On Group Decision Making, Consensus Reaching, Voting and Voting Paradoxes under Fuzzy Preferences and a Fuzzy Majority: A Survey and some Perspectives

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Abstract Group decision making, as meant in this chapter, is the following choice problem which proceeds in a multiperson setting. There is a group of individuals (decisionmakers, experts, ...) who provide their testimonies concerning an issue in question. These testimonies are assumed here to be individual preference relations over some set of option (alternatives, variants, ...). The problem is to find a solution, i.e. an alternative or a set of alternatives, from among the feasible ones, which best reflects the preferences of the group of individuals as a whole. We will survey main developments in group decision making under fuzziness. First, we will briefly outline some basic inconsistencies and negative results of group decision making and social choice, and show how they can be alleviated by some plausible modifications of underlying assumptions, mainly by introducing fuzzy preference relations and, to a lesser extent, a fuzzy majority. Then, we will concentrate on how to derive solutions under individual fuzzy preference relations, and a fuzzy majority equated with a fuzzy linguistic quantifier (e.g., most, almost all, ...) and dealt with in terms of a fuzzy logic based calculus of linguistically quantified statements or via the ordered weighted averaging (OWA) operators. We will briefly mention that one of solution concepts proposed can be a prototype for a wide class of group decision making choice functions. Then, we will discuss a related issue of how to define a "soft" degree of consensus in the group under individual fuzzy preference relations and a fuzzy majority. Finally, we will show how fuzzy preferences can help alleviate some voting paradoxes.

Key words: Fuzzy logic · Linguistic quantifier · Fuzzy preference relation · Fuzzy majority · Group decision making · Social choice · Consensus

1 Introduction

In this section we will first discus the very essence of group decision making and how fuzzy preferences and a fuzzy majority can help alleviate some inherent difficulties and make models more realistic. Then, we will briefly present some tools

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to be used, notably how to deal with linguistically quantified statements, and with a linguistic quantifier driven aggregation.

1.1 Group Decision Making

The essence of decision making, one of the most crucial and omnipresent human activities, is basically to find a best alternative (option, variant, ...) from among some feasible (relevant, available, ...) ones. It has been a subject of intensive research, notably formal, mathematical models have been devised to formalize the human rational behavior. Initially, this rationality has been equated with the maximization of some utility (value) function. Unfortunately, it has become more and more clear that the human behavior is rarely consistent with the maximization of a (expected) utility function, and some attempts to make decision making models more human consistent have been made, notably via a plausible modification of assumptions on, e.g., human preferences, axioms underlying the (expected) utility based approach, etc. – cf. Aizerman [1], many contributions in Kacprzyk and Fedrizzi [24], Kacprzyk and Roubens [51], Nurmi [34], etc. Potentials of fuzzy sets, in particular fuzzy relations, have been recognized quite early as well, cf. Blin [10], Blin and Whinston [11].

However, decision making in real world usually proceeds under multiple criteria, decisionmakers, stages, etc. In this paper we basically consider the case of multiperson decision making, more specifically group, practically from the perspective of social choice, under some fuzification of preferences and majority. We assume that there is a set of individuals who provide their testimonies assumed to be *preferences* over the set of alternatives. The problem is to find a *solution*, i.e. an alternative (or a set of alternatives) which is best acceptable by the group of individuals as a whole. For a different point of departure, involving choice sets or utility functions, we may refer the interested reader to, e.g., Kim [29], Salles [37], etc.

Since its very beginning group decision making has been plagued by negative results. Their essence is that no "rational" choice procedure satisfies all "natural", or plausible, requirements; so, each choice procedure has at least one serious drawback. By far the best known negative result is the so-called Arrow's impossibility theorem (cf. Arrow [2] or Kelly [56]). Another well known negative results are due to Gibbard and Satterthwaite, McKelvey, Schofield, etc. – cf. Nurmi [34]. Basically, all these negative results might be summarized as: no matter which group choice procedure we employed, it would satisfy one set of plausible conditions but not another set of equally plausible ones. Unfortunately, this general property pertains to all possible choice procedures, so that attempts to develop new, more sophisticated choice procedures do not seem very promising in this respect. Much more promising seems to be to modify some basic assumptions underlying the group decision making process. This line of reasoning is also basically assumed here.

A notable research direction is here based on the introduction of an *individual* and *social fuzzy preference relation*. Suppose that we have a set of $n \geq 2$ alternatives,

$S = \{s_1, \ldots, s_n\}$, and a set of $m \geq 2$ individuals, $E = \{1, \ldots, m\}$. Then, an individual's $k \in E$ individual fuzzy preference relation in $S \times S$ assigns a value in the unit interval for the preference of one alternative over another.

Normally, there are also some conditions to be satisfied, as, e.g., reflexivity, connectivity, (max-min) transitivity, etc. One should however note that it is not clear which of these "natural" properties of preference relations should be assumed. We will briefly discuss this issue in Sect. 2, but the interested reader should consult, e.g., Salles [37]. Moreover, a deep discussion is given in, e.g., Fodor and Roubens' [15], and also in De Baets et al.'s paper in this volume.

In this paper we assume that the individual and social fuzzy preference relations are defined in $S \times S$, i.e. assign to each pair of alternatives a strength of preference of one over another as a value from [0, 1]. Sometimes a better solution would be to assume the values of the strength of preference belonging to some ordered set (e.g. a set of linguistic values). This gives rise to some non-standard notions of soft preferences, orderings, etc. The best source for information on these and other related topics is Salles [37], and among the new approaches, the ones due to Herrera et al. [27, 28, 29, 30, 31, 32, 33] are particularly worth mentioning. The fuzzy preferences will be employed only instrumentally, and we will not discuss them and their properties in more detail.

Another basic element underlying group decision making is the concept of a *majority* – notice that a solution is to be an alternative (or alternatives) best acceptable by the group as a whole, that is by (at least!) *most* of its members since in practically no real nontrivial situation it would be accepted by all.

Some of the above mentioned problems, or negative result, with group decision making are closely related to too strict a representation of majority (e.g., at least a half, at least 2/3, ...). A natural line of reasoning is to somehow make that strict concept of majority closer to its real human perception by making it more vague. A good, often cited example in a biological context may be found in Loewer and Laddaga [62]:

> " ... It can correctly be said that there is a consensus among biologists that Darwinian natural selection is an important cause of evolution though there is currently no consensus concerning Gould's hypothesis of speciation. This means that there is a widespread agreement among biologists concerning the first matter but disagreement concerning the second ... "

and it is clear that a rigid majority as, e.g., more than 75% would evidently not reflect the essence of the above statement. However, it should be noted that there are naturally situations when a strict majority is necessary, for obvious reasons, as in all political elections.

A natural manifestations of such a "soft" majority are the so-called *linguistic quantifiers* as, e.g., most, almost all, much more than a half, etc. Such linguistic quantifiers can be, fortunately enough, dealt with by fuzzy-logic-based calculi of linguistically quantified statements as proposed by Zadeh [86]. Moreover, Yager's [84] ordered weighted averaging (OWA) operators can be used for this purpose (cf. Yager and Kacprzyk [85]), and also some other tools as, e.g., the Choquet integral.

In this paper we will present how fuzzy preference relations and fuzzy majorities can be employed for deriving solution of group decision making, and of degrees of consensus. We also mention some approaches to the alleviation of some voting paradoxes.

1.2 Fuzzy Linguistic Quantifiers and the Ordered Weighted Averaging (OWA) Operators for a Linguistic Quantifier Driven Aggregation

Our notation is standard. A fuzzy set A in $X = \{x\}$, will be characterized and equated with its membership function $\mu_A : X \longrightarrow [0, 1]$ such that $\mu_A(x) \in [0, 1]$ is the grade of membership of $x \in X$ in A, from full membership to full nonmembership, through all intermediate values. For a finite $X = \{x_1, \dots, x_n\}$ we write $A = \mu_A(x_1)/x_1 + \cdots + \mu_A(x_n)/x_n$. Moreover, we denote $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. Other, more specific notation will be introduced when needed.

A *linguistically quantified statement*, e.g. "most experts are convinced", may be generally written as

$$Qy\text{'s are } F \tag{1}$$

where Q is a linguistic quantifier (e.g., most), $Y = \{y\}$ is a set of objects (e.g., experts), and F is a property (e.g., convinced).

We may assign to the particular y's (objects) a different importance (relevance, competence, ...), B, which may be added to (1) yielding a *linguistically quantified statement with importance qualification* generally written as

$$QBy'\text{s are } F \tag{2}$$

which may be exemplified by "most (Q) of the important (B) experts (y's) are convinced (F)".

