Set Operations for L-Fuzzy Sets

Jouni Järvinen

Turku Centre for Computer Science (TUCS)
FI-20014 University of Turku, Finland
Jouni.Jarvinen@utu.fi

Abstract. In this paper, we introduce the operations of union, intersection, and complement for preorder-based fuzzy sets. The given operations are even capable of dealing with fuzzy sets that have membership degrees coming from different preordered sets. This enables us to handle the difficult situation in which one has different people giving judgements and they all like to use their own language and expressions.

1 Fuzzy Sets and L-Fuzzy Sets

Fuzzy sets were introduced in 1965 by Zadeh [9]. For a given universe of discourse U, a *fuzzy set* A on U is determined by a membership function $\mu_A : U \to [0, 1]$ associating with each element $x \in U$ a real number $\mu_A(x)$ which represents the grade of membership of x in A.

Zadeh also introduced the set operations of union, intersection, and complementation for fuzzy sets. These operations are important because if one looks at the logical aspect of these operations, they represent 'or', 'and', and 'not'. The *union* of two fuzzy sets A and B is a fuzzy set whose membership function is $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$. Further, the *intersection* of the fuzzy sets A and B is a fuzzy set with the membership function $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$. The *complement* of the fuzzy set A is defined by $\mu_{A'}(x) = 1 - \mu_A(x)$. The above operations are often referred to as the *standard fuzzy set operations*, but in the literature one can find numerous different ways to define the set operations; see [7], for example.

The fundamental problem with fuzzy sets is that our perceptions have to be quantized to the unit interval. In this paper, our aim is to get rid of this semi-arbitrary choosing of the proper weighting scheme. We try to move towards the methodology, called *computing with words* [10], in which the objects of computation are given by a natural language. Computing with words, in general, is inspired by the human capability to perform a wide variety of tasks without any measurements and any quantizations.

Goguen generalized fuzzy sets to L-fuzzy sets in [3]. An *L-fuzzy set* φ on U is a mapping $\varphi : U \to L$, where L is a 'transitive partially ordered set'. In this work, we assume that $(L, \leq)$ is a preordered set. Notice that it is natural to assume that the relation $\leq$ is not antisymmetric; if $x, y \in L$ are synonyms, that is, words or expressions that are used with the same meaning, then $x \leq y$ and $x \geq y$, but still x and y are distinct words.

M. Kryszkiewicz et al. (Eds.): RSEISP 2007, LNAI 4585, pp. 221–229, 2007.

Example 1. Suppose that U consists of a group of people. The L-fuzzy set, whose membership function φ is depicted in Fig. 1, describes how well the persons in U can ski. For instance, there exist people who can ski very well, some ski badly, and some are moderate skiers.

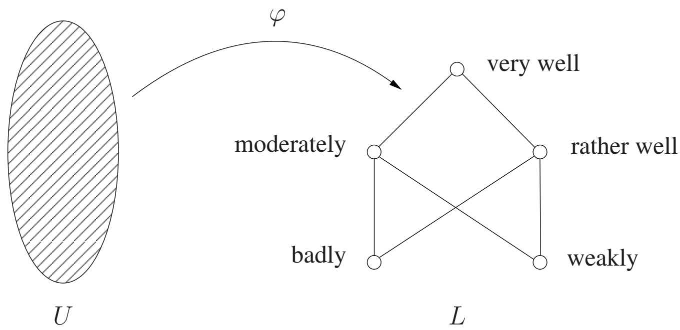

Fig. 1.

