Preferences and Consistency Issues in Group Decision Making

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Abstract A group selection of one alternative from a set of feasible ones should be based on the preferences of individuals in the group. Decision making procedures are usually based on pair comparisons, in the sense that processes are linked to some degree of credibility of preference. The main advantage of pairwise comparison is that of focusing exclusively on two alternatives at a time and on how they are related. However, it generates more information that needed and therefore inconsistent information may be generated. This paper addresses both preference representation and consistency of preferences issues in group decision making.

Different preference representation formats individuals can use to model or present their preferences on a set of alternatives in a group decision making situation are reviewed. The results regarding the relationships between these preference representation formats mean that the fuzzy preference relation "is preferred to" representing the strength of preference of one alternative over another in the scale [0, 1] can be used as the base element to integrate these different preference representation formats in group decision making situations.

Due to the complexity of most decision making problems, individuals' preferences may not satisfy formal properties that fuzzy preference relations are required to verify. Consistency is one of them, and it is associated with the **transitivity property.** Many properties have been suggested to model transitivity of fuzzy preference relations. As aforementioned, this paper provides an overview of the main results published in this area.

1 Introduction

Group Decision-Making (GDM) consists of multiple individuals interacting to reach a decision. Each decision maker (expert) may have unique motivations or goals and may approach the decision process from a different angle, but have a common interest in reaching eventual agreement on selecting the 'best' option(s) [11, 32].

Decisions depend, at least in part, on preferences [14]. Indeed, the following quotation from Fishburn [13] fully justifies the above in the context of GDM:

H. Bustince et al., (eds.), *Fuzzy Sets and Their Extensions: Representation, Aggregation and Models.*

> DEMOCRATIC THEORY is based on the premise that the resolution of a matter of social policy, group choice or collective action should be based on the desires or preferences of the individuals in the society, group or collective

In order to reach a decision, experts have to express their preferences by means of a set of evaluations over a set of alternatives. It has been common practice in research to model GDM problems in which all the experts express their preferences using the same preference representation format. However, in real practice this is not always possible because each expert has his/her unique characteristics with regard to knowledge, skills, experience and personality, which implies that different experts may express their evaluations by means of different preference representation formats. In fact, this is an issue that recently has attracted the attention of many researchers in the area of GDM, and as a result different approaches to integrating different preference representation formats have been proposed [5, 6, 7, 12, 15, 49, 50].

In many situations decision processes are based on preference relations, in the sense that processes are linked to some degree of preference of any alternative over another. The main advantage of pairwise comparison is that of focusing exclusively on two alternatives at a time and on how they are related. However, it generates more information that needed and therefore inconsistent information may be generated.

This paper addresses both preference representation formats and the consistency of preferences issues in group decision making. A review of the main results on integration of preference representation structures will be given in Sect. 2, while the problem of consistency when working with fuzzy preference relations will be reviewed in Sect. 3. In Sect. 4 we make note the existence of a conflict between the definition of a consistent multiplicative preference relation and the scale proposed to provide a such preference relation. Obviously, the same problem exists when dealing with fuzzy preference relations. In order to overcome this problem, In Sect. 5 we propose a set of conditions to be verified by a function in order to model consistency of fuzzy preferences. Finally, in Sect. 6 we draw our conclusions.

2 Preference Representation Formats: Integration

Experts may provide preferences on a set of alternatives, $X = \{x_1, x_2, \ldots, x_n\}$ $(n \geq 2)$, by using many different representation formats. Among these formats we have:

A preference ordering of the alternatives

In this case, an expert, e_k, gives his preferences on X as an individual preference ordering, $O^k = \{o^k(1), ..., o^k(n)\}$, where $o^k(\cdot)$ is a permutation function over the index set, $\{1, ..., n\}$, for the expert, e_k, [5, 42]. Therefore, according to this point of view , an ordered vector of alternatives, from best to worst, is given.

A utility function

In this case, an expert, e_k, gives his preferences on X as a set of n utility values, $U^k = \{u_i^k,\ i = 1, ..., n\}$, $u_i^k \in [0, 1]$, where u_i^k represents the utility evaluation given by the expert e_k to the alternative x_i [30, 47].

A preference relation

In the classical preference modelling, given two alternatives, an expert judges them in one of the following ways: (i) one alternative is preferred to another; (ii) the two alternatives are indifferent to him/her; (iii) he/she is unable to compare them.

According to these cases, three binary relations can be defined: (i) the strict preference relation P: $(x, y) \in P$ if and only if the expert prefers x to y; (ii) the indifference relation I: $(x, y) \in I$ if and only if the expert is indifferent between x and y; (iii) the incomparability relation J: $(x, y) \in J$ if and only if the expert unable to compare x and y.

Fishburn in [14] defines indifference as the absence of strict preference. He also points out that indifference might arise in three different ways: (a) when an expert truly feels that there is no real difference, in a preference sense, between the alternatives; (b) when the expert is uncertain as to his/her preference between the alternatives because 'he might find their comparison difficult and may decline to commit himself[/herself] to a strict preference judgement while not being sure that he[/she] regards [them] equally desirable (or undesirable)'; (c) or when both alternative are considered incomparable on a preference basis by the expert. It is obvious from the third case that Fishburn treats the incomparability relation as an indifference relation, i.e., J is empty (there is no incomparability).

A preference structure on a set of alternatives X is defined as a triplet (P, I, J) of binary relation in X that satisfy [38, 39]:

1. P is irreflexive and asymmetrical
2. I is reflexive and symmetrical
3. J is irreflexive and symmetrical
4. $P \cap I = P \cap J = I \cap J = \emptyset$
5. $P \cup P^t \cup I \cup J = A^2$

where P^t is the transpose (or inverse) of P: $(x, y) \in P \Leftrightarrow (y, x) \in P^t$. Condition 5 is called the completeness condition.

In [38] it is proved that a preference structure (P, I, J) on a se of alternatives X can be characterised by the single reflexive relation $R = P \cup I$: $(x, y) \in R$ if and only if "x is as good as y". R is called the large preference relation of (P, I, J). Conversely, given any reflexive binary relation R in X, a preference structure (P, I, J) can be constructed on it as follows: $P = R \cap (R^t)^c$, $I = R \cap R^t$, $J = R^c \cap (R^t)^c$, where R^c is the complement of R: $(x, y) \in R \Leftrightarrow (y, x) \notin R^c$.

