2 Axion Cosmology

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Abstract. The cosmological properties of axions are reviewed. We discuss the axions produced by thermal processes in the early universe, the evolution of the average axion field between the Peccei-Quinn and QCD phase-transitions, the domain-wall problem and its possible resolutions, the population of cold axions produced by vacuum realignment, string decay and domain wall decay, and, finally, axion miniclusters and axion isocurvature perturbations.

For background information on the strong CP problem and on laboratory and astrophysical axion constraints, the reader is referred to Chaps. 1 and 3. In this chapter, we will be concerned only with the cosmological properties of axions. We merely mention that different authors may use different definitions of the axion decay constant f_a. We define normalization by the action density for QCD plus an axion

$$L_{\mathrm{QCD}+a} = -\frac{1}{4} G^b_{\mu\nu} G^{b\mu\nu} + \frac{1}{2} \partial_\mu a \, \partial^\mu a + \sum_q \bar{q} \left(\mathrm{i}\gamma^\mu \partial_\mu - m_q \right) q + \frac{g_\mathrm{s}^2}{32\,\pi^2} \left(\theta + \frac{a}{f_a} \right) G^b_{\mu\nu} \tilde{G}^{b\mu\nu} \,, \quad (2.1)$$

where a is the axion field before mixing with the η and π^0 mesons. Equation (2.1) uses standard notation for the chromomagnetic field strength tensor G, the strong coupling constant g_s, and the quark fields q. The axion mass, after mixing with the η and π^0 mesons, is given in terms of f_a by

$$m_a \simeq 6\,\mu\mathrm{eV} \left(\frac{10^{12}\,\mathrm{GeV}}{f_a} \right) . \quad (2.2)$$

The axion-decay constant is related to the magnitude v_a of the vacuum expectation value that breaks the $U(1)_{\mathrm{PQ}}$ symmetry by $f_a = v_a/N$. N is an integer characterizing the color anomaly of $U(1)_{\mathrm{PQ}}$, and $N = 6$ in the original Peccei-Quinn-Weinberg-Wilczek axion model. All axion couplings are inversely proportional to f_a.

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2.1 Thermal Axions

Axions are created and annihilated during interactions among particles in the primordial soup. Let us call the population of axions established as a result of such processes "thermal axions", to distinguish them from the population of "cold axions" which we discuss later. The number density $n_a^{\rm th}(t)$ of thermal axions solves the Boltzmann equation [1]

$$\frac{{\rm d}n_a^{\rm th}}{{\rm d}t} + 3Hn_a^{\rm th} = \Gamma \left(n_a^{\rm eq} - n_a^{\rm th}\right) \ , \tag{2.3}$$

where

$$\Gamma = \sum_i n_i \langle \sigma_i v \rangle \tag{2.4}$$

is the rate at which axions are created and annihilated. $H(t)$ is the Hubble expansion rate and

$$n_a^{\rm eq} = \frac{\zeta(3)}{\pi^2} T^3 \tag{2.5}$$

is the number density of axions at thermal equilibrium, where $\zeta(3) = 1.202\ldots$ is the Riemann zeta function of argument 3. In (2.4), the sum is over processes of the type $a + i \leftrightarrow 1 + 2$, where 1 and 2 are other particle species, n_i is the number density of particle species i, σ_i is the corresponding cross section, and $\langle \ldots \rangle$ indicates averaging over the momentum distributions of the particles involved.

Unless unusual events are taking place, $T \propto R^{-1}$ where $R(t)$ is the scale factor, and (2.5) implies, therefore,

$$\frac{{\rm d}n_a^{\rm eq}}{{\rm d}t} + 3Hn_a^{\rm eq} = 0 \ . \tag{2.6}$$

Combining (2.6) and (2.3), one obtains

$$\frac{\rm d}{{\rm d}t}\left[R^3(n_a^{\rm th} - n_a^{\rm eq})\right] = -\Gamma R^3 \left(n_a^{\rm th} - n_a^{\rm eq}\right) \ . \tag{2.7}$$

This equation implies that a thermal distribution of axions is approached exponentially fast whenever the condition

$$\Gamma > H \tag{2.8}$$

is satisfied. So, we have a thermal population of axions today, provided the inequality (2.8) prevailed for a few expansion times at some point in the early universe and the thermal population of axions thus established did not subsequently get diluted away by inflation or some other cause of huge entropy release.

The least model-dependent processes for thermalizing axions in the early universe are (a) $a + q(\bar{q}) \leftrightarrow g + q(\bar{q})$, (b) $a + g \leftrightarrow q + \bar{q}$, and (c) $a + g \leftrightarrow g + g$.

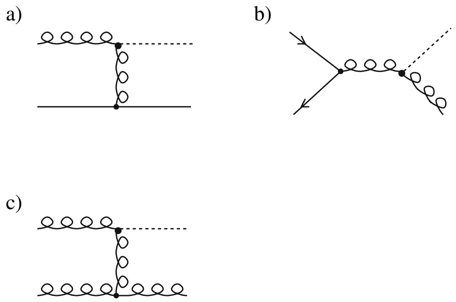

Fig. 2.1. Processes which produce thermal axions in the early universe

The corresponding diagrams are shown in Fig. 2.1. These processes involve only the coupling of the axion to gluons, present in any axion model, and the coupling of quarks to gluons.

A detailed treatment is provided in [2]. We give only a rough estimate here. The processes of Fig. 2.1 have cross sections of order

$$\sigma \sim \frac{\alpha_{\rm s}^3}{8\pi^2}\,\frac{1}{f_a^2}\,, \tag{2.9}$$

where $\alpha_s = g_s^2/4\pi$. At temperatures $T > 1\,\text{TeV}$, the densities of quarks, antiquarks, and gluons are

$$n_q = n_{\bar q} = 27\,\frac{\zeta(3)}{\pi^2}\,T^3 \quad \text{and} \quad n_g = 16\,\frac{\zeta(3)}{\pi^2}\,T^3\,. \tag{2.10}$$

The Hubble rate is given by the Friedmann equation

$$H^2 = \frac{8\pi\,G}{3}\,\varrho = \frac{8\pi\,G}{3}\,\left[\mathcal{N}_b(T) + \frac{7}{8}\,\mathcal{N}_f(T)\right]\,\frac{\pi^2}{30}\,T^4\,, \tag{2.11}$$

where $\mathcal{N}_b(T)$ and $\mathcal{N}_f(T)$ are, respectively, the total effective number of bosonic and fermionic spin degrees of freedom at temperature T. For $T > 1\,\text{TeV}$,

$$\mathcal{N} \equiv \mathcal{N}_b + \frac{7}{8}\mathcal{N}_f = 107.75 \tag{2.12}$$

if we assume no new degrees of freedom other than those of the Standard Model plus an axion. Combining everything and setting $\alpha_{\rm s} \simeq 0.03$, one finds

$$\sum_{i=1}^{3} \frac{n_i\langle\sigma_i v\rangle}{H} \sim 2\left(\frac{10^{12}\,\text{GeV}}{f_a}\right)^2\,\frac{T}{10^{12}\,\text{GeV}}\,. \tag{2.13}$$

Thus we find that the processes of Fig. 2.1 keep axions in thermal equilibrium with the primordial soup until the temperature

$$T_{\rm D} \sim 5 \times 10^{11}\,{\rm GeV} \left(\frac{f_a}{10^{12}\,{\rm GeV}}\right)^2 . \tag{2.14}$$

Note that the calculation is not valid when $T \gtrsim v_a = N f_a$, as the PQ symmetry is restored then. In particular, in view of (2.14), the processes under consideration produce a population of thermal axions only if $f_a \lesssim 2N\,10^{12}$ GeV.

We will see in Sect. 2.4 that f_a has to be less than approximately 10^{12} GeV to avoid overclosing the universe with cold axions. That bound suggests that the processes of Fig. 2.1 do produce a population of thermal axions. We should keep in mind, however, that this thermal axion population may be wiped out by a period of inflation with reheat temperature less than $T_{\rm D}$. So it is interesting to search for processes that may re-establish a thermal axion population later on. We briefly discuss two such possibilities.

First we consider the Compton-like scattering process $Q + g \leftrightarrow Q + a$, where Q may be a known quark or a new heavy quark. There are related processes in which the gluon is replaced by a photon or Z-boson, and/or the quark is replaced by a lepton. However $Q + g \leftrightarrow Q + a$ is the least model-dependent among the Compton-like processes because, in every axion model, there is at least one colored fermion Q that carries PQ charge and hence to which the axion couples. The cross section is

$$\sigma_Q \sim \frac{\alpha_s}{v_a^2} \times \begin{cases} (m_Q/T)^2 & \text{for } \; T > m_Q \; , \\ 1 & \text{for } \; T < m_Q \; . \end{cases} \tag{2.15}$$

The relevant regime is when $T > m_Q$ as the Q number density is Boltzmann suppressed for $T < m_Q$. Using $\alpha_s = 0.05$ and $\mathcal{N} = 107.75$, we have

$$\frac{n_Q \langle \sigma_Q v \rangle}{H} \sim \frac{m_Q^2}{T\,(2 \times 10^7\,{\rm GeV})} \left(\frac{10^{12}\,{\rm GeV}}{v_a}\right)^2 \tag{2.16}$$

for $T > m_Q$. So this process produces a population of thermal axions provided

$$m_Q \gtrsim 2 \times 10^7\,{\rm GeV} \left(\frac{v_a}{10^{12}\,{\rm GeV}}\right)^2 . \tag{2.17}$$

The axions decouple then at a temperature $T_{\rm D} \sim m_Q$.

Let us also consider the process $\pi + \pi \leftrightarrow \pi + a$. As the axion necessarily mixes with the π^0, this process is model-independent as well. It occurs at $T \sim 200$ MeV, after the QCD phase-transition but before the pions have annihilated. The cross section is of order

$$\sigma_\pi \sim f_a^{-2} . \tag{2.18}$$

Using $\mathcal{N} = 17.25$, we find

$$\frac{n_\pi \langle \sigma_\pi v \rangle}{H} \sim \left(\frac{3 \times 10^8\,{\rm GeV}}{f_a}\right)^2 \tag{2.19}$$

at $T \sim m_\pi$. The $\pi + \pi \leftrightarrow \pi + a$ process has the advantage of occurring very late so that any thermal axion population it establishes cannot be wiped out by inflation – inflation occurring that late would also wipe out the baryons. However, (2.19) indicates that it is ineffective unless the bound $f_a > 10^9\,\mathrm{GeV}$ from SN 1987A is saturated.