From our point of view, the main problem is to find the truth of such linguistically quantified statements, i.e. truth(Qy's are F) or truth(QBy's are F) knowing truth(y is F), for each $y \in Y$. One can use different calculi but we will consider Zadeh's [86] and Yager's [84] OWA operators based calculi only.

1.2.1 A Fuzzy-logic-based Calculus of Linguistically Quantified Statements

In Zadeh's [86] method, a fuzzy linguistic quantifier Q is assumed to be a fuzzy set defined in [0, 1]. For instance, Q = "most" may be given as

$$\mu_Q(x) = \begin{cases} 1 & \text{for } x \geq 0.8 \\ 2x - 0.6 & \text{for } 0.3 < x < 0.8 \\ 0 & \text{for } x \leq 0.3 \end{cases} \tag{3}$$

which may be meant as that if at least 80% of some elements satisfy a property, then *most* of them certainly (to degree 1) satisfy it, when less than 30% of them satisfy it, then *most* of them certainly do not satisfy it (satisfy to degree 0), and between 30% and 80% – the more of them satisfy it the higher the degree of satisfaction by *most* of the elements.

This is a *proportional* fuzzy linguistic quantifier (e.g., most, almost all, etc.), and we will deal with such quantifiers only since they are obviously more important for modeling a fuzzy majority than the absolute quantifiers (e.g., about 5, much more than 10, etc.).

Property F is defined as a fuzzy set in Y. For instance, if $Y = \{X, W, Z\}$ is the set of experts and F is a property "convinced", then F = "convinced" $= 0.1/X + 0.6/W + 0.8/Z$ which means that expert X is convinced to degree 0.1, W to degree 0.6 and Z to degree 0.8. If now $Y = \{y_1, \ldots, y_p\}$, then it is assumed that truth(y_i is F) $= \mu_F(y_i)$, $i = 1, \ldots, p$.

Then, we follow the two steps:

$$r = \frac{1}{p} \sum_{i=1}^{p} \mu_F(y_i) \tag{4}$$

$$\text{truth}(Qy\text{'s are } F) = \mu_Q(r) \tag{5}$$

In the case of importance qualification, B is defined as a fuzzy set in Y, and $\mu_B(y_i) \in [0, 1]$ is a degree of importance of y_i: from 1 for definitely important to 0 for definitely unimportant, through all intermediate values. We rewrite first "QBy's are F" as "$Q(B$ and $F)y$'s are B" which leads to the following counterparts of (4) and (5):

$$r' = \frac{\sum_{i=1}^{p} [\mu_B(y_i) \wedge \mu_F(y_i)]}{\sum_{i=1}^{p} \mu_B(y_i)} \tag{6}$$

$$\text{truth}(QBY'\text{s are } F) = \mu_Q(r') \tag{7}$$

Example 1. Let Y = "experts" $= \{X, Y, Z\}$, F = "convinced" $= 0.1/X + 0.6/Y + 0.8/Z$, Q = "most" be given by (3), B = "important" $= 0.2/X + 0.5/Y + 0.6/Z$. Then: $r = 0.5$ and $r' = 0.92$, and truth("most experts are convinced")=0.4 and truth("most of the important experts are convinced")=1.

The method presented is simple and efficient, and has proven to be useful in a multitude of cases, also in this paper.

1.2.2 Ordered Weighted Averaging (OWA) Operators

Yager [84] (see also Yager and Kacprzyk's [85]) has proposed a special class of aggregation operators, called the *ordered weighted averaging* (or OWA, for short) operators, which seem to provide an even better and more general aggregation in the sense of being able to simply and uniformly model a large class of fuzzy linguistic quantifiers.

An OWA operator of dimension p is a mapping $F : [0, 1]^p \to [0, 1]$ if associated with F is a weighting vector $W = [w_1, \ldots, w_p]^T$ such that: $w_i \in [0, 1]$, $w_1 + \cdots + w_p = 1$, and

$$F(x_1, \ldots, x_p) = w_1 b_1 + \cdots + w_p b_p \tag{8}$$

where b_i is the i-th largest element among $\{x_1, \ldots, x_p\}$. B is called an ordered argument vector if each $b_i \in [0, 1]$, and $j > i$ implies $b_i \geq b_j$, $i = 1, \ldots, p$.

Then

$$F(x_1, \ldots, x_p) = W^T B \tag{9}$$

Example 2. Let W^T = [0.2, 0.3, 0.1, 0.4], and calculate F(0.6, 1.0, 0.3, 0.5). Thus, B^T = [1.0, 0.6, 0.5, 0.3], and F(0.6, 1.0, 0.3, 0.5) = $W^T B$ = 0.55; and F(0.0, 0.7, 0.1, 0.2) = 0.21.

For us it is relevant how the OWA weights are found from the membership function of a fuzzy linguistic quantifier Q; an early approach given in Yager [84] may be used here:

$$w_k = \mu_Q(k/p) - \mu_Q((k-1)/p) \quad \text{for} \quad k = 1, \ldots, p \tag{10}$$

Some examples of the w_i's associated with the particular quantifiers are:

- If $w_p = 1$, and $w_i = 0$, for each $i \neq p$, then this corresponds to Q = "all";
- If $w_i = 1$ for $i = 1$, and $w_i = 0$, for each $i \neq 1$, then this corresponds to Q = "at least one",
- If $w_i = 1/p$, for each $i = 1, 2, \ldots, p$, then this corresponds to the arithmetic mean,

and the intermediate cases as, e.g., a half, most, much more than 75%, a few, almost all, etc. may be obtained by a suitable choice of the w_i's between the above two extremes.

Thus, we will write

$$\text{truth}(Qy'\text{s are } F) = \text{OWA}_Q(\text{truth } (y_i \text{ is } F)) = W^T B \tag{11}$$

An important, yet difficult problem is the OWA operators with importance qualification. Suppose that we have a vector of data (pieces of evidence) $A = [a_1, \ldots, a_n]$, and a vector of importances $V = [v_1, \ldots, v_n]$ such that $v_i \in [0, 1]$ is the importance of a_i, $i = 1, \ldots, n$, ($v_1 + \cdots + v_n \neq 1$, in general), and the OWA weights $W = [w_1, \ldots, w_n]^T$ corresponding to Q is determined via (10).

The case of an OWA operator with importance qualification, OWA$_V$, is unfortunately not trivial. In a popular Yager's [84] approach to be used here, the problem boils down to some redefinition of the OWA's weights w_i into $\overline{w}_i$. Then, (8) becomes

$$F_V(a_1, \ldots, a_n) = \overline{W}^T \cdot B = \sum_{j=1}^{n} \overline{w}_j b_j \tag{12}$$

We order first the pieces of evidence a_i, $i = 1, \ldots, n$, in descending order to obtain B such that b_j is the j-th largest element of $\{a_1, \ldots, a_n\}$. Next, we denote by u_j the importance of b_j, i.e. of the a_i which is the j-th largest; $i, j = 1, \ldots, n$. Finally, the new weights $\overline{W}$ are defined as

$$\overline{w}_j = \mu_Q\left(\frac{\sum_{k=1}^{j} u_k}{\sum_{k=1}^{n} u_k}\right) - \mu_Q\left(\frac{\sum_{k=1}^{j-1} u_k}{\sum_{k=1}^{n} u_k}\right) \tag{13}$$

Example 3. Suppose that $A = [a_1, a_2, a_3, a_4]$ = [0.7, 1, 0.5, 0.6], and $V = [u_1, u_2, u_3, u_4]$ = [1, 0.6, 0.5, 0.9]. Q = "most" is given by (3).

Then, $B = [b_1, b_2, b_3, b_4] = [1, 0.7, 0.6, 0.5]$, and $\overline{W} = [0.04, 0.24, 0.41, 0.31]$, and $F_I(A) = \sum_{j=1}^{4} \overline{w}_j b_j = 0.067 \cdot 1 + 0.4 \cdot 0.7 + 0.333 \cdot 0.6 + 0.2 \cdot 0.5 = 0.6468$.

We have now the necessary formal means to proceed to our discussion of group decision making and consensus formation models under fuzzy preferences and a fuzzy majority.

Finally, let us mention that OWA-like aggregation operators may be defined in an ordinal setting, i.e. for non-numeric data (which are only ordered), and we will refer the interested reader to, e.g., Delgado, Verdegay and Vila [17] or Herrera, Herrera-Viedma and Verdegay [28], and some other of their later papers.

2 Group Decision Making under Fuzzy Preferences and a Fuzzy Majority: General Remarks

Group decision making (equated here with social choice) proceeds in the following setting. We have a set of $n \geq 2$ alternatives, $S = \{s_1, \ldots, s_n\}$, and a set of $m \geq 2$ individuals, $E = \{1, \ldots, m\}$. Each individual $k \in E$ provides his or her testimony as to the alternatives in S, assumed to be individual fuzzy preference relations defined over S (i.e. in $S \times S$). Fuzzy preference relations are employed to reflect an omnipresent fact that the preferences may be not clear-cut so that conventional non-fuzzy preference relations may be not adequate (see, e.g., many articles in Kacprzyk and Roubens [51] or Kacprzyk, Nurmi and Fedrizzi [27]).