When using numerical representations of preferences on a set of alternatives X, we have [13]:

$$r_{ij} = 1 \Leftrightarrow \text{the expert prefers } x_i \text{ to } x_j \Leftrightarrow x_i \succ x_j$$
$$r_{ij} = 0 \Leftrightarrow \text{the expert prefers } x_j \text{ to } x_i \Leftrightarrow x_j \succ x_i$$

Clearly, this can be extended by adding the indifference case:

$$r_{ij} = 0.5 \Leftrightarrow \text{the expert is indifferent between } x_i \text{ and } x_j \Leftrightarrow x_i \sim x_j$$

However, if x_i is preferred to x_j and x_j to x_k, the question whether the "degree or strength of preference" of x_i over x_j exceeds, equals, or is less than the "degree or strength of preference" of x_j over x_k cannot be answered by the classical preference modelling. The implementation of the degree of preference between alternatives may be essential in many situations. Take for example the case of 3 alternatives $\{x, y, z\}$ and 2 experts. If one of the experts prefers x to y to z, and the other prefers z to y to x then using the above values it may be difficult or impossible to decide which alternative is the best. This may be not the case if intensities of preferences are allowed in the above model. As Fishburn points out in [13], if alternative y is closer to the best alternative than to the worst one for both experts then it might seem appropriate to "elect" it as the social choice, while if it is closer to the worst than to the best, then it might be excluded from the choice set. Intensity of preferences can be implemented when modelling preferences by using fuzzy preference relations [51] or multiplicative preference relations [40].

A fuzzy preference relation R on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is characterized by a membership function

$$\mu_R : X \times X \longrightarrow [0, 1].$$

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $R = (r_{ij})$ being $r_{ij} = \mu_R(x_i, x_j)\ \ \forall i, j \in \{1, \ldots, n\}$. The element $r_{ij} \in R$ is usually interpreted as the preference degree of the alternative x_i over x_j, as follows [46]: $r_{ij} = 1/2$ indicates indifference between x_i and x_j $(x_i \sim x_j)$, $r_{ij} > 1/2$ represents an uncertain preference of x_i over x_j $(x_i \succ x_j)$ with $r_{ij} = 1$ when x_i is definitely (certainly) preferred to x_j. In this case, the preference matrix, R, is usually assumed additive reciprocal, i.e.,

$$r_{ij} + r_{ji} = 1\ \ \forall i, j \in \{1, \ldots, n\}.$$

A multiplicative preference relation A on a set of alternatives X is represented by a matrix $A \subset X \text{ x } X$, $A = (a_{ij})$, being a_{ij} interpreted as the ratio of the preference intensity of alternative x_i to that of x_j, i.e., it is interpreted as x_i is a_{ij} times as good as x_j. Saaty suggests measuring a_{ij} using a ratio-scale, and precisely the 1 to 9 scale: $a_{ij} = 1$ indicates indifference between x_i and x_j, $a_{ij} = 9$ indicates that x_i is absolutely preferred to x_j, and $a_{ij} \in \{1, \ldots, 9\}$ indicates intermediate preference evaluations. In this case, the preference relation, A, is usually assumed multiplicative reciprocal, i.e.,

$$a_{ij} \cdot a_{ji} = 1 \;\; \forall i, j \in \{1, \ldots, n\}.$$

In the context of GDM with heterogeneous information, an interesting question to answer is that of the relationship between the different preference representation formats.

Preference Orderings and Utility Functions → Binary Preference Relations

Clearly, from a preference ordering on X we can derive a binary preference relation

$$x_i \succeq x_j \Leftrightarrow o(i) \leq o(j) \; \forall i, j = 1, \ldots, n,$$

Also, given an utility function on X, a preference ordering, and consequently a classical preference relation, can easily be derived as follows

$$o(i) \leq o(j) \Leftrightarrow u(x_i) \geq u(x_j) \; \forall i, j = 1, \ldots, n,$$

Binary Preference Relations → Preference Orderings and Utility Functions

Given a binary preference relation, it is not always possible to assure the existence of a unique preference ordering or an utility function verifying the above equivalence. In order to get a positive answer additional conditions have to be imposed to the preference relation.

Given the binary preference relation *is preferred to* ($\succ$) on a countable set X, with *is indifferent to* ($\sim$) defined as $x \sim y$ if neither $x \succ y$ nor $y \succ x$, a fundamental result is that there exists an utility function $u\colon X \to \mathbb{R}$ such that

$$x \succ y \Leftrightarrow u(x) > u(y)$$

if and only if $\succ$ on X is a weak order, i.e., it is transitive ($x \succ y \wedge y \succ z \Rightarrow x \succ z$), irreflexive (we never have $x \succ x$) and $\sim$ is transitive ($x \sim y \wedge y \sim z \Rightarrow x \sim z$) [15]. The utility function u is said to represent the preference relation $\succ$. Obviously, in this case, any (positive) monotonic transformation of the utility function u is also a utility function representing the same preference relation because such a transformation preserves the ranking order of the original utility numbers. So, if we begin with the utility function u and then use the (positive) monotonic transformation f to get a new function v defined as $v(x) = f(u(x))$, then v is also a utility function representing the same preference relation as u.

Fuzzy Preference Relations → Preference Orderings and Utility Functions

Given a fuzzy preference relation on a finite set of alternatives X, not necessarily reciprocal, Wang proved in [48] that if the following acyclic property was verified

$$\begin{aligned}&\forall i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, n\} : \\ &\qquad r_{i_1 i_2} > r_{i_2 i_1}, r_{i_2 i_3} > r_{i_3 i_2}, \ldots, r_{i_m i_{m-1}} > r_{i_{m-1} i_m} \Rightarrow r_{i_1 i_m} > r_{i_m i_1}\end{aligned}$$

then a total order can be produced in X, i.e, given any two arbitrary alternatives x_i and x_j in X, one of the following relations holds: $x_i \succ x_j$, $x_j \succ x_i$, $x_i \sim x_j$. A similar result was obtained in [2] when the fuzzy preference relation is reciprocal and is weakly transitive ($r_{ij} > 0.5 \wedge r_{jk} > 0.5 \Rightarrow r_{ik} > 0.5$).

With fuzzy preference relations, Orlovsky [26] proposed a rational criterion to produce a total order on X based on the strict preference relation $R^s = (r^s_{ij})$ with $r^s_{ij} = \max\{r_{ij} - r_{ji}, 0\}$ and the concept of non-dominance. Conditions that guarantee the existence of un-fuzzy non-dominated alternatives were obtained by Montero and Tejada (see [35, 36]) and by Kołodziejczyk [29]. A quantifier non-dominance degree that extended Orlovsky's non-dominance degree was proposed by Chiclana et al. in [4].

