Rough Sets and Vague Sets

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Abstract. The subject-matter of the consideration touches the problem of vagueness. The notion of the rough set, originated by Zdzisław Pawlak, was constructed under the influence of vague information and methods of shaping systems of notions leading to conceptualization and representation of vague knowledge, so also systems of their scopes as some vague sets. This paper outlines some direction of searching for a solution to this problem. In the paper, in connection to the notion of the rough set, the notion of a vague set is introduced. Some operations on these sets and their properties are discussed. The considerations intend to take into account a classical approach to reasoning, based on vague premises, and suggest finding a logic of vague sentences as a non-classical logic in which all counterparts of tautologies of classical logic are laws.

1 Introduction

Logicians and philosophers have been interested in the problem area of vague knowledge for a long time, looking for some logical bases of a theory of vague notions (terms) constituting such knowledge. Recently it has become the subject of investigations of computer scientists interested in the problems of AI, in particular, in problems of reasoning on the basis of incomplete or vague information and applications of computers to support and represent such reasoning in the computer memory. Significant results obtained by computer scientists in the scope of imprecision and vagueness: the Zadeh's fuzzy set theory [20], the Shafer's theory of evidence [17] and the Pawlak's rough sets theory [14] greatly contributed to actualization and intensification of research into vagueness.

The present paper proposes a new approach to vagueness and considers the problem of denotations of vague notions (terms) from the logical and computer sciences perspective. It yields logical foundations to a theory of vague notions (terms) and should be an essential contribution to that problem.

The paper consists of four sections. In Section 2, we introduce the notion of unit information (unit knowledge) and vague information (vague knowledge). The main notion of the vague set, inspired by the Pawlak's notion of a rough set is defined in Section 3. In Section 4 some operations on vague sets and their

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algebraic properties are given. A view on the problem of logic of vague concepts (terms) is discussed in Section 5. The paper ends with Section 6 including some final remarks.

2 Knowledge and Vague Knowledge

In the process of cognition of a definite fragment of reality, the cognitive agent (a man, an expert, a group of men or experts, a robot) attempts to discover information contained in it, and properly, about its objects. Each fragment of reality recognized by the agent can be understood as the following relational structure:

$$\Re = \langle \mathcal{U}, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n \rangle,$$

where $\mathcal{U}$, the *universe of objects of reality* $\Re$, is a nonempty set, and $\mathcal{R}_i$, for $i = 1, 2, \ldots, n$, is the set of i-ary relations on $\mathcal{U}$. One-ary relations are regarded as subsets of $\mathcal{U}$ and understood as properties of objects of $\mathcal{U}$, and multi-argument relations as relationships among its objects. Formally, every k-ary relation of $\mathcal{R}_k$ is a subset of $\mathcal{U}^k$.

We assume that reality $\Re$ is objective in relation to cognition. The objective knowledge about it consists of pieces of unit information (knowledge) about objects of $\mathcal{U}$ in relation to all particular relations of $\mathcal{R}_k$ $(k = 1, 2, \ldots, n)$.

We introduce the notion of knowledge and vague knowledge in accordance with some conceptions of the second author of this paper ([19]).

Definition 1. ***Unit information (knowledge)*** *about the object $o \in \mathcal{U}$ with respect to the relation $R \in \mathcal{R}_k$ $(k = 1, 2, \ldots, n)$ is the image $\overrightarrow{R}(o)$ of the object o with respect to the relation R*[1]*.*

Discovering unit knowledge about objects of reality $\Re$ is realized through asking questions including certain aspects called ***attributes*** of the objects of its universe $\mathcal{U}$. Then, as the universe we usually choose a finite set $U \subseteq \mathcal{U}$ and we put it forward as generalized *attribute-value system* Σ called also an *information system* (cf. Codd [3]; Pawlak [11], [13], [14]; Marek and Pawlak [9]). Its definition is the following:

Definition 2. *Σ is an* ***information system*** *iff it is an ordered system*

$$\Sigma = \langle U, A_1, A_2, \ldots, A_n \rangle,$$

where $U \subseteq \mathcal{U}$, card(U) $< \omega$ and A_k $(k = 1, 2, \ldots, n)$ is the set of k-ary attributes understood as k-ary functions, i.e.

$$\forall_{a \in A_k} a \colon U^k \to V_a,$$

where V_a is the set of all values of the attribute a.