As noted by Goguen [3], the set of all L-fuzzy sets on a set U can be equipped whatever operations L has, and these inherited operations obey any law valid in L which extends pointwise. This implies that if L is, for example, a Boolean lattice, then also the set all of L-fuzzy sets on U forms a Boolean lattice. Formally, if φ and ψ are L-fuzzy sets on U, then for any $x \in U$,

$$(\varphi \vee \psi)(x) = \varphi(x) \vee \psi(x)$$
$$(\varphi \wedge \psi)(x) = \varphi(x) \wedge \psi(x)$$
$$\varphi'(x) = \varphi(x)' .$$

In this paper, we show how to define unions, intersections, and complements of L-fuzzy sets in cases L is just a preordered set, which means that joins, meets, and complements are not defined in L. The presented approach also handles the union and the intersection of an L_1-fuzzy set φ and an L_2-fuzzy set ψ on the same universe U, but not necessarily on the same preordered set. This means that we can, for example, combine with 'or', 'and', and 'not' judgements of evaluators all wanting to use their own words and expressions. Our key idea is that the order determined by membership values is essential, not the values themselves. It should be noted that some ideas presented in this work appear already in [4,5].

2 Preorders and Alexandrov Topologies

Preorders and Alexandrov topologies have a major role in this paper. Therefore, we begin with presenting some results concerning them. This section contains also many lattice-theoretical notions which can be found in [1,2,4], for example.

Let U be any set and let R be a binary relation on U. Then, the relation R is a *preorder*, if

(i) for all $x \in U$, $x\,R\,x$ (reflexive)
(ii) for all $x, y, z \in U$, $x\,R\,y$ and $y\,R\,z$ imply $x\,R\,z$ (transitive)

The pair $(U, \leq)$ is called a *preordered set*. Note that often we say simply that 'U is a preordered set'.

We may depict preorders by Hasse diagrams as in case of partially ordered sets. The only difference is that preorders are not necessarily antisymmetric, meaning that there may exist elements $x \neq y$ such that $x \leq y$ and $x \geq y$. However, such elements can simply be represented as collections of $\approx$-equivalent elements, where the equivalence $\approx$ is defined by

$$x \approx y \text{ if and only if } x \leq y \text{ and } x \geq y.$$

This means that 'synonymous' elements are represented by a same point in a Hasse diagram, but still they all preserve their identities.

Let us denote by $\mathrm{Pre}(U)$ the set of all preorders on the set U. The set $\mathrm{Pre}(U)$ can be ordered with the usual set-inclusion relation, because relations are just sets of ordered pairs. First we recall the following well-known lemma that is clear since the intersection of any subset of $\mathrm{Pre}(U)$ is a preorder. Note that generally the union of preorders is not a preorder.

Lemma 2. *For any set U,* $\mathrm{Pre}(U)$ *is a complete lattice with respect to the set-inclusion relation.*

Since $\mathrm{Pre}(U)$ is a *closure system*, that is, a family of sets closed under arbitrary intersections, we have that for any $\mathcal{H} \subseteq \mathrm{Pre}(U)$, the meet $\bigwedge \mathcal{H}$ is the intersection $\bigcap \mathcal{H}$ and the join $\bigvee \mathcal{H}$ is the intersection of all preorders including $\bigcup \mathcal{H}$. We will present another description of joins later in this section. Furthermore, the 'all relation' $\nabla = \{(x, y) \mid x, y \in U\}$ is the greatest element and the 'identity relation' $\Delta = \{(x, x) \mid x \in U\}$ is the least element of $\mathrm{Pre}(U)$.

A *topological space* is a pair $(U, \mathcal{T})$, where U is a set and $\mathcal{T}$ is a collection of subsets of U such that

(i) $\emptyset, U \in \mathcal{T}$;
(ii) for all $\mathcal{H} \subseteq \mathcal{T}$, $\bigcup \mathcal{H} \in \mathcal{T}$;
(iii) for all $X, Y \in \mathcal{T}$, $X \cap Y \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology*.

An *Alexandrov topology* is a topology $\mathcal{T}$ that contains also all arbitrary intersections of its members. This means that for Alexandrov topologies, condition (iii) is replaced by condition

(iii)$^\circ$ for all $\mathcal{H} \subseteq \mathcal{T}$, $\bigcap \mathcal{H} \in \mathcal{T}$.