Preference Orderings and Utility Functions → Fuzzy Preference Relations

In [5, 6, 22] the following results were obtained:

Proposition 1. *Let X be a set of alternatives and λ^k_i represents an evaluation associated to alternative x_i, indicating the performance of that alternative according to a point of view (expert or criteria) e_k. Then, the intensity of preference of alternative x_i over alternative x_j, r^k_{ij}, for e_k is given by the following transformation function*

$$r^k_{ij} = \varphi(\lambda^k_i, \lambda^k_j) = \frac{1}{2} \cdot [1 + \psi(\lambda^k_i, \lambda^k_j) - \psi(\lambda^k_j, \lambda^k_i)],$$

where ψ is a function verifying

1. $\psi(z, z) = \frac{1}{2}, \ \forall z \in \mathbb{R}$.
2. *ψ is non decreasing in the first argument and non increasing in the second argument.*

Utility Values and Fuzzy Preference Relations

Corollary 1. *If $\lambda^k_i = u^k_i$ and*

$$\psi(z, y) = \begin{cases} \dfrac{s(z)}{s(z) + s(y)} & \text{if } (z, y) \neq (0, 0) \\ \dfrac{1}{2} & \text{if } (z, y) = (0, 0) \end{cases}$$

where $s : [0, 1] :\rightarrow \mathbb{R}^+$ is a non decreasing and continuous function, verifying $s(0) = 0$, then φ transforms utility values given on the basis of a ratio scale into fuzzy preference relations. In particular, if $\psi(x, y) = \frac{x^2}{x^2+y^2}$, then

$$r^k_{ij} = f^1\left(u^k_i, u^k_j\right) = \frac{\left(u^k_i\right)^2}{\left(u^k_i\right)^2 + \left(u^k_j\right)^2}$$

Preference Orderings and Fuzzy Preference Relations

Corollary 2. *If* $\lambda_i^k = o^k(i)$, *and* $\psi(\lambda_i^k, \lambda_j^k) = F(\lambda_j^k - \lambda_i^k)$, *where F is any non decreasing function, then* φ *transforms preference orderings into fuzzy preference relations.*

In particular, if $\psi(x, y) = \frac{y-x}{2(n-1)}$, *then*

$$r_{ij}^k = f^2\left(o_i^k, o_j^k\right) = \frac{1}{2}\left(1 + \frac{o_j^k - o_i^k}{n-1}\right)$$

Multiplicative Preference Relations and Fuzzy Preference Relations

Proposition 2. *Let X be a set of alternatives, and associated with it a multiplicative preference relation* $A^k = \left(a_{ij}^k\right)$. *Then, the corresponding additive fuzzy preference relation,* $R^k = \left(r_{ij}^k\right)$, *associated with* A^k *is given as follows:*

$$r_{ij}^k = g\left(a_{ij}^k\right) = \frac{1}{2}\left(1 + \log_9 a_{ij}^k\right)$$

These results may justify the choice of fuzzy preference relations as the base element to integrate these different preference representation formats in GDM context. In the following section we deal with issue of consistency of preferences.

3 Consistency of Preferences

There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations [18]:

- The first level of rationality requires indifference between any alternative and itself.
- The second one assumes the property of reciprocity in the pairwise comparison between any two alternatives.
- Finally, the third one is associated with the transitivity in the pairwise comparison among any three alternatives.

The mathematical modelling of all these rationality assumptions obviously depends on the scales used for providing the preference values [9, 16, 26, 40, 46].

A preference relation verifying the third level of rationality is usually called a *consistent preference relation* and any property that guarantees the transitivity of the preferences is called a consistency property. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, in fact crucial, to study conditions under which consistency is satisfied [16, 26, 40].

In a crisp context, where an expert provides his/her opinion on the set of alternatives X by means of a binary preference relation, R, the concept of consistency it has traditionally been defined in terms of acyclicity [43], that is the absence of sequences such as $x_1, x_2, \ldots, x_k (x_{k+1} = x_1)$ with $x_j R x_{j+1} \forall j = 1, \ldots, k$. Clearly, this condition as said before is closely related to the transitivity of the binary relation and its corresponding binary indifference relation.

In a fuzzy context, where an expert expresses his/her opinions using fuzzy preference relations, R, or multiplicative preference relations, in the case of Saaty's method, A, the traditional requirement to characterise consistency has followed the way of extending the classical requirements of binary preference relations. Thus, in these cases consistency is also based on the notion of transitivity, in the sense that if alternative x_i is preferred to alternative x_j and this one to x_k then alternative x_i should be preferred to x_k. The main difference in these cases with respect to the classical one is that transitivity has been modelled in many different ways due to the role the intensities of preference have [16, 19, 20, 21, 26, 40, 46, 52].

Due to the hierarchical structure of the three rationality assumptions for a preference relation, the verification of a particular level of rationality should be a necessary condition in order to verify the next level of rationality. This means that the third level of rationality, transitivity of preferences, should imply or be compatible with the second level of rationality, reciprocity of preferences, and the second level with the first one, indifference of any alternative with itself.

This necessary compatibility between the rationality assumptions can be used as a criterion for considering a particular condition modelling any one of the rationality levels as adequate or inadequate. In the case of fuzzy (multiplicative) preference relations, the indifference between any alternative, x_i, and itself is modelled by associating the preference value $r_{ii} = 0.5$ ($a_{ii} = 1$). The reciprocity of fuzzy (multiplicative) preferences is modelled using the property $r_{ij} + r_{ji} = 1, \ \forall i, j$ ($a_{ij} \cdot a_{ji} = 1, \ \forall i, j$). A necessary condition for a preference relation to verify reciprocity should be that indifference between any alternative and itself holds. Because reciprocity property implies the indifference of preferences, we conclude that both properties are compatible.

In the case of multiplicative preference relations Saaty means by *consistency* what he calls *cardinal transitivity* in the strength of preferences, which is a stronger condition than the traditional requirement of the transitivity of preferences [40]:

Definition 1. *A reciprocal multiplicative preference relation $A = (a_{ij})$ is consistent if $a_{ij} \cdot a_{jk} = a_{ik} \ \ \forall i, j, k = 1, \ldots, n$.*

Inconsistency for Saaty is a violation of proportionality which may not entail violation of transitivity [40]. Furthermore, consistency implies reciprocity, and therefore, they are both compatible.

In [40] Saaty shows that a reciprocal multiplicative preference relation is consistent if and only if its maximum or principal eigenvalue λ_{max} is equal to the number of alternatives n. Under this consistency property, Saaty proves that there exists a set of priorities (utilities) $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $a_{ij} = \frac{\lambda_i}{\lambda_j}$. Moreover, this set of values is unique up to positive linear transformation $f(\lambda_i) = \beta \cdot \lambda_i$ with $\beta > 0$.

Thus, if a multiplicative preference relation is consistent then it can be represented by a unique (up to positive linear transformations) utility function.

For fuzzy preference relations, there exist many properties or conditions that have been suggested as rational conditions to be verified by a consistent relation. Among these, we can citethe following:

