Discrete Duality and Its Applications to Reasoning with Incomplete Information

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Abstract. We present general principles of establishing a duality between a class of algebras and a class of relational systems such that topology is not involved. We show how such a discrete duality contributes to proving completeness of logical systems and to correspondence theory. Next, we outline applications of discrete dualities to analysis of data in information systems with incomplete information in the rough set-style, and in contexts of formal concept analysis.

1 General Principles of Discrete Duality and Duality Via Truth

Duality theory emerged from the work by Marshall Stone [Sto36] on Boolean algebras and distributive lattices in the 1930s. Jónsson and Tarski [JT51] extended Stone's results to Boolean algebras with operators. These operators are now known to be modal possibility operators. Later in the early 1970s Larisa Maksimova [Mak72, Mak75] and Hilary Priestley [Pri70, Pri72] developed analogous results for Heyting algebras, topological Boolean algebras, and distributive lattices. The latter has been extended to distributive lattices with operators by [Gol89, CLP91]. Since then establishing a duality has become an important methodological problem both in algebra and in logic. All the abovementioned classical duality results are developed using topological spaces as dual spaces of algebras.

Discrete duality is a duality where a class of abstract relational systems is a dual counterpart to a class of algebras. These relational systems are referred to as frames following the terminology of non-classical logics. A topology is not involved in the construction of these frames and hence they may be thought of as having a discrete topology. Establishing discrete duality involves the following steps. Given a class Alg of algebras (resp. a class Frm of frames) we define a class Frm of frames (resp. a class Alg of algebras). Next, for an algebra $W \in \mathsf{Alg}$ we define its canonical frame $X(W)$ and for each frame $X \in \mathsf{Frm}$ we define its complex algebra $C(X)$. Then we prove that $\mathcal{X}(W) \in \mathsf{Frm}$ and $\mathcal{C}(X) \in \mathsf{Alg}$. A duality between Alg and Frm holds provided that the following representation theorems are proved:

(D1) Every algebra $W \in \mathsf{Alg}$ is embeddable into the complex algebra of its canonical frame i.e., $\mathcal{C}(\mathcal{X}(W))$.

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(D2) Every frame $X \in \mathsf{Frm}$ is isomorphic with a substructure of the canonical frame of its complex algebra, i.e., $\mathcal{X}(\mathcal{C}(X))$.

A distinguishing feature of this framework for establishing a discrete duality is that the algebraic and the logical notions involved in the proofs are defined in an autonomous way, we do not mix the algebraic and logical methodologies.

The separation of logical and algebraic constructs enables us to view dual classes of algebras and frames as two types of semantic structures of a formal language. As a consequence we easily obtain what we call duality via truth. Given a formal language Lan, a class of frames Frm which determines a relational semantics for Lan and a class Alg of algebras which determines its algebraic semantics, a duality via truth theorem says that these two kinds of semantics are equivalent in the following sense:

(DvT) A formula $\phi \in \mathsf{Lan}$ is true in every algebra of Alg iff it is true in every frame of Frm.

In order to prove such a theorem we need to prove the following lemma referred to as a complex algebra theorem.

(CA) For every frame $X \in \mathsf{Frm}$, a formula $\phi \in \mathsf{Lan}$ is true in X iff ϕ is true in the complex algebra $\mathcal{C}(X)$.

With the theorem (CA) and the representation theorem (D1) we can prove (DvT) theorem. The right-to-left implication of (DvT) follows from the left-to-right implication of (CA) and the left-to-right implication of (DvT) follows from right-to-left implication of (CA) and (D1). In this way the discrete duality contributes to a development of a relational semantics (resp. an algebraic semantics) once an algebraic semantics (resp. a relational semantics) of a language is known.

2 Application to Completeness and Correspondence Theorems

Discrete duality contributes also to a completeness result once a deductive system for the language Lan is given. Assume that an algebraic semantics of Lan is given in terms of a class Alg of algebras and a relational semantics in terms of a class Frm of frames such that a discrete duality holds between these two classes. We assume that the algebras from Alg are based on bounded lattices. To prove completeness we define a binary relation $\approx$ in the set of formulas of Lan in terms of provability of double implication, if it is among the propositional operations of Lan, or otherwise in terms of provability of a sequent built with a pair of formulas. Next we show that this relation is an equivalence relation and a congruence with respect to all the propositional operations admitted in Lan. Then we form the Lindenbaum algebra $\mathcal{A}_{\approx}$ of Lan. Its universe consists of equivalence classes $|\phi|$ (with respect to relation $\approx$) of formulas. Then we show that the algebra $\mathcal{A}_{\approx}$ belongs to the class Alg of algebras. Now, depending on whether we are interested in completeness with respect to the algebraic or relational semantics we proceed as follows.