We have seen that, under a broad set of circumstances, a population of relic thermal axions is produced. For $f_a > 10^9\,\mathrm{GeV}$, the axion lifetime exceeds by many orders of magnitude the age of the universe. Between their last decoupling, at temperature T_D, and today, the thermal axion population is merely diluted and redshifted by the expansion of the universe. Their present number density is

$$n_a^{\mathrm{th}}(t_0) = \frac{\zeta(3)}{\pi^2}\, T_\mathrm{D}^3 \left(\frac{R_\mathrm{D}}{R_0}\right)^3 , \tag{2.20}$$

where R_D/R_0 is the ratio of scale factors between the time t_D of decoupling and today. Their average momentum is

$$\left\langle p_a^{\mathrm{th}}(t_0)\right\rangle = \frac{\pi^4}{30\zeta(3)}\, T_\mathrm{D}\, \frac{R_\mathrm{D}}{R_0} = 2.701\, T_\mathrm{D}\, \frac{R_\mathrm{D}}{R_0} . \tag{2.21}$$

If $\left\langle p_a^{\mathrm{th}}(t_0)\right\rangle \gg m_a$, the energy distribution is thermal with temperature

$$T_{a0} = T_\mathrm{D}\, \frac{R_\mathrm{D}}{R_0} . \tag{2.22}$$

If there is neither inflation nor any other form of entropy release, from t_D until the present, T_{a0} is related to the present cosmic microwave background temperature $T_{\gamma 0} = 2.735\,\mathrm{K}$ by the conservation of entropy. As electron-positron annihilation occurs after neutrino decoupling, one finds

$$T_{a0} = \left(\frac{10.75}{\mathcal{N}_\mathrm{D}}\, \frac{4}{11}\right)^{1/3} T_{\gamma 0} = 0.905\,\mathrm{K} \left(\frac{106.75}{\mathcal{N}_\mathrm{D}}\right)^{1/3} . \tag{2.23}$$

The average momentum of relic thermal axions is

$$\left\langle p_a^{\mathrm{th}}(t_0)\right\rangle = 2.1 \times 10^{-4}\,\mathrm{eV} \left(\frac{106.75}{\mathcal{N}_\mathrm{D}}\right)^{1/3} , \tag{2.24}$$

and their number density is

$$n_a^{\mathrm{th}}(t_0) = 7.5\,\mathrm{cm}^{-3}\, \frac{106.75}{\mathcal{N}_\mathrm{D}} . \tag{2.25}$$

2.2 Axion Field Evolution

The thermal axions discussed in the previous section are quantum fluctuations about the average background value of the axion field. The evolution of the average axion field, from the moment $U(1)_{\rm PQ}$ gets spontaneously broken during the PQ phase-transition to the moment the axion acquires mass during the QCD phase-transition, is the topic of this section.

The $U(1)_{\rm PQ}$ symmetry gets spontaneously broken at a critical temperature $T_{\rm PQ} \sim v_a$, where v_a is the vacuum expectation value of a complex field $\phi(x)$. The action density for this field is of the form

$$L_\phi = \frac{1}{2}\,\partial_\mu \phi^\dagger\,\partial^\mu \phi - \frac{\lambda}{4}\,\left(\phi^\dagger\phi - v_a^2\right)^2 + \dots\ , \tag{2.26}$$

where the dots represent interactions with other fields in the theory. At $T > T_{\rm PQ}$, the free energy has its minimum at $\phi = 0$. At $T < T_{\rm PQ}$, the minimum is a circle, the radius of which quickly approaches v_a as T decreases. Afterwards

$$\langle\phi(x)\rangle = v_a\,{\rm e}^{{\rm i}a(x)/v_a}\ , \tag{2.27}$$

where $a(x)$ is the axion field before mixing with the π^0 and η mesons. The field $a(x)$ has random initial conditions. In particular, at two points outside each other's causal horizon, the values of $a(x)$ are completely uncorrelated.

It is well-known that the size of the causal horizon is hugely modified during cosmological inflation. Without inflation, the size of the causal horizon is of order t, the age of the universe. But, during an inflationary epoch, the causal horizon grows exponentially fast and becomes enormous compared to t. There are two cases to consider.

- **Case 1:** Inflation occurs with reheat temperature smaller than $T_{\rm PQ}$, and the axion field is homogenized over enormous distances. The subsequent evolution of this zero-momentum mode is relatively simple.
- **Case 2:** Inflation occurs with reheat temperature larger than $T_{\rm PQ}$. In addition to the zero mode, the axion field has non-zero modes and carries strings and domain walls as topological defects.

The early universe is assumed to be homogeneous and isotropic. Its curvature is negligible. The space-time metric can therefore be written in the Robertson-Walker form

$$-\,{\rm d}s^2 = {\rm d}t^2 - R(t)^2\,{\rm d}\mathbf{x}\cdot{\rm d}\mathbf{x}\ , \tag{2.28}$$

where the $\mathbf{x}$ are co-moving spatial coordinates and $R(t)$ is the scale factor. The equation of motion for $a(x)$ in this space-time is

$$D_\mu\partial^\mu a(x) + V_a'[a(x)] = \left(\partial_t^2 + 3\,\frac{\dot R}{R}\,\partial_t - \frac{1}{R^2}\,\nabla_{\mathbf{x}}^2\right) a(x) + V_a'[a(x)] = 0\ , \tag{2.29}$$

where V_a is the effective potential for the axion field and prime indicates a derivative with respect to a. V_a results from non-perturbative QCD effects associated with instantons [3, 4]. They break the $U(1)_{\rm PQ}$ symmetry down to a $Z(N)$ discrete subgroup [5]. $V_a(a)$ is therefore periodic with period $\Delta a = 2\pi\,v_a/N = 2\pi f_a$. We may write such a potential qualitatively as

$$V_a = f_a^2\, m_a^2(t)\; [1 - \cos{(a/f_a)}]\ , \tag{2.30}$$

where the axion mass $m_a(t) = m_a[T(t)]$ is a function of temperature and hence of time. Equation (2.27) implies that the axion field has a range $0 \le a \le 2\pi v_a$. Hence there are N degenerate vacua. The discrete degeneracy implies the existence of domain walls, which will be discussed in Sect. 2.3.

Substituting (2.30) into (2.29), the equation of motion becomes

$$\left(\partial_t^2 + 3\,\frac{\dot{R}}{R}\,\partial_t - \frac{1}{R^2}\,\nabla_{\mathbf{x}}^2\right)\, a(x)\, m_a^2(t)\, f_a \sin\left(\frac{a(x)}{f_a}\right) = 0\ . \tag{2.31}$$

The non-perturbative QCD effects associated with instantons have amplitudes proportional to

$$\mathrm{e}^{-2\pi/\alpha_s(T)} \simeq \left(\frac{\Lambda_{\mathrm{QCD}}}{T}\right)^{11-2N_f/3}\ , \tag{2.32}$$

where N_f is the number of quark flavors with mass less than T. Equation (2.32) implies that the axion mass is strongly suppressed at temperatures that are large compared to the QCD scale but turns on rather abruptly when the temperature approaches Λ_{QCD}.

Because the first three terms in (2.31) are proportional to t^{-2}, the axion mass is unimportant in the evolution of the axion field until $m_a(t)$ becomes of order t^{-1}. Let us define a critical time t_1 by

$$m_a(t_1)\, t_1 = 1\ . \tag{2.33}$$

The axion mass effectively turns on at t_1. From a calculation of the effects [6, 7, 8] of QCD instantons at high temperature [9] one obtains

$$m_a(T) \simeq 4\times 10^{-9}\,\mathrm{eV}\left(\frac{10^{12}\,\mathrm{GeV}}{f_a}\right)\left(\frac{\mathrm{GeV}}{T}\right)^4\ , \tag{2.34}$$

when T is near 1 GeV. The relation between T and t follows from (2.11) and $H = 1/2t$. The total effective number $\mathcal{N}$ of thermal spin degrees of freedom is changing near 1 GeV temperature from a value near 60, valid above the quark-hadron phase transition, to a value of order 30 below that transition. Using $\mathcal{N} \simeq 60$, one has

$$m_a(t) \simeq 0.7\times 10^{20}\,\mathrm{s}^{-1}\left(\frac{t}{1\,\mathrm{s}}\right)^2\left(\frac{10^{12}\,\mathrm{GeV}}{f_a}\right)\ , \tag{2.35}$$

which implies

$$t_1 \simeq 2\times 10^{-7}\,\mathrm{s}\left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right)^{1/3}\ . \tag{2.36}$$

The corresponding temperature is

$$T_1 \simeq 1\,\mathrm{GeV}\left(\frac{10^{12}\,\mathrm{GeV}}{f_a}\right)^{1/6} . \qquad (2.37)$$

Equation (2.35) implies $\mathrm{d}\ln[m_a(t)]/\mathrm{d}t < m_a(t)$ after t_1. So, at least for a short while below 1 GeV, as long as (2.34) remains valid, the axion mass changes adiabatically. The number of axions is the adiabatic invariant. Conservation of the number of axions after t_1 allows us to estimate the energy density of axions today from an estimate of their number density at t_1. When the temperature drops well below 1 GeV, the dilute instanton gas calculations that yield (2.34) are no longer reliable. Complicated things happen such as the confinement and chiral symmetry-breaking phase-transitions. However, because $m_a \gg H$ then, it is reasonable to expect the number of axions to be conserved, at least in order of magnitude.