The pair $(U, \mathcal{T})$ is referred to as an *Alexandrov space*.

Every Alexandrov topology $\mathcal{T}$ has the property that each point $x \in U$ has a *smallest neighbourhood* $N_{\mathcal{T}}(x) = \bigcap \{X \in \mathcal{T} \mid x \in X\}$. This means that $N_{\mathcal{T}}(x)$ is the smallest set in the topology $\mathcal{T}$ containing the point x.

Let us denote by $\mathrm{Alex}(U)$ the set of all Alexandrov topologies. Obviously, also $\mathrm{Alex}(U)$ can be ordered by the set-inclusion relation. Because the intersection of Alexandrov topologies is an Alexandrov topology, we may write the next lemma.

Lemma 3. *For any set U,* Alex(U) *is a complete lattice with respect to the set-inclusion relation.*

Clearly, Alex(U) is a closure system and hence for any $\mathcal{H} \subseteq \mathrm{Alex}(U)$, $\bigwedge \mathcal{H}$ is equal to the intersection $\bigcap \mathcal{H}$ and $\bigvee \mathcal{H}$ is the intersection of all Alexandrov topologies including $\bigcup \mathcal{H}$. In addition, the 'discrete topology' $\mathcal{T}_\Delta = \{X \mid X \subseteq U\}$ is the greatest element and the 'trivial topology' $\mathcal{T}_\nabla = \{\emptyset, U\}$ is the smallest element of Alex(U).

There is a close connection between preorders and Alexandrov topologies. Let $\leq$ be a preorder on a set U. We may now define an Alexandrov topology $\mathcal{T}_\leq$ on U consisting of all upward-closed subsets of U with respect to the relation $\leq$, that is,

$$\mathcal{T}_\leq = \{X \subseteq U \mid (\forall x, y \in U)\ x \in X \ \ \&\ \ x \leq y \Longrightarrow y \in X\}.$$

Let us denote for any $x \in U$, the principal filter of x by $\uparrow x = \{y \in U \mid x \leq y\}$. Now we can give the following lemma.

Lemma 4. *If $\leq$ is a preorder on U, then the following assertions hold for all $X \subseteq U$ and $x \in U$:*

(i) $X \in \mathcal{T}_\leq$ *if and only if* $X = \bigcup\{\uparrow x \mid x \in X\}$*;*
(ii) $\uparrow x$ *is the smallest neighbourhood of x in the Alexandrov topology $\mathcal{T}_\leq$.*

Proof. (i) Assume that $X \in \mathcal{T}_\leq$. If $x \in X$, then $x \leq x$ gives $x \in \uparrow x$. Thus, $X \subseteq \bigcup\{\uparrow x \mid x \in X\}$. On the other hand, if $y \in \bigcup\{\uparrow x \mid x \in X\}$, then there exists $x \in X$ such that $x \leq y$. Since $X \in \mathcal{T}_\leq$, we obtain $y \in X$. Hence, also $\bigcup\{\uparrow x \mid x \in X\} \subseteq X$.

Conversely, suppose $X = \bigcup\{\uparrow x \mid x \in X\}$, $x \in X$, and $x \leq y$. Then $y \in \uparrow x$ and so $y \in X$. Therefore, X is upward closed and $X \in \mathcal{T}_\leq$.

(ii) It is clear that $x \in \uparrow x \in \mathcal{T}_\leq$ and if $x \in X \in \mathcal{T}_\leq$, then $\uparrow x \subseteq X$ by (i). □

By the above lemma, $\uparrow x$ is the smallest neighbourhood of the point x in the Alexandrov topology $\mathcal{T}_\leq$ and clearly $y \in \uparrow x$ if and only if $x \leq y$. This hints how we may also define preorders by means of Alexandrov topologies. If $\mathcal{T}$ is an Alexandrov topology on U, then we define a preorder $\leq_\mathcal{T}$ on U by setting

$$x \leq_\mathcal{T} y \iff y \in N_\mathcal{T}(x).$$

The following theorem by Steiner [8] is essential for our studies.