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[1] $\overrightarrow{R}(o) = \begin{cases} R, \text{ if } o \in R, \\ \emptyset, \text{ otherwise.} \end{cases}$ for $R \in \mathcal{R}_1$.

$\overrightarrow{R}(o) = \{\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \rangle : \langle x_1, \ldots, x_{i-1}, o, x_{i+1}, \ldots, x_k \rangle \in R\}$ for $R \in \mathcal{R}_k$ $(k = 2, \ldots, n)$.

Example 1. Let us consider the following information system:

$$\mathbf{S} = \langle S, A_1, A_2 \rangle,$$

where $S = \{s_1, s_2, \ldots, s_5\}$ is a set of 5 scientists and $A_1 = \{$PUBLICATION ACTIVITY (PA), QUOTATIONS (Q)$\}$, $A_2 = \{$SCIENTIFIC COLLABORATION (SC)$\}$. The attribute PA is a function which assigns to every scientist of S a number of papers published by him. We assume that $V_{PA} = \{1, 2, \ldots, 1000\}$. The value of the attribute Q for any scientist of S is the number of quotations of his papers. We assume that $V_Q = \{0, 2, \ldots, 2000\}$. We also assume that $V_{SC} = \{0, 1, 2, 3\}$, where 0 is a value for cases, when arguments of the function SC are the same, and for any different s_n and s_m from S, 1 means that they do not collaborate, 2 means that they collaborate but they have not published any common paper, 3 means that they collaborate and have at least one common paper published.

The information system $\mathbf{S}$ can be clearly presented in the following tables:

	PA	Q
s_1	203	250
s_2	145	245
s_3	198	200
s_4	105	150
s_5	203	245

SC	s_1	s_2	s_3	s_4	s_5
s_1	0	1	3	1	3
s_2	1	0	2	3	1
s_3	3	2	0	2	2
s_4	1	3	2	0	1
s_5	3	1	2	1	0

Every attribute of the information system Σ and every value of this attribute explicitly indicate a relation belonging to the so-called ***relational system determined by*** Σ. The unit information (knowledge) about an object $o \in U$ should be considered with respect to relations of the system.

Definition 3. $\Re(\Sigma)$ *is a* ***system determined by the information system*** Σ *iff*

$$\Re(\Sigma) = \langle U, \{R_{a,W} : a \in A_1, \emptyset \neq W \subseteq V_a\}, \ldots, \{R_{a,W} : a \in A_n, \emptyset \neq W \subseteq V_a\} \rangle,$$

where $R_{a,W} = \{(o_1, o_2, \ldots, o_k) \in U^k : a((o_1, o_2, \ldots, o_k)) \in W\}$ *for any* $k \in \{1, 2, \ldots, n\}$, $a \in A_k$, $\emptyset \neq W \subseteq V_a$.

Let us see that $\bigcup\{R_{a,\{v\}} : a \in A_1, v \in V_a\} = U$, i.e. the family $\{R_{a,\{v\}} : a \in A_1, v \in V_a\}$ is a covering of U.

It is easy to see that

Fact 1. *The system* $\Re(\Sigma)$ *is uniquely determined by the system* Σ.

Example 2. Let $\mathbf{S}$ is the above given information system. Then the system determined by this system is $\Re(\mathbf{S}) = \langle U, R_{A_1}, R_{A_2} \rangle$, where $R_{A_1} = \{R_{PA,S}\}_{\emptyset \neq S \subseteq V_{PA}} \cup \{R_{Q,S}\}_{\emptyset \neq S \subseteq V_Q}$ and $R_{A_2} = \{R_{SC,S}\}_{\emptyset \neq S \subseteq V_{SC}}$.