2.2.1 Zero Mode

In case 1, where inflation occurs after the PQ phase-transition, the axion field is homogenized over enormous distances. Equation (2.31) becomes [6, 7, 8]

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{3}{2t}\frac{\mathrm{d}}{\mathrm{d}t}\right) a(t) + m_a^2(t)\, f_a \sin\left(\frac{a(t)}{f_a}\right) = 0 , \qquad (2.38)$$

where we used $R(t) \propto t^{1/2}$. For $t \ll t_1$, we may neglect m_a. The solution is then

$$a(t) = a_0 + a_{1/2}\, t^{-1/2} , \qquad (2.39)$$

where a_0 and $a_{1/2}$ are constants. Equation (2.39) implies that the expansion of the universe slows the axion field down to a constant value.

When t approaches t_1, the axion field starts oscillating in response to the turn-on of the axion mass. We will assume that the initial value of a is sufficiently small that $f_a \sin(a/f_a) \simeq a$. Let us define ψ by

$$a(t) \equiv t^{-3/4}\psi(t) . \qquad (2.40)$$

The equation for $\psi(t)$ is

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2(t)\right] \psi(t) = 0 , \qquad (2.41)$$

where

$$\omega^2(t) = m_a^2(t) + \frac{3}{16\, t^2} . \qquad (2.42)$$

For $t > t_1$, we have $\mathrm{d}\ln\omega/\mathrm{d}t \ll \omega \simeq m_a$. That regime is characterized by the adiabatic invariant $\psi_0^2(t)\,\omega(t)$, where $\psi_0(t)$ is the changing oscillation amplitude of $\psi(t)$. We have, therefore,

$$\psi(t) \simeq \frac{C}{\sqrt{m_a(t)}} \cos\left[\int^t \mathrm{d}t'\,\omega(t')\right] , \tag{2.43}$$

where C is a constant. Hence

$$a(t) = A(t) \cos\left[\int^t \mathrm{d}t'\omega(t')\right] , \tag{2.44}$$

with

$$A(t) = \frac{C}{\sqrt{m_a(t)}}\, t^{-3/4} . \tag{2.45}$$

Hence, during the adiabatic regime,

$$A^2(t)\, m_a(t) \propto t^{-3/2} \propto R(t)^{-3} . \tag{2.46}$$

The zero-momentum mode of the axion field has energy density $\varrho_a = \frac{1}{2}\, m_a^2\, A^2$ and describes a coherent state of axions at rest with number density $n_a = \frac{1}{2}\, m_a A^2$. Equation (2.46) states, therefore, that the number of zero-momentum axions per co-moving volume is conserved. The result holds as long as the changes in the axion mass are adiabatic.

We estimate the number density of axions in the zero-momentum mode at late times later by saying that the axion field has a random initial value $a(t_1) = f_a\alpha_1$ and evolves according to (2.44) and (2.45) for $t > t_1$, where α_1 is called the "initial misalignment angle." As the effective potential for a is periodic with period $2\pi f_a$, the relevant range of α_1 values is $-\pi$ to $+\pi$. The number density of zero-momentum axions at time t_1 is then [6, 7, 8]

$$n_a^{\mathrm{vac},0}(t_1) \sim \frac{1}{2}\, m_a(t_1)\, a^2(t_1) = \frac{f_a^2}{2t_1}\, \alpha_1^2 , \tag{2.47}$$

where we used (2.33). We will use (2.47) in Sect. 2.4 to estimate the zero-mode contribution to the cosmological energy density of cold axions.

A more precise treatment would solve (2.38) for $t \sim t_1$, e.g. by numerical integration, to obtain the exact interpolation between the sudden ($t < t_1$) and adiabatic ($t > t_1$) regimes. An additional improvement is to solve (2.38) without linearizing the sine function, thus allowing large values of α_1. Although these improvements are desirable, they would still leave the number of axions unknown in case 1 because the initial misalignment angle α_1 is unknown. In case 2, the zero-mode contribution to the axion number density is also given by (2.47), but the misalignment angle α_1 varies randomly from one horizon to the next.

2.2.2 Non-Zero Modes

In case 2, where there is no inflation after the PQ phase-transition, the axion field is spatially varying. Axion strings are present as topological defects, and non-zero momentum modes of the axion field are excited. We first consider a region of the universe that happens to be free of strings. Strings will be added in the next subsection.

The axion field satisfies (2.31). We neglect the axion mass until $t \sim t_1$. The solution of (2.31) is a linear superposition of eigenmodes with definite co-moving wavevector $\mathbf{k}$:

$$a(\mathbf{x}, t) = \int \mathrm{d}^3\mathbf{k}\, a(\mathbf{k}, t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} , \tag{2.48}$$

where $a(\mathbf{k}, t)$ satisfies

$$\left(\partial_t^2 + \frac{3}{2t}\,\partial_t + \frac{k^2}{R^2}\right) a(\mathbf{k}, t) = 0 . \tag{2.49}$$

Equations (2.28) and (2.48) imply that the wavelength $\lambda(t) = 2\pi R(t)/k$ of each mode is stretched by the Hubble expansion. There are two qualitatively different regimes in the evolution of a mode, depending on whether its wavelength is outside $[\lambda(t) > t]$ or inside $[\lambda(t) < t]$ the horizon.

For $\lambda(t) \gg t$, only the first two terms in (2.49) are important and the most general solution is

$$a(\mathbf{k}, t) = a_0(\mathbf{k}) + a_{-1/2}(\mathbf{k})\, t^{-1/2} . \tag{2.50}$$

Thus, for wavelengths larger than the horizon, each mode goes to a constant; the axion field is "frozen by causality".

For $\lambda(t) \ll t$, let $a(\mathbf{k}, t) = R^{-3/2}(t)\,\psi(\mathbf{k}, t)$. Equation (2.49) becomes

$$\left[\partial_t^2 + \omega^2(t)\right]\, \psi(\mathbf{k}, t) = 0 , \tag{2.51}$$

where

$$\omega^2(t) = \frac{k^2}{R^2(t)} + \frac{3}{16\,t^2} \simeq \frac{k^2}{R^2(t)} . \tag{2.52}$$

As $\mathrm{d}\ln\omega/\mathrm{d}t \ll \omega$, this regime is again characterized by the adiabatic invariant $\psi_0^2(\mathbf{k}, t)\,\omega(t)$, where $\psi_0(\mathbf{k}, t)$ is the oscillation amplitude of $\psi(\mathbf{k}, t)$. Hence the most general solution is

$$a(\mathbf{k}, t) = \frac{C}{R(t)} \cos\left[\int^t \mathrm{d}t'\omega(t')\right] , \tag{2.53}$$

where C is a constant. The energy density and the number density behave, respectively, as $\varrho_{a,\mathbf{k}} \sim C^2\omega^2 R^{-2}(t) \propto R^{-4}(t)$ and $n_{a,\mathbf{k}} \sim \omega^{-1}\varrho_{a,\mathbf{k}} \propto R^{-3}(t)$, indicating that the number of axions in each mode is conserved. This is as expected because the expansion of the universe is adiabatic for modes with $\lambda(t)\, t \ll 1$.

Let us call $\mathrm{d}n_a(\omega, t)/\mathrm{d}\omega$ the number density, in physical and frequency space, of axions with wavelength $\lambda = 2\pi/\omega$, for $\omega > t^{-1}$. The axion number density in physical space is thus

$$n_a(t) = \int_{t^{-1}} \mathrm{d}\omega \, \frac{\mathrm{d}n_a}{\mathrm{d}\omega}(\omega, t) \; , \tag{2.54}$$

whereas the axion energy density is

$$\varrho_a(t) = \int_{t^{-1}} \mathrm{d}\omega \, \omega \, \frac{\mathrm{d}n_a}{\mathrm{d}\omega}(\omega, t) \; . \tag{2.55}$$

Under the Hubble expansion, axion energies redshift according to $\omega' = \omega\,(R/R')$ and volume elements expand according to $\Delta V' = \Delta V\,(R'/R)^3$, whereas the number of axions is conserved mode by mode. Hence

$$\frac{\mathrm{d}n_a}{\mathrm{d}\omega}(\omega, t) = \left(\frac{R'}{R}\right)^2 \frac{\mathrm{d}n_a}{\mathrm{d}\omega}(\omega\, R/R', t') \; . \tag{2.56}$$

Moreover, the size of $\mathrm{d}n_a/\mathrm{d}\omega$ for $\omega \sim t^{-1}$ is determined in order of magnitude by the fact that the axion field typically varies by $v_a = N f_a$ from one horizon to the next. Thus

$$\omega \, \frac{\mathrm{d}n_a}{\mathrm{d}\omega}(\omega, t)\, \Delta\omega \Big|_{\omega \sim \Delta\omega \sim t^{-1}} \sim \frac{\mathrm{d}n_a}{\mathrm{d}\omega}(t^{-1}, t)\, t^{-2} \sim \frac{1}{2}\, (\boldsymbol{\nabla} a)^2 \sim \frac{1}{2}\, \frac{N^2 f_a^2}{t^2} \; . \tag{2.57}$$

From (2.56), (2.57) and $R \propto t^{1/2}$, we have [10]

$$\frac{\mathrm{d}n_a}{\mathrm{d}\omega}(\omega, t) \sim \frac{N^2 f_a^2}{2t^2 \omega^2} \; . \tag{2.58}$$

This equation holds until the axion acquires mass during the QCD phase-transition.

2.2.3 Strings

In case 2, axion strings are present as topological defects in the axion field from the PQ to the QCD phase-transitions [11]. The energy per unit length of an axion string is

$$\mu = \pi v_a^2 \ln(v_a L) \; . \tag{2.59}$$

L is an infra-red cutoff, which in practice equals the distance to the nearest neighbor string. Because they are strongly coupled to the axion field, the strings decay very efficiently into axions. We will see that practically all axions produced by string-decay are non-relativistic after t_1. Because each such axion contributes m_a to the present energy density, it is important to evaluate the *number* density of axions emitted in string-decay. This is our main goal in this subsection.

At a given time t, there is at least on the order of one string per horizon. Indeed the axion field is completely uncorrelated over distances larger than t. Hence there is non-zero probability that the random values of $a(\mathbf{x}, t)$ wander from zero to $2\pi\, v_a$ along a closed path in physical space, if that closed path has size larger than t. When this is the case, a string perforates the surface subtended by the closed path.