Theorem 5. *For any set U, the complete lattice* Pre(U) *of all preorders on U is dually isomorphic to* Alex(U)*, the complete lattice of all Alexandrov topologies on U; in symbols* $(\mathrm{Pre}(U), \subseteq) \cong (\mathrm{Alex}(U), \supseteq)$.

A nice property of set unions and intersections is that they distribute over each other. Therefore, it is a natural question to ask whether joins and meets defined in a particular lattice have analogous properties. Formally, a lattice L is *distributive* if it satisfies either (and therefore both) of the distributive laws:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Furthermore, L is *modular* if

$$x \leq z \Longrightarrow x \vee (y \wedge z) = (x \vee y) \wedge z.$$

Trivially, each distributive lattice is modular.

Steiner noted that $\mathrm{Pre}(U)$ and $\mathrm{Alex}(U)$ are distributive if U has fewer than three elements. If U has three or more elements, $\mathrm{Pre}(U)$ and $\mathrm{Alex}(U)$ are not even modular.

Next we present a simpler way to determine the joins in $\mathrm{Pre}(U)$ and $\mathrm{Alex}(U)$. Recall that in $(\mathrm{Alex}(U), \subseteq)$, the meet is the intersection of Alexandrov topologies. Thus, the join in its dual $(\mathrm{Alex}(U), \supseteq)$ is the intersection of Alexandrov topologies, that is,

$$\mathcal{T}_1 \vee \mathcal{T}_2 \ = \ \mathcal{T}_1 \cap \mathcal{T}_2.$$

By Theorem 5, $(\mathrm{Pre}(U), \subseteq) \cong (\mathrm{Alex}(U), \supseteq)$, which implies that in $(\mathrm{Pre}(U), \subseteq)$,

$$\leq_1 \vee \leq_2 = \leq_{(\mathcal{T}_1 \cap \mathcal{T}_2)},$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are the Alexandrov topologies determined by $\leq_1$ and $\leq_2$. Similarly, in $(\mathrm{Alex}(U), \subseteq)$,

$$\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{T}_{(\sqsubseteq_1 \cap \sqsubseteq_2)},$$

where $\sqsubseteq_1$ and $\sqsubseteq_2$ are the preorders of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Next we study complementation in these isomorphic lattices. A *lattice-complement* of a preorder R is a preorder R' such that $R \vee R' = \nabla$ and $R \wedge R' = \Delta$. The next important theorem is also proved by Steiner [8].

Theorem 6. *The lattice* $\mathrm{Pre}(U)$ *is complemented.*

It is trivial that the set-theoretical complement R^c of a preorder R cannot serve as the lattice-theoretical complement, because R^c is not a preorder and $R \wedge R^c = \emptyset \neq \Delta$. Next we describe the lattice-theoretical complement R' of R in $\mathrm{Pre}(U)$. Let R^E be the smallest equivalence including R. Further, let $\{X_i \mid i \in I\}$ be the set of equivalence classes of R^E. By the Axiom of Choice we may pick an element from each equivalence class. Let us denote the representative of the class X_i by x_i. Next we derive two new relations R_1 and R_2 from R by setting

$$R_1 = \{(y,x) \mid x \, R \, y \ \& \ (y,x) \notin R\} \cup \Delta$$

and

$$R_2 = \{(x_i, x_j) \mid i, j \in I\} \cup \Delta.$$

It is easy to see that R_1 and R_2 are preorders. The lattice-theoretical complement R' of R is defined by