An *individual fuzzy preference relation* of individual k, R_k, is given by its membership function $\mu_{R_k} : S \times S \longrightarrow [0, 1]$ such that

$$\mu_{R_k}(s_i, s_j) = \begin{cases} 1 & \text{if } s_i \text{ is definitely preferred to } s_j \\ c \in (0.5, 1) & \text{if } s_i \text{ is slightly preferred to } s_j \\ 0.5 & \text{in the case of indifference} \\ d \in (0, 0.5) & \text{if } s_j \text{ is slightly preferred to } s_i \\ 0 & \text{if } s_j \text{ is definitely preferred to } s_i \end{cases} \quad (14)$$

We will also use a special type of an individual fuzzy preference relation, a *fuzzy tournament*, but this will be explained later on.

If card S is small enough (as assumed here), an individual fuzzy preference relation of individual k, R_k, may conveniently be represented by an $n \times n$ matrix $R_k = [r_{ij}^k]$, such that $r_{ij}^k = \mu_{R_k}(s_i, s_j)$; $i, j = 1, \ldots, n$; $k = 1, \ldots, m$. R_k is commonly assumed (also here) to be reciprocal in that $r_{ij}^k + r_{ji}^k = 1$; moreover, it is also normally assumed that $r_{ii}^k = 0$, for all i, k; for a different, more justified convention, cf. García-Lapresta and Llamazares [16]. Notice that we do not mention here other properties of (individual) fuzzy preference relations which are often discussed (cf. Salles [37]) but which will not be relevant to our discussion. Moreover, we will not use here a more sophisticated concept of a fuzzy preference systems proposed by De Baets et al. which is presented in their paper in this volume. The reasoning is in this case principally the same.

Basically, two lines of reasoning may be followed here (cf. Kacprzyk [36, 37, 38, 39, 19, 41]):

- a direct approach: $\{R_1, \ldots, R_m\} \longrightarrow$ solution, that is, a solution is derived directly (without any intermediate steps) just from the set of individual fuzzy preference relations, and
- an indirect approach: $\{R_1, \ldots, R_m\} \longrightarrow R \longrightarrow$ solution, that is, from the set of individual fuzzy preference relations we form first a social fuzzy preference relation, R (to be defined later), which is then used to find a solution.

A solution is here, unfortunately, not clearly understood – see, e.g., Nurmi [33, 69, 70, 71, 34] for diverse solution concepts. In this paper we will only sketch the derivation of some more popular solution concepts, and this will show to the reader not only the essence of the particular solution concept but how a fuzzification may be performed so that the reader can eventually fuzzify other crisp solution concepts that may be found in the literature. More specifically, we will show the derivation of some fuzzy cores and minimax sets for the direct approach, and some fuzzy consensus winners for the indirect approach. In addition to fuzzy preference relations, which are usually employed, we will also use a fuzzy majority represented by a linguistic quantifier as proposed by Kacprzyk [36, 37, 38, 39, 19, 41].

First, we will consider the case of fuzzy preferences only, and then we will add a fuzzy majority which is a more interesting case for our purposes.

3 Group Decision Making under Fuzzy Preferences

In this section we will only assume that we have individual fuzzy preferences and a non-fuzzy majority. We will present some solution concepts that are derived using the above mentioned direct and indirect approach, i.e. directly from individual fuzzy preference relations or via a social preference relation.

3.1 Solutions Based on Individual Fuzzy Preference Relations

Let us first consider solution concepts that do not require any preference aggregation at all. One of the best solution concepts is that of a core or a set of undominated alternatives. Suppose that the nonfuzzy required majority be r (e.g., at least 50%).

Definition 1. An alternative $x \in S$ belongs to the *core* iff there is no other alternative $y \in S$ that defeats x by the required majority r.

We can extend the notion of a core to cover fuzzy individual preference relations by defining *the fuzzy α-core* as follows (cf. Nurmi [33]):

Definition 2. An alternative $s_i \in S$ belongs to the *fuzzy α-core* S_α iff there exists no other alternative $s_j \in S$ such that $r_{ji} > \alpha$ for at least r individuals.

It is easy to see that if the nonfuzzy core is nonempty, so is S_α for some $\alpha \in (0, 1]$. In other words, $\exists \alpha \in (0, 1]$: core $\subset S_\alpha$. Moreover, for any two values $\alpha_1, \alpha_2 \in (0, 1]$ such that $\alpha_1 < \alpha_2$, we have:

$$S_{\alpha_1} \subseteq S_{\alpha_2}$$

The intuitive interpretation of the fuzzy α-core is obvious: an alternative is a member of S_α iff a sufficient majority of voters does not feel strongly enough against it.

Another nonfuzzy solution concept with much intuitive appeal is a minimax set. In a nonfuzzy setting it is defined as follows:

Definition 3. For each $x, y \in S$ denote the number of individuals preferring x to y by $n(x, y)$. Then define

$$v(x) = \max_y n(y, x)$$

and

$$n* = \min_x v(x)$$

Now the minimax set is

$$Q(n*) = \{x \mid v(x) = n*\}$$

Thus, $Q(n*)$ consists of those alternatives that in pairwise comparison with any other alternative are defeated by no more than $n*$ votes. Obviously, if $n* < m/2$, where m is the number of individuals, then $Q(n*)$ is singleton and $x \in Q(n*)$ is the core if the simple majority rule is being applied.

Analogously, we can define a *the minimax degree set* $Q(\beta)$ as follows. Given $s_i, s_j \in S$ and let, for individuals $k = 1, \ldots, m$:

$$v_D^k(x_j) = \max_i r_{ij}$$

We now define

$$v_D(x_j) = \max_k v_D^k(x_j)$$

Let $\min_j v_D(x_j) = \beta$. Then

$$Q(\beta) = \{x_j \mid v_D(x_j) = \beta\}$$

For properties of the minimax degree set, we refer to Nurmi [33, 69, 70].

Another concept that is analogous to the nonfuzzy minimax set is a *minimax opposition set*. Let n_{ij} be the number of those individuals for whom $r_{ij} > r_{ji}$ and let $v_f(x_j) = \max_i n_{ij}$. Denote by $\bar{v}_f$ the minimum of $v_f(x_j)$ with respect to j, i.e.

$$\bar{v}_f = min_j v_f(x_j)$$

Then: $Q(v_f) = \{x_j \mid v_f(x_j) = \bar{v}_f\}$.

But, clearly, $Q(v_f) = Q(n*)$ since $r_{ij} > r_{ji}$ implies that the individual prefers alternative x_i to x_j. Similarly, the preference of x_i over x_j implies that $r_{ij} > r_{ji}$. Consequently, the minimax opposition set does not take into account the intensity of preferences as expressed in the individual preference relation matrices.

A more general solution concept, the *α-minimax set* (cf. Nurmi [33]) denoted $Q^\alpha(v_f^\alpha)$, is defined as follows. Let $n_\alpha(x_i, x_j)$ be the number of individuals for whom $r_{ij} \leq \alpha$ for some value of $\alpha \in [0, 0.5)$. We now define $\forall x_i \in S : v_f^\alpha(x_i) = \max_j n_\alpha(x_i, x_j)$ and $\bar{v}_f^\alpha = \min_i v_f^\alpha(x_i)$. Then

$$Q^\alpha(v_f^\alpha) = \{x_i \mid v_f^\alpha(x_i) = \bar{v}_f^\alpha\}$$

It can be shown that $Q^\alpha(v_f^\alpha) \subseteq Q(n*)$ (see [33]).

3.1.1 Fuzzy Tournaments

One purpose of studying fuzzy tournaments is to overcome the difficulties inherent in the use of conventional solution concepts, namely the fact that the latter tend to produce too large solution sets and are therefore not decisive enough. Another purpose of our discussion is to apply analogues of the nonfuzzy solutions to contexts where the opinions of individuals can be represented by more general constructs than just connected and transitive preference relations (cf., e.g., [51]).

Let us take a look at a few solution concepts of nonfuzzy tournaments, mostly those proposed by Nurmi and Kacprzyk [35].

Definition 4. Given the alternative set S, a tournament P on S is a complete and asymmetric relation on S.

In the context of group decision making P can be viewed as a strict preference relation. When S is of small cardinality, P can be expressed as a matrix $[p_{ij}]$, $p_{ij} \in \{0, 1\}$ so that $p_{ij} = 1$ if the alternative represented by row i is preferred to that represented by column j, and $p_{ij} = 0$ if the alternative represented by column j is preferred to that represented by row i.