For any attribute a of system $\mathbf{S}$ and any $i, j \in N$ we can accept the following notation:

$S_i^j = \{v \in V_a : i \leq v \leq j\}$, $S^j = \{v \in V_a : v \leq j\}$, $S_i = \{v \in V_a : i \leq v\}$.

Then, in particular, we can easily state that: $R_{PA,S_{145}^{145}} = R_{PA,\{145\}} = \{s_2\}$, $R_{PA,S_{145}^{200}} = \{s_2, s_3\}$, $R_{PA,S^{210}} = \{s_1, s_2, s_3, s_4, s_5\}$, $R_{Q,S_{150}} = \{s_1, s_2, s_3, s_4, s_5\}$, $R_{Q,S_{200}} = \{s_1, s_2, s_3, s_5\}, R_{Q,S_{245}} = \{s_1, s_2, s_5\}, R_{Q,S_{250}} = \{s_1\}$ and $R_{SC,\{2\}} = \{(s_2, s_3), (s_3, s_2), (s_3, s_4), (s_4, s_3), (s_3, s_5), (s_5, s_3)\}$, $R_{SC,\{0\}} = \{(s_i, s_i)\}_{i=1,\ldots,5}$.

The notion of knowledge about the attributes of the system Σ depends on the cognitive agent discovering the fragment of reality Σ. According to Skowron's understanding a notion, of knowledge determined by any unary attribute (cf. Pawlak [12], Skowron et all [18], Demri, Orlowska [5] pp.16–17), we can accept the following definition of the notion of ***knowledge*** K_a ***about any k-ary attribute a***:

Definition 4. *Let Σ be the information system and $a \in A_k$ $(k = 1, 2, \ldots, n)$. Then*

(a) $K_a = \{((o_1, o_2, \ldots, o_k), V_{a,u}) : u = (o_1, o_2, \ldots, o_k) \in U^k\}$,

where $V_{a,u} \subseteq P(V_a)$, $V_{a,u}$ is the family of all sets of possible values of the attribute a for the object u from the point of view of the agent and $P(V_a)$ is the family of all subsets of V_a.

(b) *The knowledge K_a of the agent about the attribute a and its value for the object u is*

- (0) ***empty*** *if card$(V_{a,u}) = 0$,*
- (1) ***definite*** *if card$(V_{a,u}) = 1$,*
- (> 1) ***imprecise****, in particular* ***vague****, if card$(V_{a,u}) > 1$.*

Let us observe that the vague knowledge about some attribute of the information system Σ is connected with assignation of a ***vague value*** to the object u.

Example 3. Let us consider again the information system **S**. The knowledge K_{PA}, K_Q, K_{SC} of the agent about the attributes from the information system **S** can be characterized by means of the following tables:

	$V_{PA,s}$	$V_{Q,s}$
s_1	$\{S_{150}^{200}, S_{170}^{220}, S^{220}\}$	$\{S_{250}, S_{300}, S_{350}, S_{400}\}$
s_2	$\{S_{100}^{150}, S_{100}^{200}, S_{150}^{180}\}$	$\{S_{200}^{250}, S_{250}^{300}, S_{200}^{300}\}$
s_3	$\{S_{150}^{160}, S_{160}^{170}, S_{170}^{180}, S_{180}^{190}, S_{190}^{200}\}$	$\{S_{150}, S_{170}, S_{190}, S_{210}, S_{230}, S_{250}, S_{300}\}$
s_4	$\{S_{105}^{105}\}$	$\{S_{200}, S_{250}^{500}, S_{400}^{800}, S_{500}\}$
s_5	$\{S_{180}^{220}, S_{200}^{240}\}$	$\{S_{200}^{250}\}$