At first, the strings are stuck in the primordial plasma and are stretched by the Hubble expansion. During that time, because $R(t) \propto t^{1/2}$, the density of strings grows to be much larger than one per horizon. However, expansion dilutes the plasma, and at some point, the strings become unstuck. The temperature at which strings start to move freely is of order [12]

$$T_* \sim 2 \times 10^7 \,\text{GeV} \left(\frac{f_a}{10^{12}\,\text{GeV}} \right)^2 . \tag{2.60}$$

Below T_*, there is a network of axion strings moving at relativistic speeds. Axions are radiated very efficiently by collapsing string loops and by oscillating wiggles on long strings. By definition, long strings stretch across horizons. They move and intersect one another. When strings intersect, there is a high probability of reconnection, i.e., of rerouting of the topological flux [13]. Because of such "intercommuting," long strings produce loops which then collapse freely. In view of this efficient decay mechanism, the average density of long strings is expected to be of order the minimum consistent with causality, namely, one long string per horizon. Hence the energy density in long strings is

$$\varrho_{\text{str}}(t) = \xi \, \frac{\tau}{t^2} \simeq \xi \, \pi \, \frac{(f_a N)^2}{t^2} \, \ln(v_a t) \,, \tag{2.61}$$

where ξ is a parameter of order one.

The equations governing the number density $n_a^{\text{str}}(t)$ of axions radiated by axion strings are [12]

$$\frac{\mathrm{d}\varrho_{\text{str}}}{\mathrm{d}t} = -2H \varrho_{\text{str}} - \frac{\mathrm{d}\varrho_{\text{str}\to a}}{\mathrm{d}t} \tag{2.62}$$

and

$$\frac{\mathrm{d}n_a^{\text{str}}}{\mathrm{d}t} = -3H n_a^{\text{str}} + \frac{1}{\omega(t)} \, \frac{\mathrm{d}\varrho_{\text{str}\to a}}{\mathrm{d}t} \tag{2.63}$$

where $\omega(t)$ is defined by

$$\frac{1}{\omega(t)} = \left(\frac{\mathrm{d}\varrho_{\text{str}\to a}}{\mathrm{d}t} \right)^{-1} \int \frac{\mathrm{d}k}{k} \, \frac{\mathrm{d}^2 \varrho_{\text{str}\to a}}{\mathrm{d}t \, \mathrm{d}k} \,, \tag{2.64}$$

where $k = |\mathbf{k}|$. Here, $\mathrm{d}\varrho_{\text{str}\to a}(t)/\mathrm{d}t$ is the rate at which energy density gets converted from strings to axions at time t, and $\mathrm{d}^2\varrho_{\text{str}\to a}(t,k)/\mathrm{d}t\,\mathrm{d}k$ is the spectrum of the axions produced. Therefore, $\omega(t)$ is the average energy of axions radiated in string-decay processes at time t. The term $-2H\varrho_{\text{str}} = +H\varrho_{\text{str}} - 3H\varrho_{\text{str}}$ in (2.62) takes account of the fact that the Hubble expansion both stretches ($+H\varrho_{\text{str}}$) and dilutes ($-3H\varrho_{\text{str}}$) long strings. Integrating (2.61)–(2.63), setting $H = 1/2t$, and neglecting terms of order one versus terms of order $\ln(v_a t)$, one obtains

$$n_a^{\rm str}(t) \simeq \frac{\xi \pi f_a^2 N^2}{t^{3/2}} \int_{t_{\rm PQ}}^{t} {\rm d}t' \, \frac{\ln(v_a t')}{t'^{\,3/2} \omega(t')} \; , \tag{2.65}$$

where $t_{\rm PQ}$ is the time of the PQ transition.

To obtain $n_a^{\rm str}(t)$, we need to know $\omega(t)$, the average energy of axions radiated at time t. If $\omega(t)$ is large, the number of radiated axions is small, and vice versa. Axions are radiated by wiggles on long strings and by collapsing string loops. Consider a process that starts at $t_{\rm in}$ and ends at $t_{\rm fin}$ and converts an amount of energy E from strings to axions. The times $t_{\rm in}$ and $t_{\rm fin}$ are both taken to be of order t. It is useful to define the quantity [14]

$$N_{\rm ax}(t) \equiv \int {\rm d}k \, \frac{{\rm d}E}{{\rm d}k}(t) \, \frac{1}{k} \; , \tag{2.66}$$

where k is the wavevector and ${\rm d}E/{\rm d}k$ is the wavevector spectrum of the a field. At the start ($t = t_{\rm in}$), only strings contribute to the integral in (2.66). At the end ($t = t_{\rm fin}$), only axions contribute. In between, both axions and strings contribute. The number of axions radiated is $N_a = N_{\rm ax}(t_{\rm fin})$, and their average energy is $\omega = E/N_a$. The energy stored in strings has the spectrum ${\rm d}E/{\rm d}k \propto k^{-1}$ for $k_{\rm min} < k < k_{\rm max}$ where $k_{\rm max}$ is of order v_a and $k_{\rm min}$ of order $2\pi/L \sim 2\pi/t$. If $\ell \equiv E/\mu$ is the length of string converted to axions, we have

$$N_{\rm ax}(t_{\rm in}) = \frac{E}{\ln(t v_a)\, k_{\rm min}} \; . \tag{2.67}$$

Hence

$$\omega^{-1} = \frac{r}{\ln(v_a t)\, k_{\rm min}} \; , \tag{2.68}$$

where r is the relative change in $N_{\rm ax}(t)$ during the process in question,

$$r \equiv \frac{N_{\rm ax}(t_{\rm fin})}{N_{\rm ax}(t_{\rm in})} \; . \tag{2.69}$$

$L \simeq 2\pi/k_{\rm min}$ is the loop size in the case of collapsing loops and the wiggle wavelength in the case of bent strings. L is at most of order t but may be substantially smaller if the string network has a lot of small-scale structures. To parameterize our ignorance in this matter, we define χ such that the suitably averaged $k_{\rm min} \equiv \chi\, 2\pi/t$. Combining (2.65) and (2.68) we find

$$n_a^{\rm str}(t) \simeq \frac{\xi \bar{r} N^2}{\chi} \, \frac{f_a^2}{t} \; , \tag{2.70}$$

where $\bar{r}$ is the weighted average of r over the various processes that convert strings to axions. One can show [31] that the population of axions that were radiated between $t_{\rm PQ}$ and t have a spectrum ${\rm d}n_a/{\rm d}k \propto k^{-2}$ for $t^{-1} \lesssim k \lesssim (t_{\rm PQ} t)^{-1/2}$, irrespective of the shape of ${\rm d}^2 \varrho_{{\rm str}\to a}/{\rm d}t\, {\rm d}k$, provided $t \gg t_{\rm PQ}$.

At time t_1, each string becomes the edge of N domain walls, and the process of axion radiation by strings stops. Since their momenta are of order

t_1^{-1} at time t_1, the axions radiated by strings become non-relativistic soon after they acquire mass. We discuss the string-decay contribution to the axion energy density in Sect. 2.4. For the moment we turn our attention to the domain walls that appear at t_1 in case 2.

2.3 The Domain-Wall Problem

Axion models have an exact, spontaneously broken, discrete $Z(N)$ symmetry. $Z(N)$ is the subgroup of $U(1)_{\mathrm{PQ}}$ that does not get broken by non-perturbative QCD effects [5]. The spontaneously broken $Z(N)$ symmetry implies an N fold degeneracy of the vacuum. The N vacua are at equidistant points on the circle at the bottom of the "Mexican hat" potential for the Peccei-Quinn field ϕ. An axion domain wall is the minimum-energy field configuration that interpolates between neighboring vacua. Note that there are axion domain walls even when $N = 1$. In this case, both sides of the domain wall are in the same vacuum (indeed there is only one vacuum) but the interpolating field configuration winds around the bottom of the Mexican hat potential once. The properties of walls in $N = 1$ models are for most purposes identical to those of walls in $N \geq 2$ models. The only seemingly important difference is that the walls of $N = 1$ models are quantum-mechanically unstable. Even so, their decay-rate per unit surface and time is exponentially small and negligible in the context of our discussion.

When the axion mass turns on, at time t_1, each axion string becomes the edge of N domain walls. The domain walls produce a cosmological disaster unless there is inflation after the PQ phase-transition (case 1) or unless $N = 1$. Indeed, let us consider the implications of case 2 if $N \geq 2$. Since there are two or more exactly degenerate vacua and they have identical properties, the vacua chosen at points outside each other's causal horizon are independent of one another. Hence there is at least on the order of one domain wall per causal horizon at any given time. In case 2, the size of the causal horizon is of order t, the age of the universe. Thus the energy density in domain walls is

$$\varrho_{\mathrm{w}}(t) \gtrsim \frac{\sigma}{t} \,, \tag{2.71}$$

where σ is the wall energy per unit surface, given by [5, 15]

$$\sigma \simeq 9 f_a^2 m_a \simeq 5.5 \times 10^{10}\,\mathrm{GeV}^3 \left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right) \,. \tag{2.72}$$

The energy density in axion domain walls today ($t_0 \simeq 14\,\mathrm{Gyr}$)

$$\varrho_{\mathrm{w}}(t_0) \gtrsim \frac{\sigma}{t_0} \simeq 2 \times 10^{-14}\,\mathrm{g\,cm}^{-3} \left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right) \tag{2.73}$$

would exceed by many orders of magnitude the critical energy density, of order $10^{-29}\,\mathrm{g\,cm^{-3}}$, for closing the universe. This would be grossly inconsistent with observation. Let us see what would happen.