Suppose that each individual has a complete, transitive and asymmetric preference relation over S, and that the number of individuals is odd. Then a tournament can be constructed through pairwise comparisons of alternatives. In the ensuing tournament alternative s_i is preferred to s_j iff the number of individuals preferring the former to the latter is larger than the number of individual preferring s_j to s_i.

Perhaps the best-known solution concept of tournaments is the Condorcet winner.

Definition 5. The *Condorcet winner* is an alternative which is preferred to all other alternatives by a majority.

The main problem with this solution concept is that it does not always exist.

Definition 6. The *Copeland winning set* UC_C consists of those alternatives that have the largest number of 1s in their corresponding rows in the tournament matrix.

In other words, the Copeland winners defeat more alternatives than any other alternatives do.

Definition 7. The uncovered set is defined by means of a binary relation of covering. An alternative s_i *covers* another alternative s_j iff s_i defeats s_j and everything that s_j defeats. The *uncovered set* consists of those alternatives that are covered by no alternatives.

Definition 8. The *Banks set* is the set of end-points of Banks chains. Starting from any alternative s_i the *Banks chain* is constructed as follows. First one looks for an alternative that defeats s_i. Suppose that such an alternative exists and is s_j (if one does not exist, then of course s_i is the Condorcet winner). Next one looks for another alternative that defeats both s_i and s_j, etc. Eventually, no alternative can be found that would defeat all previous ones in the chain starting from s_i. The last alternative which defeats all previous ones is the end-point of the Banks chain starting from s_i. The Banks set is then the set of all those end points.

The following relationships hold between the above mentioned solutions (cf. [34]):

- all solutions converge to the Condorcet winner when one exists,
- the uncovered set includes the Copeland winning set and the Banks set,

- when S contains less than 7 elements, the uncovered set and the Banks set coincide, and
- when the cardinality of S exceeds 12, the Banks set and the Copeland winning set may be distinct; however, they both always belong to the uncovered set.

Given a group E of m individuals, a collective fuzzy tournament $F = [r_{ij}]$ can be obtained through pairwise comparisons of alternatives so that

$$r_{ij} = \frac{card\{k \in E \mid s_i P_k s_j\}}{m}$$

where P_k is a nonfuzzy tournament representing the preferences of individual k.

Let us now define a *strong fuzzy covering relation* $C_S \subset S \times S$ as follows

$$\forall i, j, l \in \{1, \ldots, n\} : s_i C_S s_j \Leftrightarrow r_{il} \geq r_{jl} \quad \& \quad r_{ij} > r_{ji}$$

Clearly, the strong fuzzy covering relation implies the nonfuzzy covering relation, but not *vice versa*. The set of C_S-undominated alternatives is denoted by UC_S.

Let us first define:

Definition 9. The *weak fuzzy covering relation* $C_W \subset S \times S$ is defined as follows:

$$\begin{aligned} &\forall s_i, s_j \in S : \\ &\quad s_i C_W s_j \Leftrightarrow r_{ij} > r_{ji} \\ &\quad \& \quad card\{s_l \in S : r_{il} > r_{jl}\} \geq card\{s_p \in S : r_{jp} > r_{ip}\} \end{aligned}$$

Obviously, $s_i C_S s_j$ implies $s_i C_W s_j$, but not conversely. Thus, the set of C_W-undominated alternatives, UC_W, is always a subset of UC_S. Moreover, the Copeland winning set is always included in UC_S, but not necessarily in UC_W (see [35]).

If one is looking for a solution that is a plausible subset of the uncovered set, then UC_W is not appropriate since it is possible that UC_C is not always a subset of the uncovered set, let alone the Banks set.

Another solution concept, the α-uncovered set, is based on the individual fuzzy preference tournament matrices. One first defines the fuzzy domination relation D and an α-covering relation $C_\alpha \subseteq S \times S$ as follows.

Definition 10. $s_i D s_j$ iff at least 50% of the individuals prefer s_i to s_j to a degree of at least 0.5.

Definition 11. If $s_i C_\alpha s_j$, then $s_i D s_j$ and $s_i D_\alpha s_k$, for all $s_k \in S$ for which $s_j D_\alpha s_k$.

Definition 12. The α-uncovered set consists of those alternatives that are not α-covered by any other alternative.

An obvious candidate for a plausible solution concept for fuzzy tournaments is an α-uncovered set with the smallest value of α.

Other fuzzy solution concepts analogous to their nonfuzzy counterparts can be defined (see Nurmi and Kacprzyk [35]). For example, the *α-Banks set* can be constructed by imposing the restriction that the majority of voters prefer the next alternative to the previous one in the Banks chain with intensity of at least α.

3.2 Solutions Based on a Social Fuzzy Preference Relation

The derivation of these solution concepts requires first a derivation of a social fuzzy preference relation.

Bezdek, Spillman and Spillman [8, 9] discuss the problem of finding the set of undominated alternatives or other stable outcomes given a collective fuzzy preference ordering over the alternative set; see also Nurmi [33].

We now define a couple of solution concepts for voting games with fuzzy collective preference relation.

Definition 13. The set S_α of *α-consensus winners* is defined as: $s_i \in S_\alpha$ iff $\forall s_j \neq s_i : r_{ij} \geq \alpha$, with $0.5 < \alpha \leq 1$

Whenever S_α is nonempty, it is a singleton, but it does not always exist. Thus, it may be useful to find other solution concepts that specify a nonempty alternative sets even when S_α is empty. One possible candidate is a straightforward extension of Kramer's minimax set. We call it a set of *minimax consensus winners*, denote it by S_M and define as follows.

Definition 14. Let $\bar{r}_j = \max_i r_{ij}$ and $\bar{r} = \min_j \max_i r_{ij}$. Then $s_i \in S_M$ (the set of minimax consensus winners) iff $\bar{r}_i = \bar{r}$.

Clearly S_M is always nonempty, but not necessarily a singleton. As a solution set it has the same interpretation as Kramer's minimax set: it consists of those alternatives which, when confronted with their toughest competitors, fare best, i.e. win by the largest score (if $\bar{r} \leq 0.5$) or lose by the smallest one (if $\bar{r} > 0.5$).

These solution concepts are based on the social preference relation matrix. Other ones can be obtained in several ways. For instance, one may start from a preference profile over a set of alternatives and construct the $[r_{ij}]$ matrix as follows:

$$r_{ij} = \begin{cases} \frac{1}{m} \Sigma_{k=1}^{m} a_{ij}^{k} & \text{for } i \neq j \\ r_{ij} = 0 & \text{for } i = j \end{cases}$$

where $a_{ij}^k = 1$ if s_i is strictly preferred to s_j by voter k, and $a_{ij}^k = 0$ otherwise.

There is nothing "fuzzy" in the above solutions. As the method of constructing the social preference relation matrix suggests, the starting point can just be the ordinary preference profile as well.

4 Group Decision Making under Fuzzy Preferences and a Fuzzy Majority

In this section we will consider some solution concepts of group decision making but when we both have fuzzy preference relations and a fuzzy majority, We will also follow here the two directions, i.e. by using the individual fuzzy preference relations only (a direct approach), and by deriving first a social fuzzy preference relation, and using it to derive solutions (an indirect approach).

4.1 Direct Derivation of a Solution

We will first employ the direct approach, i.e. $\{R_1, \ldots, R_m\} \longrightarrow$ solution to derive two popular solution concepts: fuzzy cores and minimax sets.

4.1.1 Fuzzy Cores

The core is a very intuitively appealing and often used solution concept. Conventionally, the core is defined as a set of *undominated alternatives*, i.e. those not defeated in *pairwise comparisons* by a required majority (strict!) $r \leq m$, i.e.

$$C = \{s_j \in S : \neg \exists s_i \in S \text{ such that } r_{ij}^k > 0.5 \text{ for at least } r \text{ individuals}\} \tag{15}$$

The first attempt at a fuzzification of the core is due to Nurmi [33] who has extended it to the *fuzzy* α*-core* defined as

$$C_\alpha = \{s_j \in S : \neg \exists s_i \in S \text{ such that } r_{ij}^k > \alpha \geq 0.5 \text{ for at least } r \text{ individuals}\} \tag{16}$$

that is, as a set of alternatives not sufficiently (at least to degree α) defeated by the required (still strict!) majority $r \leq m$.

As we have already indicated, in many group decision making related situations is may be more adequate to assume that the required majority is imprecisely specified as, e.g., given by a fuzzy linguistic quantifier as, say, *most*. This concept of a fuzzy majority has been proposed by Kacprzyk [36, 37, 38, 39, 19, 41], and it has turned out that it can be quite useful and adequate.