$V_{SC,(s,s')}$	s_1	s_2	s_3	s_4	s_5
s_1	$\{\{0\}\}$	$\{\{1\},\{2\}\}$	$\{\{3\}\}$	$\{\{1\}\}$	$\{\{3\}\}$
s_2	$\{\{1\},\{2\}\}$	$\{\{0\}\}$	$\{\{2\}\}$	$\{\{1,3\}\}$	$\{\{1\}\}$
s_3	$\{\{3\}\}$	$\{\{2\}\}$	$\{\{0\}\}$	$\{\{2\}\}$	$\{\{1,3\},\{2,3\}\}$
s_4	$\{\{1\}\}$	$\{\{1,3\}\}$	$\{\{2\}\}$	$\{\{0\}\}$	$\{\{1\}\}$
s_5	$\{\{3\}\}$	$\{\{1\}\}$	$\{\{1,3\},\{2,3\}\}$	$\{\{1\}\}$	$\{\{0\}\}$

From Definitions 1 and 3 we get:

Fact 2. *Unit information (knowledge) about the object $o \in U$ with respect to a relation R of the system $\Re(\Sigma)$ is the image $\overrightarrow{R}(o)$ of the object o with respect to the relation R, from the point of view of the agent.*

Contrary to the objective unit knowledge $\overrightarrow{R}(o)$ about the object o of U in the reality $\Re$ with regard to its relation R, the subjective unit knowledge about the object o of U in the reality $\Re(\Sigma)$ depends on an attribute of Σ determining the relation R and its possible values from the point of view of knowledge of the agent discovering $\Re(\Sigma)$. The subjective unit knowledge $\overrightarrow{R}(o)$ from the point of view of the agent depends on his ability to solve the following equation:

$$\overrightarrow{R}(o) = x, \tag{e}$$

where x is an unknown quantity.

Solutions of (e) for k-ary relation R should be images of the object o with respect to k-ary relations $R_{a,W}$ from $\Re(\Sigma)$, where $\emptyset \neq W \in V_{a,o}$. Let us note, that for unary relation R solutions of (e) are unary relations from $\Re(\Sigma)$.

A solution of the equation (e) can be correct (then the agent's knowledge about object o is ***exact***). If the knowledge is ***inexact*** then at least one solution of (e) is not the image of the object o with respect to relation R.

Definition 5. *Unit knowledge about the object $o \in U$ in $\Re(\Sigma)$ with respect to its relation R is*

(0) ***empty*** *iff the equation (e) does not have a solution for the agent (the agent knows nothing about the value of the function $\overrightarrow{R}$ for the object o),*

(1) ***definite*** *iff the equation (e) has exactly one solution for the agent (either the agent's knowledge is exact – the agent knows the value of the function $\overrightarrow{R}$ for the object o – or he accepts only one, but not necessarily proper, value of the function),*

(> 1) ***imprecise*** *iff the equation (e) has at least two solutions for the agent (the agent allows at least two possible values of the function $\overrightarrow{R}$ for the object o).*

From Definitions 4 and 5 it follows that:

Fact 3. *The unit knowledge about the object $o \in U$ in $\Re(\Sigma)$ with respect to its relation R is*

(0) *empty if the knowledge K_a of the agent about the attribute a and its value for the object o is empty,*

(1) *definite if the knowledge K_a of the agent about the attribute a and its value for the object o is definite,*

(> 1) *imprecise if the knowledge K_a of the agent about the attribute a and its value for the object o is imprecise.*

When the unit knowledge of the agent about the object o is imprecise then most often we replace the unknown quantity x in (e) by a vague value.