$$R' = R_1 \vee R_2.$$

It is known that if a lattice is distributive, the complements – if they exist – are unique. We have already mentioned that $\mathrm{Pre}(U)$ is not distributive when $|U| \geq 3$. This implies that the complements are not necessarily unique. Namely, if R is a preorder such that R^E has at least two equivalence classes of which at least one is non-singleton, then the complement of R depends on the choice function $U/R^E \to U$. On the other hand, if R^E has only one equivalence class U, then $R_2 = \Delta$ and the complement of R is R_1 which clearly is unique. In such a case, the Hasse diagram of R' is just the Hasse diagram of R turned upside down with its equivalent elements being separated. Note also that R^E has only one equivalence class if and only if R is *connected*, that is, for any $x, y \in U$, there exists a sequence $a_0, a_1, \ldots, a_n$ of elements of U such that $a_0 = x$, $a_n = y$, and $a_i \, R \, a_{i+1}$ or $a_{i+1} \, R \, a_i$ for $i = 0, \ldots, n-1$.

We end this section by noting that Theorem 6 has the following obvious corollary.

Corollary 7. *The lattice* $\mathrm{Alex}(U)$ *is complemented.*

3 *L*-Fuzzy Sets and Their Operations

In this section our aim is to define set operations for L-fuzzy sets.

Let U be a set and let L be an arbitrary preordered set. Any L-fuzzy set φ on U determines naturally a preorder on U, as suggested by Kortelainen in [6]. A preorder $\lesssim_\varphi$ is defined by setting for all $x, y \in U$,

$$x \lesssim_\varphi y \iff \varphi(x) \leq \varphi(y).$$

By Theorem 5 there is one-to-one correspondence between preorders and Alexandrov topologies on U. This implies directly that each L-fuzzy set induces also an Alexandrov topology $\mathcal{T}_\varphi$ consisting of upward-closed subsets of $\lesssim_\varphi$. Let us denote the principal filter $\uparrow x$ of x with respect to the preorder $\lesssim_\varphi$ by $N_\varphi(x)$, that is, $N_\varphi(x) = \{y \mid \varphi(x) \leq \varphi(y)\}$. By Lemma 4, it is clear that

$$X \in \mathcal{T}_\varphi \iff X = \bigcup \{N_\varphi(x) \mid x \in U\}$$

and $N_\varphi(x)$ is the smallest neighbourhood of x in the Alexandrov topology $\mathcal{T}_\varphi$.

Next we show how Alexandrov topologies determine fuzzy sets. Let $\mathcal{T}$ be an Alexandrov topology on a set U. Let us denote by $\mathcal{T}^{\mathrm{op}}$ the ordered set $(\mathcal{T}, \supseteq)$. Now the mapping

$$\varphi_{\mathcal{T}} \colon U \to \mathcal{T}^{\mathrm{op}}, \ x \mapsto N_{\mathcal{T}}(x)$$

is a $\mathcal{T}^{\mathrm{op}}$-fuzzy set. It is also easy to observe that if φ is an L-fuzzy set on U, then $\varphi^* \colon U \to {\mathcal{T}_\varphi}^{\mathrm{op}}, \ x \mapsto N_\varphi(x)$ is a fuzzy set such that the preorder $\lesssim_\varphi$ of φ is equal to the preorder $\lesssim_{\varphi^*}$ determined by φ^*. Furthermore, $\varphi^{**} = \varphi^*$. Thus, φ^* can be identified as a *canonical representation* of φ, as is done in [5].

Let us denote by $\mathrm{Fuzzy}(U)$ the class of all fuzzy sets on U, that is, the collection of all such mappings $\varphi \colon U \to L$ that L is any arbitrary preordered set. We noted in the previous section that $(\mathrm{Pre}(U), \subseteq)$ is a complemented lattice. Because each element in Fuzzy(U) determines a unique preorder, we may now define the *union*, the *intersection*, and the *complement* for any elements $\varphi: U \to L_1$ and $\psi: U \to L_2$ of Fuzzy(U) as follows:

$$\varphi \cup \psi := \lesssim_\varphi \vee \lesssim_\psi \tag{1}$$
$$\varphi \cap \psi := \lesssim_\varphi \wedge \lesssim_\psi \tag{2}$$
$$\varphi^c := \lesssim_\varphi{}' . \tag{3}$$