To employ a fuzzy majority to extend (fuzzify) the core, we start by denoting

$$h_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k < 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{17}$$

where here and later on in this section, if not otherwise specified, $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$.

Thus, h_{ij}^k just reflects if alternative s_j defeats (in pairwise comparison) alternative s_i ($h_{ij}^k = 1$) or not ($h_{ij}^k = 0$).

Then, we calculate

$$h_j^k = \frac{1}{n-1} \sum_{i=1, i \neq j}^{n} h_{ij}^k \tag{18}$$

which is clearly the extent, from 0 to 1, to which individual k is not against alternative s_j, where 0 standing for definitely against to 1 standing for definitely not against, through all intermediate values.

Next, we calculate

$$h_j = \frac{1}{m} \sum_{k=1}^{m} h_j^k \tag{19}$$

which expresses to what extent, from 0 to 1 as in the case of (18), *all* the individuals are not against alternative s_j.

And, finally, we calculate

$$v_Q^j = \mu_Q(h_j) \tag{20}$$

is to what extent, from 0 to 1 as before, Q (say, most) individuals are not against alternative s_j.

The *fuzzy Q-core* is now defined (cf. Kacprzyk [36, 37, 38, 39, 19, 41]) as a fuzzy set

$$C_Q = v_Q^1/s_1 + \cdots + v_Q^n/s_n \tag{21}$$

i.e. as a fuzzy set of alternatives that are not defeated by Q (say, most) individuals.

Notice that in the above basic definition of a fuzzy Q-core we do not take into consideration to what degrees those defeats of one alternative by another are. They can be accounted for in a couple of plausible ways.

First and most straightforward is the introduction of a threshold into the degree of defeat in (17), for instance by denoting

$$h_{ij}^k(\alpha) = \begin{cases} 1 \text{ if } r_{ij}^k < \alpha \leq 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{22}$$

where, again, $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$. Thus, $h_{ij}^k(\alpha)$ just reflects if alternative s_j sufficiently (i.e. at least to degree $1 - \alpha$) defeats (in pairwise comparison) alternative s_i or not.

We can also explicitly introduce the strength of defeat into (17). Namely, we can introduce a function exemplified by

$$\hat{h}_{ij}^k = \begin{cases} 2(0.5 - r_{ij}^k) & \text{if } r_{ij}^k < 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

where, again, $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$. Thus, $\hat{h}_{ij}^k$ just reflects how strongly (from 0 to 1) alternative s_j defeats (in pairwise comparison) alternative s_i.

Then, by following the same steps (18) – (21), we can derive an α/Q-fuzzy core and an s/Q-fuzzy core.

Example 4. Suppose that we have four individuals, $k = 1, 2, 3, 4$, whose individual fuzzy preference relations are:

$R_1 =$

	$j=1$	2	3	4
$i=1$	0	0.3	0.7	0.1
2	0.7	0	0.6	0.6
3	0.3	0.4	0	0.2
4	0.9	0.4	0.8	0

$R_2 =$

	$j=1$	2	3	4
$i=1$	0	0.4	0.6	0.2
2	0.6	0	0.7	0.4
3	0.4	0.3	0	0.1
4	0.8	0.6	0.9	0

$R_3 =$

	$j=1$	2	3	4
$i=1$	0	0.5	0.7	0.1
2	0.5	0	0.8	0.4
3	0.3	0.2	0	0.2
4	1	0.6	0.8	0

$R_4 =$

	$j=1$	2	3	4
$i=1$	0	0.4	0.7	0.8
2	0.6	0	0.4	0.3
3	0.3	0.6	0	0.1
4	0.7	0.7	0.9	0

Suppose now that the fuzzy linguistic quantifier is $Q =$ "most" defined by (3). Then, say:

$$C_{\text{“most”}} \cong 0.06/s_1 + 0.56/s_2 + 1/s_4$$
$$C_{0.3/\text{“most”}} \cong 0.56/s_4$$
$$C_{s/\text{“most”}} \cong 0.36/s_4$$

to be meant as follows: in case of $C_{\text{“most”}}$ alternative s_1 belongs to to the fuzzy Q-core to the extent 0.06. s_2 to the extent 0.56, and s_4 to the extent 1, and analogously for the $C_{0.3/\text{“most”}}$ and $C_{s/\text{“most”}}$. Notice that though the results obtained for the particular cores are different, for obvious reasons, s_4 is clearly the best choice which is evident if we examine the given individual fuzzy preference relations.

Clearly, the fuzzy linguistic quantifier based aggregation of partial scores in the above definitions of the fuzzy Q-core, α/Q-core and s/Q-core, may be replaced by an ordered weighted averaging (OWA) operator based aggregation given by (10) and (11). This was proposed by Fedrizzi, Kacprzyk and Nurmi [19], and then followed by some other authors. The results obtained by using the OWA operators are similar to those for the usual fuzzy linguistic quantifiers.

Finally, let us notice that the individuals and alternatives may be assigned variable importance (competence) and relevance, respectively, and then the OWA based aggregation with importance qualification may be used. This will not change however the essence of the fuzzy cores defined above, and will not be discussed here for lack of space.

4.1.2 Minimax Sets

Another intuitively justified solution concept may be the minimax (opposition) set which may be defined for our purposes as follows.

Let $w(s_i, s_j) \in \{1, 2, \ldots, m\}$ be the number of individuals who prefer alternative s_j to alternative s_i, i.e. for whom $r_{ij}^k < 0.5$.

If now

$$v(s_i) = \max_{j=1,\dots,n} w(s_i, s_j) \tag{24}$$

and

$$v^* = \min_{i=1,\dots,n} v(s_i) \tag{25}$$

then the *minimax set* is defined as

$$M(v^*) = \{s_i \in S : v(s_i) = v^*\} \tag{26}$$

i.e. as a (nonfuzzy) set of alternatives which in pairwise comparisons with any other alternative are defeated by no more than v^* individuals, hence by the least number of individuals.

Nurmi [33] extends the minimax set, similarly in spirit to his extension of the core (16), to the *α-minimax set* as follows. Let $w_\alpha(s_i, s_j) \in \{1, 2, \dots, m\}$ be the number of individuals who prefer alternative s_j to alternative s_i at least to degree $1 - \alpha$, i.e. for whom $r_{ij}^k < \alpha \leq 0.5$.

If now

$$v_\alpha(s_i) = \max_{j=1,\dots,n} w_\alpha(s_i, s_j) \tag{27}$$

and

$$v_\alpha^* = \min_{i=1,\dots,n} v_\alpha(s_i) \tag{28}$$

then the *α-minimax set* is defined as

$$M_\alpha(v_\alpha^*) = \{s_i \in S : v_\alpha(s_i) = v_\alpha^*\} \tag{29}$$

i.e. as a (nonfuzzy) set of alternatives which in pairwise comparisons with any other alternative are defeated (at least to degree $1 - \alpha$) by no more than v^* individuals, hence by the least number of individuals.

A fuzzy majority was introduced into the above definitions of minimax sets by Kacprzyk [36, 37, 38, 39, 19] as follows.

We start with (17), i.e.

$$h_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k < 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{30}$$

and

$$h_i^k = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} h_{ij}^k \tag{31}$$

is the extent, between 0 and 1, to which individual k is against alternative s_i.

Then

$$h_i = \frac{1}{m} \sum_{k=1}^{m} h_i^k \tag{32}$$

is the extent, between 0 and 1, to which all the individuals are against alternative s_i.

Next

$$t_i^Q = \mu_Q(h_i) \tag{33}$$

is the extent, from 0 to 1, to which Q (say, most) individuals are against alternative s_i, and

$$t_Q^* = \min_{i=1,\ldots,n} t_i^Q \tag{34}$$

is the least defeat of any alternative by Q individuals.

Finally, the Q-minimax set is

$$M_Q(t_Q^*) = \{s_i \in S : t_i^Q = t_Q^*\} \tag{35}$$

And analogously as for the α/Q-core and s/Q-core, we can explicitly introduce the degree of defeat $\alpha < 0.5$ and s into the definition of the Q-minimax set.

Example 5. For the same four individual fuzzy preference relations $R_1, \ldots, R_4$ as in Example 4, we obtain for instance:

$$M_{\text{“most”}}(0) = \{s_4\}$$
$$M_{0.3/\text{“most”}}(0) = \{s_1, s_2, s_4\}$$
$$M_{s/\text{“most”}} = \{s_1, s_2, s_4\}$$

The OWA based aggregation can also be employed for the derivation of fuzzy minimax sets given above. And, again, the results obtained by using the OWA based aggregation are similar to those obtained by directly employing Zadeh's [86] calculus of linguistically quantified statements.