Let t_w be the age of the universe when the domain walls start to dominate the energy density. The condition $H^2 \sim 8\pi G \varrho_\mathrm{w}/3$ and (2.71) imply

$$t_\mathrm{w} \lesssim \frac{3}{32\pi G\sigma} \simeq 53\,\mathrm{s}\left(\frac{10^{12}\,\mathrm{GeV}}{f_a}\right) . \tag{2.74}$$

Domain walls are gravitationally repulsive [16, 17, 18]. They accelerate away from each other with acceleration $2\pi G\sigma$ and, after a time of order $(2\pi G\sigma)^{-1}$, recede at the speed of light. By averaging over volumes containing many cells separated by walls, the equation of state of a wall dominated universe is

$$p_\mathrm{w} = -\frac{2}{3}\,\varrho_\mathrm{w} . \tag{2.75}$$

Conservation of energy

$$\mathrm{d}\left(\varrho_\mathrm{w} R^3\right) = -p_\mathrm{w}\mathrm{d}\left(R^3\right) , \tag{2.76}$$

where R is the scale factor, then implies $\varrho_\mathrm{w} \propto R^{-1}$. This scaling-law and the Friedmann equation

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\,\varrho_\mathrm{w} \tag{2.77}$$

imply that a domain-wall-dominated universe expands according to

$$R \propto t^2 . \tag{2.78}$$

The domain-wall-dominated universe has an accelerated expansion.

One may be tempted to attribute the present-day accelerated expansion of the universe [19, 20] to domain walls. However, a domain-wall-dominated universe is far less homogeneous than today's. It would be divided into cells separated by rapidly expanding walls. Inside each cell, concentrated near the cell's center, would be a clump of matter and radiation with total energy of order

$$M \sim \varrho(t_\mathrm{w})\,t_\mathrm{w}^3 \sim 10^{11}\,M_\odot\left(\frac{m_a}{\mathrm{eV}}\right) . \tag{2.79}$$

One cannot identify these clumps with galaxies because neighboring clumps are receding from one another at close to the speed of light.

There are three solutions to the axion domain-wall problem. The first solution is to have inflation with reheating temperature less than the PQ phase-transition temperature, i.e., postulate case 1. The axion field is then homogenized by inflation, and there are no strings or domain walls. The second solution is to postulate $N = 1$. The third solution is to postulate a small explicit breaking of the $Z(N)$ symmetry. The viability of the second and third solutions is less obvious. We will discuss them in succession.

2.3.1 $N = 1$

The above arguments, showing the existence of a domain-wall problem, are valid only when the vacuum is multiply degenerate. They do not apply to the $N = 1$ case. On the other hand, as $N = 1$ models contain domain walls too, it is not immediately clear that they are free of difficulties. However, $N = 1$ is a solution [11, 21, 22], as we now discuss.

In the circumstances under consideration (case 2), axion strings are present in the early universe from the time of the PQ phase-transition to that of the QCD phase-transition. At temperature T_1 each string becomes the boundary of a single domain wall. To see what the network of walls bounded by strings looks like, a cross section of a finite but statistically significant volume of the universe near time t_1 was simulated [22, 10]. The simulation shows that there are no infinite domain walls that are not cut up by any string. The reason for this is easily understood. An extended domain wall has some probability to be cut up by a string in each successive horizon it traverses. The probability that no string is encountered after traveling a distance l along the wall decreases exponentially with l.

The question now is: what happens to the network of walls bounded by strings? The walls are transparent to the thermalized particles in the primordial soup, whose typical momentum is of order 1 GeV, but have a large reflection coefficient for non-relativistic axions such as the cold axions that were produced by vacuum realignment and wall-decay [15]. The drag on the motion of the walls that results from their reflecting cold axions is important at time t_1 but turns off soon afterwards as the cold axions are diluted by the expansion of the universe [10]. The wall energy per unit surface $\sigma(t) \simeq 8 m_a(t) f_a^2$ is time-dependent. [Equation (2.72) is for zero temperature.] The string at the boundary of a wall is embedded into the wall. Hence its infra-red cutoff L, in the sense of (2.59), is of order m_a^{-1}, the wall thickness [5]. The energy per unit length of such string is therefore

$$\mu \simeq \pi f_a^2 \ln(f_a/m_a) \; . \tag{2.80}$$

The surface energy E_σ of a typical (size $\sim t_1$) piece of wall bounded by a string is $\sigma(t)\, t_1^2$ whereas the energy in the boundary is $E_\mu \sim \mu\, t_1$. There is a critical time t_2 when the ratio

$$\frac{E_\sigma(t)}{E_\mu} \sim \frac{8\, m_a(t)\, t_1}{\pi \ln(f_a/m_a)} \tag{2.81}$$

is of order one. Using (2.35) and (2.36), one estimates

$$\begin{aligned} t_2 &\simeq 10^{-6}\,\mathrm{s} \left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right)^{1/3} , \\ T_2 &\simeq 600\,\mathrm{MeV} \left(\frac{10^{12}\,\mathrm{GeV}}{f_a}\right)^{1/6} . \end{aligned} \tag{2.82}$$

After t_2, the dynamics of the walls bounded by string is dominated by the energy in the walls, whereas, before t_2, it is dominated by the energy in the string. A string attached to a wall is pulled by the wall's tension. For a straight string and flat wall, the acceleration is

$$a_{\rm s}(t) = \frac{\sigma(t)}{\mu} \simeq \frac{8\, m_a(t)}{\pi \ln(f_a/m_a)} \simeq \frac{m_a(t)}{23} \simeq \frac{1}{t_1} \frac{m_a(t)}{m_a(t_2)} \ . \tag{2.83}$$

Therefore, after t_2, each string typically accelerates to relativistic speeds, in the direction of the wall to which it is attached, in less than a Hubble time. The string will then unzip the wall, releasing the stored energy in the form of barely relativistic axions. We will estimate in Sect. 2.4.3 how much do walls bounded by string contribute to the present cosmological axion energy density.

A very small portion of the domain-wall energy density is in walls that are not bounded by string and that form closed surfaces such as spheres or torii [10, 22]. Such closed walls do not decay by the process just described. Instead, they oscillate and emit gravitational waves [11]. Using the quadrupole formula, we may estimate the gravitational-wave power emitted by a closed wall of size ℓ oscillating with frequency $\omega \sim \ell^{-1}$ as

$$P \sim -\frac{{\rm d}(\sigma\ell^2)}{{\rm d}t} \sim G\,(\sigma\ell^4)^2\omega^6 \sim G\,\sigma^2\ell^2 \ . \tag{2.84}$$

This result implies the lifetime

$$\tau_{\rm grav} \sim (G\sigma)^{-1} \simeq 2\times 10^3\,{\rm s}\left(\frac{10^{12}\,{\rm GeV}}{f_a}\right) \ , \tag{2.85}$$

independently of size.

2.3.2 Small Breaking of the PQ Symmetry

A third solution to the domain-wall problem is to postulate a small explicit breaking of the $Z(N)$ symmetry and hence of the PQ symmetry [5]. The symmetry breaking must lift completely the degeneracy of the vacuum and be large enough so that the unique true vacuum takes over before the walls dominate the energy density. On the other hand, it must be small enough so that the PQ mechanism still works. This solution does not appear very attractive, and we will see below that there is little room in parameter space for it to occur, but it is a logical possibility.

To get rid of the walls, we add to the RHS of (2.26) a tiny $U(1)_{\rm PQ}$ breaking term that lifts the vacuum degeneracy completely, e.g.

$$\delta V = -\xi\,\left(\phi\,{\rm e}^{-{\rm i}\delta} + {\rm h.c.}\right) \ . \tag{2.86}$$

To add such a term by hand to an axion model seems rather unnatural. However, it is conceivable that a small $U(1)_{\rm PQ}$-breaking term is in fact a natural property of the ultimate theory. This would be the case, for example, if the low energy effective theory at some energy scale has an automatic PQ symmetry that is broken in the full theory. Be that as it may, (2.86) yields a small correction to the effective potential for the axion field of

$$\delta V_a = -2 v_a \xi \cos\left(\frac{a}{v_a} - \delta\right) . \tag{2.87}$$

The unique true vacuum is the one for which $|\delta - a/v_a|$ is smallest. Its energy density is lowered by an amount of order ξv_a relative to the other now quasi-vacua. As a result, the walls at the boundary of a region in the true vacuum are subject to an outward pressure of order ξv_a. As the walls are typically a distance t apart, the volume energy $\xi\, v_a\, t^3$ associated with the lifting of the vacuum degeneracy grows more rapidly than the energy σt^2 in the walls. At a time $\tau \sim \sigma/\xi v_a$, the pressure favoring the true vacuum starts to dominate the wall dynamics and the true vacuum takes over, i.e., the walls disappear. The energy stored in the network of walls and strings decays into gravitational waves [10]. The true vacuum must take over before the walls dominate the energy density. Using (2.74), we obtain

$$\tau \sim \frac{\sigma}{\xi v_a} \lesssim 10^2\,\mathrm{s}\left(\frac{10^{12}\,\mathrm{GeV}}{f_a}\right) . \tag{2.88}$$

On the other hand ξ is bounded from above by the requirement that δV does not upset the PQ mechanism. δV shifts the minimum of the effective potential for the axion field, inducing $\bar\theta \sim \xi/m_a^2 f_a$. The requirement $\bar\theta < 10^{-10}$ implies

$$\tau \gtrsim \frac{10\,\mathrm{s}}{N}\left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right) . \tag{2.89}$$

These equations indicate that there is little room in parameter space for this solution to the axion domain-wall problem, but it is not ruled out.

2.4 Cold Axions

We now turn to estimating the cosmological energy density in cold axions and their velocity dispersion. In the previous two sections, we identified several sources for cold axions, i.e., vacuum realignment for the zero-momentum and higher-momentum modes, string decay, and domain-wall decay. In case 1, only the contribution from vacuum realignment with zero momentum is present. In case 2, all sources contribute.

2.4.1 Vacuum Realignment

Zero-Momentum Mode

Let us start with case 1 where the universe was homogenized by inflation after the PQ phase-transition and that is easiest to analyze. The axion number density at time t_1 is given by (2.47) in terms of the initial vacuum misalignment angle α_1 that has the same value everywhere. We saw in Sect. 2.2 that, barring any sudden changes in the axion mass during the chiral symmetry-breaking phase-transition, the number of axions is an adiabatic invariant after t_1. Hence, the axion density from the vacuum realignment zero mode at a later time t is

$$n_a^{\mathrm{vac},0}(t) \sim \frac{f_a^2}{2t_1}\,\alpha_1^2 \left(\frac{R_1}{R}\right)^3 , \tag{2.90}$$

where R_1/R is the ratio of scale factors between t_1 and t. Thus, we find

$$\varrho_a^{\mathrm{vac},0}(t_0) \sim \frac{m_a f_a^2}{2t_1}\,\alpha_1^2 \left(\frac{R_1}{R_0}\right)^3 \tag{2.91}$$

for the axion energy density today.