Note that there always exists a fuzzy set in Fuzzy(U) corresponding to the results of these operations. For example, let us consider the union $\varphi \cup \psi$. As we have shown, the Alexandrov topology $\mathcal{T}_{\varphi\cup\psi}$ determines a fuzzy set

$$(\varphi \cup \psi)^*: U \to \mathcal{T}_{\varphi\cup\psi}{}^{\mathrm{op}},\ x \mapsto N_{(\varphi\cup\psi)}(x).$$

Using preorders as results of set operations is useful also because in applications we are often interested in the order of elements with respect to aggregation of some criteria.

Example 8. Assume that $U = \{x, y, z, w\}$ consists of four applicants of a certain academic position and that $\varphi: U \to L_1$ and $\psi: U \to L_2$ represent how two experts evaluate the suitability of the applicants by using some expressions and attributes L_1 and L_2 of their own languages. The fuzzy sets φ and ψ are given in Fig. 2 of page 228. The induced preorders are

$$\lesssim_\varphi = \{(y, x), (z, x), (w, x), (w, y), (w, z)\} \cup \Delta$$

and

$$\lesssim_\psi = \{(w, x), (z, x), (z, y)\} \cup \Delta.$$

These preorders and the canonical representations $\varphi^*: U \to \mathcal{T}_\varphi^{\mathrm{op}}$ and $\psi^*: U \to \mathcal{T}_\psi^{\mathrm{op}}$ are also depicted in Fig. 2. We define the union, the intersection, and the complements as described in (1)–(3). The results of these operations can be found in Fig. 2 as well.

Now $\varphi \cap \psi$ can be viewed as an order that takes into account the opinions of both the experts. The applicants x and y must be considered as suitable for the open position, but z and w should not be selected, since the both experts have the opinion that they are weaker than x. Let us consider the applicants in the view of the union $\varphi \cup \psi$. According to it, the applicant x should be chosen, because there exists one expert evaluating x as the best candidate, and this is not true for the others. The complements φ^c and ψ^c can be considered as orders totally opposite to the opinions of the expert.

Notice also that the De Morgan laws do not hold, because

$$\varphi^c \cap \psi^c \neq (\varphi \cup \psi)^c \quad \text{and} \quad \varphi^c \cup \psi^c \neq (\varphi \cap \psi)^c.$$