Example 4. If in the system $\Re(\mathbf{S})$ we consider the relation $R = R_{Q,S_{200}}$, i.e. the set of all scientists of S that have at least 200 quotations of their papers (the property of possessing at least 200 quotations) then the unit knowledge about the scientist s_3 with respect to R can be the following vague information:

$$\overrightarrow{R}(s_3) = NUMEROUS, \tag{e_1}$$

where $NUMEROUS$ is an unknown, indefinite, vague quantity, and the unit information about s_3 with respect to R, from the point of view of the agent, is certainly *imprecise* and *vague* if (e_1) has for him different solutions in different situations. Then the agent points to the scientist s_3 non-uniquely, possibly from his point of view different images $\overrightarrow{R}(s_3)$ of the scientist s_3 with respect to the relation R. Then the equation (e_1) has usually, for him, at least two solutions. On the basis of *Example 3* a solution of (e_1) can be each relation $R_{Q,S_{150}}, R_{Q,S_{170}}, R_{Q,S_{190}}, R_{Q,S_{210}}, R_{Q,S_{230}}, R_{Q,S_{250}}, R_{Q,S_{300}}$. Let us observe that $R_{Q,S_{150}} = \{s_1, s_2, s_3, s_4, s_5\}$, $R_{Q,S_{170}} = R_{Q,S_{190}} = \{s_1, s_2, s_3, s_5\}$, $R_{Q,S_{210}} = R_{Q,S_{230}} = \{s_1, s_2, s_5\}$, $R_{Q,S_{250}} = \{s_1\}$, $R_{Q,S_{300}} = \emptyset$.

3 Vague Sets and Rough Sets

In order to simplify the further considerations, we will limit ourselves to the unary relation R (property) – a subset of U.

Let $\Re(\Sigma)$ be the system determined by the information system Σ, R be its unary relation and $o \in U$.

Definition 6. *The* ***unit knowledge*** *about the object o in $\Re(\Sigma)$ with respect to R is* ***inexact*** *iff the equation (e) has the form:*

$$\overrightarrow{R}(o) = X, \tag{ine}$$

where X is an unknown quantity from the point of view of the agent, and (ine) has for him at least one solution and at least one of the solutions is not the image $\overrightarrow{R}(o)$.

The equation (ine) can be called the *equation of inexact knowledge of the agent.* All solutions of (ine) are unary relations in the system $\Re(\Sigma)$.

Definition 7. *The* ***unit knowledge*** *about the object o in $\Re(\Sigma)$ with respect to R is* ***vague*** *iff the equation (ine) has the form:*

$$\overrightarrow{R}(o) = VAGUE, \tag{ve}$$

where $VAGUE$ is an unknown quantity and (ve) has at least two different solutions for the agent.

The equation (ve) can be called the *equation of vague knowledge of the agent.*

Fact 4. *Vague unit knowledge is a particular case of inexact unit knowledge.*

Definition 8. *The family of all solutions (sets) of* (*ine*)*, respectively* (*ve*)*, such that at least one of them includes* R*, is called the* ***vague set for the object o approximated by*** R*, respectively the* ***proper vague set for the object o approximated by*** R*.*

Example 5. The family of all solutions of (e_1) from *Example 4* is a vague set $\mathbf{V}_{s_3}$ for the scientist s_3 approximated by $R_{Q,S_{200}}$ and $\mathbf{V}_{s_3} = \{R_{Q,S_{150}}, R_{Q,S_{170}}, R_{Q,S_{190}}, R_{Q,S_{210}}, R_{Q,S_{230}}, R_{Q,S_{250}}, R_{Q,S_{300}}\}$.

Vague sets, so also proper vague sets, determined by a set R are here some generalizations of sets approximated by representations (see Bonikowski [2]). They are non-empty families of unary relations from $\Re(\Sigma)$ (such that at least one of them includes R) and subfamilies of the family $P(U)$ of all subsets of the set U, determined by the set R. They have the greatest lower bound (the *lower limit*) and the least upper bound (the *upper limit*) in $P(U)$ with respect to inclusion. We will denote the greatest lower bound of any family $\mathbf{X}$ by $\underline{\mathbf{X}}$. The least upper bound of $\mathbf{X}$ will be denoted by $\overline{\mathbf{X}}$. So, we can note

Fact 5. *For each vague set* $\mathbf{V}$ *approximated by the set (property)* R

$$\mathbf{V} \subseteq \{Y \in P(U) : \underline{\mathbf{V}} \subseteq Y \subseteq \overline{\mathbf{V}}\}.$$

The idea of vague sets was conceived upon the idea of Pawlak's rough sets [14], who defined them by means of the operations of the *lower approximation*: $\underline{\ \ }$, and the upper approximation: $\overline{\ \ }$, defined on subsets of U. The lower approximation of a set is defined as a union of indiscernibility classes of a given relation in U^2, which are included in this set, whereas the upper approximation of a set is defined as a union of the indiscernibility classes of the relation, which have non-empty intersection with this set.