4.2 Indirect Derivation of a Solution – the Consensus Winner

Now we follow the scheme: $\{R_1, \ldots, R_m\} \longrightarrow R \longrightarrow$ solution i.e. from the individual fuzzy preference relations we determine first a social fuzzy preference relation, R, which is similar in spirit to its individual counterpart but concerns the whole group of individuals, and then find a solution from such a social fuzzy preference relation.

It is easy to notice that the above direct derivation scheme involves in fact two problems:

- how to find a social fuzzy preference relation from the individual fuzzy preference relations, i.e.

$$\{R_1, \ldots, R_m\} \longrightarrow R$$

- how to find a solution from the social fuzzy preference relation, i.e.

$$R \longrightarrow \text{ solution}$$

In this paper we will not deal in more detail with the first step, i.e. $\{R_1, \ldots, R_m\} \longrightarrow R$, and assume a (most) straightforward alternative that the social fuzzy preference relation $R = [r_{ij}]$ is given by

$$r_{ij} = \begin{cases} \frac{1}{m} \sum_{k=1}^{m} a_{ij}^k & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

where

$$a_{ij}^k = \begin{cases} 1 \text{ if } r_{ij}^k > 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{37}$$

Notice that R obtained via (36) need not be reciprocal, i.e. $r_{ij} \neq 1 - r_{ji}$, but it can be shown that $r_{ij} \leq 1 - r_{ji}$, for each $i, j = 1, \ldots, n$.

We will discuss now the second step, i.e. $R \longrightarrow$ solution, i.e. how to determine a solution from a social fuzzy preference relation.

A solution concept of much intuitive appeal is here the consensus winner (cf. Nurmi [33]) which will be extended under a social fuzzy preference relation and a fuzzy majority.

We start with

$$g_{ij} = \begin{cases} 1 \text{ if } r_{ij} > 0.5 \\ 0 \text{ otherwise} \end{cases} \tag{38}$$

which expresses whether alternative s_i defeats (in the whole group's opinion!) alternative s_j or not.

Next

$$g_i = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} g_{ij} \tag{39}$$

which is a mean degree to which alternative s_i is preferred, by the whole group, over all the other alternatives.

Then

$$z_Q^i = \mu_Q(g_i) \tag{40}$$

is the extent to which alternative s_i is preferred, by the whole group, over Q (e.g., most) other alternatives.

Finally, we define the *fuzzy Q-consensus winner* as

$$W_Q = z_Q^1/s_1 + \cdots + z_Q^n/s_n \tag{41}$$

i.e. as a fuzzy set of alternatives that are preferred, by the whole group, over Q other alternatives.

And analogously as in the case of the core, we can introduce a threshold $\alpha \geq 0.5$ and s into (38) and obtain a *fuzzy* α/Q*-consensus winner* and a *fuzzy* s/Q*-consensus winner*, respectively.

Example 6. For the same individual fuzzy preference relations as in Example 4, and using (36) and (37), we obtain the following social fuzzy preference relation

$R =$

	$j=1$	2	3	4
$i=1$	0	0	1	0.25
2	0.75	0	0.75	0.25
3	0	0.25	0	0
4	1	0.75	1	0

If now the fuzzy majority is given by $Q =$ "most" defined by (3) and $\alpha = 0.8$, then we obtain

$$W_{\text{"most"}} = \tfrac{1}{15}/s_1 + \tfrac{11}{15}/s_2 + 1/s_4$$
$$W_{0.8/\text{"most"}} = \tfrac{1}{15}/s_1 + \tfrac{11}{15}/s_4$$
$$W_{s/\text{"most"}} = \tfrac{1}{15}/s_1 + \tfrac{1}{15}/s_2 + 1/s_4$$

which is to be read similarly as for the fuzzy cores in Example 4. Notice that here once again alternative s_4 is clearly the best choice which is obvious by examining the social fuzzy preference relation.

One can also use here an OWA based aggregation defined by (10) and (11) as proposed by proposed by Fedrizzi and Kacprzyk [19] and Kacprzyk and Fedrizzi [47].

This concludes our brief exposition of how to employ fuzzy linguistic quantifiers to model the fuzzy majority in group decision making. We did not present some other solution concepts as, e.g., minimax consensus winners (cf. Nurmi [33], Kacprzyk [38]) or those based on fuzzy tournaments which have been proposed by Nurmi and Kacprzyk [35] and are mentioned earlier in this paper.

We will finish this section with a remark that in a number of recent papers by Kacprzyk and Zadrożny [52, 53] it has been shown that the concept of Kacprzyk's [36, 37] fuzzy Q-core can be a general (prototypical) choice function in group decision making and voting, for instance those of: a "consensus solution", Borda's rule, the minimax degree set, the plurality voting, the qualified plurality voting, the approval voting-like, the "consensus + approval voting", Condorcet's rule, the Pareto rule, Copeland's rule, Nurmi's minimax set, Kacprzyk's Q-minimax, the Condorcet looser, the Pareto inferior alternatives, etc. This result, as interesting as it is, is however beyond the scope of this paper.

5 Degrees of Consensus under Fuzzy Preferences and a Fuzzy Majority

In this section fuzzy linguistic quantifiers as representations of a fuzzy majority will be employed to define a degree of consensus as proposed in Kacprzyk [19]), and then advanced in Kacprzyk and Fedrizzi [21, 22], and Kacprzyk, Fedrizzi and Nurmi [46, 47], see also Kacprzyk, Nurmi and Fedrizzi [27, 50] and Zadrożny [65]. This degree is meant to overcome some "rigidness" of the conventional concept of consensus in which (full) consensus occurs only when "all the individuals agree as to all the issues". This may often be counterintuitive, and not consistent with a real human perception of the very essence of consensus (see, e.g., the citation from a biological context given in the beginning of the paper).

The new degree of consensus proposed can be therefore equal to 1, which stands for full consensus, when, say, "most of the individuals agree as to almost all (of the relevant) issues (alternatives, options)".

Our point of departure is again a set of individual fuzzy preference relations which are meant analogously as in Sect. 2 [see, e.g., (17)].

The degree of consensus is now derived in three steps:

- first, for each pair of individuals we derive a degree of agreement as to their preferences between *all* the pairs of alternatives,
- second, we aggregate these degrees to obtain a degree of agreement of each pair of individuals as to their preferences between Q_1 (a linguistic quantifier as, e.g., "most", "almost all", "much more than 50%", ...) pairs of relevant alternatives, and
- third, we aggregate these degrees to obtain a degree of agreement of Q_2 (a linguistic quantifier similar to Q_1) pairs of important individuals as to their preferences between Q_1 pairs of relevant alternatives, and this is meant to be the *degree of consensus* sought.

Notice that we assume here, as opposed to Sect. 2, that both the individuals and alternatives are assigned different degrees of importance and relevance. However, this may be useful in the context of consensus reaching, and a basic case with the same importance and relevance for all the individuals and alternatives will just be a special case of the one adopted in this paper.

The above derivation process of a degree of consensus may be formalized by using Zadeh's [86] calculus of linguistically quantified statements and Yager's [84] OWA based aggregation.

We start with the degree of strict agreement between individuals k_1 and k_2 as to their preferences between alternatives s_i and s_j

$$v_{ij}(k_1, k_2) = \begin{cases} 1 \text{ if } r_{ij}^{k_1} = r_{ij}^{k_2} \\ 0 \text{ otherwise} \end{cases} \tag{42}$$

where here and later on in this section, if not otherwise specified, $k_1 = 1, \ldots, m-1$; $k_2 = k_1 + 1, \ldots, m$; $i = 1, \ldots, n-1$; $j = i+1, \ldots, n$.

The relevance of alternatives is assumed to be given as a fuzzy set defined in the set of alternatives S such that $\mu_B(s_i) \in [0, 1]$ is a *degree of relevance* of alternative s_i, from 0 for fully irrelevant to 1 for fully relevant, through all intermediate values.

The relevance of a pair of alternatives, $(s_i, s_j) \in S \times S$, may be defined, say, as

$$b_{ij}^B = \frac{1}{2}[\mu_B(s_i) + \mu_B(s_j)] \tag{43}$$

which is clearly the most straightforward option; evidently, $b_{ij}^B = b_{ji}^B$, and b_{ii}^B do not matter; for each i, j.

And analogously, the *importance of individuals*, I, is defined as a fuzzy set in the set of individuals such that $\mu_I(k) \in [0, 1]$ is a *degree of importance* of individual k, from 0 for fully unimportant to 1 for fully important, through all intermediate values.