In case 2, the initial misalignment angle α_1 is different from one QCD horizon to the next. As the average of α_1^2 over many QCD horizons is of order one, we have

$$\varrho_a^{\mathrm{vac},0}(t_0) \sim \frac{m_a f_a^2}{2t_1} \left(\frac{R_1}{R_0}\right)^3 . \tag{2.92}$$

However in case 2, there are additional contributions.

Higher-Momentum Modes

In case 2, the axion field is not constant even within each horizon volume. It has wiggles inherited from earlier epochs when the horizon was smaller and the axion field was inhomogeneous on correspondingly shorter scales. The associated density of axions in physical and frequency space is given in (2.58). Integrating over $\omega > t_1^{-1}$, we find the contribution from vacuum realignment involving higher-momentum modes

$$n_a^{\mathrm{vac},1}(t_1) \sim \frac{N^2 f_a^2}{2t_1} . \tag{2.93}$$

Almost all these axions are non-relativistic after t_1. Hence

$$\varrho_a^{\mathrm{vac},1}(t_0) \sim \frac{m_a N^2 f_a^2}{2t_1} \left(\frac{R_1}{R_0}\right)^3 . \tag{2.94}$$

Note that, except for the factor N^2, the contributions from the zero- and higher-momentum modes are similar.

2.4.2 String Decay

The evolution of axion strings between the PQ and QCD phase-transitions was discussed in Sect. 2.2.3. The number density of axions emitted in string decay is given by (2.70) in terms of the quantities ξ, $\bar{r}$, and χ that we introduced to parameterize our ignorance of various aspects of string evolution and decay.

Because their spectrum $\mathrm{d}n_a/\mathrm{d}k \propto k^{-2}$ in the range $t^{-1} \lesssim k \lesssim (t_{\mathrm{PQ}}t)^{-1/2}$, the bulk of axions emitted in string-decay have momenta of order t_1^{-1} at time t_1 and become non-relativistic soon after they acquire mass. Therefore, the string-decay contribution to the axion energy density today is

$$\varrho_a^{\mathrm{str}}(t_0) = m_a n_a^{\mathrm{str}}(t_1) \left(\frac{R_1}{R_0}\right)^3 \simeq m_a \,\frac{\xi\,\bar{r}}{\chi}\,\frac{N^2 f_a^2}{t_1}\,\left(\frac{R_1}{R_0}\right)^3 . \tag{2.95}$$

We now discuss the factors on the right hand side (RHS) of this equation that are specific to the string-decay contribution.

The parameter ξ determines the density of the string network (2.61) with $\xi = 1$ corresponding to a density of one long string per horizon. In [14] it was argued that $\xi \simeq 1$ because global strings can decay efficiently into axions, and, therefore, the number density of long strings should be close to the minimum consistent with causality. A numerical simulation of global string networks in an expanding universe [23] found that indeed $\xi \simeq 1$. So there appear to be good grounds for using $\xi \simeq 1$.

The parameter χ defines the low-wavevector edge of the $\mathrm{d}n_a/\mathrm{d}k \propto k^{-2}$ spectrum through $k_{\mathrm{min}} \equiv \chi\, 2\pi/t$. The parameters χ and ξ are related as the average interstring distance controls both. On dimensional grounds, $\chi \propto \xi^{1/2}$. So the effect of small-scale structure in the axion string network partially cancels out in the RHS of (2.95). We expect χ to be of order one, but the uncertainty on this is at least a factor two.

The parameter $\bar{r}$ defines the average energy of the axions emitted in string-decay through (2.68) and (2.69). It is the unknown on which most of the debate has focused in the past. Two basic scenarios have been put forth, which we call A and B. The question is: what is the spectrum of axions radiated by strings? The main source is closed loops of size $L \sim t$. Scenario A postulates that a bent string or closed loop oscillates many times, with period of order L, before it has released its excess energy and that the spectrum of radiated axions is concentrated near $2\pi/L$. In that case, one has $\bar{r} \sim \ln(v_a t_1) \simeq 67$. Scenario B postulates that the bent string or closed loop releases its excess energy very quickly and that the spectrum of radiated axions is $\mathrm{d}E/\mathrm{d}k \propto k^{-1}$ with a high-frequency cutoff of order $2\pi v_a$ and a low-frequency cutoff of order $2\pi L$. In scenario B, the initial and final spectra $\mathrm{d}E/\mathrm{d}k$ of the energy stored in the axion field are qualitatively the same and hence $\bar{r} \sim 1$. In scenario A, the string-decay contribution dominates over the vacuum-realignment contribution by the factor $\ln(v_a t_1)$, whereas in scenario B the contributions from string-decay and vacuum-realignment have the same order of magnitude.

Many authors [24, 25, 26, 27, 28, 29, 30] have argued in favor of scenario A, adopting the point of view that global strings are similar to local strings and their coupling to the axion field can be treated perturbatively. My collaborators and I [12, 14] have argued in support of scenario B, emphasizing that the dynamics of global strings is dominated by the energy stored in the axion field and there is no reason to believe that this energy would behave in the same way as the energy stored in the string core. The numerical simulations of the motion and decay of axion strings in [14, 31] give strong support to scenario B. These simulations are of oscillating strings with ends held fixed, of collapsing circular loops and non-circular closed loops with angular momentum. Over the range of $\ln(v_a L)$ accessible with present technology, $2.5 \lesssim \ln(v_a L) \lesssim 5.0$, it was found that $\bar{r} \simeq 0.8$ for closed loops and $\bar{r} \simeq 1.07$ for oscillating strings with ends held fixed. No dependence of $\bar{r}$ on $\ln(v_a L)$ was found for closed loops, and for bent strings with ends held fixed, $\bar{r}$ was found to slightly *decrease* with increasing $\ln(v_a L)$, whereas scenario A predicts $\bar{r}$ to be proportional to $\ln(v_a L)$.

2.4.3 Wall Decay

The final contribution to the cold axion cosmological energy density in case 2 is from the decay into non-relativistic axions of axion walls bounded by string. We assume here that $N = 1$. Indeed, if $N > 1$, the domain-wall problem is presumably solved by introducing a small breaking of the PQ symmetry, as described in Sect. 2.3.2. In that case, the axion walls decay predominantly into gravitational radiation [10].

Let t_3 be the time when the decay effectively takes place and $\gamma \equiv \omega'/m_a(t_3)$ the average Lorentz γ factor of the axions produced, where ω' is their average energy. The density of walls at time t_1 was estimated to be of order 0.7 per horizon volume [10]. Hence the average energy density in walls is

$$\varrho_{\rm wd}(t) \sim 0.7\,\frac{\sigma(t)}{t_1}\left(\frac{R_1}{R}\right)^3 \sim 0.7 \times 8 \times m_a(t)\,\frac{f_a^2}{t_1}\left(\frac{R_1}{R}\right)^3 \tag{2.96}$$

between t_1 and t_3. We assume that the energy in walls simply scales as $\sigma(t)$. After time t_3, the number density of axions produced in the decay of walls bounded by strings is of order

$$n_a^{\rm wd}(t) \sim \frac{\varrho_{\rm wd}(t_3)}{\omega'}\left(\frac{R_3}{R}\right)^3 \sim \frac{6}{\gamma}\,\frac{f_a^2}{t_1}\left(\frac{R_1}{R}\right)^3 . \tag{2.97}$$

Note that the dependence on t_3 drops out of our estimate of $n_a^{\rm wd}$. In the simulations of the motion and decay of walls bounded by string [10], it was found that $\gamma \simeq 7$ for $\ln(v_a/m_a) \sim 4.6$ but γ increases approximately linearly with $\ln(\lambda^{1/2} v_a/m_a)$. If this behavior is extrapolated all the way to $\ln(\lambda^{1/2} v_a/m_a) \simeq 60$, which is the value in axion models of interest, then $\gamma \simeq 60$. In that case the contribution from wall decay is subdominant relative to those from vacuum realignment and string decay.

2.4.4 Cosmological Cold-Axion Energy Density

To estimate the cosmological energy density of cold axions in case 2, we neglect the contribution from wall decay and assume that scenario B is correct for the string contribution. By adding the RHS of (2.92), (2.94) and (2.95) with $N = \bar{r} = \xi = \chi = 1$, we find

$$\varrho_a(t_0) \sim 2\,\frac{f_a^2}{t_1}\left(\frac{R_1}{R_0}\right)^3 m_a\ . \tag{2.98}$$

Equation (2.91) gives the cosmological cold-axion energy density in case 1. To determine the ratio of scale factors R_1/R_0, we assume conservation of entropy from time t_1 until the present. The number $\mathcal{N}_1$ of effective thermal degrees of freedom at time t_1 is of order 60. Keeping in mind that neutrinos decouple before electron-positron annihilation, one finds

$$\left(\frac{R_1}{R_0}\right)^3 \simeq 0.063\left(\frac{T_{\gamma,0}}{T_1}\right)^3\ . \tag{2.99}$$

Combining (2.36), (2.91),(2.98) and (2.99), and dividing by the critical density $\varrho_{\rm crit} = 3H_0^2/8\pi G$, we find

$$\Omega_a \sim \left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right)^{7/6}\left(\frac{0.7}{h}\right)^2 \times \begin{cases} 0.15\,\alpha_1^2 & \text{for Case 1,} \\ 0.7 & \text{for Case 2,} \end{cases} \tag{2.100}$$

where h is defined as usual by $H_0 = h\,100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

Equations (2.100) are subject to uncertainty from many sources, apart from the uncertainty about the contribution from string-decay. The axion energy density may be diluted by the entropy release from heavy particles which decouple before the QCD epoch but decay afterwards [32, 33, 34], or by the entropy release associated with a first order QCD phase-transition. On the other hand, if the QCD phase-transition is first order [35, 36, 37, 38], an abrupt change of the axion mass at the transition may increase Ω_a. A model has been put forth [39] in which the axion decay constant f_a is time-dependent, the value $f_a(t_1)$ during the QCD phase-transition being much smaller than the value f_a today. This yields a suppression of the cosmological axion energy density by a factor $[f_a(t_1)/f_a]^2$ compared to the usual case. Finally, it has been proposed that the axion density is diluted by "coherent de-excitation," i.e., adiabatic level crossing of $m_a(t)$ with the mass of some other pseudo-Nambu-Goldstone boson which mixes with the axion [40].