1. *Triangle condition [30]:* $r_{ij} + r_{jk} \geq r_{ik} \ \ \forall i, j, k$.
   This condition can be geometrically interpreted considering alternatives $x_i,\ x_j$, x_k as the vertices of a triangle with length sides $r_{ij},\ r_{jk}$ and r_{ik} [30], and therefore the length corresponding to the vertices $x_i,\ x_k$ should not exceed the sum of the lengths corresponding to the vertices $x_i,\ x_j$ and $x_j,\ x_k$.
2. *Weak transitivity [46]:* $r_{ij} \geq 0.5,\ r_{jk} \geq 0.5 \Rightarrow r_{ik} \geq 0.5 \ \ \forall i, j, k$.
   The interpretation of this condition is the following: If x_i is preferred to x_j and x_j is preferred to x_k, then x_i should be preferred to x_k. This kind of transitivity is the usual transitivity condition (x_i is preferred to alternative x_j and this one to x_k then alternative x_i should be preferred to x_k) a logical and consistent person should use if he/she does not want to express inconsistent opinions, and therefore it is the minimum requirement condition that a consistent fuzzy preference relation should verify.
3. *Max-min transitivity [9, 52]:* $r_{ik} \geq \min(r_{ij}, r_{jk}) \ \ \forall i, j, k$.
   The idea represented here is that the preference value obtained by a direct comparison between two alternatives should be equal to or greater than the minimum partial values obtained when comparing both alternatives with an intermediate one. This kind of transitivity has been the traditional requirement to characterise consistency in the case of fuzzy preference relations [52], although it is a very strong concept that it could not be verified even when a fuzzy preference relation is considered perfectly consistent from a practical point of view. For example, let us consider a set of three alternatives $X = \{x_1, x_2, x_3\}$, such that $x_1 \prec x_2 \prec x_3$. Suppose that the opinions about these alternatives are given by the following fuzzy preference relation

   $$R = \begin{pmatrix} 0.5 & 0.1 & 0 \\ 0.9 & 0.5 & 0.4 \\ 1 & 0.6 & 0.5 \end{pmatrix}.$$

   On the one hand, this matrix reflects the fact that $x_1 \prec x_2 \prec x_3$; it verifies weak transitivity and the triangle condition. On the other hand, it does not verifies max-min transitivity because $r_{13} < \min\{r_{12}, r_{23}\}$.
4. *Max-max transitivity [9, 52]:* $r_{ik} \geq \max(r_{ij}, r_{jk}) \ \ \forall i, j, k$.
   This concept represents the idea that the preference value obtained by a direct comparison between two alternatives should be equal to or greater than the maximum partial values obtained when comparing both alternatives using an intermediate one. This is a stronger concept than max-min transitivity and therefore if a fuzzy preference relation does not verify the latter neither verifies the former.
5. *Restricted max-min transitivity [46]:* $r_{ij} \geq 0.5,\ r_{jk} \geq 0.5 \Rightarrow r_{ik} \geq \min(r_{ij}, r_{jk})$ $\forall i, j, k$.

When a fuzzy preference relation verifies this condition it is modelled the concept that when an alternative x_i is preferred to x_j with a value p_{ij} and x_j is preferred to x_k with a value r_{ij}, then x_i should be preferred to x_k with at least an intensity of preference r_{ik} equal to the minimum of the above values. The inequality should becomes equality only when there exist indifference between at least two of the three alternatives. A consistent fuzzy preference relation has to verify this condition, which goes a step further than weak transitivity because add an extra requirement about the degrees of preferences involved. This transitivity condition is therefore stronger than weak transitivity but it is milder than max-min transitivity. It is easy to prove that the above fuzzy preference relation R verifies restricted max-min transitivity.

6. *Restricted max-max transitivity [46]:* $r_{ij} \geq 0.5,\ r_{jk} \geq 0.5 \Rightarrow r_{ik} \geq \max(r_{ij}, r_{jk})$ $\forall i, j, k$.
   In this case it is modelled the concept that when an alternative x_i is preferred to x_j with a value r_{ij} and x_j is preferred to x_k with a value r_{ij}, then x_i should be preferred to x_k with at least an intensity of preference r_{ik} equal to the maximum of the above values. As in the previous case, the the equality should hold only when there exist indifference between at least two of the three alternatives, in which case, restricted max-max transitivity and restricted max-min transitivity coincide. It is clear that this concept is, on the one hand, stronger than restricted max-min transitivity and, on the other hand, milder than max-max transitivity. This concept has been considered by Tanino [46] as a compulsory condition to be verified by a consistent fuzzy preference relation. It is easy to prove that the fuzzy reciprocal preference relation R, given above, verifies restricted max-max transitivity.
7. *Multiplicative transitivity [46]*: $\frac{r_{ji}}{r_{ij}} \cdot \frac{r_{kj}}{r_{jk}} = \frac{r_{ki}}{r_{ik}}$ $\forall i, j, k$.
   Tanino in [46] introduced this concept of transitivity only in the case of being $r_{ij} > 0\ \ \forall i, j$, and interpreting r_{ij}/r_{ji} as a ratio of the preference intensity for x_i to that of x_j, i.e., x_i is r_{ij}/r_{ji} times as good as x_j. Multiplicative transitivity includes restricted max-max transitivity [45, 46], and rewritten as $r_{ij} \cdot r_{jk} \cdot r_{ki} = r_{ik} \cdot r_{kj} \cdot r_{ji}\ \ \forall i, j, k$. In the case of a reciprocal fuzzy preference relation this expression can be expressed in the following form:

$$
\begin{aligned}
&\forall i, j, k : \\
&\qquad r_{ij} \cdot r_{jk} \cdot (1 - r_{ik}) = r_{ik} \cdot r_{kj} \cdot r_{ji} &&\Leftrightarrow \\
&\qquad r_{ij} \cdot r_{jk} - r_{ij} \cdot r_{jk} \cdot r_{ik} = r_{ik} \cdot r_{kj} \cdot r_{ji} &&\Leftrightarrow \\
&\qquad r_{ik} \cdot r_{kj} \cdot r_{ji} + r_{ij} \cdot r_{jk} \cdot r_{ik} = r_{ij} \cdot r_{jk} &&\Leftrightarrow \\
&\qquad r_{ik} \cdot (r_{kj} \cdot r_{ji} + r_{ij} \cdot r_{jk}) = r_{ij} \cdot r_{jk} &&\Leftrightarrow \\
&\qquad r_{ik} = \frac{r_{ij} \cdot r_{jk}}{r_{ij} \cdot r_{jk} + r_{ji} \cdot r_{jk}} &&\Leftrightarrow \\
&\qquad r_{ik} = \frac{r_{ij} \cdot r_{jk}}{r_{ij} \cdot r_{jk} + (1 - r_{ij}) \cdot (1 - r_{jk})}
\end{aligned}
$$

   This expression is a well known andlike uninorm which is self-dual with respect to the negator operator $N(x) = 1 - x$ (for more details see [17, 28]).

This type of transitivity has also been studied by De Baets et al. in [8] within a general framework of transitivity of reciprocal fuzzy preference relations, the cycle-transitivity, under the name of 'isostochastic transitivity'. This is also a symmetric sum in the sense of Silvert [44] that has been applied for information combination in approximate reasoning (see [10] and cites within it).