To prove completeness of the deduction system with respect to the relational semantics we consider the canonical frame $\mathcal{X}(\mathcal{A}_{\approx})$ of the Lindenbaum algebra. Its universe consists of prime filters of $\mathcal{A}_{\approx}$. Then we form a model $M_{\approx}$ based on this frame. Preservation of operations by the mapping that provides an embedding of $\mathcal{A}_{\approx}$ into $\mathcal{C}(\mathcal{X}(\mathcal{A}_{\approx}))$ guaranteed by theorem (D1) enables us to prove the truth lemma saying that satisfaction of a formula ϕ in model $M_{\approx}$ by a filter F is equivalent to $|\phi| \in F$. From this lemma the completeness follows in the usual way.

To prove completeness of the deduction system with respect to the algebraic semantics we define a valuation of atomic formulas of Lan in $\mathcal{A}_{\approx}$ as $v(p) = |p|$ and we prove that it extends to all the formulas so that $v(\phi) = |\phi|$. Then we show that provability of a formula ϕ is equivalent to $v(\phi) = 1_{\approx}$, where $1_{\approx}$ is the unit element of the lattice reduct of $\mathcal{A}_{\approx}$. Then the completeness follows.

Discrete duality is also relevant for the correspondence theory which aims at finding relationships between truth of formulas in a frame and properties of relations in the frame. Typically, a correspondence has the following form:

(Cps) A formula $\phi \in \mathsf{Lan}$ is true in a frame $\mathcal{X}$ iff the relations of the frame have a certain property.

Given the classes Alg and Frm for which a discrete duality and duality via truth theorem with respect to a language Lan hold, we may consider the following correspondences:

(Cps1) The relations of a frame $\mathcal{X} \in \mathsf{Frm}$ have a certain property iff a formula $\phi \in \mathsf{Lan}$ is true in the complex algebra $\mathcal{C}(X)$.

(Cps2) A formula $\phi \in \mathsf{Lan}$ is true in an algebra $W \in \mathsf{Alg}$ iff the relations of the canonical frame $\mathcal{X}(W)$ have a certain property.

It is known that these corespondences are related to the classical correspondence (Cps). The left-to-right implication of (Cps1) and the right-to-left implication of (CA) imply the right-to-left implication of (Cps). The right-to-left implication of (Cps1) and the left-to-right implication of (CA) imply left-to-right implication of (Cps). Examples of the correspondences of these types can be found in [JO06]

3 Applications to Reasoning with Incomplete Information and Data Analysis

The general framework of discrete duality and duality via truth outlined above may be applied to various classes of lattices with operators. In [ORD06] a duality via truth framework is presented and illustrated with four case studies. The classical dualities for Boolean algebras with a possibility operator and for

Boolean algebras with a sufficiency operator are formulated in the form of a discrete duality and duality via truth. Then these results are extended to discrete dualities and dualities via truth for two classes of information algebras arising from information systems. The class of weak similarity algebras is an axiomatic extension of the class of Boolean algebras with a family of possibility operators and the class of strong right orthogonality (or in other words strong disjointness) algebras is an axiomatic extension of the class of Boolean algebras with a family of sufficiency operators. These two classes of information algebras provide a formal means for reasoning about and computing generalized approximation operations in the rough set style [Paw91] determined by similarity relations or their complements.

The framework of discrete duality is also relevant for formal concept analysis [Wil82, GaW99]. The theory of formal concept analysis provides a means of data analysis and discovery of concepts from data structures which are referred to as contexts. Contexts may be identified with information systems whose attributes are binary, in the sense of being features an object may or may not have. In [ORe07b] a class of sufficiency algebras derived from contexts is introduced and referred to as context algebras. A discrete duality and duality via truth for the class of context algebras is developed. These results provide the tools for solving various problems that can be specified within the framework of formal concept analysis e.g., finding extensions (resp. intensions) of concepts once their intensions (resp. extensions) are given; proving implications of sets of attributes; proving entailment of implications.

Many other discrete duality and duality via truth results can be found in the literature. Most of them concern not necessarily distributive lattices with operators. Various types of modal operators (possibility, necessity, sufficiency, dual sufficiency) are dealt with in [OV05]. These operators may be seen as generalizations of rough set style approximation operators. Several kinds of negations on lattices are treated in [DOvA06a, DOvA06b]. Relation algebra operators on lattices are studied in [DOR06]. Residuated lattices and their axiomatic extensions corresponding to substructural logics and some fuzzy logics are studied within the framework of discrete duality in [OR06, OR07]. In the field of distributive lattices, discrete dualities for Heyting algebras with operators (various types of modal operators and negations) are presented in [ORe07a].

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