Then, the importance of a pair of individuals, (k_1, k_2), b_{k_1,k_2}^I, may be defined in various ways, e.g., analogously as (19), i.e.

$$b_{k_1,k_2}^I = \frac{1}{2}[\mu_I(k_1) + \mu_I(k_2)] \tag{44}$$

The degree of agreement between individuals k_1 and k_2 as to their preferences between *all* the relevant pairs of alternatives is [cf. (6)]

$$v_B(k_1, k_2) = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [v_{ij}(k_1, k_2) \wedge b_{ij}^B]}{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij}^B} \tag{45}$$

The degree of agreement between individuals k_1 and k_2 as to their preferences between Q_1 relevant pairs of alternatives is

$$v^B_{Q_1}(k_1, k_2) = \mu_{Q_1}[v_B(k_1, k_2)] \tag{46}$$

In turn, the degree of agreement of *all* the pairs of important individuals as to their preferences between Q_1 pairs of relevant alternatives is

$$v^{I,B}_{Q_1} = \frac{2}{m(m-1)} \frac{\sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} [v^B_{Q_1}(k_1, k_2) \wedge b^I_{k_1,k_2}]}{\sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} b^I_{k_1,k_2}} \tag{47}$$

and, finally, the degree of agreement of Q_2 pairs of important individuals as to their preferences between Q_1 pairs of relevant alternatives, called the *degree of $Q1/Q2/I/B$-consensus*, is

$$con(Q_1, Q_2, I, B) = \mu_{Q_2}(v^{I,B}_{Q_1}) \tag{48}$$

Since the strict agreement (42) may be viewed too rigid, we can use the degree of sufficient agreement (at least to degree $\alpha \in (0, 1]$) of individuals k_1 and k_2 as to their preferences between alternatives s_i and s_j, defined by

$$v^\alpha_{ij}(k_1, k_2) = \begin{cases} 1 \text{ if } \mid r^{k_1}_{ij} - r^{k_2}_{ij} \mid \leq 1 - \alpha \leq 1 \\ 0 \text{ otherwise} \end{cases} \tag{49}$$

where, $k_1 = 1, \ldots, m-1; k_2 = k_1 + 1, \ldots, m; i = 1, \ldots, n-1; j = i + 1, \ldots, n$.

The degree of sufficient (at least to degree α) agreement between individuals k_1 and k_2 as to their preferences between all the relevant pairs of alternatives is

$$v^\alpha_B(k_1, k_2) = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [v^\alpha_{ij}(k_1, k_2) \wedge b^B_{ij}]}{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b^B_{ij}} \tag{50}$$

The degree of sufficient (at least to degree α) agreement between the individuals k_1 and k_2 as to their preferences between Q_1 relevant pairs of alternatives is

$$v^{B,\alpha}_{Q_1}(k_1, k_2) = \mu_{Q_1}[v^\alpha_B(k_1, k_2)] \tag{51}$$

In turn, the degree of sufficient (at least to degree α) agreement of *all* the pairs of important individuals as to their preferences between Q_1 relevant pairs of alternatives is

$$v^{I,B,\alpha}_{Q_1} = \frac{2}{m(m-1)} \frac{\sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} [v^{B\alpha}_{Q_1}(k_1, k_2) \wedge b^I_{k_1,k_2}]}{\sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} b^I_{k_1,k_2}} \tag{52}$$

and, finally, the degree of sufficient (at least to degree α) agreement of Q_2 pairs of important individuals as to their preferences between Q_1 relevant pairs of alternatives, called the *degree of $\alpha/Q1/Q2/I/B$-consensus*, is

$$con^{\alpha}(Q_1, Q_2, I, B) = \mu_{Q_2}(v_{Q_1}^{I,B,\alpha}) \tag{53}$$

We can also explicitly introduce the strength of agreement into (42), and analogously define the degree of strong agreement of individuals k_1 and k_2 as to their preferences between alternatives s_i and s_j, e.g., as

$$v_{ij}^{s}(k1, k2) = s(|\ r_{ij}^{k_1} - r_{ij}^{k_2}\ |) \tag{54}$$

where $s : [0, 1] \longrightarrow [0, 1]$ is some function representing the degree of strong agreements as, e.g.,

$$s(x) = \begin{cases} 1 & \text{for } x \leq 0.05 \\ -10x + 1.5 & \text{for } 0.05 < x < 0.15 \\ 0 & \text{for } x \geq 0.15 \end{cases} \tag{55}$$

such that $x' < x'' \Longrightarrow s(x') \geq s(x'')$, for each $x', x'' \in [0, 1]$, and there is such an $x \in [0, 1]$ that $s(x) = 1$.

The degree of strong agreement between individuals k_1 and k_2 as to their preferences between *all* the pairs of alternatives is then

$$v_{B}^{s}(k_1, k_2) = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [v_{ij}^{s}(k_1, k_2) \wedge b_{ij}^{B}]}{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij}^{B}} \tag{56}$$

The degree of strong agreement between individuals k_1 and k_2 as to their preferences between Q_1 relevant pairs of alternatives is

$$v_{Q_1}^{B,s}(k_1, k_2) = \mu_{Q_1}[v_{B}^{s}(k_1, k_2)] \tag{57}$$

In turn, the degree of strong agreement of *all* the pairs of important individuals as to their preferences between Q_1 relevant pairs of alternatives is

$$v_{Q_1}^{I,B,s} = \frac{2}{m(m-1)} \frac{\sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} [v_{Q_1}^{B,s}(k_1, k_2) \wedge b_{k_1,k_2}^{I}]}{\sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} b_{k_1,k_2}^{I}} \tag{58}$$

and, finally, the degree of agreement of Q_2 pairs of important individuals as to their preferences between Q_1 relevant pairs of alternatives, called the *degree of $s/Q1/Q2/I/B$-consensus*, is

$$con^{s}(Q_1, Q_2, I, B) = \mu_{Q_2}(v_{Q_1}^{I,B,s}) \tag{59}$$

Example 7. Suppose that $n = m = 3$, $Q_1 = Q_2 =$ "most" are given by (3), $\alpha = 0.9$, $s(x)$ is defined by (55), and the individual preference relations are:

$$R_1 = [r^1_{ij}] = \begin{array}{r|ccc} & j=1 & 2 & 3 \\ \hline i=1 & 0 & 0.1 & 0.6 \\ 2 & 0.9 & 0 & 0.7 \\ 3 & 0.4 & 0.3 & 0 \end{array} \qquad R_2 = [r^2_{ij}] = \begin{array}{r|ccc} & j=1 & 2 & 3 \\ \hline i=1 & 0 & 0.1 & 0.7 \\ 2 & 0.9 & 0 & 0.7 \\ 3 & 0.3 & 0.3 & 0 \end{array}$$

$$R_3 = [r^3_{ij}] = \begin{array}{r|ccc} & j=1 & 2 & 3 \\ \hline i=1 & 0 & 0.2 & 0.6 \\ 2 & 0.8 & 0 & 0.7 \\ 3 & 0.4 & 0.3 & 0 \end{array}$$

If we assume the relevance of the alternatives to be $B = \{b^B_i/s_i\} = 1/s_1 + 0.6/s_2 + 0.2/s_3$, the importance of the individuals to be $I = \{b^I_k/k\} = 0.8/1 + 1/2 + 0.4/3$, $\alpha = 0.9$ and $Q =$ "most$''$ given by (3), then we obtain the following degrees of consensus:

$$con(\text{“most”}, \text{“most”}, I, B) \cong 0.35$$
$$con^{0.9}(\text{“most”}, \text{“most”}, I, B) \cong 0.06$$
$$con^{s}(\text{“most”}, \text{“most”}, I, B) \cong 0.06$$

And, similarly as for the group decision making solutions shown in Sect. 2, the aggregation via Zadeh's [86] calculus of linguistically quantified propositions employed above may be replaced by the OWA based aggregation given by (10) and (11). The procedure is analogous as that presented in Sect. 2, and will not be repeated here.

For more information on these degrees of consensus, see, e.g., works by Kacprzyk, Fedrizzi, Nurmi and Zadrożny [19, 19, 41, 21, 22, 23, 24, 46], etc.

6 Remarks on some Voting Paradoxes and their Alleviation

Voting paradoxes are an interesting and very relevant topic that has a considerable theoretical and practical relevance. In this paper we will just give some simple examples of well known paradoxes and indicate some possibilities of how to alleviate them by using some elements of fuzzy preferences and a fuzzy majority. The paper is based on the works by Nurmi [74, 75], and Nurmi and Kacprzyk [77].

Table 1 presents an instance of Condorcet's paradox where there are three voter groups of equal size having preferences over alternatives A, B and C as indicated

Table 1 Condorcet's paradox

Group I	*Group II*	*Group III*
A	*B*	*C*
B	*C*	*A*
C	*A*	*B*

by the rank order indicated below each group. In fact, the groups need not be of equal size. What is essential for the paradox is that any two of them constitutes a majority. Clearly, a collective preference relation formed on the basis of comparing alternatives in pairs and using majority rule, results in a cycle: A is preferred to B, B is preferred to C and C is preferred to A.