2.4.5 Velocity Dispersions

The axions produced by vacuum realignment, string decay, and wall decay all have extremely small velocity dispersion today. In case 1, where the axions are produced in a zero-momentum state, the velocity dispersion is zero.

(This ignores the small quantum-mechanical fluctuations created during the inflationary epoch, which will be discussed in Sect. 2.6.)

In case 2, we distinguish two sub-populations of cold axions, pop. I and pop. II, with the second kind having velocity dispersion larger than the first typically by a factor 10^3 to 10^4. The pop. I axions are those produced by vacuum realignment or string decay and which escaped being hit by moving domain walls. They have a typical momentum $p_{\rm I}(t_1) \sim t_1^{-1}$ at time t_1 because they are associated with axion field configurations that are inhomogeneous on the horizon scale at that time. Their velocity dispersion is of order

$$\beta_{\rm I}(t) \sim \frac{1}{m_a t_1} \left(\frac{R_1}{R}\right) \simeq 3 \times 10^{-17} \left(\frac{10^{-5}\,{\rm eV}}{m_a}\right)^{5/6} \frac{R_0}{R} \, . \qquad (2.101)$$

The corresponding effective T is of order $0.5 \times 10^{-34}\,{\rm K}\,(10^{-5}\,{\rm eV}/m_a)^{2/3}$ today. This is very cold, indeed!

Pop. II are axions produced in the decay of domain walls and axions that were hit by moving domain walls. Axions produced in the decay of domain walls have typical momentum $p_{\rm II}(t_3) \sim \gamma\, m_a(t_3)$ at time t_3 when the walls effectively decay. Their velocity dispersion is therefore of order

$$\beta_{\rm II}(t) \sim \gamma \, \frac{m_a(t_3)}{m_a} \, \frac{R_3}{R} \simeq 10^{-13} \, q \left(\frac{10^{-5}\,{\rm eV}}{m_a}\right)^{1/6} \frac{R_0}{R} \, , \qquad (2.102)$$

where $q \equiv \gamma\, [m_a(t_3)/m_a](R_3/R_1)$ parameterizes our ignorance of the wall decay process. We expect q to be of order one but with very large uncertainties. There is however a lower bound on q that follows from the fact that the time t_3 when the walls effectively decay must be after t_2 when the energy density in walls starts to exceed the energy density in strings. Using (2.82), we have

$$q = \frac{\gamma\, m_a(t_3)}{m_a} \, \frac{R_3}{R_1} > \frac{\gamma\, m_a(t_2)}{m_a} \, \frac{R_2}{R_1} \simeq \frac{\gamma}{130} \left(\frac{10^{-5}\,{\rm eV}}{m_a}\right)^{2/3} \, . \qquad (2.103)$$

As computer simulations suggest γ is of order 60, pop. II axions have much larger velocity dispersion than pop. I, by a factor of 10^3 or more. Whereas pop. II axions are relativistic or near-relativistic at the end of the QCD phase-transition, pop. I axions are definitely non-relativistic at that time as $m_a \gg t_1^{-1}$. The axions that were produced by vacuum-realignment or string-decay, but were hit by relativistically moving walls at some time between t_1 and t_3 should be included in pop. II as they are relativistic just after getting hit. The next section will highlight the differences in the behaviors of the two populations of cold axions.

The very low velocity dispersion of cold axions and their extremely weak couplings imply that these particles behave as collisionless cold dark matter (CDM). CDM particles lie at all times on a 3-dim hypersurface in 6-dim phase-space [41, 42]. As a result, CDM forms discrete flows and caustics. The number of discrete flows at a given physical location is the number of times the 3-dim hypersurface covers physical space at that location. At the boundaries between regions with differing number of flows, the 3-dim hypersurface is tangent to velocity space. The dark matter density is very large on these surfaces, which are called caustics. The density diverges at the caustics in the limit of zero velocity dispersion.

2.5 Axion Miniclusters

If there is no inflation after the PQ phase-transition (case 2); the initial misalignment angle α_1 changes by $\mathcal{O}(1)$ from one QCD time horizon to the next. Hence, the fluid of cold axions produced by vacuum realignment is inhomogeneous with $\delta\varrho_a/\varrho_a = \mathcal{O}(1)$ at the time of the QCD phase-transition. As will be shown shortly, the streaming length of pop. I axions is too short for these inhomogeneities to get erased by free streaming before the time $t_{\rm eq}$ of equality between matter and radiation, when density perturbations start to grow in earnest by gravitational instability. At time $t_{\rm eq}$, the $\delta\varrho_a/\varrho_a = \mathcal{O}(1)$ inhomogeneities in the axion fluid promptly form gravitationally bound objects, called axion miniclusters [10, 43, 44, 45]. Their properties are of concern to experimentalists attempting the direct detection of dark matter axions on Earth. Indeed, those experiments would become even more challenging, if most of the cold axions condense into miniclusters and they withstand tidal disruption afterwards. Of course, these issues only arise in case 2. There are no axion miniclusters in case 1.

As described above, there are two populations of cold axions with velocity dispersions given by (2.101) and (2.102) respectively. Both populations are inhomogeneous at the time of the QCD phase-transition. The free-streaming length from time t_1 to $t_{\rm eq}$ is

$$\ell_{\rm f} = R(t_{\rm eq}) \int_{t_1}^{t_{\rm eq}} {\rm d}t\, \frac{\beta(t)}{R(t)} \simeq \beta(t_1)\,(t_1 t_{\rm eq})^{1/2}\, \ln\left(\frac{t_{\rm eq}}{t_1}\right) . \tag{2.104}$$

The time of equality and the corresponding temperature are, respectively, $t_{\rm eq} \simeq 2.3 \times 10^{12}\,{\rm s}$ and $T_{\rm eq} \simeq 0.77\,{\rm eV}$. The free-streaming length should be compared with the size

$$\ell_{\rm mc} \sim t_1 \frac{R_{\rm eq}}{R_1} \simeq (t_1 t_{\rm eq})^{1/2} \simeq 2 \times 10^{13}\,{\rm cm} \left(\frac{10^{-5}\,{\rm eV}}{m_a}\right)^{1/6} \tag{2.105}$$

of axion inhomogeneities at $t_{\rm eq}$. Using (2.101) we find for pop. I

$$\frac{\ell_{\rm f,I}}{\ell_{\rm mc}} \simeq \frac{1}{t_1 m_a} \ln\left(\frac{t_{\rm eq}}{t_1}\right) \simeq 2 \times 10^{-2} \left(\frac{10^{-5}\,{\rm eV}}{m_a}\right)^{2/3} . \tag{2.106}$$

Hence, in the axion mass range of interest, pop. I axions do not homogenize. At $t_{\rm eq}$ most pop. I axions condense into miniclusters. The typical size of axion miniclusters is $\ell_{\rm mc}$, and their typical mass is [10, 45]

$$M_{\rm mc} \sim \eta\, \varrho_a(t_{\rm eq})\, \ell_{\rm mc}^3 \sim \eta\; 5 \times 10^{-13}\, M_\odot \left(\frac{10^{-5}\,{\rm eV}}{m_a}\right)^{5/3} , \tag{2.107}$$

where η is the fraction of cold axions that are pop. I. We assume that all pop. I axions condense into miniclusters and use (2.100, Case 2) to estimate $\varrho_a(t_{\rm eq})$.

Using (2.102), we find for pop. II

$$\frac{\ell_{\rm f,II}}{\ell_{\rm mc}} \sim q\, \ln\left(\frac{t_{\rm eq}}{t_3}\right) \simeq 42\, q\ . \tag{2.108}$$

Using (2.103) and assuming the range $\gamma \sim 7$–60, suggested by the numerical simulations [10], we conclude that pop. II axions do homogenize and hence the axion energy density has a smooth component at $t_{\rm eq}$.

However, pop. II axions may get gravitationally bound to miniclusters later on. It seems rather difficult to model this process reliably. A discussion is given in [10]. It is concluded there that the accretion of pop. II axions results in miniclusters that have an inner core of pop. I axions with density of order $10^{-18}\,{\rm g\,cm^{-3}}$ and a fluffy envelope of pop. II axions with density of order $10^{-25}\,{\rm g\,cm^{-3}}$.

When a minicluster falls onto a galaxy, tidal forces are apt to destroy it. If a minicluster falls through the inner parts of the Galaxy ($r < 10\,{\rm kpc}$), where the density is of order $10^{-24}\,{\rm g\,cm^{-3}}$, its fluffy envelope of pop. II axions will likely be pulled off immediately. This is helpful for direct searches of dark matter axions on Earth as it implies that a smooth component of dark-matter axions with density of order the halo density permeates us whether or not there is inflation after the PQ phase-transition. Even the central cores of pop. I axions may eventually get destroyed. When a minicluster passes by an object of mass M with impact parameter b and velocity v, the internal energy per unit mass ΔE given to the minicluster by the tidal gravitational forces from that object is of order [43]

$$\Delta E \sim \frac{G^2 M^2 \ell_{\rm mc}^2}{b^4 \beta^2} , \tag{2.109}$$

whereas the binding energy per unit mass of the minicluster $E \sim G\varrho_{\rm mc}\ell_{\rm mc}^2$. If the minicluster travels a length $L = \beta\, t$ through a region where objects of mass M have density n, the relative increase in internal energy is

$$\frac{\Delta E}{E} \sim \frac{G\varrho_M^2 t^2}{\varrho_{\rm mc}} , \tag{2.110}$$

where $\varrho_M = Mn$. Equation (2.110) follows from the fact that ΔE is dominated by the closest encounter and the latter has impact parameter $b_{\rm min}$ given by $\pi b_{\rm min}^2 nL = \mathcal{O}(1)$. Note that $\Delta E/E$ is independent of M. A minicluster inner core that has spent most of its life in the central part of our galaxy only barely survived as $\Delta E/E \sim 10^{-2}$ in that case.