Definition 9. *A* ***rough set*** *determined by a set* $R \subseteq U$ *is a family* $\mathbf{P}$ *of all sets satisfying the following condition:*

$$\mathbf{P} = \{Y \in P(U) : \underline{Y} = \underline{R} \wedge \overline{Y} = \overline{R}\}^2.$$

Let us observe that because $R \subseteq R \in \mathbf{P}$, the family $\mathbf{P}$ is a non-empty family of sets such that at least one of them includes R (cf. Definition 8). By analogy to Fact 5 we have

Fact 6. *For each rough set* $\mathbf{P}$ *determined by the set (property)* R

$$\mathbf{P} \subseteq \{Y \in P(U) : \underline{R} \subseteq Y \subseteq \overline{R}\}.$$

It is obvious that

Fact 7. *If* $\mathbf{V}$ *is a vague set and* $\underline{X} = \underline{\mathbf{V}}$ *and* $\overline{X} = \overline{\mathbf{V}}$ *for any* $X \in \mathbf{V}$*, then* $\mathbf{V}$ *is a subset of a rough set determined by any set of* $\mathbf{V}$*.*

[2] Some authors define a rough set as a pair of sets (lower approximation, upper approximation)(cf. e.g. Iwiński [7], Pagliani [10]).

For every rough set $\mathbf{P}$ determined by R we have: $\underline{\mathbf{P}} = \underline{R}$ and $\overline{\mathbf{P}} = \overline{R}$. So we can consider the following generalization of the notion of the rough set:

Definition 10. *A non-empty family* $\mathbf{G}$ *of subsets of* U *is called a* ***generalized rough set*** *determined by a set* R *iff it satisfies the following condition:*

$$\underline{\mathbf{G}} = \underline{R} \text{ and } \overline{\mathbf{G}} = \overline{R}.$$

It is easily seen that

Fact 8. *Every rough set determined by a set* R *is a generalized rough set determined by* R.

Fact 9. *If* $\mathbf{V}$ *is a vague set and there exists a set* $X \subseteq U$ *such, that* $\underline{X} = \underline{\mathbf{V}}$ *and* $\overline{X} = \overline{\mathbf{V}}$, *then* $\mathbf{V}$ *is a generalized rough set determined by the set* X.

4 Operations on Vague Sets

Let us denote by $\mathcal{V}$ the family of all vague sets approximated by relations in system $\Re(\Sigma)$. In the family $\mathcal{V}$ we can define an operation of negation $\neg$ on vague sets, a union operation $\oplus$ and an intersection operation $\odot$ on any two vague sets.

Definition 11. *Let* $\mathbf{V_1} = \{R_i\}_{i \in I}$ *and* $\mathbf{V_2} = \{R_j\}_{j \in J}$ *be vague sets determined by sets* $R \subseteq U$ *and* $S \subseteq U$, *respectively.*

(a) $\mathbf{V_1} \oplus \mathbf{V_2} = \{R_i\}_{i \in I} \oplus \{R_j\}_{j \in J} = \{R_i \cup R_j\}_{i \in I, j \in J}$,
(b) $\mathbf{V_1} \odot \mathbf{V_2} = \{R_i\}_{i \in I} \odot \{R_j\}_{j \in J} = \{R_i \cap R_j\}_{i \in I, j \in J}$,
(c) $\neg \mathbf{V_1} = \neg \{R_i\}_{i \in I} = \{U \setminus R_i\}_{i \in I}$.