8. *Additive transitivity [45, 46]:* $(r_{ij} - 0.5) + (r_{jk} - 0.5) = (r_{ik} - 0.5)\ \forall i, j, k$, or equivalently $r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}\ \forall i, j, k$.
This kind of transitivity has the following interpretation: suppose we want to establish a ranking between three alternatives x_i, x_j and x_k, and that the information available about these alternatives suggests that we are in an indifference situation, i.e. $x_i \sim x_j \sim x_k$. When giving preferences this situation would be represented by $r_{ij} = r_{jk} = r_{ik} = 0.5$. Suppose now that we have a piece of information that says $x_i \prec x_j$, i.e. $r_{ij} < 0.5$. This means that r_{jk} or r_{ik} have to change, otherwise there would be a contradiction, because we would have $x_i \prec x_j \sim x_k \sim x_i$. If we suppose that $r_{jk} = 0.5$ then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must then conclude that x_k has to be preferred to x_i. Furthermore, as $x_j \sim x_k$ then $r_{ij} = r_{ik}$, and so $(r_{ij} - 0.5) + (r_{jk} - 0.5) = (r_{ij} - 0.5) = (r_{ik} - 0.5)$. We have the same conclusion if $r_{ik} = 0.5$. In the case of $r_{jk} < 0.5$, then we have that x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value r_{ik} has to be equal to or lower than r_{ij}, being equal only in the case of $r_{jk} = 0.5$ as we have already shown. Interpreting the value $r_{ji} - 0.5$ as the intensity of strict preference of alternative x_j over x_i, then it seems reasonable to suppose that the intensity of preference of x_i over x_k should be equal to the sum of the intensities of preferences when using an intermediate alternative x_j, that is, $r_{ik} - 0.5 = (r_{ij} - 0.5) + (r_{jk} - 0.5)$. The same reasoning can be applied in the case of $r_{jk} > 0.5$ [26, 45, 46]. The fuzzy preference relation R, given above, verifies additive transitivity.

The following diagram shows all logical relationships between the defined transitivity conditions. We note that there is no relationship between weak-transitivity and triangle condition [30],

$$
\begin{array}{ccccccccc}
 & & & & (7) & & & & \\
 & & & & \Downarrow & & & & \\
(1) & \Leftarrow & (8) & \Rightarrow & (6) & \Rightarrow & (5) & \Rightarrow & (2) \\
 & & & & \Uparrow & & \Uparrow & & \\
 & & & & (4) & \Rightarrow & (3) & &
\end{array}
$$

In the following, we will show that max-max transitivity is not compatible with the reciprocity property. If a fuzzy preference relation verifies max-max transitivity and reciprocity then $r_{ik} \geq \max\{r_{ij}, r_{jk}\}\ \forall i, j, k$ and $r_{ij} = 1 - r_{ji}\}\ \forall i, j$, which implies:

$$1 - r_{ik} \leq 1 - \max\{r_{ij}, r_{jk}\}\ \forall i, j, k \Rightarrow r_{ki} \leq \min\{r_{kj}, r_{ji}\}\ \forall i, j, k$$

which contradicts max-max transitivity. The same conclusion can be obtained regarding max-min transitivity. Therefore both properties are not adequate properties to model the transitivity for fuzzy preference relations.

If we examine the relationship between restricted max-max transitivity and reciprocity, then we conclude that the fuzzy preference relation also has to verify the complementary restricted min-min transitivity, that is,

$$\forall i, j, k: \ \min\{r_{ij}, r_{jk}\} \leq 0.5 \Rightarrow r_{ik} \geq \min\{r_{ij}, r_{jk}\}.$$

However, nor restricted max-max transitivity nor restricted min-min transitivity imply reciprocity. For example, the following fuzzy preference relation

$$R = \begin{pmatrix} 0.5 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.7 \\ 0.1 & 0.3 & 0.5 \end{pmatrix}$$

verifies both restricted transitivity properties but it is not reciprocal. This does not imply that they are incompatible with the reciprocity property. In fact, a fuzzy preference relation can be reciprocal and still verify both restricted transitivity properties, as the one we would have obtained by changing the values r_{13} for 0.9 or the value r_{31} for 0.2.

If we examine the compatibility between the additive consistency property and reciprocity then we conclude that the first one implies the second one. Firstly, we show that additive consistency property implies indifference of preferences. Indeed, when $i = j = k$ additive consistency property reduces to $r_{ii} + r_{ii} + r_{ii} = 1.5 \ \forall i$ which implies $r_{ii} = 0.5 \ \forall i$. Secondly, we show that additive consistency property implies reciprocity property. If $k = i$ then additive consistency reduces to $r_{ij} + r_{ji} + r_{ii} = 1.5 \ \forall i, j$ and because we already have that $r_{ii} = 0.5 \ \forall i$ then $r_{ij} + r_{ji} = 1$ $\forall i, j$.

As shown in [26], additive transitivity for fuzzy preference relations can be seen as the parallel concept of Saaty's consistency property for multiplicative preference relations [41]:

Proposition 3. *Let $A = (a_{ij})$ be a consistent multiplicative preference relation, then the corresponding reciprocal fuzzy preference relation, $R = g(A)$ verifies additive transitivity property.*

In such a way, the following definition of a consistent fuzzy preference relation may be given:

Definition 2. *A reciprocal fuzzy preference relation $R = (r_{ij})$ is additive consistent if $r_{ij} + r_{jk} + r_{ki} = \frac{3}{2} \ \forall \, i, j, k = 1, \ldots, n.$*

In [26], Herrera-Viedma et. al. gave a characterisation of the consistency property defined by the additive transitivity property of a fuzzy preference relation $R^k = (r^k_{ij})$. Using this characterization method, a procedure was given to construct a consistent fuzzy preference relation $\widetilde{R}^k$ from a non-consistent fuzzy preference relation R^k. As in the case of multiplicative preference relations, if a fuzzy preference relation is additive transitivity then it can be represented by a unique (up to positive linear transformations) utility function. Additive transitivity has been used to obtain more consistent fuzzy preference relation from a given one (see [31]) and as shown in [1, 24, 25] it is also a valuable concept for incomplete fuzzy preference relations as it reduces experts' uncertainty when choosing values to estimate their unknown ones, which is not the case if other types of transitivity conditions were to be used.

4 Conflict Between Additive and Multiplicative Consistency Properties and Scales

There are many reasons that point in the direction of considering additive consistency as an adequate property to model transitivity of fuzzy preferences. However, a conflict between the additive consistency property and the scale used for providing the preference values, i.e., the closed interval [0, 1], can appear. To show this, we will use a simple example.

Let us suppose a set of three alternatives $\{x_1, x_2, x_3\}$ for which we have the following information: alternative x_1 is considerably more important than alternative x_2 and this one is demonstrably or very considerably more important than alternative x_3. Suppose that these statements are modelled using the following values $r_{12} = 0.75$ and $r_{23} = 9$ respectively. If we want to maintain the additive consistency property then we would obtain a negative value $r_{13} = 1.5 - r_{12} - r_{23} = -0.15$.

This conflict between the additive consistency property and the scale used for providing preference values suggests that a modification of this property where it acts incoherently has to be made. Because restricted max-max transitivity is the minimum condition required for a reciprocal fuzzy preference relation to be considered consistent, then the modification to introduce in the additive consistency property should maintain restricted max-max transitivity and, by reciprocity, the complementary restricted min-min transitivity.

Obviously, the same problem exists when dealing with multiplicative preference relations.The following simple example will show that there exists a conflict between the scales used to associate multiplicative preference values to judgements and the definition of consistency given by Saaty. Let us suppose a set of three alternatives $\{x_1, x_2, x_3\}$ on which an expert provides the following judgements: alternative x_1 is considerably more important than alternative x_2 and this one demonstrably or overwhelming more important than alternative x_3. In such a case, using Saaty's 1–9 scale, we would have the values $a_{12} = 5$ and $a_{23} = 7$.