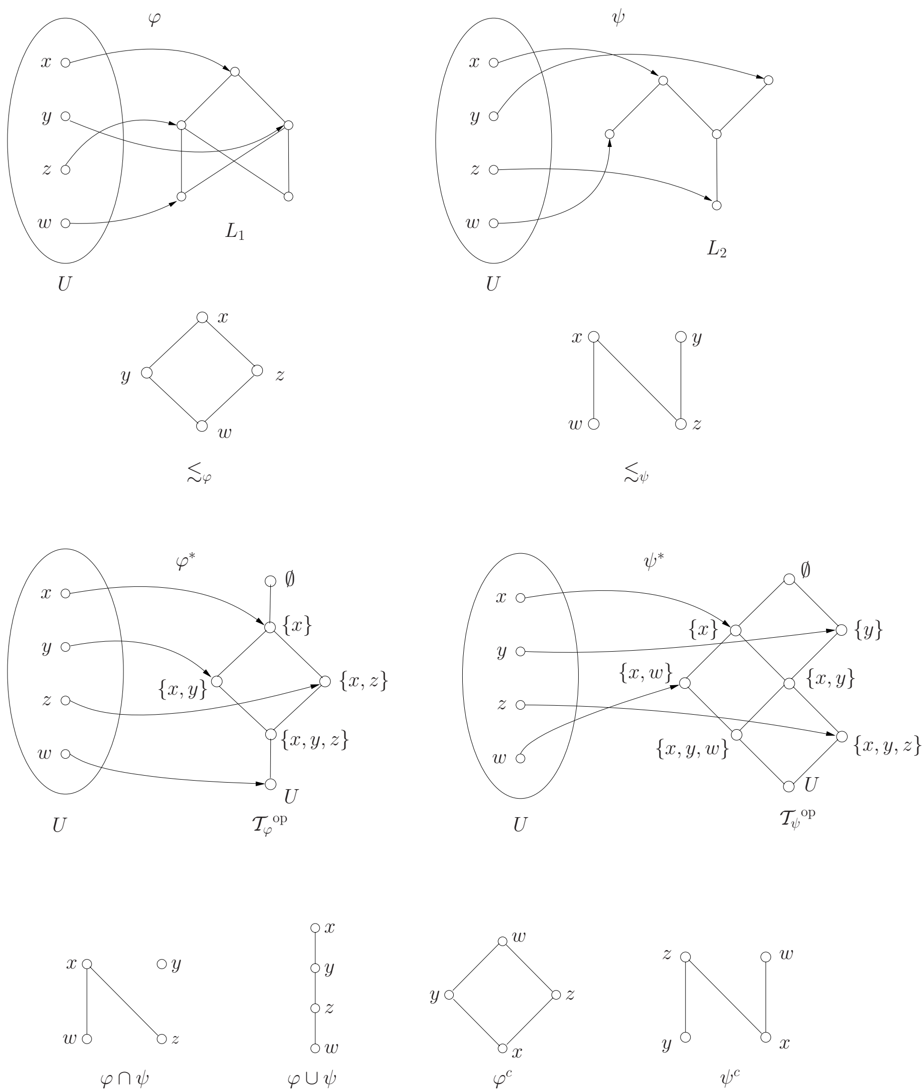

Fig. 2.

Some Concluding Remarks and Acknowledgements

In this paper we have introduced unions, intersections and complements for preorder-based fuzzy sets on a given universe U. Our work was based on the observation that each preorder-based fuzzy set determines a preorder and an Alexandrov topology on U. We have described how the results of these set operations can be easily formed. Importantly, the presented approach can handle the union and the intersection of an L_1-fuzzy set φ and an L_2-fuzzy set ψ of the universe U also in the case L_1 and L_2 are different preordered sets. This enables us to cope with the common situation in which one has different people giving judgements and they all like to use their own language and expressions.

The author thanks the anonymous referees for their comments and suggestions that helped to improve the paper.

References

1. Birkhoff, G.: Lattice Theory, Colloquim publications, 3rd edn. vol. XXV, American Mathematical Society (AMS), Providence, Rhode Island (1995)
2. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order, 2nd edn. Cambridge University Press, Cambridge (2002)
3. Goguen, J.A.: L-fuzzy sets. Journal of Mathematical Analysis and Applications 18, 145–174 (1967)
4. Järvinen, J.: Lattice theory for rough sets. In: Transactions on Rough Sets VI. LNCS, vol. 4374, pp. 400–498. Springer, Heidelberg (2007)
5. Järvinen, J., Kortelainen, J.: A unifying study between modal-like operators, topologies, and fuzzy sets. Fuzzy Sets and Systems 158, 1217–1225 (2007)
6. Kortelainen, J.: On relationship between modified sets, topological spaces and rough sets. Fuzzy Sets and Systems 61, 91–95 (1994)
7. Lowen, R.: Fuzzy Set Theory: Basic Concepts, Techniques and Bibliography. Kluwer Academic Publishers, Norwell, MA, USA (1996)
8. Steiner, A.K.: The lattice of topologies: structure and complementation. Transactions of the American Mathematical Society 122, 379–398 (1966)
9. Zadeh, L.A.: Fuzzy sets. Information and Control 8, 338–353 (1965)
10. Zadeh, L.A.: From computing with numbers to computing with words – from manipulation of measurements to manipulation of perceptions. International Journal of Applied Mathematics and Computer Science 12, 307–324 (2002)