The family $\mathbf{V_1} \oplus \mathbf{V_2}$ *is called the union of vague sets* $\mathbf{V_1}$ *and* $\mathbf{V_2}$ *determined by relation* $R \cup S$, *the family* $\mathbf{V_1} \odot \mathbf{V_2}$ *is called the intersection of vague sets* $\mathbf{V_1}$ *and* $\mathbf{V_2}$ *determined by relation* $R \cap S$ *and the family* $\neg \mathbf{V_1}$ *is called the negation of vague set* $\mathbf{V_1}$ *determined by relation* $U \setminus R$.

Theorem 1. *Let* $\mathbf{V_1} = \{R_i\}_{i \in I}$ *and* $\mathbf{V_2} = \{R_j\}_{j \in J}$ *be vague sets determined by sets* R *and* S, *respectively.*

(a) $\underline{\mathbf{V_1} \oplus \mathbf{V_2}} = \underline{\mathbf{V_1}} \cup \underline{\mathbf{V_2}} = \bigcap \{R_i \cup R_j\}_{i \in I, j \in J}$ *and*
$\overline{\mathbf{V_1} \oplus \mathbf{V_2}} = \overline{\mathbf{V_1}} \cup \overline{\mathbf{V_2}} = \bigcup \{R_i \cup R_j\}_{i \in I, j \in J}$,
(b) $\underline{\mathbf{V_1} \odot \mathbf{V_2}} = \underline{\mathbf{V_1}} \cap \underline{\mathbf{V_2}} = \bigcap \{R_i \cap R_j\}_{i \in I, j \in J}$ *and*
$\overline{\mathbf{V_1} \odot \mathbf{V_2}} = \overline{\mathbf{V_1}} \cap \overline{\mathbf{V_2}} = \bigcup \{R_i \cap R_j\}_{i \in I, j \in J}$,
(c) $\underline{\neg \mathbf{V_1}} = U \setminus \overline{\mathbf{V_1}}$ *and* $\overline{\neg \mathbf{V_1}} = U \setminus \underline{\mathbf{V_1}}$.

Theorem 2. *The structure* $\mathfrak{B} = (\mathcal{V}, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1})$ *is a Boolean algebra, where* $\mathbf{0} = \{\emptyset\}$ *and* $\mathbf{1} = \{U\}$.

We can easily observe that the above-defined operations on vague sets differ from Zadeh's operations on fuzzy sets, from standard operations in any field of sets and, in particular, also from operations on rough sets defined in papers of

Pomykala [16] and Bonikowski [1]. In the last cases the family of all rough sets with operations defined in these papers is Stone algebra.

5 On Logic of Vague Terms

How to solve the problem of logic of vague terms, logic of vague sentences (*vague logic*) based on the vague sets characterized in the previous sections? An answer to this question requires describing briefly the problem of language representation of unit knowledge.

On the basis of our examples let us consider two pieces of unit information about the scientist s_3, with respect to the set R of all scientists that have at least 200 quotations of their papers:

first, exact unit knowledge

$$\overrightarrow{R}(s_3) = \{s_1, s_2, s_3, s_5\}, \tag{ee}$$

next, vague unit knowledge:

$$\overrightarrow{R}(s_3) = \mathit{NUMEROUS}. \tag{e_1}$$

Let s_3 be the designator of the proper name a, R – denotation (extension) of the name-predicate P ('*a scientist who has at least 200 quotations of his papers*') and the vague name-predicate V ('*a scientist who has numerous quotations of his papers*') be a language representation of the vague quantity *NUMEROUS*. Then a representation of the first equation (ee) is the logical atomic sentence

$$a \text{ is } P \tag{re}$$

and a representation of the second equation (e_1) is the vague sentence

$$a \text{ is } V. \tag{re_1}$$

In an equivalent way we can represent, respectively, (ee) and (e_1) by means of a logical atomic sentence:

$$aP \text{ or } P(a), \tag{re'}$$

where P is the predicate (*'has at least 200 quotations of his papers'*) and by means of a vague sentence

$$aV \text{ or } V(a), \tag{re'_1}$$

where V is the vague predicate (*'has numerous quotations of his papers'*).