On the one hand, if we want to maintain the multiplicative consistency property then, according to Saaty [40], we would have to assign the value $a_{13} = a_{12} \cdot a_{23} = 35$, and the only solution would be using the following consistent reciprocal multiplicative preference relation

$$A = \begin{pmatrix} 1 & 5 & 35 \\ 1/3 & 1 & 7 \\ 1/35 & 1/5 & 1 \end{pmatrix}.$$

Therefore, to avoid such a type of conflict we could proceed by choosing a different scale for providing judgements or by modifying the above definition. With respect to the first question, the use of any other scale of the form $[1/a, a],\ a \in \mathbb{R}^+$, would not make this conflict disappear, which means that the the only possible solution to overcome this conflict would consist of using the scale of pairwise comparison from 0 to $+\infty$. However, as Saaty points out in [40], this may not be useful at all because it assumes that the human judgement is capable of comparing the relative dominance of any two objects, which is not the case.

On the other hand, we note that if $a_{13} \in [7, 9]$ transitivity still holds. We analyze this fact by means of the measure of consistency proposed by Saaty. In [40] Saaty shows that a reciprocal multiplicative preference relation is consistent if and only if its maximum or principal eigenvalue λ_{max} is equal to the number of alternatives n. However, because perfect consistency is difficult to obtain in practice, especially when measuring preferences on a set with a large number of alternatives, Saaty defined a *consistency index ($CI = \lambda_{max} - n$)* that reflects the deviation from consistency of all the a_{ij} of a particular reciprocal multiplicative preference relation from the estimated ratio of priorities w_i/w_j.

A measure of inconsistency independent of the order of the reciprocal multiplicative preference relation is defined as the *consistency ratio (CR)*. This is obtained by taking the ratio of CI to the *random index (RI)*, which is an average consistency index of a sample set of randomly generated reciprocal matrices from the scale 1 to 9 (size 500 up to 11 by 11 matrices, and size 100 for squares matrices of orders 12, 13, 14 and 15). For this consistency measure, he proposed a threshold of 0.10 to accept the reciprocal multiplicative preference relation as consistent. When the *CR* is greater than 0.10 then, in order to improve consistency, those judgements with a greater difference a_{ij} and w_i/w_j, are usually modified and a new priority vector is derived.

In our previous example we observe that the conflict between the multiplicative consistency property and the scale used by Saaty arises because if we impose consistency then we get values outside the range $[1/9, 9]$. If we restrict the possible values of a_{13} to be in $[1/9, 9]$, then it is clear than in this case alternative x_1 should be considered as overwhelming more important than alternative x_3, and thus the value of a_{13} should be greater or equal to 7. If $a_{13} = 7$ we get a *CR* value of 0.25412, with $a_{13} = 8$ a *CR* value of 0.212892 and with $a_{13} = 9$ a *CR* value of 0.179714, all of them greater than the minimum 0.10 for considering any reciprocal multiplicative preference relation consistent in this situation.

All these considerations mean that if we do not change the scale used to associate preference values to judgement or want to have a homogeneous scale when working in a group decision context, then the above definitions of consistency of preference relations should be modified.

In the next section, we set out the properties to be verified by a $f\colon [0,1] \times [0,1] \to [0,1]$ so that it can be used to obtained r_{ik} from the pair of values $\left(r_{ij}, r_{jk}\right)$, that is, $r_{ik} = f\left(r_{ij}, r_{jk}\right)$.

5 Consistency Function of Preferences: Conditions to Verify

The assumption of experts being able to quantify their preferences in the domain [0,1] instead of {0, 1} or a set with finite cardinality, as it may be a set of linguistic labels [4, 27, 33, 34], underlies unlimited computational abilities and resources from the experts. Taking these unlimited computational abilities and resources into account we may formulate that an expert's preferences are consistent when for any three alternatives x_i, x_j, x_k their preference values are related in the exact form

$$r_{ik} = f(r_{ij}, r_{jk})$$

being f a function $f\colon [0,1] \times [0,1] \to [0,1]$. In what follows we will set out a set of conditions or properties to be verified by such a function f.

The above equality can be interpreted as the equation to solve in a a situation when we do want to compare the alternatives x_i and x_k, but cannot do it directly, but we have an alternative x_j of which we know the exact values of r_{ij} and r_{jk}. In this situation, we can establish a broad comparison between alternatives x_i and x_k on the basis of the values r_{ij} and r_{jk}. Indeed, we can distinguish the following cases:

Case 1. $r_{ij} = 0.5$ ($r_{jk} = 0.5$) which means that $x_i \sim x_j$ ($x_j \sim x_k$) and as a consequence the strength of preference between x_i and x_k should be the same as the one between x_j and x_k. We then have: $r_{ik} = r_{jk}$ ($r_{ik} = r_{jk}$).

Case 2. $r_{ij} > 0.5$ and $r_{jk} > 0.5$. In this case, alternative x_i is preferred to alternative x_j ($x_i \succ x_j$) and alternative x_j is preferred to alternative x_k ($x_j \succ x_k$). We then have that $x_i \succ x_j \succ x_k$ which implies $x_i \succ x_k$ and therefore $r_{ik} > 0.5$. Furthermore, in these cases restricted max-max transitivity should be imposed, which means that x_i should be preferred to x_k with a degree of intensity at least equal to the maximum of the intensities r_{ij} and r_{jk}: $r_{ik} \geq max\{r_{ij}, r_{jk}\}$, where the equality holds only when there exists indifference between at least one of the alternatives and x_j, i.e., $r_{ij} = 0.5$ or $r_{jk} = 0.5$, as we have said in case 1. As a result, in this case $r_{ik} > \max\{r_{ij}, r_{jk}\}$ should be verified.

Case 3. When $r_{ij} < 0.5$ and $r_{jk} < 0.5$, a similar argument to the one of case 2 leads to $r_{ik} < \min\{r_{ij}, r_{jk}\}$.

Case 4. One reference value is greater than 0.5 and the other is lower than 0.5. Suppose that $r_{ij} > 0.5$ and $r_{jk} < 0.5$. This is equivalent to $r_{ij} > 0.5$ and $r_{kj} = 1 - r_{jk} > 0.5$, that is: $x_i \succ x_j$ and $x_k \succ x_j$. The comparison of alternatives x_i and x_j is done by comparing the intensities of preferences of them over the alternative x_j. An indifference situation between x_i and x_k would exist only when both alternatives are preferred over x_j with the same intensity, while the alternative with greater intensity of preference over x_j should be preferred to the other one. This is summarized in the following way:

$$\left\{\begin{array}{l} x_i \sim x_k \text{ if } r_{ij} = r_{kj} \Leftrightarrow r_{ij} + r_{jk} = 1 \\ x_i \succ x_k \text{ if } r_{ij} > r_{kj} \Leftrightarrow r_{ij} + r_{jk} > 1 \\ x_i \prec x_k \text{ if } r_{ij} < r_{kj} \Leftrightarrow r_{ij} + r_{jk} < 1 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} r_{ik} = 0.5 \text{ if } r_{ij} + r_{jk} = 1 \\ r_{ik} > 0.5 \text{ if } r_{ij} + r_{jk} > 1 \\ r_{ik} < 0.5 \text{ if } r_{ij} + r_{jk} < 1 \end{array}\right\}$$

It is obvious that the greater the value $|r_{ij} + r_{jk} - 1|$ the greater $|r_{ik} - 0.5|$.