An instance of Borda's paradox, in turn, is given in Table 2, where alternative A would win by a plurality of votes and, yet, both B and C would beat A, should pairwise majority comparisons be conducted.

A common feature in these classic paradoxes is an incompatibility of several intuitively plausible requirements regarding social choices. In the case of Condorcet's paradox the result obtained by using majority rule on a set of complete and transitive preferences is intransitive. In the case of Borda's paradox, the winner in the plurality sense is different from the winner in another sense, i.e. in the sense that requires the winner to beat all the other alternatives in binary contests.

Let us try to solve the above paradoxes using some fuzzy tools. The solutions presented are very much in the spirit of Sen's idea of broadening the amount of information about individuals. In particular, we shall take our point of departure in the notion of fuzzy individual preference relation. We consider the set E of individuals and the set S of decision alternatives. Each individual $i \in E$ is assumed to possess a fuzzy preference relation $R_i(x, y)$ over S. For each $x, y \in S$ the value $R_i(x, y)$ indicates the degree in which x is preferred to y by i with 1 indicating the strongest preference of x to y, 0.5 indifference between the two and value 0 the strongest preference of y to x. Obviously, the assumption that the voters be endowed with fuzzy preference relations is precisely the kind of broadening of the information about individuals that Sen discusses. Some properties of fuzzy preference relations are defined in the following [15, 81].

Definition 15. *Connectedness. A fuzzy preference relation R is connected if an only if* $R(x, y) + R(y, x) \geq 1, \forall x, y \in S$.

Definition 16. *Reflexivity. A fuzzy preference relation R is reflexive if an only if* $R(x, x) = 1, \forall x \in S$.

Definition 17. *Max-min transitivity. A fuzzy connected and reflexive relation R is max-min transitive if and only if* $R(x, z) \geq min[R(x, y), R(y, z)], \forall x, y, z \in S$.

For the case of the Condorcet paradox, given the broadening of information concerning voter preferences represented by fuzzy preference relations, we can solve it very much in the spirit of its "father", Marquis de Condorcet (cf. Nurmi [75]). A way out of cyclical collective preferences is to look at the sizes of majorities supporting various collective preferences. For example, if the number of voters preferring a to b is 5 out of 9, while that of voters preferring b to c is 7 out of 9, then, according

Table 2 Borda's paradox

voters 1–4	*voters 5–7*	*voters 8,9*
A	*B*	*C*
B	*C*	*B*
C	*A*	*A*

to Condorcet, the latter preference is stronger than the former. By cutting the cycle of collective majority preferences at its weakest link, one ends up with a complete and transitive relation. Clearly, with nonfuzzy preference relation this method works only in cases where not all of the majorities supporting various links in the cycle are of same size. With fuzzy preferences one can form the collective preference between any x and $y \in S$ using a variation of the average rule (cf. Intilligator [34]), i.e.

$$R(x, y) = \frac{\sum_i R_i(x, y)}{m} \tag{60}$$

where $R(x, y)$ is the degree of collective fuzzy preference of x over y.

Now, supposing that a preference cycle is formed on the basis of collective fuzzy preferences, one could simply ignore the link with weakest degree of preference and thus possibly end up with a ranking. In general one can proceed by eliminating weakest links in collective preference cycles until a ranking results.

The above method of successive elimination of weakest links in preference cycles thus works with fuzzy and nonfuzzy preferences. When individual preferences are fuzzy each voter is assumed to report his/her preferences so that the following matrix can be formed:

$$R_i = \begin{pmatrix} - & r_{12} & \dots & r_{1n} \\ r_{21} & - & \dots & r_{2n} \\ \dots & \dots & \dots & \dots \\ r_{n1} & r_{n2} & \dots & - \end{pmatrix} \tag{61}$$

Here r_{ij} indicates the degree in which i prefers the i-th alternative to the j-th one. By averaging over the voters we obtain:

$$\bar{R} = \begin{pmatrix} - & \bar{r}_{12} & \dots & \bar{r}_{1n} \\ \bar{r}_{21} & - & \dots & \bar{r}_{2n} \\ \dots & \dots & \dots & \dots \\ \bar{r}_{n1} & \bar{r}_{n2} & \dots & - \end{pmatrix} \tag{62}$$

Apart from the successive elimination method one can use another straightforward method to resolve Condorcet's paradox, once the $\bar{R}$-matrix is given. It proceeds as follows. One first computes the row sums of the matrix:

$$\bar{r}_i = \sum_j \bar{r}_{ij} \tag{63}$$

These represent the total fuzzy preference weight assigned to the i-th alternative in all pairwise preference comparisons, when the weight in each comparison is the average fuzzy preference value. Let now

Table 3 Fuzzy Condorcet's paradox

voter 1				*voter 2*				*voter 3*			
	A	*B*	*C*		*A*	*B*	*C*		*A*	*B*	*C*
A	-	.6	.8	*A*	-	.9	.3	*A*	-	.6	.3
B	.4	-	.6	*B*	.1	-	.7	*B*	.4	-	.1
C	.2	.4	-	*C*	.7	.3	-	*C*	.7	.9	-

$$p_i = \frac{\bar{r}_i}{\sum_i \bar{r}_i}. \tag{64}$$

Clearly $p_i \geq 0$ and $\sum_i p_i = 1$. Thus, p_i has the natural interpretation of choice probability. An obvious way to utilize this is to form the collective preference ordering on the basis of these choice probabilities. The result is necessarily a complete and transitive relation. Hence we can use the information broadening provided by fuzzy preferences to solve Condorcet's paradox (cf. Table 3).

For illustration, consider the example of Table 1 again and assume that each group consists of just one voter. Assume, furthermore, that the fuzzy preferences underlying the preference rankings are as follows:

The $\bar{R}$- matrix is now:

$$\bar{R} = \begin{pmatrix} - & .7 & .5 \\ .3 & - & .5 \\ .5 & .5 & - \end{pmatrix}$$

Now, $P_A = 0.4$, $P_B = 0.3$, $P_C = 0.3$.

Obviously, the solution is based on somewhat different fuzzy preference relations over the three alternatives. Should the preference relations be identical, we would necessarily end up with identical choice probabilities.

With fuzzy individual preference relations we can resolve Borda's paradox. To do that, we simply apply the same procedure as in the resolution of Condorcet's paradox.

Let us take a look at a fuzzy Borda's paradox for illustration. Assume that the fuzzy preferences underlying Table 2 are those indicated in Table 4.

Table 4 A fuzzy Borda's paradox

4 voters				*3 voters*				*2 voters*			
	A	*B*	*C*		*A*	*B*	*C*		*A*	*B*	*C*
A	-	.6	.8	*A*	-	.9	.3	*A*	-	.2	.1
B	.4	-	.6	*B*	.1	-	.7	*B*	.8	-	.3
C	.2	.4	-	*C*	.7	.3	-	*C*	.9	.7	-

The matrix of average preference degrees is then the following:

$$\bar{R} = \begin{pmatrix} - & .6 & .5 \\ .4 & - & .6 \\ .5 & .4 & - \end{pmatrix}$$

The choice probabilities of A, B and C are, thus, $0.37, 0.33, 0.30$. We see that the choice probability of A is the largest. In a sense, then, the method does not solve Borda's paradox in the same way as the Borda count does since also plurality method ends up with A being chosen instead of the Condorcet winner alternative B. Note, however, that fuzzy preference relations give a richer picture of voter preferences than the ordinary preference rankings. In particular, A is strongly preferred to B and C by both the 4 and 3 voter groups. Hence, it is to be expected that its choice probability is the largest.

For additional information on voting paradoxes and some ways to solve them using fuzzy logic, we refer the reader to Nurmi and Kacprzyk [77].

7 Concluding Remarks

In this paper we have briefly presented the use of fuzzy preference relations and fuzzy majorities in the derivation of group decision making (social choice) solution concepts and degrees of consensus. First, we briefly discussed some more general issues related to the role fuzzy preference relations and a fuzzy majority may play as a tool to alleviate difficulties related to negative results in group decision making exemplified by Arrow's impossibility theorem. Though very important for a conceptual point of view, these analyses are of a lesser practical relevance to the user who wishes to employ those fuzzy tools to constructively solve the problems considered.

Therefore, emphasis has been on the use of fuzzy preference relations and fuzzy majorities to derive more realistic and human-consistent solutions of group decision making. Reference has been given to other approaches and works in this area, as well as to the authors' previous, more foundational works in which an analysis of basic issues underlying group decision making and consensus formation has been included.

It is hoped that this work will provide the interested reader with some tools to constructively solve group decision making and consensus formation problems when both preferences and majorities are imprecisely specified or perceived, and may be modeled by fuzzy relations and fuzzy sets.

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