The sentence (re_1) (res. the sentence (re'_1)) is not a logical sentence, but it can be treated as a *sentential form*, which represents all logical sentences, in particular the sentence (re) (respectively sentence (re')) that arises by replacing the vague name-predicate (res. vague predicate) V by allowable sharp name-predicates (res. sharp predicates), whose denotations (extensions) constitute the vague set $\mathbf{V}_{s_3}$ that is the denotation of V and simultaneously the set of solutions the equation (e_1) from the agent's point of view.

By analogy we can consider every atomic vague sentence with the form $V(a)$, where a is an individual term and V — its vague predicate, as a *sentential form with V as a vague variable*, run over all denotations of sharp predicates that can be substituted for V in order to get precise, true or false, logical sentences from the form $V(a)$. Then, the scope of the variable V is the vague set $\mathbf{V}_o$ determined by the designator o of the term a.

All the above remarks lead to a 'conservative', classical approach in searching for logic of vague terms or vague sentences, called here *vague logic* (cf. Fine [6], Cresswell [4]). It is easy to see that all counterparts of laws of classical logic are laws of *vague logic*, even if for the fact that vague sentences have an interpretation in Boolean algebra $\mathfrak{B}$ of vague sets (see Theorem 2).

It should be noticed that sentential connectives for *vague logic* should not satisfy *standard conditions* (see Malinowski [8]). For example, an alternative of two vague sentences $V(a)$ and $V(b)$ can be a 'true' vague sentence (sentential form) despite the fact that its arguments $V(a)$ and $V(b)$ are neither 'true' or 'false' sentential form, i.e. they represent in certain cases true and in other cases false sentences. It is not contrary to the statement that all vague sentential forms which we obtain by suitable substitution of sentential variables (resp. predicate variables) by vague sentences (resp. vague predicates) in laws of classical logic always represent true sentences. Thus they are laws of vague logic.

6 Final Remarks

1. The concept of vagueness was defined here as a certain indefinite, vague quantity or property corresponding to the agent knowledge discovering a fragment of reality. It was given by means of the *equation of inexact knowledge of the agent*. A vague set was defined as a set (a family) of all possible solutions (sets) of this equation and although our considerations were limited to the case of unary relations, they can easily be generalized to the cases of any k-ary relations.
2. The idea of *vague sets* was taken here from the idea of rough sets originating from Zdzisław Pawlak, because Pawlak's theory of rough sets takes a non-numerical, qualitative approach, to the issue vagueness, as opposed to the quantitative characteristics of vagueness phenomenon by Lotfi Zadeh.
3. Vague sets, like rough sets, are based on the idea of a set approximation by two sets called the lower and the upper limits of this set. These two kinds of sets are families of sets approximated by suitable limits.
4. Pawlak's approach and the approach discussed in this paper are connected with a reference to the concept of a cognitive agent's knowledge about the objects of the investigated reality (see Pawlak [15]) This knowledge is determined by the system of concepts, that is determined by a system of their extensions (denotations). When the concept is vague, its denotation, in Pawlak's sense, is a rough set, while in the authors' sense – a vague set which at some conditions is a subset of the rough set.
5. In language representation the *equation of inexact, vague knowledge of the agent* can be expressed by means of vague sentences containing a vague predicate. Its denotation (extension) is a family of all scopes of sharp predicates which can be substituted for the vague predicate from the point of view of the agent. The denotation is simultaneously the vague set of all solutions of the equation of the vague agent's knowledge.

6. Because vague sentences can be treated as sentential forms in which variables are vague predicates, all counterparts of tautologies of classical logic are laws of *vague logic* (logic of vague sentences).
7. *Vague logic* is based on classical logic but it is many-valued logic, because its sentential connectives are intensional.

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