The following modification of the additive consistency property where it acts incoherently meet the above conditions:

$$f(x, y) = \begin{cases} \min\{x, y\} & x, y \in [0, 0.5] \\ \max\{x, y\} & x, y \in [0.5, 1] \\ x + y - 0.5 & \text{otherwise} \end{cases}$$

However, this function is not associative which is a necessary requirement for a function to be considered consistent in this context. Indeed, we have hat:

$$r_{ik} = f(r_{ij}, r_{jk}) = f\left(r_{ij}, f(r_{il}, r_{lk})\right); \quad r_{ik} = f(r_{il}, r_{lk}) = f\left(f(r_{ij}, r_{jl}), r_{lk}\right)$$

and therefore it is true that:

$$f\left(f(r_{ij}, r_{jl}), r_{lk}\right) = f\left(r_{ij}, f(r_{il}, r_{lk})\right)$$

In terms of function f, case 1 implies $f(0.5, x) = f(x, 0.5) = x \ \forall x \in [0, 1]$, which means that function f has neutral element 0.5. In particular, $f(0.5, 0.5) = 0.5$ which means that the neutral element of f is idempotent. This property in conjunction with case 2 mean that function f behaves in $[0.5, 1] \times [0.5, 1]$ as a t-conorm, while in conjunction with case 3 mean that function f behaves in $[0, 0.5] \times [0, 0.5]$ as a t-norm. Clearly, other properties desirable to be verified by such a function f include that of being continuous except maybe in the points (0, 1) and (1,0); increasing with respect to both arguments x and y; and commutative.

We make note that uninorm operators present all the above properties, which may suggest that function f may belong to the class of uninorms operators. As said in Sect. 3, multiplicative transitivity is a uninorm, and therefore it may be taken as the condition to be verified for a fuzzy preference relation to be considered as consistent.

6 Conclusions

In a GDM problem experts may provide their preferences by means of different preference representation formats. The integration of heterogeneous information is therefore an important issue to be addresses in these situations. A review of the main results regarding the relationships between the numerical representation formats was provided, and it was suggested that the fuzzy preference relation “is preferred to” representing the strength of preference of one alternative over another in the scale [0, 1] could be used as the base element to integrate these different preference representation formats in group decision making situations.

Once preferences are provided by the expert, the problem of measuring the consistency of these preferences becomes crucial to get ‘good’ solutions. While for crisp and multiplicative preference relations there might exist an agreement on the properties to be satisfied in order to be considered consistent, this is not the case for fuzzy preference relations. Indeed, for a fuzzy preference relation to be considered consistent, many different properties have been suggested. Most of these properties are related in some way to the traditional concept of transitivity. One of this properties is the additive transitivity property, which is equivalent to Saaty’s consistency property for multiplicative preference relations. However, both consistency properties are in conflict with the corresponding scale used for providing the preferences. I order to overcome this conflict, a set of conditions have been set for reciprocal fuzzy preference relations to be considered ‘fully consistent.’ These set of conditions suggest that consistency might be represented by a uninorm operator. One of the suggested properties to model consistency for fuzzy preference relations, the multiplicative consistency, introduced by Tanino in 1988, is an example of a such operator.

References

1. S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, J. Alcalá-Fernández and C. Porcel, “A consistency based procedure to estimate missing pairwise preference values,” **Int. J. Intell. Syst.,** accepted for publication, 2005.
2. L. Basile, “Ranking alternatives by weak transitivity relations,” in J. Kacprzyk, M. Fedrizzi (Eds.) **Multiperson Decision Making using Fuzzy Sets and Possibility Theory,** Kluwer Academic Publishers, Dordrecht, pp. 105 – 112, 1990.
3. P.P. Bonissone and K.S. Decker, “Selecting uncertainty calculi and granularity: an experiment in trading-off and complexity”. In Kanal, Lemmer (Eds.) *Uncertainty in Artificial Intelligence* (North-Holland, 1986) 217–247.
4. F. Chiclana, F. Herrera, E. Herrera-Viedma and M.C. Poyatos, “A classification method of alternatives for preference ordering criteria based on fuzzy majority,” **The Journal of Fuzzy Mathematics** Vol. 4, No. 4, pp. 801–813, 1996.
5. F. Chiclana, F. Herrera and E. Herrera-Viedma, “Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations,” **Fuzzy Sets Syst.,** Vol. 97, No. 1, pp. 33–48, July 1998.
6. F. Chiclana, F. Herera and E. Herera-Viedma, “Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations,” **Fuzzy Sets Syst.,** Vol. 122, No. 2, pp. 277–291, Sept. 2001.
7. F. Chiclana, F. Herrera and E. Herrera-Viedma, “A note on the internal consistency of various preference representations,” **Fuzzy Sets Syst.,** Vol. 131, No. 1, pp. 75–78, Oct. 2002.
8. B. De Baets, H. De Meyer and D. De Schuymer, “Cyclic evaluation of transitivity of reciprocal relations,” **Soc. Choice Welfare,** Vol.26, pp. 217–238, 2006.
9. D. Dubois and H. Prade, **Fuzzy sets and systems: Theory and application.** Academic Press, New York, 1980.