The direct encounter of a minicluster with Earth would be rare, happening only every 10^4 years or so. The encounter would last for about 3 days during which the local axion density would increase by a factor of order 10^6.

2.6 Axion Isocurvature Perturbations

We now turn to isocurvature perturbations [32, 46, 47, 48, 49, 50] produced if inflation occurs after the PQ phase-transition and derive the constraints on axion parameters from the absence of isocurvature fluctuations in cosmic microwave background radiation (CMBR) observations.

If the reheat temperature after inflation is less than the temperature $T_{\rm PQ}$ at which $U(1)_{\rm PQ}$ is restored (case 1), the axion field is present during inflation and is subject to quantum-mechanical fluctuations, just like the inflaton. In fact, since the axion field is massless and weakly-coupled like the inflaton, it has the same fluctuation spectrum [51, 52, 53, 54]

$$P_a(k) = \int \frac{{\rm d}^3\mathbf{x}}{(2\pi)^3} \, \langle \delta a(\mathbf{x}, t) \, \delta a(\mathbf{x}', t) \rangle \, {\rm e}^{-{\rm i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \left(\frac{H_{\rm I}}{2\pi}\right)^2 \frac{2\pi^2}{k^3} \,, \quad (2.111)$$

where $H_{\rm I}$ is the expansion rate during inflation. As before, $\mathbf{x}$ are co-moving spatial coordinates. The axion fluctuations described by (2.111) are commonly written in shorthand notation as $\delta a = H_{\rm I}/2\pi$. The fluctuation in each axion field mode is "frozen in" after $R(t)/k$ exceeds the horizon length $H_{\rm I}^{-1}$.

We do not consider here the possibility of fluctuations in the axion decay constant f_a during inflation. Such fluctuations are discussed in [55, 56, 57, 58].

At the start of the QCD phase-transition, the local value of the axion field $a(\mathbf{x}, t)$ determines the local number density of cold axions produced by the vacuum realignment mechanism [see (2.47)],

$$n_a(\mathbf{x}, t_1) = \frac{f_a^2}{2t_1} \, \alpha(\mathbf{x}, t_1)^2 \quad (2.112)$$

where $\alpha(\mathbf{x}, t_1) = a(\mathbf{x}, t_1)/f_a$ is the misalignment angle. The fluctuations in the axion field produce perturbations in the cold axion density

$$\frac{\delta n_a^{\rm iso}}{n_a} = \frac{2\delta a}{a_1} = \frac{H_{\rm I}}{\pi f_a \alpha_1} \,, \quad (2.113)$$

where $a_1 = a(t_1) = f_a\alpha_1$ is the value of the axion field at the start of the QCD phase-transition, common to our entire visible universe. These perturbations obey

$$\delta\varrho_a^{\rm iso}(t_1) = -\delta\varrho_{\rm rad}^{\rm iso}(t_1) \tag{2.114}$$

as the vacuum-realignment mechanism converts energy stored in the quark-gluon plasma into axion rest mass energy. In contrast, the density perturbations produced by the fluctuations in the inflaton field [54, 59, 60, 61, 62] satisfy

$$\frac{\delta\varrho_{\rm matter}}{\varrho_{\rm matter}} = \frac{3}{4}\frac{\delta\varrho_{\rm rad}}{\varrho_{\rm rad}} \,. \tag{2.115}$$

Density perturbations that satisfy (2.115) are called "adiabatic," whereas that do not satisfy (2.115) are called "isocurvature". Isocurvature perturbations, such as the density perturbations of (2.114), make a different imprint on the cosmic microwave background than do adiabatic ones. The CMBR observations are consistent with pure adiabatic perturbations. This places a constraint on axion models if the PQ phase-transition occurs before inflation.

Before we derive this constraint, two comments are in order. The first is that if the PQ transition occurs *after* inflation, axion models still predict isocurvature perturbations, but not on length scales relevant to CMBR observations. Indeed, we saw in the previous section that in this case (case 2), the axion field fluctuates by order f_a from one QCD horizon to the next. Those fluctuations produce isocurvature perturbations on the scale of the QCD horizon, which is much smaller than the length scales observed in the CMBR. Their main phenomenological implication is the axion miniclusters, which were discussed in Sect. 2.5. The second comment is that, if the PQ phase-transition occurs before inflation (case 1), the density perturbations in the cold axion fluid have both adiabatic and isocurvature components. The adiabatic perturbations, $\delta\varrho_a^{\rm ad}/3\varrho_a = \delta\varrho_{\rm rad}^{\rm ad}/4\varrho_{\rm rad} = \delta T/T$, are produced by the quantum-mechanical fluctuations of the inflaton field during inflation, whereas the isocurvature perturbations are produced by the quantum-mechanical fluctuations of the axion field during that same epoch. The adiabatic and axion isocurvature components are uncorrelated.

The upper bound from CMBR observations and large-scale structure data on the fraction of CDM perturbations that are isocurvature is of order 30% in amplitude (10% in the power spectrum) [63, 64, 65, 66, 67, 68]. Allowing for the possibility that only part of the cold dark matter is axions, the bound on isocurvature perturbations implies

$$\frac{\delta\varrho_a^{\rm iso}}{\varrho_{\rm CDM}} = \frac{\delta\varrho_a^{\rm iso}}{\varrho_a}\frac{\varrho_a}{\varrho_{\rm CDM}} = \frac{H_{\rm I}}{\pi f_a \alpha_1}\frac{\Omega_a}{\Omega_{\rm CDM}} < 0.3\,\frac{\delta\varrho_{\rm m}}{\varrho_{\rm m}} \,, \tag{2.116}$$

where we used (2.113). Here, $\delta\varrho_{\rm m}/\varrho_{\rm m}$ is the amplitude of the primordial spectrum of matter perturbations. It is related to the amplitude of low multipole CMBR anisotropies through the Sachs-Wolfe effect [69, 70, 71]. The observations imply $\delta\varrho_{\rm m}/\varrho_{\rm m} \simeq 4.6 \times 10^{-5}$ [72].

In terms of α_1, the cold axion energy density is given by (2.100). We rewrite that equation here, assuming $h \simeq 0.7$,

$$\Omega_a \simeq 0.15 \left(\frac{f_a}{10^{12}\,\text{GeV}}\right)^{7/6} \alpha_1^2\ . \tag{2.117}$$

It has been remarked by many authors, starting with S.-Y. Pi [73], that it is possible for f_a to be much larger than 10^{12} GeV because α_1 may be accidentally small in our visible universe. The requirement that $\Omega_a < \Omega_{\rm CDM} = 0.22$ implies

$$\left|\frac{\alpha_1}{\pi}\right| < 0.4 \left(\frac{10^{12}\,\text{GeV}}{f_a}\right)^{7/12}\ . \tag{2.118}$$

Since $-\pi < \alpha_1 < +\pi$ is the a-priori range of α_1 and no particular value is preferred over any other, $|\alpha_1/\pi|$ may be taken to be the "probability" that the initial misalignment angle has magnitude less than $|\alpha_1|$. (Strictly speaking, the word probability is not appropriate here as there is only one universe in which α_1 may be measured.) If $|\alpha_1/\pi| = 2\times10^{-3}$, for example, f_a may be as large as 10^{16} GeV, which is often thought to be the "grand unification scale".

The presence of isocurvature perturbations constrains the small α_1 scenario in two ways [50]. First, it makes it impossible to have α_1 arbitrarily small. Using (2.111), one can show that the fluctuations in the axion field cause the latter to perform a random walk [74] characterized by the property

$$\frac{1}{V}\int_V d^3\mathbf{x} \left\langle [\delta a(\mathbf{x},t) - \delta a(\mathbf{0},t)]^2 \right\rangle = 4\pi H_{\rm I}^2 \ln(Rk_{\rm max})\ . \tag{2.119}$$

The integral is over a sphere of volume $V = 4\pi R^3/3$ centered at $\mathbf{x} = \mathbf{0}$ and $k_{\rm max}$ is a cutoff on the wavevector spectrum. a_1^2 cannot be smaller than the RHS of (2.119) with R equal to the size of the present universe and $k_{\rm max}$ equal to the Hubble rate at the QCD time, redshifted down to the present. As $\Omega_a < 0.22$, this implies a bound on $H_{\rm I}$. Translated to a bound on the scale of inflation $\Lambda_{\rm I}$, defined by $H_{\rm I}^2 = 8\pi G\Lambda_{\rm I}^4/3$, it is

$$\Lambda_{\rm I} < 5\times10^{14}\,\text{GeV} \left(\frac{f_a}{10^{12}\,\text{GeV}}\right)^{5/24}\ . \tag{2.120}$$

Second, one must require axion isocurvature perturbations to be consistent with CMBR observations. Combining (2.116) and (2.117), and setting $\Omega_{\rm CDM} = 0.22$, $\delta\varrho_{\rm m}/\varrho_{\rm m} = 4.6\times10^{-5}$, one obtains

$$\Lambda_{\rm I} < 10^{13}\,\text{GeV}\,\Omega_a^{-1/4} \left(\frac{f_a}{10^{12}\,\text{GeV}}\right)^{5/24}\ . \tag{2.121}$$

Let us keep in mind that the bounds (2.120) and (2.121) pertain only if the reheat temperature $T_{\rm RH} < T_{\rm PQ}$. One may, for example, have $\Omega_a = 0.22$, $f_a \simeq 10^{12}$ GeV, and $\Lambda_{\rm I} \simeq 10^{16}$ GeV, provided $T_{\rm RH} \gtrsim 10^{12}$ GeV, which is possible if reheating is efficient enough.

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