (possibly empty) upper sequence σ_u of e. Let x' be the endpoint of s_e on ℓ'. Replace s_e and σ_u with an arc whose endpoints are x' and $t(v_i)$.

Let $e = (v_i, v_j)$ be an edge of $E_{b,r}$. Edge e is represented in Γ_i as an arc c_e with endpoints $s(v_i)$ and $s(v_j)$. Arc c_e is replaced by the straight-line segment $\overline{t(v_i)s(v_j)}$; see also Figure 3.

Let $e_j = (v_i, v_{i_j})$ $(0 \leq j \leq h-1)$ be the edges of $E_{t,r}$ with $i_j < i_{j+1}$ $(0 \leq j < h-1)$. Let s' be a point on ℓ such that s' is to the right of $s(v_i)$ and no vertex or spine crossing is between $s(v_i)$ and s' on ℓ. Choose h points $p_0, p_1, \ldots, p_{h-1}$ such that each p_j $(0 \leq j \leq h-1)$ is between $s(v_i)$ and s' on ℓ and p_j is to the left of p_{j+1} along ℓ $(0 \leq j < h-1)$. Edge e_j is represented in Γ_i as an arc c_{e_j} with endpoints $s(v_i)$ and $s(v_{i_j})(0 \leq j \leq h-1)$. Arc c_{e_j} is replaced by the segment $\overline{t(v_i)p_j}$ and the arc with endpoints p_j and $s(v_{i_j})$; see also Figure 3.

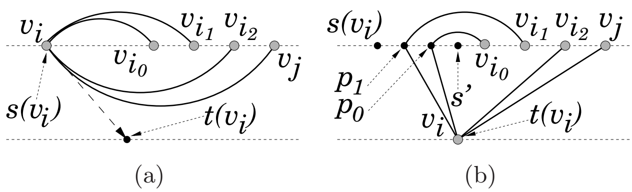

Fig. 3. Transformation of the shape of the edges incident on v_i

After n steps have been executed, and hence all vertices have been moved to their target positions, we obtain a drawing Γ_n where all vertices are aligned and have a left-to-right order coincident with λ. It can be proved that the drawing Γ_{i+1} obtained after the execution of Step i is a 2-spine drawing of G. It follows that Γ_n is a 2-spine drawing (and hence a topological book embedding) of G. □

By means of Lemma 3 and Theorem 2 the following results can be proved.

Lemma 4. *Let G be an n-colored planar graph with n vertices and let σ be an n-colored sequence compatible with G. G admits an augmenting n-colored Hamiltonian path consistent with σ and inducing at most $3n-3$ flat division vertices and at most 2 pointy division vertices per edge.*

Theorem 4. *Let G be a k-colored planar graph with n vertices such that $1 \leq k \leq n$ and let S be a k-colored set of points compatible with G. There exists an $O(n^2 \log n)$-time algorithm that computes a k-colored point-set embedding of G on S having curve complexity at most $3n+2$.*

Since by Theorem 1 k-colored point-set embeddings can have a linear number of edges each requiring a linear number of bends, the upper bound on the curve complexity expressed by Theorem 4 is asymptotically tight. However, as the next theorem shows, there can be special colorings of the input graph which guarantee a curve complexity that depends on k and does not depend on n.

Theorem 5. *Let G be a k-colored planar graph with n vertices such that: (i) $1 \leq k < n$; (ii) $|V_i| = 1$ for every $0 \leq i \leq k-2$; (iii) $|V_{k-1}| = n-k+1$. Let S*

be a k-colored set of points compatible with G. There exists an $O(n^2 \log n)$-time algorithm that computes a k-colored point-set embedding of G on S having curve complexity at most $9k - 1$.

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