10. D. Dubois and H. Prade, "A review of fuzzy sets aggregation connectives," **Information Sciences,** Vol. 36,pp. 85–121, 1985.
11. T. Evangelos, **Multi-Criteria Decision Making Methods: A Comparative Study**, Dordrecht: Kluwer Academic Publishers, 2000.
12. Z.-P. Fan, S.-H. Xiao and G.-H Hu, "An optimization method for integrating two kinds of preference information in group decision-making," **Comput. Ind. Eng.,** Vol. 46, pp. 329–335, 2004.
13. P.C. Fishburn, **The Theory of Social Choice,** Princeton University Press, 1973.
14. P.C. Fishburn, **Utility Theory for Decision Making,** Robert E. Krieger Publishing Company, 1979.
15. P.C. Fishburn, "Preference structures and their numerical representations," **Theoretical Computer Science** Vol. 217, pp. 359–383, 1999.
16. J.C. Fodor and M. Roubens, "Fuzzy preference modelling and multicriteria decision support." Kluwer, Dordrecht, 1994.
17. J.C. Fodor, R.R. Yager and A. Rybalov, "Structure of Uninorms," **Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems,** Vol. 5, pp. 411–127, 1997.
18. J.L. García-Lapresta and R.A. Marques, "Constructing reciprocal and stable aggregation operators," Proceedings of AGOP'2003, University of Alcalá de Henares, Madrid, Spain, pp. 73–78, 2003.
19. J.L. García-Lapresta L.C. Meneses, "An empirical analysis of transitivity with four scaled preferential judgment modalities," **Review of Economic Design,** Vol. 8, pp. 335–346, 2003.
20. J.L. García-Lapresta L.C. Meneses, " Individual valued preferences and their aggregation: Analysis of consistency in a real case," **Fuzzy Sets Syst.,** Vol. 151, pp. 269–284, 2005.
21. J.L. García-Lapresta and J. Montero, "Consistency in preference modelling." In: **Modern Information Processing: From Theory to Applications** (eds. B. Bouchon-Meunier, G. Coletti, R.R. Yager), Elsevier, pp. 87–97, 2006.
22. F. Herrera, E. Herrera-Viedma and F. Chiclana, "Multiperson decision making based on multiplicative preference relations," **Eur. J. Oper. Res.,** Vol. 129, pp. 372–385, 2001.
23. F. Herera, L. Martínez and P.J. Sánchez, "Managing non-homogeneous information in group decision making," **Eur. J. Oper. Res.,** Vol. 166, pp. 115–132, 2005.
24. E. Herrera-Viedma, S. Alonso, F. Chiclana and F. Herrera, "A consensus model for group decision making with incomplete fuzzy preference relations," **IEEE Trans. Fuzzy Syst.,** accepted for publication 2006.
25. E. Herrera-Viedma, F. Chiclana, F. Herrera and S. Alonso, "A group decision-making model with incomplete fuzzy preference relations based on additive consistency," **IEEE Trans. Syst., Man, Cybern. B: Cybernetics,** accepted for publication 2006.
26. E. Herera-Viedma, F. Herrera, F. Chiclana and M. Luque, "Some issues on consistency of fuzzy preference relations," **Eur. J. Oper. Res.,** Vol. 154, No. 1, pp. 98–109, Apr. 2004.
27. E. Herrera-Viedma, L. Martínez, F. Mata and F. Chiclana, "A consensus support system model for group decision-making problems with multi-granular linguistic preference relations," **IEEE Trans. Fuzzy Syst.,** Vol. 13, No. 5, pp. 644–658, 2005.
28. E.P. Klement, R. Mesiar and E. Pap, "On the relationship of associative compensatory operators to triangular norms and conorms,"**Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems,** Vol. 4, pp. 129–144, 1996.
29. W. Kołodziejczyk, "Orlovsky's concept of decision-making with fuzzy preference relations – Further results," **Fuzzy Sets Syst.,** Vol. 19, pp. 11–20, 1986.
30. R.D. Luce and P. Suppes, "Preferences, Utility and Subject Probability," in: R.D. Luce et al. (Eds.) *Handbook of Mathematical Psychology*, Vol. III, Wiley, New York, pp. 249–410, 1965.
31. J. Ma, Z-P. Fan,Y-P. Jiang, J.Y. Mao and L. Ma, "A method for repairing the inconsistency of fuzzy preference relations," **Fuzzy Sets Syst.,** Vol. 157, No. 1, pp. 20–33, 2006.
32. G.H. Marakas, **Decision Support Systems in the 21th Century,** 2nd edition, New Jersey: Pearson Education, Inc., 2003.
33. Marimin, M. Umano, I. Hatono and H. Tamure, "Linguistic labels for expressing fuzzy preference relations in fuzzy group decision making", **IEEE Trans. Syst., Man, Cybern. B: Cybernetics,** Vol. 28, pp. 205–218, 1998.

34. G.A. Miller, "The magical number seven or minus two: some limits on our capacity of processing information", *Psychological Rev.,* Vol. 63, pp. 81–97, 1956.
35. F.J. Montero and J. Tejada, "Some problems on the definition of fuzzy preference relations," **Fuzzy Sets Syst.,** Vol. 20, pp. 45–53, 1986.
36. F.J. Montero and J. Tejada, "A necessary and sufficient condition for the existence of Orlovsky's choice set," **Fuzzy Sets Syst.,** Vol. 26, pp. 121–125, 1988.
37. S.A. Orlovski, "Decision-making with fuzzy preference relations," **Fuzzy Sets Syst.,** Vol. 1, pp. 155–167, 1978.
38. S. Ovchinnikov and M. Roubens, "On fuzzy stric preference, indifference, and incomparability relations," **Fuzzy Sets Syst.,** Vol. 49, No. 1, pp. 15–20, 1992.
39. P. Perny and M. Roubens, "Fuzzy preference modeling." In R. Slowinski (Edt.) **Fuzzy sets in decision analysis, operation research and statistics** (Massachusetts: Kluwer Academic Publishers, 1998) pp. 3–30.
40. Th. L. Saaty, **The Analytic Hierarchy Process,** McGraw-Hill, New York, 1980.
41. Th.L. Saaty, **Fundamentals of Decision Making and Priority Theory with the AHP,** Pittsburg: RWS Publications, 1994.
42. F. Seo and M. Sakawa, "Fuzzy Multiattribute Utility Analysis for Collective Choice," *IEEE Transactions on Systems, Man, and Cybernetics,* Vol. 15, No. 1, pp. 45–53, 1985.
43. A.K. Sen, "Social Choice Theory : A Re-Examination," **Econometrica,** Vol. 45, pp. 53–89, 1977.
44. W. Silvert, "Symmetric summation: A class of operations on fuzzy sets," *IEEE Transactions on Systems, Man, and Cybernetics,* Vol. 9, pp. 657–659, 1979.
45. T. Tanino, "Fuzzy preference orderings in group decision making," **Fuzzy Sets Syst.,** Vol. 12, pp. 117–131, 1984.
46. T. Tanino, "Fuzzy preference relations in group decision making," in J. Kacprzyk and M. Roubens (Eds.) **Non-Conventional Preference Relations in Decision Making,** Springer-Verlag, Berlin, 1988, pp. 54–71.
47. T. Tanino, "On Group Decision Making Under Fuzzy Preferences," in: J. Kacprzyk and M. Fedrizzi (Eds.) *Multiperson Decision Making Using Fuzzy Sets and Possibility Theory*, Kluwer Academic Publishers, Dordrech, pp. 172–185, 1990.
48. X. Wang, "An investigation into relations between some transitivity-related concepts," **Fuzzy Sets Syst.,** Vol. 89, pp. 257–262, 1997.
49. Q. Zhang, J.C.H. Chen, Y.-Q. He, J. Ma and D.-N Zhou, "Multiple attribute decision making: approach integrating subjective and objective information," **Int. J. Manuf. Technol. Manag.,** Vol. 5, pp. 338–361, 2003.
50. Q. Zhang, J.C.H. Chen and P.P. Chong, "Decision consolidation: criteria weight determination using multiple preference formats," **Decis. Support Syst.,** Vol. 38, pp. 247–258, 2004.
51. L. A. Zadeh, "Similarity relations and fuzzy orderings," **Information Sciences** Vol. 3, pp. 177–200, 1971.
52. H.-J. Zimmermann, **Fuzzy Set Theory and Its Applications.** Kluwer, Dordrecht, 1991.